CHAPTER XX.

RECTIFICATION OF TWISTED CURVES.

706. Let PQ be any elementary arc δs of the curve. Let the coordinates of P and Q be respectively

$$(x, y, z) \quad \text{and} \quad (x+\delta x, y+\delta y, z+\delta z)$$

with regard to any three fixed rectangular axes Ox, Oy, Oz. Then

$$(\text{chord } PQ)^2 = \delta x^2 + \delta y^2 + \delta z^2.$$

Now, if Q be made to travel along the curve so as to approach indefinitely near to P, the chord PQ and the arc PQ ultimately differ by an infinitesimal of higher order than the arc PQ itself, *i.e.* the chord PQ and the arc PQ ultimately vanish in a ratio of equality.* Hence we have to the second order of small quantities,

$$\delta s^2 = \delta x^2 + \delta y^2 + \delta z^2. \quad \dots\dots\dots\dots\dots\dots\dots(1)$$

Now suppose the curve to be specified in one of the two usual ways,

(*a*) as the line of intersection of two specified surfaces

$$f(x, y, z) = 0, \quad F(x, y, z) = 0,$$

or (*b*) the coordinates of any point x, y, z upon it expressed in terms of some fourth variable t, and defined by the equations $\quad x = f_1(t), \quad y = f_2(t), \quad z = f_3(t).$

The First Case.

In Case (*a*) choice must be made of one of the three variables x, y, z to be considered as the independent variable, say x, and the equations $f=0$, $F=0$ are then to be solved to find

*For a discussion of this point see De Morgan, *Differential and Integral Calculus*, p. 445. See also *Diff. Calc.*, Art. 34, for a plane curve.

the other two, y and z, in terms of x. Then differentiating, we express $\frac{dy}{dx}$ and $\frac{dz}{dx}$ in terms of x; say

$$\frac{dy}{dx}=\phi(x), \quad \frac{dz}{dx}=\psi(x).$$

We then have
$$s=\int\sqrt{dx^2+dy^2+dz^2}$$
$$=\int\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2\right\}}\,dx$$
$$=\int\sqrt{[1+\{\phi(x)\}^2+\{\psi(x)\}^2]}\,dx.$$

And when the integration has been effected, the length of the arc between the points specified by any particular limits which may be assigned to x, will have been obtained.

707. A more Symmetrical Mode of Procedure.

We might also proceed as follows:

Along the line of intersection of $f=0$ and $F=0$ we have

$$f_x\,dx+f_y\,dy+f_z\,dz=0$$

and
$$F_x\,dx+F_y\,dy+F_z\,dz=0,$$

giving
$$\frac{dx}{J_1}=\frac{dy}{J_2}=\frac{dz}{J_3}=\frac{ds}{\sqrt{J_1^2+J_2^2+J_3^2}}=\frac{ds}{\sqrt{\phi}}, \text{ say,}$$

J_1, J_2, J_3 being the Jacobians

$$\begin{vmatrix} f_y, & f_z \\ F_y, & F_z \end{vmatrix}, \quad \begin{vmatrix} f_z, & f_x \\ F_z, & F_x \end{vmatrix}, \quad \begin{vmatrix} f_x, & f_y \\ F_x, & F_y \end{vmatrix},$$

i.e.
$$\frac{\partial(f, F)}{\partial(y, z)}, \quad \frac{\partial(f, F)}{\partial(z, x)}, \quad \frac{\partial(f, F)}{\partial(x, y)}.$$

Then
$$s=\int\sqrt{\phi}\frac{dx}{J_1} \text{ or } \int\sqrt{\phi}\frac{dy}{J_2} \text{ or } \int\sqrt{\phi}\frac{dz}{J_3},$$

making use of the one which is most convenient; and whichever is used, both the dependent variables occurring must be expressed in terms of the independent one before integration.

708. The Second Case.

In Case (b) we have

$$x=f_1(t), \quad y=f_2(t), \quad z=f_3(t)$$

and
$$\frac{dx}{dt}=f_1'(t), \quad \frac{dy}{dt}=f_2'(t), \quad \frac{dz}{dt}=f_3'(t),$$

whence
$$s=\int\sqrt{[\{f_1'(t)\}^2+\{f_2'(t)\}^2+\{f_3'(t)\}^2]}\,dt;$$

and we obtain the arc by integration, as before, between any two points corresponding to the limits assigned for the variable t.

709. If the equations of the curve be presented in the form

$$\frac{x}{f_1(t)}=\frac{y}{f_2(t)}=\frac{z}{f_3(t)}=\frac{1}{f(t)},$$

we have $$\frac{dx}{dt}=\frac{f_1'(t)f(t)-f_1(t)f'(t)}{\{f(t)\}^2}=\frac{J_1}{f^2}, \text{ say.}$$

Similarly $$\frac{dy}{dt}=\frac{J_2}{f^2},\ \frac{dz}{dt}=\frac{J_3}{f^2},$$

where J_2 and J_3 have meanings corresponding to J_1.

Hence $$\frac{dx}{J_1}=\frac{dy}{J_2}=\frac{dz}{J_3}=\frac{dt}{f^2}=\frac{ds}{\sqrt{\phi}},$$

where $$\phi=J_1^2+J_2^2+J_3^2.$$

Hence $$s=\int\frac{\sqrt{J_1^2+J_2^2+J_3^2}}{f^2}\,dt=\int\frac{\sqrt{\phi}}{f^2}\,dt.$$

710. The rectification of a curve therefore depends upon the possibility of performing the integration $\int\frac{\sqrt{\phi}}{f^2}\,dt$.

When f_1, f_2, f_3, f are rational integral and algebraic functions of t, we have the case of a unicursal twisted curve.

The advanced student is referred to the very important memoir by Mr. R. A. Roberts, "On the Rectification of Certain Curves," in vol. xviii. of the *Proceedings of the London Mathematical Society*, which has already been referred to in other places.

711. Ex. 1. Find the length of an arc of the curve which is the line of intersection of the parabolic cylinder $y^2=4ax$ and the cylinder

$$z=\sqrt{x(x-a)}-\frac{a}{2}\cosh^{-1}\frac{2x-a}{a}.$$

Here we take x as the independent variable and obtain

$$\frac{dy}{dx}=\sqrt{\frac{a}{x}},$$

$$\frac{dz}{dx}=\frac{2x-a}{2\sqrt{x(x-a)}}-\frac{1}{\sqrt{\left(\frac{2x-a}{a}\right)^2-1}}=\frac{2x-a}{2\sqrt{x(x-a)}}-\frac{a}{2\sqrt{x(x-a)}}=\sqrt{\frac{x-a}{x}};$$

$$\therefore \left(\frac{ds}{dx}\right)^2 = 1 + \frac{a}{x} + 1 - \frac{a}{x} = 2;$$

$$\therefore s = \sqrt{2}\int_{x_1}^{x_2} dx = \sqrt{2}(x_2 - x_1),$$

where x_1 and x_2 are the lower and upper limits of integration.

Hence, in this curve any portion of the arc is $\sqrt{2}$ times its projection upon the x-axis. In other words, at every point of this curve the tangent makes an angle of $\frac{\pi}{4}$ with the x-axis.

Taking the same curve, let us put

$$x = \frac{a}{2}(1 + \cosh u), \quad i.e.\ a\cosh^2\frac{u}{2}.$$

Then $$y = 2a\cosh\frac{u}{2}, \quad z = \frac{a}{2}(\sinh u - u);$$

we then have a case such as that discussed in (*b*) of the preceding article, having expressed x, y and z in terms of an auxiliary fourth variable u.

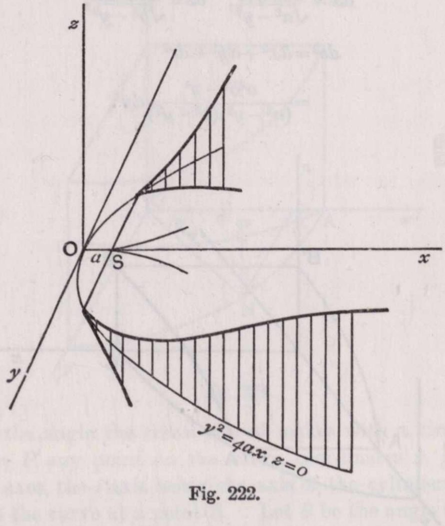

Fig. 222.

Then $$\frac{dx}{du} = \frac{a}{2}\sinh u, \quad \frac{dy}{du} = a\sinh\frac{u}{2}, \quad \frac{dz}{du} = \frac{a}{2}(\cosh u - 1).$$

Therefore $$\left(\frac{ds}{du}\right)^2 = \frac{a^2}{4}\left[\sinh^2 u + 4\sinh^2\frac{u}{2} + (\cosh u - 1)^2\right]$$

$$= \frac{a^2}{4}[\sinh^2 u + 2(\cosh u - 1) + (\cosh u - 1)^2]$$

$$= \frac{a^2}{2}\sinh^2 u;$$

whence $$s = \left[\frac{a}{\sqrt{2}} \cosh u + C\right]$$

$$= \frac{a}{\sqrt{2}}\left[\frac{2x}{a} - 1\right]_{x_1}^{x_2} = \sqrt{2}(x_2 - x_1) \text{ as before.}$$

The curve of intersection of the two cylinders is represented in Fig. 222.

Ex. 2. To find an expression in the form of an integral for the rectification of the line of intersection of two right circular cylinders whose axes intersect at right angles.

If we take the axes of the cylinders as the axes of z and x respectively, we may write the equations of the cylinders as

$$x^2 + y^2 = a^2, \quad y^2 + z^2 = b^2.$$

Let us take $a > b$.

From the equations

$$x = \sqrt{a^2 - y^2}, \quad z = \sqrt{b^2 - y^2},$$

we have $$dx = \frac{-y\,dy}{\sqrt{a^2 - y^2}}, \quad dz = \frac{-y\,dy}{\sqrt{b^2 - y^2}},$$

and $$ds^2 = dx^2 + dy^2 + dz^2$$

$$= \frac{a^2 b^2 - y^4}{(a^2 - y^2)(b^2 - y^2)}\,dy^2.$$

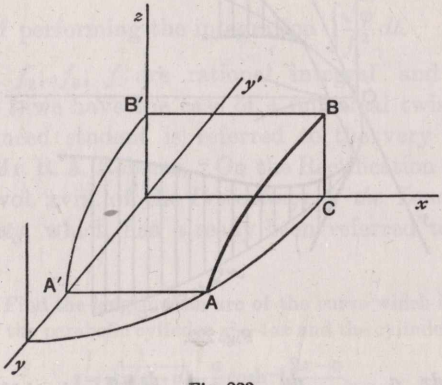

Fig. 223.

Put $y = b \sin\theta$, and let $b = ka$, $k < 1$.

Then $$ds^2 = \frac{k^2 a^4 - k^4 a^4 \sin^4\theta}{(a^2 - k^2 a^2 \sin^2\theta)}\,d\theta^2,$$

$$s = b\int_0^\theta \frac{\sqrt{1 - k^2 \sin^4\theta}}{\sqrt{1 - k^2 \sin^2\theta}}\,d\theta.$$

When the cylinders are of equal radius, $k=1$, and this becomes

$$s=b\int\sqrt{1+\sin^2\theta}\,d\theta$$
$$=\int\sqrt{(b\sqrt{2})^2\sin^2\theta+b^2\cos^2\theta}\,d\theta,$$

i.e. the result of Art. 573, for an ellipse whose axes are in the ratio $\sqrt{2}:1$, to which the curve of intersection then reduces.

It is interesting in this connexion to note more generally that when the axes of two equal cylinders cut at right angles, and a sphere rolls completely round in contact with both cylinders, the locus of its centre is two ellipses. In our case the rolling sphere has a zero radius.

712. In the **"right circular Helix"** or **"Helicoidal curve,"** which is an ordinary thread on a screw, we have a curve traced on a right circular cylinder and cutting all the generators of the cylinder at the same angle.

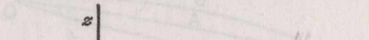

Fig. 224.

Let α be the angle the screw-thread makes with a circular section of the cylinder, P any point on the curve, coordinates x, y, z referred to rectangular axes, the z-axis being the axis of the cylinder and the x-axis taken to cut the curve at a point A. Let θ be the angle the plane OPN through P and the axis makes with the plane of xz, and let a be the radius of the cylinder.

We have $\quad x=a\cos\theta,\quad y=a\sin\theta,\quad z=a\theta\tan\alpha.$

Hence $\quad ds^2=dx^2+dy^2+dz^2=a^2\sec^2\alpha\,d\theta^2$

and $\quad s=a\theta\sec\alpha.$

This is obvious from the fact that in this case the surface may be developed into a plane, and the triangle ANP becomes a right-angled triangle with sides $a\theta$, $a\theta\tan\alpha$ and s, with one of its acute angles α.

713. Since the curve develops into a straight line when the surface is developed into a plane, the surface itself being supposed entirely inextensible, the distance between any two points which it connects upon the cylinder is a minimum distance on the cylinder between those two points. Such lines of minimum length on any surface are termed **Geodesics** (see Smith's *Solid Geom.*, Art. 259).

Hence geodesic lines on a right circular cylinder are helices.

714. A Property of Geodesic Lines.

It is an obvious property of such curves that if P, Q be any points upon a geodesic line upon any surface, the path from P to Q *via* this line being less than from P to Q *via* any contiguous supposititious paths from P to Q, viz. PBQ, or PCQ, on opposite sides of it and of the same length, and the three

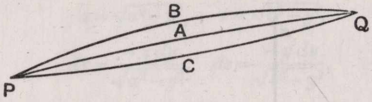

Fig. 225.

lengths PAQ the geodesic, and PBQ, PCQ the supposititious paths being unaltered in length by any deformation of the surface on which they are drawn, supposed inextensible, the deformed path to which PAQ is changed will still be in length intermediate between the lengths of the contiguous paths to which PBQ and PCQ are changed and which are equal. Hence, in the limit when PBQ and PCQ and their deformed lengths are made to close up to ultimate coincidence with PAQ and its deformed length, it will be clear that the deformed PAQ is still a line of minimum length on the deformed surface, being entrapped between two supposititious paths which are both of greater length on opposite sides of it. Thus geodesics on inextensible surfaces remain geodesics after any deformation of the surface on which they are drawn.

715. It follows that a right circular helix remains a right circular helix if the paper on which it is drawn be transferred from the cylinder upon which it was wrapped to a cylinder of different radius. Let a and b be the radii of the first and second cylinders and β the angle the new helix makes with the circular section. Then $s=\dfrac{a}{\cos\alpha}\theta=\dfrac{b}{\cos\beta}\theta'$, where θ' is the angle in the new helix corresponding to θ in the original one;

$$\therefore\ \theta'=\frac{a\cos\beta}{b\cos\alpha}\theta,$$

and the new coordinates of P can be written down, the axes being placed as described for the first helix.

716. Cylindrical Coordinates.

For many cases, particularly for curves drawn upon cylinders, it is desirable to use cylindrical coordinates, viz. r, θ, z, *i.e.* the ordinary Cartesians are transformed to the polar system as regards the x, y plane, and the z-coordinate is left unaltered.

Taking r, θ, z and $r+\delta r, \theta+\delta\theta, z+\delta z$ as the coordinates of contiguous points P, Q on a curve, we have, since δr, $r\,\delta\theta$, δz are mutually perpendicular elements,

$$PQ^2=\delta r^2+(r\,\delta\theta)^2+\delta z^2.$$

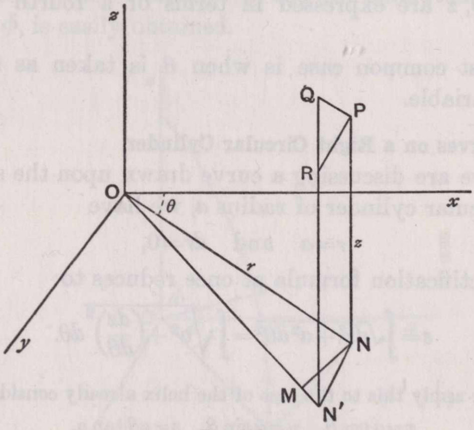

Fig. 226.

For if N, N' be the feet of the perpendiculars from P, Q upon the plane of x-y, we have, to the second order,

$$NN'^2=(r\,\delta\theta)^2+\delta r^2,$$

and plainly $$PQ^2=NN'^2+\delta z^2.$$

Hence, if the distance measured along the arc PQ be δs, we have, to the second order,

$$\delta s^2=\delta r^2+(r\,\delta\theta)^2+\delta z^2,$$

whence $$s=\int\sqrt{dr^2+(r\,d\theta)^2+dz^2},$$

which we may write in any of the forms

$$s=\int\sqrt{\left[\left(\frac{dr}{d\theta}\right)^2+r^2+\left(\frac{dz}{d\theta}\right)^2\right]}\,d\theta,$$

$$s=\int\sqrt{\left[1+r^2\left(\frac{d\theta}{dr}\right)^2+\left(\frac{dz}{dr}\right)^2\right]}\,dr$$

or
$$s=\int\sqrt{\left[\left(\frac{dr}{dz}\right)^2+r^2\left(\frac{d\theta}{dz}\right)^2+1\right]}\,dz,$$

according as it is convenient to take θ, r or z as the independent variable; or we may also write it, as in Cartesians, as

$$s=\int\sqrt{\left[\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]}\,dt,$$

in case r, θ, z are expressed in terms of a fourth auxiliary variable t.

The most common case is when θ is taken as the independent variable.

717. Curves on a Right Circular Cylinder.

When we are discussing a curve drawn upon the surface of a right circular cylinder of radius a, we have

$$r=a \quad \text{and} \quad dr=0,$$

and the rectification formula at once reduces to

$$s=\int\sqrt{dz^2+a^2\,d\theta^2}=\int\sqrt{a^2+\left(\frac{dz}{d\theta}\right)^2}\,d\theta.$$

718. If we apply this to the case of the helix already considered, viz.

$$x=a\cos\theta, \quad y=a\sin\theta, \quad z=a\theta\tan\alpha,$$

we have
$$r=a, \quad z=a\theta\tan\alpha,$$

$$s=\int a\sqrt{1+\tan^2\alpha}\,d\theta=a\theta\sec\alpha, \quad \text{as before (Art. 712).}$$

It will be at once remarked, however, that in all cases of curves drawn upon a right circular cylinder, the length of the arc may as readily be considered by first developing the cylindrical surface into a plane, and in fact the formula above is merely the Cartesian formula

$$\int\sqrt{dz^2+dx^2}$$

for the developed surface, dx replacing $a\,d\theta$.

719. Ex. Find the length of an arc of the curve of intersection of the cylinders

$$x^2+y^2=a^2, \quad xe^{\frac{z}{a}}=a.$$

Putting $x=a\cos\theta$, we have

$$y=a\sin\theta \quad \text{and} \quad z=a\log\sec\theta.$$

Hence $$\frac{dz}{d\theta}=a\tan\theta \quad \text{and} \quad \frac{ds}{d\theta}=a\sec\theta;$$

whence $$s=a\,\mathrm{gd}^{-1}\theta \quad \text{or} \quad s=a\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

In this case the developed curve is the Catenary of Equal Strength, viz. $\zeta=a\log\sec\frac{\xi}{a}$, in which $\xi=a\psi$ and $s=a\,\mathrm{gd}^{-1}\psi$ (see Ex. 5, Art. 519).

720. General Polar Formulae.

The general polar formula for rectification in terms of the radius vector r, the co-latitude θ, and the azimuthal angle, or longitude, ϕ, is easily obtained.

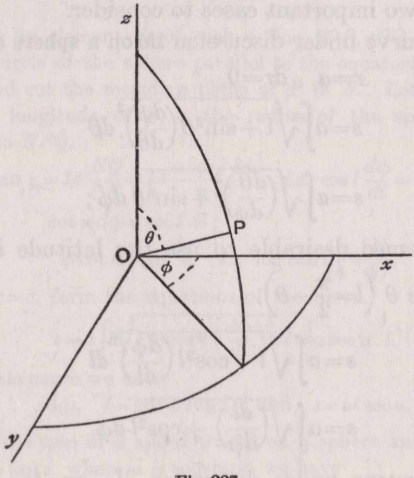

Fig. 227.

In passing from the point $P(r, \theta, \phi)$ to a contiguous point $Q(r+\delta r, \theta+\delta\theta, \phi+\delta\phi)$ along an elementary arc δs of a curve, the projections of the chord PQ in the three directions,

(a) along the radius vector, increasing r;

(b) in the meridian plane, increasing θ;

(c) perpendicular to the meridian plane, increasing ϕ,

are respectively $\quad \delta r, \quad r\,\delta\theta, \quad r\sin\theta\,\delta\phi;$

and these being mutually perpendicular elements we have, to the second order,

$$\delta s^2 = \delta r^2 + r^2\,\delta\theta^2 + r^2 \sin^2\theta\,\delta\phi^2,$$

and as either r, θ, ϕ or a fourth variable t can be regarded as the independent variable to suit circumstances, we have

$$s=\int\sqrt{\left[1+r^2\left(\frac{d\theta}{dr}\right)^2+r^2\sin^2\theta\left(\frac{d\phi}{dr}\right)^2\right]}\,dr,$$

or $$s=\int\sqrt{\left[\left(\frac{dr}{d\theta}\right)^2+r^2+r^2\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right]}\,d\theta,$$

or $$s=\int\sqrt{\left[\left(\frac{dr}{d\phi}\right)^2+r^2\left(\frac{d\theta}{d\phi}\right)^2+r^2\sin^2\theta\right]}\,d\phi,$$

or $$s=\int\sqrt{\left[\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2+r^2\sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right]}\,dt.$$

721. **Modification for Curves on the Sphere and the Cylinder.** There are two important cases to consider.

(1) If the curve under discussion **lie on a sphere of radius** a,

$$r=a, \quad dr=0,$$

and $$s=a\int\sqrt{1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}\,d\theta$$

or $$s=a\int\sqrt{\left(\frac{d\theta}{d\phi}\right)^2+\sin^2\theta}\,d\phi;$$

or if it be deemed desirable to use the latitude l instead of the co-latitude θ $\left(l=\frac{\pi}{2}-\theta\right)$,

$$s=a\int\sqrt{1+\cos^2 l\left(\frac{d\phi}{dl}\right)^2}\,dl$$

or $$s=a\int\sqrt{\left(\frac{dl}{d\phi}\right)^2+\cos^2 l}\,d\phi.$$

(2) If the curve under discussion lie on the surface of **a right circular** cone whose semivertical angle is a, and whose axis is the z-axis and vertex the origin, we have

$$\theta=a, \quad d\theta=0,$$

$$s=\int\sqrt{1+r^2\sin^2 a\left(\frac{d\phi}{dr}\right)^2}\,dr$$

or $$s=\int\sqrt{\left(\frac{dr}{d\phi}\right)^2+r^2\sin^2 a}\,d\phi.$$

722. Ex. 1. "**Rhumb**" **Line** or "**Loxodrome**" on a sphere.

This is a curve on the surface of a sphere which cuts all the meridians at the same angle.

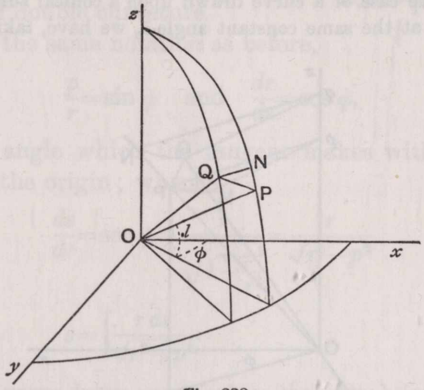

Fig. 228.

Let PQ be an element ds of such a line, zOP, zOQ meridian planes. Let a small circle of the sphere parallel to the equatorial plane x-y pass through Q and cut the meridian plane of P in N. Let l and ϕ be the latitude and longitude of P, a the radius of the sphere and α the constant angle NPQ.

Then $\tan\alpha = Lt\,\dfrac{NQ}{PN} = Lt\,\dfrac{a\cos l\,\delta\phi}{a\,\delta l}$, *i.e.* $\cos l\,\dfrac{d\phi}{dl} = \tan\alpha$,

or $\cot\alpha\,d\phi = \sec l\,dl$;

whence $\phi\cot\alpha = \mathrm{gd}^{-1}l$, *i.e.* $\log\tan\left(\dfrac{\pi}{4}+\dfrac{l}{2}\right)$,

which, with $r=a$, form the equations of the curve.

Also $$s = a\int\sqrt{1+\cos^2 l\left(\frac{d\phi}{dl}\right)^2}\,dl = a\sec\alpha\,.\,l.$$

Hence in this curve we have
$$r=a,\quad l=\mathrm{gd}(\phi\cot\alpha)\quad\text{and}\quad s=al\sec\alpha.$$

Ex. 2. In the case of a spiral traced on a sphere and defined by the equation $l=\phi\tan\alpha$, where α is constant, we have
$$\begin{aligned} s &= a\int\sqrt{1+\cos^2 l\left(\frac{d\phi}{dl}\right)^2}\,dl \\ &= a\int\sqrt{1+\cot^2\alpha\cos^2 l}\,dl \\ &= a\int\sqrt{\operatorname{cosec}^2\alpha-\cot^2\alpha\sin^2 l}\,dl \\ &= a\operatorname{cosec}\alpha\int\sqrt{1-\cos^2\alpha\sin^2 l}\,dl \\ &= a\operatorname{cosec}\alpha\,E(l,\cos\alpha), \end{aligned}$$

and the arc of this spiral is therefore expressible as an arc of an ellipse of semi-major axis $a \operatorname{cosec} \alpha$ and eccentricity $\cos \alpha$ (see Art. 567).

Ex. 3. In the case of a curve drawn upon a conical surface to cut all the generators at the same constant angle α, we have, taking the origin

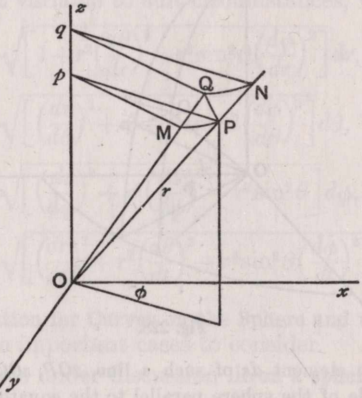

Fig. 229.

at the vertex and the axis of the cone as the z-axis and β for the semi-vertical angle of the cone,

$$\frac{r \sin \beta \, d\phi}{dr} = \tan \alpha,$$

as in Example (1), for the sphere, and therefore

$$\frac{dr}{r} = \sin \beta \cot \alpha \, d\phi \, ;$$

whence $$r = Ae^{\phi \sin \beta \cot \alpha},$$

where A is an arbitrary constant, determinable when some one point on the curve is specified.

The projection of the curve upon the x-y plane is therefore an equiangular spiral of angle $\cot^{-1}(\sin \beta \cot \alpha)$.

We also have $$s = \int \sqrt{1 + r^2 \sin^2\beta \left(\frac{d\phi}{dr}\right)^2} \, dr$$

$$= \int \sqrt{1 + \tan^2\alpha} \, dr = \Big[r \sec \alpha \Big]_{r_1}^{r_2}$$

between limits r_1, r_2.

If the spiral passes through the origin, and s be measured from that point, $$s = r \sec \alpha,$$
which is also obvious from the consideration that if the curve be developed upon a plane it will become an equiangular spiral of angle α.

723. The p, r Formula.

The p, r formula of Art. 547, viz. $s=\int\frac{r\,dr}{\sqrt{r^2-p^2}}$, still holds for curves of double curvature.

For, with the same notation as before,

$$\frac{p}{r}=\sin\phi \quad\text{and}\quad \frac{dr}{ds}=\cos\phi,$$

ϕ being the angle which the tangent makes with the radius vector from the origin; whence

$$\frac{ds}{dr}=\sec\phi=\frac{1}{\sqrt{1-\frac{p^2}{r^2}}}=\frac{r}{\sqrt{r^2-p^2}}$$

and
$$s=\int\frac{r\,dr}{\sqrt{r^2-p^2}}.$$

For cases of curves drawn upon a sphere, the centre being at the origin, the formula is useless. For in that case, the tangent being necessarily at all points at right angles to the radius vector, $\frac{dr}{ds}=0$ and $p=r$ throughout.

In the case of a curve drawn upon a right circular cone whose vertex is at the origin, we may use the formula with advantage; but it is to be remembered that we are doing no more than if we regarded the conical surface as developed upon a plane.

Ex. For the case already considered of a curve cutting all the generators of a cone at a constant angle α, we have at once $p=r\sin\alpha$ and $s=\int\frac{dr}{\cos\alpha}=r\sec\alpha$, as in the last article.

There are but few curves of double curvature, however, for which the p, r relation is known, with the exception of course of such as, having been originally plane curves, have been laid upon a developable surface. For such cases the formula is useful, as also of course whenever the relation can be readily found.

724. Ex. Let $BAA'B'$ be a strip of thin inextensible ribbon lying upon a plane. Let OAA' be a perpendicular from any point O of the plane upon AB and $A'B'$ and OPP' any other radius vector from O.

Let $OA=l_0$, $OP=l$, $PA=s$.

Then obviously $$l^2=s^2+l_0^2.$$

Now imagine this ribbon wrapped tightly without folding or creasing upon a right circular cone of vertex O with OAA' as a generator, the semivertical angle being α, the wrapping commencing with OA in contact with the cone. When the wrapping has been completed, OP coming into contact and becoming a generator, let us unwrap the triangle from the cone, keeping OP in contact and starting the unwrapping with

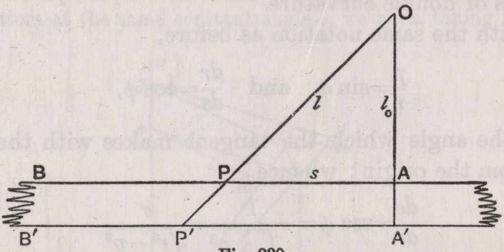

Fig. 230.

releasing the generator OA, keeping O fixed at the vertex; when the unwrapping is just complete, the triangle has taken the position OYP, and is the same triangle as we started with, $O\hat{Y}P$ being a right angle.

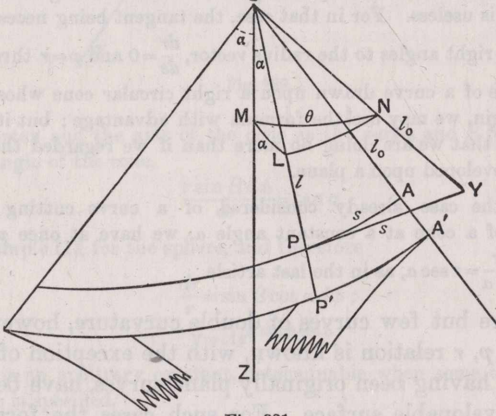

Fig. 231.

It appears

(1) that the arc AP upon the cone has a length $\sqrt{l^2-l_0{}^2}$;

(2) that the arc AP upon the cone is a geodesic;

(3) that the locus of Y in the unwrapping lies on a sphere of radius l_0 and vertex at O;

(4) that the p, r equation of this geodesic on the cone is $p=l_0$, for this is so on the plane from which it was constructed;

(5) the formula $s=\int\dfrac{r\,dr}{\sqrt{r^2-p^2}}$ is merely

$$s=\int\frac{r\,dr}{\sqrt{r^2-p_0{}^2}}=\Big[\sqrt{r^2-p_0{}^2}\Big]_{l_0}^{l}=\sqrt{l^2-l_0{}^2};$$

(6) the Y locus is an involute of the geodesic;

(7) taking a sphere of any radius with centre at O, cutting the axis OZ at M, the generator OP at L and OY the perpendicular on the tangent at N, LMN is a right-angled spherical triangle, where

$$ML=\alpha, \quad LN=\tan^{-1}\frac{s}{l_0} \quad \text{and} \quad M\hat{L}N=\frac{\pi}{2};$$

whence $$\cos MN=\cos\alpha\cos LN$$

and $$\sin\alpha=\cot L\hat{M}N\tan LN=\frac{s}{l_0}\cot L\hat{M}N.$$

If ϕ be the angle between the plane ZOY and the plane ZOA, and θ the angle ZOY, we have thus shown that

$$\cos\theta=\cos\alpha\cos LN, \quad \text{and therefore} \quad \frac{s}{l_0}=\frac{\sqrt{\cos^2\alpha-\cos^2\theta}}{\cos\theta}.$$

Now, if we take a circle on the plane OPY with centre O and radius OP, and consider the arc bounded by OP and OY produced, this arc will wrap upon the cone and will coincide with the corresponding arc of the circular section of the cone through P; whence if χ be the angle between the plane ZOP and the plane ZOA,

$$l\sin\alpha\,.\,\chi=l\times\text{angle }POY,$$

and $$\chi\sin\alpha=\tan^{-1}\frac{s}{l_0}.$$

Hence $$\phi=\chi-L\hat{M}N=\frac{1}{\sin\alpha}\tan^{-1}\frac{s}{l_0}-\tan^{-1}\frac{s}{l_0\sin\alpha},$$

i.e. $$\phi=\frac{1}{\sin\alpha}\tan^{-1}\frac{\sqrt{\cos^2\alpha-\cos^2\theta}}{\cos\theta}-\tan^{-1}\frac{1}{\sin\alpha}\,\frac{\sqrt{\cos^2\alpha-\cos^2\theta}}{\cos\theta},$$

is the equation of a cone which by its intersection with the sphere of radius l_0 and centre O gives the Y locus, which is also an involute of the geodesic on the cone.

725. Inversion.

The process of inversion may sometimes be employed with advantage. This is particularly the case when a twisted curve lies on the surface of a sphere. By inverting with regard to a point on the surface of the sphere, the spherical surface is inverted into a plane and the twisted curve into a plane curve, and *vice versa*.

Let O be the pole of inversion and k the constant, and let the diameter OA of the sphere meet the plane into which the sphere inverts at C. Then $OA\,.\,OC=k^2$,

$$OC=\frac{k^2}{OA}=c, \text{ say.}$$

Let the element PQ, viz. δs, of a twisted curve on the spherical surface invert into $P'Q'$, viz. $\delta s'$, an element of the plane inverse curve.

Then $$PQ = k^2 \frac{P'Q'}{OP' . OQ'},$$

or ultimately $$ds = k^2 \frac{ds'}{OP'^2}.$$

Let $$CP' = r.$$

Then $$s = k^2 \int \frac{ds'}{c^2 + r^2},$$

and if this integral for the plane curve can be found, the rectification of the twisted curve on the sphere will have been effected.

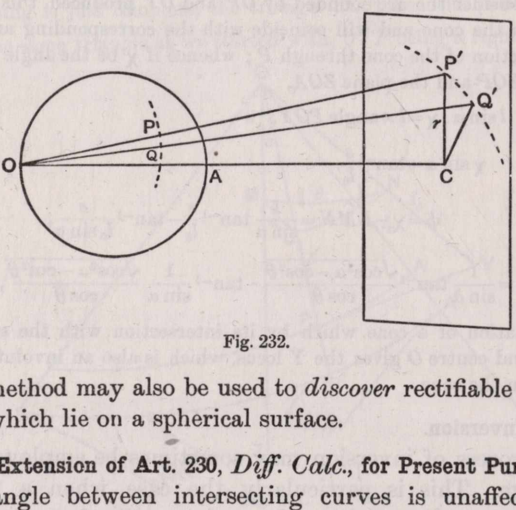

Fig. 232.

The method may also be used to *discover* rectifiable twisted curves which lie on a spherical surface.

726. Extension of Art. 230, *Diff. Calc.*, for Present Purposes.

The angle between intersecting curves is unaffected by inversion. (Extension of Art. 230 of *Diff. Calc.*)

If two planes $QPP'Q'$, $RPP'R'$ intersect in the line PP' and if PQ, $P'Q'$ make the same angle with PP' in opposite directions as also PR and $P'R'$, then the angle $Q\hat{P}R = Q'\hat{P'}R'$. For, take distances PN and $P'N'$ equal to each other in opposite directions from P and P' respectively on PP' produced, and let two planes perpendicular to the line PP' be drawn through N and N' to cut PQ and PR at Q and R, and to cut $P'Q'$ and $P'R'$ in Q' and R' respectively.

Then, from the congruent pairs of triangles PNQ and $P'N'Q'$, and PNR and $P'N'R'$ respectively, we have $NQ=N'Q'$ and $NR=N'R'$, whilst $Q\hat{N}R=Q'\hat{N}'R'$, and therefore the triangles QNR, $Q'N'R'$ are congruent and $QR=Q'R'$; whence the angles $Q\hat{P}R$, $Q'\hat{P}'R'$ are also equal.

It follows therefore that if PQ, $P'Q'$ be the directions of the tangents at P and P' to inverse elements of curves in the

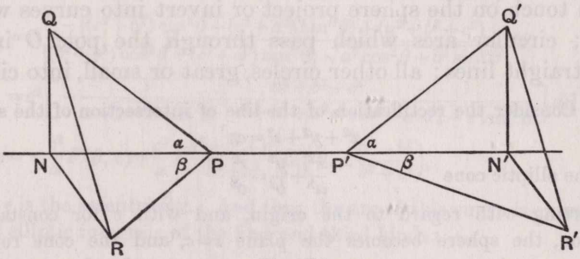

Fig. 233.

plane $PP'Q'Q$ and PR, $P'R'$ be the directions of the tangents at P and P' to inverse elements of curves in the plane $PP'R'R$, then, as in this case PQ and $P'Q'$ make equal angles with PP' in opposite directions, as also do PR and $P'R'$ (as proved in *Differential Calculus*, Art. 229, for curves in a plane), it will follow that the angle between two curves meeting at P is equal to the angle between the inverses meeting at P'. Hence the result of Art. 230 of *Diff. Calc.* is now extended to any case of inversion, the curves not being necessarily plane, and the pole of inversion now lying anywhere.

727. **Stereographic Projection, etc.**

If we take as constant of inversion the diameter of the sphere, and the pole of inversion a point O on the sphere, the sphere inverts into the tangent plane at the opposite end of the diameter through the pole.

If the constant of inversion be taken as

$$\frac{\text{diameter}}{\sqrt{2}}, \quad \textit{i.e.}\ \sqrt{2}\,.\,\text{radius},$$

the sphere inverts into the equatorial plane of which the origin of inversion is a pole.

In all such cases the inversion amounts to a conical projection with the origin O as pole of projection.

When the projection is upon an equatorial plane with O for pole, it is called a Stereographic Projection.

In any of these cases, the angles of intersection of any spherical curves project or invert into equal angles of intersection of the projected or inverted curve. Orthogonal intersection remains orthogonal intersection in the projected curves; curves which touch on the sphere project or invert into curves which touch; circular arcs which pass through the pole O invert into straight lines; all other circles, great or small, into circles.

Ex. Consider the rectification of the line of intersection of the sphere

$$x^2+y^2+z^2=cz$$

with the elliptic cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}.$$

Inverting with regard to the origin, and with c for constant of inversion, the sphere becomes the plane $z=c$, and the cone remains unaltered, but cutting the plane $z=c$ in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

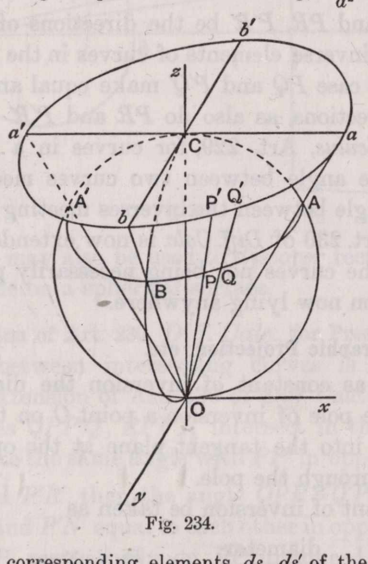

Fig. 234.

If PQ, $P'Q'$ be corresponding elements ds, ds' of the original and the inverse curves,

$$ds=c^2\frac{ds'}{OP'^2}=\frac{c^2}{c^2+r^2}ds',$$

where r is the central radius vector of the ellipse to the point P'.

Hence, taking θ as the complement of the eccentric angle of P', we have for the ellipse,

$$x=a\sin\theta,\quad y=b\cos\theta,\quad r^2=a^2\sin^2\theta+b^2\cos^2\theta,$$
$$ds'^2=(a^2\cos^2\theta+b^2\sin^2\theta)\,d\theta^2.$$

and

$$ds=\frac{c^2\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}{(c^2+b^2)\cos^2\theta+(c^2+a^2)\sin^2\theta}\,d\theta$$
$$=c^2\frac{a^2\cos^2\theta+b^2\sin^2\theta}{\{(c^2+b^2)\cos^2\theta+(c^2+a^2)\sin^2\theta\}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}\,d\theta$$
$$=c^2\frac{-\{(c^2+b^2)\cos^2\theta+(c^2+a^2)\sin^2\theta\}+(a^2+b^2+c^2)}{\{(c^2+b^2)\cos^2\theta+(c^2+a^2)\sin^2\theta\}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}\,d\theta$$
$$=c^2\left[-\frac{1}{a\sqrt{1-e^2\sin^2\theta}}+\frac{a^2+b^2+c^2}{c^2+b^2+a^2e^2\sin^2\theta}\,\frac{1}{a\sqrt{1-e^2\sin^2\theta}}\right]d\theta;$$
$$\therefore\ s=-\frac{c^2}{a}F(\theta,e)+\frac{c^2}{a}\cdot\frac{a^2+b^2+c^2}{b^2+c^2}\,\Pi\left(\theta,e,\frac{a^2-b^2}{b^2+c^2}\right),$$

where e is the eccentricity. And thus the arc of this curve is expressible by the elliptic integrals of the first and third kinds.

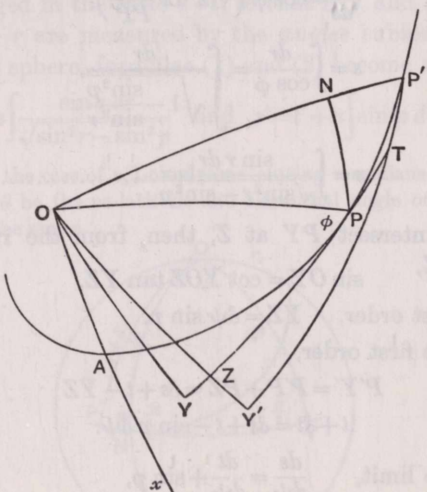

Fig. 235.

728. Curves on Spherical Surfaces in particular. Formulae for the Rectification of Curves on a Spherical Surface, analogous to the p, r and p, ψ Formulae for a Plane Curve.

Let APP' be any curve drawn upon the surface of a sphere of radius unity. Let P, P' be contiguous points, and let arc $PP' = \delta s$. Let O be any fixed pole on the sphere, and let $PY, P'Y'$ be the great circle tangents at P and P'; OY, OY' the great circle perpendiculars to them from O, and OAx a fixed great circle cutting the curve at A, the point from which s is measured.

Let $x\hat{O}Y = \psi$, $\quad Y\hat{O}Y' = \delta\psi$, $\quad OY = p$, $\quad OY' = p + \delta p$,

$$PY = t, \quad P'Y' = t + \delta t.$$

Let OP, OP' be the great circle radii vectores of P and P', and let $O\hat{P}Y = \phi$.

Then, from the spherical triangle OYP, we have

$$\cos r = \cos p \cos t \quad \text{and} \quad \sin p = \sin r \sin \phi.$$

Let PN be the great circle perpendicular upon OP'. Thus, as in plane geometry, we have

$$\frac{dr}{ds} = \cos\phi \left(\text{viz. } Lt\, \frac{NP'}{PP'}\right),$$

and
$$s = \int \frac{dr}{\cos\phi} = \int \frac{dr}{\sqrt{1 - \dfrac{\sin^2 p}{\sin^2 r}}},$$

i.e.
$$s = \int \frac{\sin r\, dr}{\sqrt{\sin^2 r - \sin^2 p}}. \qquad \text{.........................(1)}$$

Let OY' intersect PY at Z, then, from the right-angled triangle YOZ,
$$\sin OY = \cot YOZ \tan YZ,$$

i.e. to the first order, $\quad YZ = \delta\psi \sin p$.

Also to the first order,

$$P'Y' = P'P + PZ = \delta s + t - YZ$$

i.e.
$$t + \delta t = \delta s + t - \sin p\, \delta\psi.$$

And in the limit,
$$\frac{ds}{d\psi} = \frac{dt}{d\psi} + \sin p,$$

i.e.
$$s = t + \int \sin p\, d\psi. \qquad \text{.......................(2)}$$

Formulae (1) and (2) are analogous to

$$s = \int \frac{r\, dr}{\sqrt{r^2 - p^2}} \quad \text{and} \quad s = t + \int p\, d\psi$$

for plane curves.

729. Convention of Sign of t. Closed Oval.

In regard to t it is necessary to make a convention with regard to sign. It will be in agreement with the convention for plane curves, Art. 531, if we fix that t is to be reckoned positive when, as in the case of Fig. 185, PY is measured from the point of contact in the direction opposite to that of increase of the arc s.

As in plane curves, it appears that if the curve considered be a closed oval on the sphere, t returns to its original value when integration is taken round the oval. Hence for a closed curve surrounding the pole, encircling it once,

$$s=\int_0^{2\pi}\sin p\,d\psi.$$

If the radius of the sphere be a instead of unity, which has been taken for convenience, the absolute length of the arc will be changed in the ratio $a:1$, so that if s' and t' be *lengths*, whilst p and r are measured by the *angles* subtended at the centre of the sphere, formulae (1) and (2) become respectively

$$s'=a\int\frac{\sin r\,dr}{\sqrt{\sin^2 r-\sin^2 p}}\quad\text{and}\quad s'=t'+a\int\sin p\,d\psi.$$

730. Ex. In the case of a **Loxodrome** cutting meridians at a constant angle α, let r, θ be the co-latitude and azimuthal angle of any current point P upon the curve.

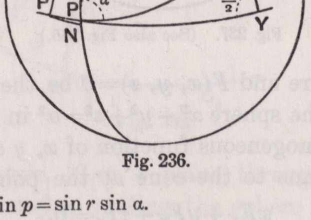

Fig. 236.

Then $\phi=\alpha$ and $\sin p=\sin r\sin\alpha$.

Hence $$s'=a\int\frac{\sin r\,dr}{\sqrt{\sin^2 r-\sin^2 p}}=\frac{a}{\cos\alpha}\cdot r,$$

a being the radius of the sphere, *i.e.*

Arc of curve measured from the pole $= \dfrac{\text{arcual radius vector } OP}{\cos\alpha}$, ...(a)

as in the case of the equiangular spiral upon a plane. (See also Art. 548.)

We also have in this curve

$$Lt\, \frac{a \sin r\, \delta\theta}{a\, \delta r} = \tan\alpha,$$

i.e.
$$\frac{dr}{\sin r} = \cot\alpha\, d\theta,$$

i.e.
$$\log\left(\frac{\tan\frac{r}{2}}{\tan\frac{r_0}{2}}\right) = \theta\cot\alpha,$$

if $r = r_0$, when $\theta = 0$, *i.e.* $\tan\frac{r}{2} = \tan\frac{r_0}{2} e^{\theta\cot\alpha}$, ...(b)

which is another form of the property $l = \mathrm{gd}(\theta\cot\alpha)$, ...(c)

already established in Art. 722, a relation between the latitude and longitude analogous to that between y and x in a Cartesian equation.

731. To find $\sin p$.

The expression for $\sin p$ in terms of ψ which is required in the integration of Art. 729 may be found as follows. Take the z-axis through O, the pole of the curve. Let C be the

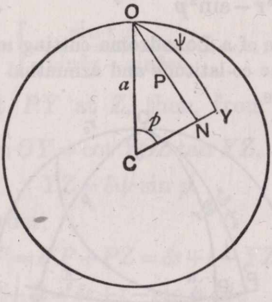

Fig. 237. (See also Fig. 235.)

centre of the sphere and $F(x, y, z) = 0$ be the equation of the cone which cuts the sphere $x^2 + y^2 + z^2 = a^2$ in the given curve.

Then F is a homogeneous function of x, y and z.

The tangent plane to the cone at the point x', y', z' of the curve is
$$xF_{x'} + yF_{y'} + zF_{z'} = 0.$$

The equation of a perpendicular plane COY through the z-axis is
$$xF_{y'} - yF_{x'} = 0.$$

Hence $\tan\psi = \dfrac{F_{y'}}{F_{x'}}$, *i.e.* $\dfrac{F_{x'}}{\cos\psi} = \dfrac{F_{y'}}{\sin\psi}$. ....(A)

And the perpendicular P ($= ON$, Fig. 237), upon the tangent plane from the pole O, whose coordinates are $(0, 0, a)$, is

$$P = \frac{aF_{z'}}{\sqrt{F_{x'}^{2} + F_{y'}^{2} + F_{z'}^{2}}}. \qquad \text{(B)}$$

From $F = 0$ and equations (A) and (B), the ratios $x' : y' : z'$ are to be eliminated, and there will result a relation between P and ψ, say,

$$P = af(\psi).$$

Again,

$$\frac{P}{a} = \sin p.$$

Hence the relation required is $\sin p = f(\psi)$.

732. Relation with the Polar Curve.

Let any curve be drawn upon a sphere of centre O and radius r, and let the cone with vertex O, and passing through

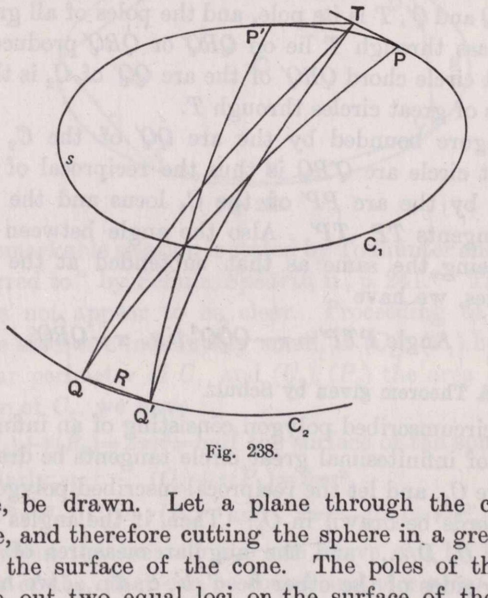

Fig. 238.

the curve, be drawn. Let a plane through the centre of the sphere, and therefore cutting the sphere in a great circle, roll upon the surface of the cone. The poles of this plane then trace out two equal loci on the surface of the sphere. Either of these equal and similar loci is called the polar curve of the given curve. The great circle arcs which are the lines of intersection of the sphere and the plane touch the curve as the plane rolls, and are great circle tangents.

Let Q, Q' be two positions of one of the poles corresponding to the great circles PT, $P'T$, intersecting at T and touching a curve C_1 drawn upon the sphere. Let the curve locus of Q be referred to as the curve C_2. Drawing the great circles PQ, TQ, TQ', $P'Q'$, we have

$$PQ = TQ, \text{ both quadrants,}$$

$$TQ' = P'Q', \text{ both quadrants,}$$

and $$TQ = TQ', \text{ both being quadrants.}$$

Hence, in the limit when P' and P are indefinitely close, T ultimately lies upon C_1, and is the pole of a tangent plane to the cone with vertex at O, which cuts the sphere in C_2. Hence the relation between the two curves is reciprocal. Each one is the locus of the poles of tangent planes of the cone which defines the other. If QRQ' be the great circle arc joining Q and Q', T is its pole, and the poles of all great circles which pass through T lie on QRQ' or QRQ' produced, that is the great circle chord QRQ' of the arc QQ' of C_2 is the path of the poles of great circles through T.

The figure bounded by the arc QQ' of the C_2 locus and the great circle arc $Q'RQ$ is thus the reciprocal of the figure bounded by the arc PP' of the C_1 locus and the two great circle tangents TP, TP'. Also the angle between two great circles being the same as that subtended at the centre by their poles, we have

$$\text{Angle } PTP' = \pi - Q\hat{O}Q', \textit{ i.e. } \pi - Q\hat{R}Q'.$$

733. A Theorem given by Schulz.

Let a circumscribed polygon consisting of an infinitely large number of infinitesimal great circle tangents be drawn to the one curve C_1, and let the reciprocal inscribed polygon of great circle chords be drawn in C_2. Then, if the angles of the one be $A, B, C, D, \dots$, and the angular measures of the corresponding sides of the other be $a', b', c', d', \dots$, we have

$$A = \pi - a', \quad B = \pi - b', \text{ etc.}$$

We have Area of the polygon $ABCD... = [\Sigma A - (n-2)\pi]r^2$ (Todhunter and Leathem, *Spherical Trigonometry*, Art. 129)

$$= [\Sigma(\pi - a') - (n-2)\pi]r^2$$
$$= (2\pi - \Sigma a')r^2$$
$$= 2(\pi - s')r^2,$$

if s' be the angular semiperimeter of the polygon $A'B'C'D'...$.

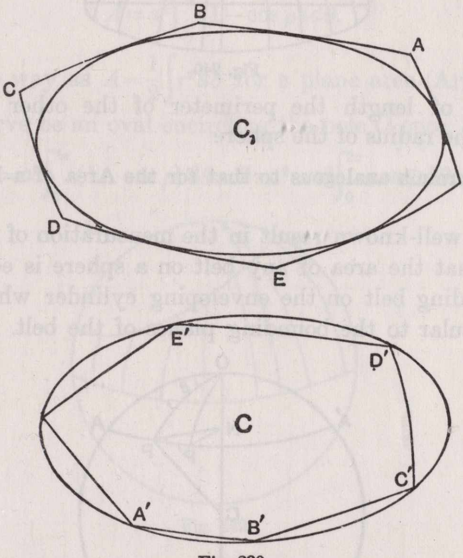

Fig. 239.

This remarkable relation is stated by Todhunter and Leathem as "referred to" by Schulz, *Sphärik*, ii., p. 241.* The authorship does not appear to be clear. Proceeding to the limit when the sides are indefinitely small, if (C_1), (P_1) be the area and linear perimeter of C_1, and (C_2), (P_2) the area and linear perimeter of C_2, we have

$$(C_1) + r(P_2) = 2\pi r^2 = \text{half the surface of the sphere,}$$

and similarly $\qquad (C_2) + r(P_1) = 2\pi r^2,$

that is $\quad 2\pi r^2 - (C_1) = r(P_2) \quad$ and $\quad 2\pi r^2 - (C_2) = r(P_1)$.

Thus when the area of the one curve can be found, the perimeter of the other can be found and *vice versa*.

* See also Williamson's *Integral Calculus*, Art. 188.

It appears also that the area included between either curve and any great circle which it does not cut is equal to a

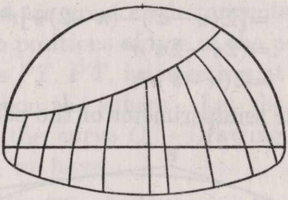

Fig. 240.

rectangle of length the perimeter of the other curve and breadth the radius of the sphere.

734. Formula analogous to that for the Area of a Plane Curve in Polars.

It is a well-known result in the mensuration of a spherical surface that the area of any belt on a sphere is equal to the corresponding belt on the enveloping cylinder whose axis is perpendicular to the bounding planes of the belt. Let APA'

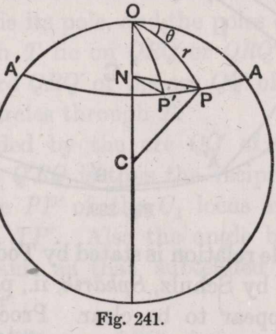

Fig. 241.

be any small circle of a sphere of radius a. Let O be the pole of the circle and OP any great circle radius vector from O of length r, subtending an angle ρ at the centre. Then the area of the spherical cap cut off by the small circle

$$=2\pi a(a-a\cos\rho)=2\pi a^2(1-\cos\rho).$$

Let the azimuthal angle of OP be θ. Then we have for the area between OP and OP' for which θ is increased to $\theta+\delta\theta$,

$$\text{Area } OPP'=\frac{\delta\theta}{2\pi}\times 2\pi a^2(1-\cos\rho)$$
$$=a^2(1-\cos\rho)\delta\theta,$$

analogous to the result $\frac{1}{2}r^2\delta\theta$ for a plane (and indeed becoming $\frac{1}{2}r^2\delta\theta$ when we put $\frac{r}{a}$ for ρ and the radius a becomes ∞).

Hence, taking ρ, θ as coordinates, we have for the area of any portion of the spherical surface bounded by a curve on the sphere, and the meridians $\theta=\theta_1$, $\theta=\theta_2$,

$$A=a^2\int_{\theta_1}^{\theta_2}(1-\cos\rho)\,d\theta,$$

in the same way as $A=\frac{1}{2}\int r^2 d\theta$ for a plane area (Art. 407).

If the curve be an oval encircling the pole O once,

$$A=a^2\int_0^{2\pi}(1-\cos\rho)\,d\theta=2\pi a^2-a^2\int_0^{2\pi}\cos\rho\,d\theta.$$

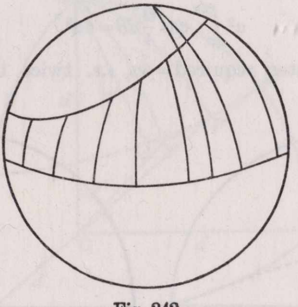

Fig. 242.

The area therefore between the curve and the equatorial plane of O is

$$a^2\int_0^{2\pi}\cos\rho\,d\theta,$$

or if we use l for the latitude, *i.e.* the complement of ρ, and θ for the longitude or azimuthal angle,

$$\text{Area}=a^2\int_0^{2\pi}\sin l\,d\theta.$$

If, then, this integral be evaluated for the polar or reciprocal curve C_2, the result will be aP_1, *i.e.*

$$\text{Perimeter}=P_1=a\int_0^{2\pi}\sin l\,d\theta,$$

$(l,\ \theta)$ being the latitude and longitude of a point on the reciprocal curve.

Illustrative Examples.

Ex. 1. To test this result in a known case, take C_1 as a small circle with pole at O and of angular radius ρ. Its perimeter is obviously

$$2\pi a \sin \rho.$$

The polar curve is another small circle of angular radius $\frac{\pi}{2}-\rho$, and therefore the latitude of any point on it is ρ, in this case a constant. The formula gives

$$P_1 = a\int_0^{2\pi} \sin\rho \, d\theta = 2\pi a \sin\rho,$$

which is in agreement with the stated result.

Ex. 2. Find the length of the spiral, traced on a sphere, whose reciprocal is defined by the equation $4\rho=\theta$ corresponding to limits for θ from 0 to 2π, ρ and θ having the meanings assigned to them in Art. 734.

The area between the reciprocal spiral and the equatorial plane is

$$a^2\int_0^{2\pi} \cos\frac{\theta}{4}\, d\theta = 4a^2.$$

Hence the perimeter required $=4a$, *i.e.* twice the diameter of the sphere. (Fig. 243.)

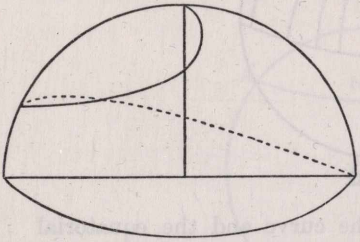

Fig. 243.

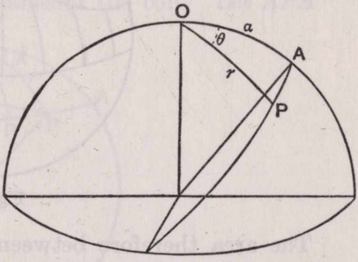

Fig. 244.

Ex. 3. To find the area bounded by any arc of a great circle and two spherical radii vectores.

Let the plane of the great circle be at right angles to the plane of the paper and cut the meridian in the plane of the paper at a point A whose co-latitude is α. (Fig. 244.)

Then the equation of the great circle is

$$\cos\theta = \cot\rho \tan\alpha,$$

from the spherical triangle OPA, right angled at A.

Then we have

$$\text{Area} = a^2\int_{\theta_1}^{\theta_2}(1-\cos\rho)\, d\theta$$

$$= a^2(\theta_2-\theta_1) - a^2\int_{\theta_1}^{\theta_2} \frac{\cos\theta \cot\alpha}{\sqrt{\cos^2\theta\cot^2\alpha+1}}\, d\theta,$$

and the integral

$$\int \frac{\cos\theta}{\sqrt{\cot^2\alpha\cos^2\theta+1}}\,d\theta=\int\frac{d\sin\theta}{\sqrt{\operatorname{cosec}^2\alpha-\cot^2\alpha\sin^2\theta}}$$
$$=\tan\alpha\int\frac{d\sin\theta}{\sqrt{\sec^2\alpha-\sin^2\theta}}$$
$$=\tan\alpha\sin^{-1}(\sin\theta\cos\alpha).$$

Hence the area between two radii making angles θ_1 and θ_2 with the meridian in the plane of the paper is

$$a^2(\theta_2-\theta_1)-a^2[\sin^{-1}(\sin\theta_2\cos\alpha)-\sin^{-1}(\sin\theta_1\cos\alpha)].\quad \text{(See Art. 781.)}$$

735. The Case of a Sphero-conic.

DEF. A sphero-conic is the line of intersection of a cone of the second degree with a sphere whose centre is at the vertex of the cone.

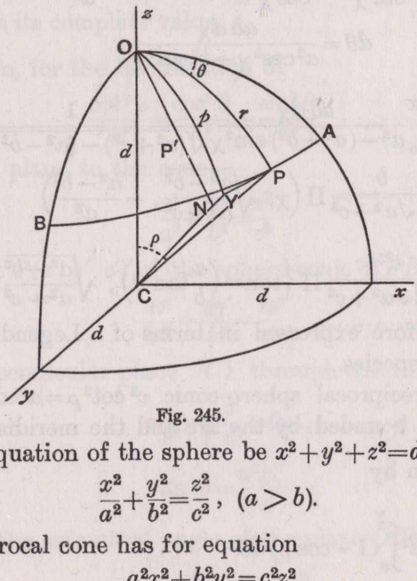

Fig. 245.

Let the equation of the sphere be $x^2+y^2+z^2=d^2$, and that of the cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2},\ (a>b).$$

The reciprocal cone has for equation

$$a^2x^2+b^2y^2=c^2z^2.$$

Putting ρ for the co-latitude and θ for the azimuthal angle of any point, we have $x=d\sin\rho\cos\theta$, $y=d\sin\rho\sin\theta$, $z=d\cos\rho$, and the equations of the sphero-conic and its reciprocal become respectively

$$\frac{\cot^2\rho}{c^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\quad\text{and}\quad c^2\cot^2\rho=a^2\cos^2\theta+b^2\sin^2\theta$$

in ρ, θ coordinates.

The area A_1 bounded by the arc of the sphero-conic

$$\frac{\cot^2\rho}{c^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}$$

and the meridians $\theta=0$, $\theta=\theta$ is given by

$$A_1=d^2\int_0^\theta(1-\cos\rho)\,d\theta$$

$$=d^2\int_0^\theta\left\{1-c\sqrt{\frac{a^2\sin^2\theta+b^2\cos^2\theta}{a^2(b^2+c^2)\sin^2\theta+b^2(a^2+c^2)\cos^2\theta}}\right\}d\theta$$

$$=d^2(\theta-cI_1),\text{ say};$$

and putting

$$\frac{a\sin\theta}{\sin\chi}=\frac{b\cos\theta}{\cos\chi},\quad \textit{i.e.}\ \tan\theta=\frac{b}{a}\tan\chi,$$

$$d\theta=\frac{ab\,d\chi}{a^2\cos^2\chi+b^2\sin^2\chi};$$

whence

$$I_1=\int_0^\chi\frac{ab\,d\chi}{a^2-(a^2-b^2)\sin^2\chi}\,\frac{1}{\sqrt{(a^2+c^2)-(a^2-b^2)\sin^2\chi}}$$

$$=\frac{b}{a\sqrt{a^2+c^2}}\Pi\left(\chi,\ \sqrt{\frac{a^2-b^2}{a^2+c^2}},\ -\frac{a^2-b^2}{a^2}\right)$$

and

$$A_1=d^2.\,\theta-\frac{d^2bc}{a\sqrt{a^2+c^2}}\Pi\left\{\tan^{-1}\left(\frac{a}{b}\tan\theta\right),\ \sqrt{\frac{a^2-b^2}{a^2+c^2}},\ -\frac{a^2-b^2}{a^2}\right\},$$

and is therefore expressed in terms of a Legendrian integral of the third species.

For the reciprocal sphero-conic $c^2\cot^2\rho=a^2\cos^2\theta+b^2\sin^2\theta$ the area A_2 bounded by the arc and the meridians $\theta=\theta$ and $\theta=\frac{\pi}{2}$ is given by

$$A_2=d^2\int_\theta^{\frac{\pi}{2}}(1-\cos\rho)\,d\theta$$

$$=d^2\int_\theta^{\frac{\pi}{2}}\left\{1-\sqrt{\frac{a^2\cos^2\theta+b^2\sin^2\theta}{(a^2+c^2)\cos^2\theta+(b^2+c^2)\sin^2\theta}}\right\}d\theta$$

$$=d^2\left(\frac{\pi}{2}-\theta-\Big[\,I_2\,\Big]_\theta^{\frac{\pi}{2}}\right),\text{ say};$$

and putting

$$\frac{b\sin\theta}{\cos\chi}=\frac{a\cos\theta}{\sin\chi},\quad \textit{i.e.}\ \tan\theta=\frac{a}{b}\cot\chi,$$

we have

$$\Big[I_2\Big]_\theta^{\frac{\pi}{2}}=\frac{b^2}{a\sqrt{b^2+c^2}}\Pi\left(\chi,\ \frac{c}{a}\sqrt{\frac{a^2-b^2}{b^2+c^2}},\ -\frac{a^2-b^2}{a^2}\right)\quad\text{(Art. 388, Ex. 7)};$$

whence

$$A_2=d^2\left[\frac{\pi}{2}-\theta-\frac{b^2}{a\sqrt{b^2+c^2}}\Pi\left\{\tan^{-1}\left(\frac{a}{b}\cot\theta\right),\ \frac{c}{a}\sqrt{\frac{a^2-b^2}{a^2+b^2}},\ -\frac{a^2-b^2}{a^2}\right\}\right];$$

and the area of the same curve from $\theta=0$ to $\theta=\theta$ is

$$d^2\left(\theta-\Big[I_2\Big]_0^\theta\right)=d^2\{\theta-(\Pi_1-\Pi)\},$$

where Π is the same elliptic integral as occurs in the value of A_2 and Π_1 is its complete value.

736. Again, for the Rectification of

$$\frac{\cot^2\rho}{c^2}=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2},$$

the tangent plane to the cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$$

at any point $P(x', y', z')$ of the sphero-conic APB (Fig. 245) is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=\frac{zz'}{c^2},\ \text{.............................}(1)$$

and the perpendicular plane OCY through the z-axis is

$$x\frac{y'}{b^2}-y\frac{x'}{a^2}=0,\ \text{...............................}(2)$$

giving $$\tan\psi=\frac{a^2y'}{b^2x'},$$

where ψ is the azimuthal angle of the plane OCY, *i.e.*

$$\frac{x'}{a^2\cos\psi}=\frac{y'}{b^2\sin\psi}.$$

Also the perpendicular ON $(=P')$ from the pole O upon the tangent plane at P, viz. CPY, is given by

$$\frac{c}{d}P'=\frac{\dfrac{z'}{c}}{\sqrt{\dfrac{x'^2}{a^4}+\dfrac{y'^2}{b^4}+\dfrac{z'^2}{c^4}}}=\sqrt{\frac{\dfrac{x'^2}{a^2}+\dfrac{y'^2}{b^2}}{\dfrac{x'^2}{a^4}+\dfrac{y'^2}{b^4}+\left(\dfrac{x'^2}{a^2}+\dfrac{y'^2}{b^2}\right)\dfrac{1}{c^2}}}$$

Therefore, if p be the angle OCY subtended at C by the great circle arc OY, $P'=d\sin p$, and we have

$$\sin p=\sqrt{\frac{a^2\cos^2\psi+b^2\sin^2\psi}{(a^2+c^2)\cos^2\psi+(b^2+c^2)\sin^2\psi}};$$

$$\therefore \int_\psi^{\frac{\pi}{2}}\sin p\,d\psi=\frac{b^2}{a\sqrt{b^2+c^2}}\Pi\left(\psi,\ \frac{c}{a}\sqrt{\frac{a^2-b^2}{b^2+c^2}},\ -\frac{a^2-b^2}{a^2}\right).$$

(Art. 388, Ex. 7.)

Hence, if s and t be the *lengths* of the arcs of the spheroconic from P to B, and of the 'tail' PY respectively (Fig. 245),

$$s=t+d\int_\psi^{\frac{\pi}{2}}\sin p\,d\psi$$

and
$$s=t+\frac{b^2d}{a\sqrt{b^2+c^2}}\Pi\left(\psi,\ \frac{c}{a}\sqrt{\frac{a^2-b^2}{b^2+c^2}},\ -\frac{a^2-b^2}{a^2}\right),$$

and t remains to be found.

Now t is the arcual measure of the great circle arc PY.

The equations of CY, CP (C being the centre of the sphere) are, from (1) and (2),

$$\frac{x}{\frac{x'}{a^2}}=\frac{y}{\frac{y'}{b^2}}=\frac{z}{\frac{c^2}{z'}\left(\frac{x'^2}{a^4}+\frac{y'^2}{b^4}\right)}\quad\text{and}\quad\frac{x}{x'}=\frac{y}{y'}=\frac{z}{z'}.$$

Hence

$$\cos\left(\frac{t}{d}\right)=\cos Y\hat{C}P=\frac{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}+c^2\left(\frac{x'^2}{a^4}+\frac{y'^2}{b^4}\right)}{d\sqrt{\frac{x'^2}{a^4}+\frac{y'^2}{b^4}}\sqrt{1+\frac{c^4}{z'^2}\left(\frac{x'^2}{a^4}+\frac{y'^2}{b^4}\right)}}$$

$$=\frac{c^2\left(\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}\right)}{d\sqrt{\frac{x'^2}{a^4}+\frac{y'^2}{b^4}}\cdot\frac{c^2}{z'}\sqrt{\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}}}$$

$$=\frac{z'}{d}\sqrt{\frac{\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}}{\frac{x'^2}{a^4}+\frac{y'^2}{b^4}}}.$$

Also $\dfrac{\frac{x'}{a}}{a\cos\psi}=\dfrac{\frac{y'}{b}}{b\sin\psi}=\dfrac{\frac{z'}{c}}{\sqrt{a^2\cos^2\psi+b^2\sin^2\psi}}$

$$=\frac{d}{\sqrt{a^2(a^2+c^2)\cos^2\psi+b^2(b^2+c^2)\sin^2\psi}};$$

$$\therefore\ \cos\left(\frac{t}{d}\right)=\sqrt{a^2\cos^2\psi+b^2\sin^2\psi}\sqrt{\frac{(a^2+c^2)\cos^2\psi+(b^2+c^2)\sin^2\psi}{a^2(a^2+c^2)\cos^2\psi+b^2(b^2+c^2)\sin^2\psi}}.$$

Hence t is found, viz.

$$t=-d\cos^{-1}\left\{\sqrt{a^2\cos^2\psi+b^2\sin^2\psi}\sqrt{\frac{(a^2+c^2)\cos^2\psi+(b^2+c^2)\sin^2\psi}{a^2(a^2+c^2)\cos^2\psi+b^2(b^2+c^2)\sin^2\psi}}\right\},$$

the negative sign being prefixed because PY is measured from P in the direction of the measurement of the arc increasing from P to B. (See Art. 729.) Finally then we have

$$\frac{\text{arc } PB}{d}=-\cos^{-1}\left\{\sqrt{a^2\cos^2\psi+b^2\sin^2\psi}\times\sqrt{\frac{(a^2+c^2)\cos^2\psi+(b^2+c^2)\sin^2\psi}{a^2(a^2+c^2)\cos^2\psi+b^2(b^2+c^2)\sin^2\psi}}\right\}$$
$$+\frac{b^2}{a\sqrt{b^2+c^2}}\Pi\left\{\psi,\ \frac{c}{a}\sqrt{\frac{a^2-b^2}{b^2+c^2}},\ -\frac{a^2-b^2}{a^2}\right\}.$$

737. Mr. Burstall's Theorem.

A remarkable property of the curve is established by Mr. Burstall, in vol. xviii. of the *Proceedings of the London Mathematical Society*, giving a result analogous to that of Fagnano for the ellipse.

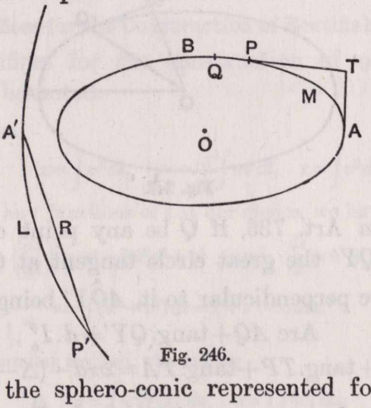

Fig. 246.

Let AB be the sphero-conic represented for convenience upon a plane, and let $A'B'$ be an arc of the reciprocal sphero-conic, A being an end of the major axis of the one and A' being the corresponding point on the reciprocal curve. Let P and P' be corresponding points of the sphero-conic and its reciprocal; and let $A'RP'$ be the great circle chord of the reciprocal sphero-conic; and AT, PT the great circle arcs tangential at A and P.

Then, since the areas $ATPMA$ and $A'LP'RA'$ are reciprocal areas, we have

$$d(\text{Arc } AP + \text{tangent } PT + \text{tangent } TA) = 2\pi d^2 - \text{area of } A'LP'RA'.$$

Now, putting Δ and Δ' for the spherical areas $OA'LP'$ and $OA'RP'$ respectively, $A'\hat{O}P' = \theta'$, and

$$I = \int \frac{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}{\sqrt{(a^2+c^2)\cos^2\theta + (b^2+c^2)\sin^2\theta}}\, d\theta,$$

the same indefinite Legendrian integral that has occurred both in the rectification above and in the quadrature of the reciprocal curve with specified limits, we have

$$\text{Arc } AP + \text{tang. } PT + \text{tang. } TA = 2\pi d - (\Delta - \Delta')/d,$$

and

$$\Delta = d^2 \int_0^{\theta'} (1 - \cos\rho)\, d\theta, \quad \text{where } c^2\cot^2\rho = a^2\cos^2\theta + b^2\sin^2\theta,$$

$$= d^2(\theta' - I_0^{\theta'}),$$

whilst Δ' can be found free from elliptic integrals (Art. 734, Ex. 3).

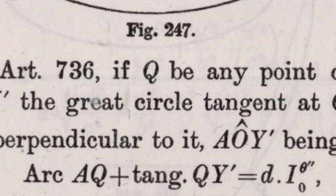

Fig. 247.

Again, as in Art. 736, if Q be any point of the original sphero-conic, QY' the great circle tangent at Q, and OY' the great circle arc perpendicular to it, $A\hat{O}Y'$ being θ'',

$$\text{Arc } AQ + \text{tang. } QY' = d\,.\,I_0^{\theta''},$$

and $\quad \text{Arc } AP + \text{tang. } TP + \text{tang. } TA = 2\pi d - (\Delta - \Delta')/d$

$$= 2\pi d + \frac{\Delta'}{d} - d\,.\,\theta' + d\,.\,I_0^{\theta'}.$$

If then we take the angles $AOQ(\theta'')$ and $A'OP'(\theta')$ equal and eliminate the integral, we have

$$\text{Arc } AP + \text{tang. } TA + \text{tang. } TP + d\,.\,\theta' - \frac{\Delta'}{d}$$
$$= 2\pi d + \text{arc } AQ + \text{tang. } QY',$$

or

$$\text{Arc } AQ - \text{arc } AP = \text{tang. } TA + \text{tang. } TP - \text{tang. } QY' + d\,.\,\theta' - \Delta'/d$$
$$- \text{circumf. of a great circle},$$

giving the difference of two arcs in terms of certain arcs of circles and Δ'.

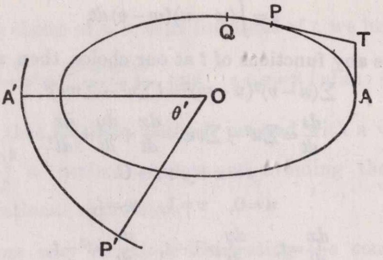

Fig. 248.

Hence we have the difference of the arcs AP, AQ expressed in terms of elementary functions, free from elliptic integrals, which is Mr. Burstall's result, and in its peculiarity resembles Fagnano's result for a plane ellipse.

738. **Artifices for the Construction of Rectifiable Twisted Curves.**

Some artifices for the construction of rectifiable twisted curves may be noted.

1. If we take

$$x = \int u^2\,dt, \quad y = \sqrt{2}\int uv\,dt, \quad z = \int v^2\,dt,$$

where u, v are any functions of t at our choice, we have

$$\left(\frac{ds}{dt}\right)^2 = u^4 + 2u^2v^2 + v^4 \quad \text{and} \quad \frac{ds}{dt} = u^2 + v^2.$$

Hence $$s = \int (u^2+v^2)\,dt = x + z + \text{const.*}$$

* For a very similar method, viz. taking

$$y = \int \sqrt{2f(x)}\,dx, \quad z = \int f(x)\,dx,$$

see Williamson's *Int. Calc.*, p. 244.

E.g. consider the line of intersection of the cylinders

$$3x=z^3, \quad y\sqrt{2}=z^2.$$

Putting
$$z=t, \quad y=\sqrt{2}\frac{t^2}{2}, \quad x=\frac{t^3}{3},$$
$$\frac{dx}{dt}=t^2, \quad \frac{dy}{dt}=\sqrt{2}t, \quad \frac{dz}{dt}=1,$$
we have the case
$$u=t, \quad v=1 \quad \text{and} \quad s=x+z+\text{const.}$$

2. If we take
$$x=\int(u-v)(u-w)\,dt,$$
$$y=\int(v-w)(v-u)\,dt,$$
$$z=\int(w-u)(w-v)\,dt,$$
where u, v, w are any functions of t at our choice, then, since
$$\sum(u-v)^2(u-w)^2=(\sum u^2-\sum vw)^2,$$
we have
$$\frac{ds}{dt}=\sum u^2-\sum vw=\frac{dx}{dt}+\frac{dy}{dt}+\frac{dz}{dt},$$
and
$$s=x+y+z+C.$$

E.g. taking
$$u=0, \quad v=1, \quad w=t,$$
$$\frac{dx}{dt}=t, \quad \frac{dy}{dt}=1-t, \quad \frac{dz}{dt}=t^2-t,$$
$$x=\frac{t^2}{2}, \quad y=t-\frac{t^2}{2}, \quad z=\frac{t^3}{3}-\frac{t^2}{2};$$
whence we have
$$x+y=t\,; \quad x+z=t^3/3\,;$$
$$\therefore \quad (x+y)^2=2x, \quad 3(z+x)=(x+y)^3,$$
the equations of the curve.

And for the rectification,
$$s=\frac{t^3}{3}-\frac{t^2}{2}+t+\text{const.}=x+y+z+C,$$
and any specified limits may be taken.

3. Again, if we take
$$x=\int(v-w)^2dt, \quad y=\int(w-u)^2dt, \quad z=\int(u-v)^2dt,$$
we have
$$\left(\frac{ds}{dt}\right)^2=\sum(v-w)^4=2(\sum u^2-\sum vw)^2,$$
and
$$s=\sqrt{2}\left[\int\sum u^2dt-\int\sum vw\,dt\right]+\text{const.}$$
$$=\frac{1}{\sqrt{2}}\int\left(\frac{dx}{dt}+\frac{dy}{dt}+\frac{dz}{dt}\right)dt+C$$
$$=\frac{1}{\sqrt{2}}(x+y+z)+C,$$
and the values of u, v, w are at our choice, as before.

In all these cases if u, v, w be chosen as rational integral algebraic functions of t, the equations of the curve can be found and its length between any specified limits.

4. Similarly, other algebraic identities which express the sum of three squares as a constant multiple of the square of a fourth expression may be used in the same manner to construct rectifiable twisted curves.

$$E.g.\quad \left[u^2-\left(\frac{v^2+w^2}{2u}\right)^2\right]^2+(v^2-w^2)^2+4v^2w^2=\left[u^2+\left(\frac{v^2+w^2}{2u}\right)^2\right]^2.$$

Hence, putting

$$\frac{dx}{dt}=u^2-\left(\frac{v^2+w^2}{2u}\right)^2,\quad \frac{dy}{dt}=v^2-w^2,\quad \frac{dz}{dt}=2vw,$$

with any arbitrary choice of u, v, w as functions of t, we have

$$\frac{ds}{dt}=u^2+\left(\frac{v^2+w^2}{2u}\right)^2 \quad\text{and}\quad s+x=2\int u^2\,dt.$$

It will be noted that all these methods proceed with a view to making $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2$ a perfect square and avoiding the necessity of integrating an irrational expression.

5. One type more may be given illustrating the construction of a rectifiable twisted curve upon the same plan, but of non-algebraic character. Taking u, v, w any arbitrary functions of t, put

$$x=\int\frac{du}{dt}\sin v\sin w\,dt,\quad y=\int\frac{du}{dt}\sin v\cos w\,dt,\quad z=\int\frac{du}{dt}\cos v\,dt.$$

Then $$\frac{ds}{dt}=\frac{du}{dt}\quad\text{and}\quad s=u+\text{const.}$$

E.g. taking $$v=w=t\quad\text{and}\quad u=\frac{t^2}{2},$$

$$\frac{dx}{dt}=t\sin^2 t,\quad \frac{dy}{dt}=t\sin t\cos t,\quad \frac{dz}{dt}=t\cos t.$$

Then $$\frac{ds}{dt}=t,\quad s=\frac{t^2}{2}+C,$$

the curve being
$$\left.\begin{aligned}8x&=2t^2-2t\sin 2t-\cos 2t,\\ 8y&=\quad\;\; -2t\cos 2t+\sin 2t,\\ z&=\qquad\qquad t\sin t+\cos t.\end{aligned}\right\}$$

Methods 1, 2, 3, 4 either give rise to unicursal twisted curves, viz. those in which the coordinates x, y, z can be expressed as rational algebraic functions of a single parameter t or may be made to give rise to curves in which x, y, z and s are irrational functions of t, this depending upon the choice made for u, v, w.

739. Generalised Formulae.

If the Cartesian coordinates of a point x, y, z be expressed as functions of any other three independent parameters u, v, w, as

$$x=f_1(u, v, w), \quad y=f_2(u, v, w), \quad z=f_3(u, v, w),$$

then $$dx=\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv+\frac{\partial x}{\partial w}dw, \quad dy=\text{etc.}, \quad dz=\text{etc.}$$

And if we write

$$a\equiv\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2, \quad b\equiv\left(\frac{\partial x}{\partial v}\right)^2++, \quad c\equiv\left(\frac{\partial x}{\partial w}\right)^2++,$$

$$f\equiv\frac{\partial x}{\partial v}\frac{\partial x}{\partial w}+\frac{\partial y}{\partial v}\frac{\partial y}{\partial w}+\frac{\partial z}{\partial v}\frac{\partial z}{\partial w}, \quad g=\frac{\partial x}{\partial w}\frac{\partial x}{\partial u}++, \quad h=\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}++,$$

we have, for the element of distance ds between x, y, z and $x+dx, y+dy, z+dz$,

$$ds^2=a\,du^2+b\,dv^2+c\,dw^2+2f\,dv\,dw+2g\,dw\,du+2h\,du\,dv,$$

and for *two* assigned relations between u, v and w, defining a linear path for x, y, z, we have the rectification formula

$$s=\int[a\,du^2+b\,dv^2+c\,dw^2+2f\,dv\,dw+2g\,dw\,du+2h\,du\,dv]^{\frac{1}{2}}.$$

740. If *one relation only* between u, v and w be assigned, x, y, z travels on an assigned surface. Let the relation be

$$\chi(u, v, w)=0.$$

Then $$\frac{\partial\chi}{\partial u}du+\frac{\partial\chi}{\partial v}dv+\frac{\partial\chi}{\partial w}dw=0,$$

and this being a linear relation between du, dv, dw, one of the letters u, v, w, and one of the differentials du, dv, dw may be eliminated, and the square of the linear element ds may then be expressed as

$$ds^2=E\,du^2+2F\,du\,dv+G\,dv^2$$

where the forms of x, y, z are now

$$x=\phi_1(u, v), \quad y=\phi_2(u, v), \quad z=\phi_3(u, v).$$

The values of E, F, G derived from these equations are

$$E=\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2, \quad F=\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}+\frac{\partial z}{\partial u}\frac{\partial z}{\partial v},$$

$$G=\left(\frac{\partial x}{\partial v}\right)^2+\left(\frac{\partial y}{\partial v}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2.$$

741. The quantity $EG-F^2$ is essentially positive.

For $$EG-F^2=\begin{vmatrix} \frac{\partial y}{\partial u}, & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v}, & \frac{\partial z}{\partial v} \end{vmatrix}^2 + \text{two similar expressions}$$

$$=\left\{\frac{\partial(y, z)}{\partial(u, v)}\right\}^2+\left\{\frac{\partial(z, x)}{\partial(u, v)}\right\}^2+\left\{\frac{\partial(x, y)}{\partial(u, v)}\right\}^2$$

$$=J_1^2+J_2^2+J_3^2, \text{ say, and is positive.}$$

742. Eliminating du, dv from the equations

$$dx=\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv, \quad dy=\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv, \quad dz=\frac{\partial z}{\partial u}du+\frac{\partial z}{dv}dv,$$

we have $J_1dx+J_2dy+J_3dz=0$, identically, viz. the differential equation of the surface on which the curve lies.

743. Dr. Salmon (*Solid Geom.*, p. 252) shows that the differential equation of the lines of curvature is

$$\begin{vmatrix} dx & dy & dz \\ J_1 & J_2 & J_3 \\ dJ_1 & dJ_2 & dJ_3 \end{vmatrix}=0,$$

and obtains in terms of u and v a formula for the evaluation of the principal radii of curvature.

744. Now ds^2 is the square of the linear element connecting the point u, v with the point $u+\delta u$, $v+\delta v$, and lies on the surface
$$x=\phi_1(u, v), \quad y=\phi_2(u, v), \quad z=\phi_3(u, v).$$

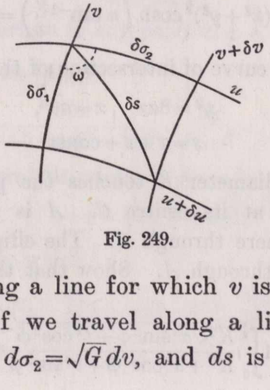

Fig. 249.

If we travel along a line for which v is constant, we have $d\sigma_1=\sqrt{E}\,du$, and if we travel along a line for which u is constant, we have $d\sigma_2=\sqrt{G}\,dv$, and ds is the corresponding diagonal of the infinitesimal parallelogram whose adjacent edges are $d\sigma_1$, $d\sigma_2$. Let ω be the angle between them.

Then $$ds^2 = E\,du^2 + 2\sqrt{EG}\,du\,dv\cos\omega + G\,dv^2;$$

whence it appears that

$$\cos\omega = \frac{F}{\sqrt{EG}} \quad \text{and} \quad \therefore\ \sin\omega = \frac{\sqrt{EG-F^2}}{\sqrt{EG}},$$

and that the area of the elementary parallelogram

$$= d\sigma_1\,d\sigma_2\sin\omega = \sqrt{EG-F^2}\,du\,dv.$$

We therefore have also a formula for the quadrature of the surface, viz.

$$S = \iint\sqrt{EG-F^2}\,du\,dv$$

$$= \iint\sqrt{J_1^2+J_2^2+J_3^2}\,du\,dv.$$

When the two families of curves on the surface, viz. $u=$const., $v=$const., cut orthogonally, we have

$$\cos\omega = 0 \quad \text{and} \quad F=0,$$

and $$s = \int\sqrt{E\,du^2+G\,dv^2}, \quad S = \iint\sqrt{EG}\,du\,dv.$$

This will necessarily be so, for instance, when $u=$const., $v=$const. are the equations of the lines of curvature on the surface.

PROBLEMS.

1. Show that the equations of a Rhumb line on a sphere of radius r may be written as

$$x^2+y^2+z^2=r^2,$$

$$(x^2+y^2)^{\frac{1}{2}}\cosh\left(n\tan^{-1}\frac{y}{x}\right) = r.$$

2. Show that the curve of intersection of the cylinders

$$y^2 = 8ax, \quad x = ae^{\frac{z}{a}},$$

is given by $$s = x + z + \text{const.}$$

3. A sphere of diameter K touches the plane of an ellipse of principal axes a, b at its centre C. A is the other end of the diameter of the sphere through C. The ellipse is projected on to the sphere by lines through A. Show that the length of the curve so described will be

$$4\int_0^{\frac{\pi}{2}}\frac{K^2\sqrt{a^2\sin^2\phi+b^2\cos^2\phi}}{K^2+a^2\cos^2\phi+b^2\sin^2\phi}\,d\phi.$$ [ST. JOHN'S, 1884.]

4. A curve is drawn upon the surface of a sphere such that

$$\phi \sin \theta = \text{const.},$$

ϕ being the longitude and θ the co-latitude of any point.

Show that $s = a \log\left(\dfrac{\tan \theta_1/2}{\tan \theta_2/2}\right)$ is the length of the arc between points where $\theta = \theta_1$ and $\theta = \theta_2$, and a is the radius of the sphere.

Give a sketch showing the nature of the curve $\phi \sin \theta = 1$ upon the sphere $r = a$.

5. Show that the line of intersection of the sphere

$$r = c \cos \theta$$

and the cone $$\tan \theta = \frac{a}{c} e^{\phi \cot \alpha}$$

is rectifiable, and that $$s = c\theta \sec \alpha.$$

Also show that the conical projection of this curve on the sphere upon the tangent plane at the end of the diameter remote from the origin, the origin being the pole of projection, is an equiangular spiral. Hence deduce the same result by inversion.

6. Show that the curve of intersection of the sphere

$$x^2 + y^2 + z^2 - 2az = 0$$

and the cone $$(2x^2 + 2y^2 + zx)^2 = z^2(x^2 + y^2)$$

projects conically from the origin into a cardioide upon the plane $z = 2a$. Hence obtain the rectification of the twisted curve.

7. Show that the length of the arc of intersection of the cylinders

$$\left.\begin{aligned} x^2 &= 2y, \\ x^3 &= 6z, \end{aligned}\right\}$$

measured from the origin to any point x, y, z, is $x + z$.

8. Show that for the curve

$$\frac{60x}{-45 - 40t^2 - 12t^4} = \frac{3y}{3 - t^2} = \frac{z}{t} = t,$$

the arc measured from the origin, is given by

$$s + x = \tfrac{1}{2}\sqrt{z} + \text{const.}$$

9. In the curve for which

$$\frac{dx}{dt} = (1-t)(1-t^2), \quad \frac{dy}{dt} = -t(1-t)^2, \quad \frac{dz}{dt} = t(1-t)^2(1+t),$$

show that $$s = x + y + z + C.$$

10. Show that in the curve of intersection of

$$r = a\cos\theta \quad \text{and} \quad \cos 2\theta = \tan^2\phi,$$

$$s = \frac{a}{2\sqrt{2}}\left[2E\left(\chi, \frac{1}{\sqrt{2}}\right) - \frac{\sin\chi\cos\chi}{\sqrt{1-\frac{1}{2}\sin^2\chi}}\right],$$

where $$\sin\chi = \sqrt{2}\sin\phi.$$

Show that the inverse of this curve with regard to the origin is a lemniscate, the constant of inversion being a.

11. Show that the rectification of the line of intersection of

$$x^2 + y^2 + z^2 = cz \quad \text{and} \quad y^2 = 4cx$$

is given by

$$s = \frac{c}{2^{\frac{1}{4}}3^{\frac{1}{2}}}\left\{\frac{1}{\sqrt{\cos\frac{\pi}{12}}}\tan^{-1}\left(\frac{\sin\theta}{2^{\frac{1}{4}}\sqrt{\cos\frac{\pi}{12}}}\right) + \frac{1}{\sqrt{\sin\frac{\pi}{12}}}\left(\tanh^{-1}\frac{\sin\theta}{2^{\frac{1}{4}}\sqrt{\sin\frac{\pi}{12}}}\right)\right\}$$

where $$\tan\theta = \sqrt{\frac{x}{c}},$$

and show that this curve can be inverted into a parabola lying upon a tangent plane to the sphere.

12. A Loxodrome is drawn on a sphere to cut all the meridians at the same constant angle α; show that the area of the surface of the sphere, included between any arc of this curve and the two meridians through its ends, is

$$a^2\tan\alpha\log\frac{1+\sin\psi_1}{1+\sin\psi_2},$$

where ψ_1 and ψ_2 are the latitudes of the ends of the arc and a is the radius of the sphere. [Ox. II. P., 1900.]