ANSWERS TO EXAMPLES AND PROBLEMS.

VOLUME I.

CHAPTER I

PAGE 12.

1. $\dfrac{b^2-a^2}{2}, \dfrac{b^3-a^3}{3}$. 2. $\frac{8}{3}a^2$. 4. $\frac{1}{3}\pi h^3 \tan^2\alpha$.

5. Gradient at $x=15$, $36^\circ\ 20'$; slope $=\cdot 735$. Slope at 9·5 is $\dfrac{y}{x}$,

$$\int_{11}^{15} y\,dx = 17\cdot 4 \text{ square units.}$$

PAGE 15

1. $\dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{6}, \dfrac{1}{n+1}$. 2. $1, 1, 1, \sqrt{2}-1$. 3. $\dfrac{\pi}{4}, \dfrac{\pi}{2}, \log 2, e-1$.

PAGE 25.

1. $\frac{4}{3}\pi ab^2$. 3. $\bar{x}=\dfrac{m+1}{m+2}a$. 4. $\dfrac{m+1}{m+3}$. Mass . a^2.

6. Using paper ruled to 10ths and 5 inches to represent unity on each of the axes, the area $=\cdot 78500$. As this should be $\dfrac{\pi}{4}$, we have the approximation $\pi=3\cdot 1400$, the true value being 3·141592..., showing an error of about ·05 per cent.

PAGE 28.

1. Harmonic oscillation. 2. $\int_{x_0}^{x} y\,dx$. 4. $\frac{4}{3}\pi a^2 b$. 5. $2\pi ah^2$. 7. c^t/t.

10. Mean by trapezoidal rule with unit increments $=23\cdot 78$.
True result $=23\cdot 026\ldots$ (Unit increments are, however, too large for a very exact result.)

$$\int_1^{10} 10x^{-0\cdot 9}\,dx = 25\cdot 9\,; \quad \int_1^{10} 10x^{-1\cdot 1}\,dx = 20\cdot 6\,;$$

$$\int_1^{10} 10x^{-0\cdot 99}\,dx = 23\,; \quad \int_1^{10} 10x^{-1\cdot 01}\,dx = 22.$$

13. About 141,550 cubic yards.

14. (1) $\bar{x}=\frac{2}{3}a$, Mom. In. $=M\frac{a^2}{2}$; (2) $\bar{x}=\frac{3}{4}a$, Mom. In. $=\frac{3}{5}Ma^2$, where $M\equiv$ mass, $a\equiv$ length.

15. $M=\dfrac{2\pi\rho_0 a^2}{n+2}$ if $\rho_0=$ density at the edge. Mom. In. $=\dfrac{n+2}{n+4}Ma^2$.

17. About 213 tons. **20.** 13,863 foot-lbs., 10,574 foot-lbs.

25. Taking ordinates at 10° intervals and four figure tables, the trapezoidal rule gave ·2501π, the true value being $\dfrac{\pi}{4}$.

29. $$\text{Area}=\frac{A}{3}(c^3-a^3)+\frac{B}{2}(c^2-a^2)+C(c-a),$$
where $A=-\Sigma(b-c)y_1/\Pi$, $B=\Sigma(b^2-c^2)y_1/\Pi$, $C=-\Sigma bcy_1/\Pi$, $\Pi=(b-c)(c-a)(a-b)$.

30. True values (1) $=25\pi$ and (2) $100+25\pi$. **33.** 59 c.c., q.p.

35. $\dfrac{\pi}{4}a^2c+(b-a)ac+\dfrac{\pi}{8}c(b-a)^2$ cubic inches, 3438·3 cubic inches.

36. Binomial Expansion to 3 terms gives ·1204, q.p.
Graphically with $\frac{1}{10}=1$ linear inch, the trapezoidal rule gave ·1178. When this was corrected for curvature of the arcs by the approximate addition of small squares, the approximation was ·1203.

40. 8465·7 **41.** Perimeter $=30·1026$ cm., q.p.

42. The true value is $\dfrac{\pi}{2}$. This will appear later.

43. When t is large I becomes $\dfrac{V}{R}$ and Q becomes $\dfrac{V}{R}t-\dfrac{VL}{R^2}$.

44. $Q=at+\dfrac{bt^2}{2}-c\dfrac{t^3}{3}$, $V=aR+bL+(bR-2cL)t-cRt^2$.

45. In the 'Otto Cycle' of operations there is one explosion for two revolutions. About 16 H.P.

46. Weddle's rule gives $-1·08873$; true value $-1·08878$.

48. $5\frac{1}{15}$ miles. **53.** ·821, q.p.

CHAPTER II.

Page 51.

1. $\dfrac{x^{11}}{11}$, $-\dfrac{x^{-9}}{9}$, x, C, $\dfrac{5}{12}x^{\frac{12}{5}}$, $\dfrac{7}{2}x^{\frac{2}{7}}$, $3x^{\frac{1}{3}}$, $2\sqrt{x}$, $\dfrac{2}{13}x^{\frac{13}{2}}$.

2. $\dfrac{2}{3}ax^{\frac{3}{2}}+2bx^{\frac{1}{2}}$, $\dfrac{p}{p+1}ax^{\frac{p+1}{p}}+\dfrac{p}{p-1}bx^{\frac{p-1}{p}}$,
$\dfrac{pqacx^{(p+q+pq)/pq}}{p+q+pq}+\dfrac{padx^{(p+1)/p}}{p+1}+\dfrac{qbcx^{(q+1)/q}}{q+1}+bdx$, $ax+b\log x-\dfrac{c}{x}$.

3. $ac\dfrac{x^2}{2}+b(a+c)x+(a^2+b^2+c^2)\log x-\dfrac{b}{x}(a+c)-\dfrac{ac}{2x^2}$,

$-\log(a-x)$, $\dfrac{1}{a-x}$, $\dfrac{(a-x)^{1-p}}{p-1}$.

4. $\log\dfrac{a+x}{a-x}$, $x-a\log(a+x)$, $\dfrac{2x}{a^2-x^2}$, $x+\dfrac{a^4}{3x^3}$.

5. $\frac{20}{13}\cdot 2^{\frac{3}{10}}=1{\cdot}894\ldots$, $\frac{3}{2}\left(5^{\frac{2}{3}}-3^{\frac{2}{3}}\right)$, $\frac{1}{2}\log\frac{7}{5}$. **6.** $832421\frac{2}{3}$.

7. $\dfrac{a}{2}(7+\log 4)$. **8.** In 5 seconds at a distance of 25 feet.

9. $400\log_e 2$. The integration is that of finding the work done in allowing a gas to expand according to Boyle's law from $v=10$ to $v=20$. If p and v be in lbs.-wt. per sq. foot and in cubic feet respectively, the result is in foot-lbs.

10. $8\frac{11}{30}$, $-\frac{19}{30}$, $\frac{11}{30}$, $-\frac{19}{30}$, $8\frac{11}{30}$. The portions are alternately above and below the x-axis.

11. $\dfrac{(ae^x+b)^{n+1}}{(n+1)a}$, $\dfrac{c}{a}\log(ae^x+b)$, $\dfrac{1}{n+1}\left(ax+\dfrac{b}{x}+c\right)^{n+1}$, $\dfrac{(ax^p+be^x)^{n+1}}{n+1}$.

12. $\log(e^{ax}+e^{bx})$, $\frac{1}{2}\log\sin 2x$, $\log\cosh x$, $\dfrac{(ax^{2n}+bx^n+c)^{1-p}}{(1-p)n}$.

13. $\log\tan^{-1}x$, $-\dfrac{1}{n-1}\dfrac{1}{(\tan^{-1}x)^{n-1}}$, $\dfrac{(\sin^{-1}x)^{n+1}}{n+1}$, $\log\sin^{-1}x$, $\log\operatorname{vers}^{-1}x$.

14. $\log\log x$, $\log\log\log x$, $\dfrac{(\log\log\log x)^{1-n}}{1-n}$, $\dfrac{(l^{r+1}x)^{1-n}}{1-n}$.

PAGE 53.

1. $\log(x+1)$, $x-2a\log(x+a)$, $\dfrac{1}{2}\log(x^2+a^2)$, $\dfrac{1}{2}\log(x^2+a^2)+\tan^{-1}\dfrac{x}{a}$,

$\dfrac{1}{3}\log(x^3+a^3)$, $\dfrac{1}{n}\log(x^n+a^n)$.

2. $\dfrac{2^x}{\log 2}$, x^2, $2\log x$, $\dfrac{x^3}{3}$, $\dfrac{x^4}{4}+\dfrac{3^x}{\log 3}$, $ax+\dfrac{b^x}{\log b}+\dfrac{c^{2x}}{2\log c}+\dfrac{d^{3x}}{3\log d}$.

3. $\dfrac{x+\sin x}{2}$, $\dfrac{x-\sin x}{2}$, $\log\tan x$, $\log\sin x-\operatorname{cosec} x$.

4. $\sin^{-1}\dfrac{x}{3}$, $\sinh^{-1}\dfrac{x}{3}$, $\cosh^{-1}\dfrac{x}{3}$, $\dfrac{1}{3}\tan^{-1}\dfrac{x}{3}$, $\dfrac{1}{6}\log\dfrac{3+x}{3-x}\equiv\dfrac{1}{3}\tanh^{-1}\dfrac{x}{3}$, $\dfrac{1}{6}\log\dfrac{x-3}{x+3}$.

5. $\dfrac{1}{2}\sec^{-1}\dfrac{x}{2}$, $\cosh^{-1}\dfrac{x}{2}+\dfrac{1}{2}\operatorname{sech}^{-1}\dfrac{x}{2}$, $-a\sqrt{c^2-x^2}+b\sin^{-1}\dfrac{x}{c}$,

$a\sqrt{x^2-c^2}+b\cosh^{-1}\dfrac{x}{c}$, $a\sqrt{x^2+c^2}+b\sinh^{-1}\dfrac{x}{c}$.

6. $\sin^{-1}(2x-1)$, $\dfrac{1}{3\sqrt{3}}\sec^{-1}\dfrac{x}{3}$, $\dfrac{1}{\sqrt{3}}\sin^{-1}\dfrac{x}{3}$, $x-4\tan^{-1}\dfrac{x}{2}$,

$x+2\log\dfrac{x-2}{x+2}\equiv\log\left\{e^x\left(\dfrac{x-2}{x+2}\right)^2\right\}$.

7. (i) $-\frac{1}{2}\operatorname{cosec}^2 x$, (ii) $\log\tan x$, (iii) $\dfrac{(e^x+a)^{n+1}}{n+1}$,

(iv) $\dfrac{1}{3(n+1)}(x^3+a^3)^{n+1}$, (v) $\dfrac{1}{n+1}(ax^3+bx+c)^{n+1}$.

8. (i) $\log\tan^{-1}x$, (ii) $-\dfrac{1}{\sin^{-1}x}$, (iii) $-\dfrac{1}{2(\log x)^2}$.

9. (i) $\log\frac{7}{3}$, (ii) $\frac{4}{21}$.

11. (i) $\dfrac{1}{4}(e^{4x}-1)$, (ii) $\dfrac{2}{n}(e^{nx}-1)$, (iii) $e-e^{-1}$, (iv) $\dfrac{b^2-a^2}{4}+\dfrac{1}{2}\log\dfrac{b}{a}$.

12. (i) 1, (ii) $\dfrac{\pi}{4}$, (iii) $\dfrac{1}{2}$, (iv) $\sinh x+\sin x$.

13. (i) $\dfrac{1}{n}$, (ii) $\sqrt{2}-1$, (iii) $\dfrac{\pi}{12}$, (iv) $\dfrac{\pi}{2}$.

14. (i) $\dfrac{x^n}{n}+\dfrac{ax^{n-1}}{n-1}+\dfrac{a^2x^{n-2}}{n-2}+\ldots+\dfrac{a^{n-1}x}{1}$, (ii) Last result $+a^n\log(x-a)$,

(iii) $\dfrac{x^5}{5}+\dfrac{x^4}{4}+\dfrac{x^3}{3}+\dfrac{x^2}{2}-\dfrac{x}{1}$, (iv) $\dfrac{x^5}{5}+\dfrac{x^4}{4}+\dfrac{x^3}{3}+\dfrac{x^2}{2}+x+\log(x-1)$, (v) $\dfrac{x^2}{2}-3x$.

Page 56.

1. (1) $\log x-\dfrac{(a+b+c)}{x}-\dfrac{ab+bc+ca}{2x^2}-\dfrac{abc}{3x^3}$.

(2) $\dfrac{x^3}{3}-(a+b)\dfrac{x^2}{2}+(a^2-ab+b^2)x$. (3) x.

(4) $\log(a\sin x+b\cos x+c)$. (5) $\dfrac{x^{a+1}}{a+1}+\dfrac{a^x}{\log a}$.

(6) $\dfrac{1}{6}\left(\tan^{-1}\dfrac{x}{3}\right)^2$. (7) $\log\tan x$. (8) $-\operatorname{cosec}x+\log\sin x$.

(9) $-\cot\dfrac{x}{2}$. (10) $-\cos\left(x+\dfrac{\pi}{4}\right)$. (11) $\tan x-\tan^{-1}x$.

(12) $\tan x+\log\sec x$. (13) $\sec x+\log\sec x$.

(14) $a\sec x-b\operatorname{cosec}x$. (15) $-2(\operatorname{cosec}x+\sec x)$.

(16) $\frac{1}{3}\tan^3x+\dfrac{a+b}{2}\tan^2x+ab\tan x$.

(17) $\tan^{-1}\log x$ (18) $\sin\log x$.

(19) $\dfrac{x^5}{5}+\dfrac{x^4}{4}+\dfrac{x^3}{3}+\dfrac{x^2}{2}+x+2\log(x-1)$. (20) $\dfrac{1}{a}\tan^{-1}(ae^x)$.

18. $\frac{1}{3}$ of a mile. **19.** $\frac{4}{3}a^3b$; about 9 feet. **20.** $\dfrac{dx}{dt}=-ax$, $\dfrac{dy}{dt}=ax-by$.

22. $\dfrac{dz}{dx}=$ the ordinate PQ; $\dfrac{d^2z}{dx^2}=$ tangent of angle the tangent at Q makes with OK; $y=a\sec^2\dfrac{x}{a}$.

23. $y=ae^{\frac{x-h}{h}}$, $y=14{\cdot}778\ldots$.

24. Approx. value given by formula ·122422. True value ·122416.

26. $-\dfrac{x^4}{4!}e^{-x}$, $e^x\displaystyle\int_0^x\dfrac{a^{n-1}}{(n-1)!}e^{-a}\,da$. **27.** True value of integral $=\pi$.

28. (1) $p_1v_1\log\frac{v_2}{v_1}$; $\frac{p_2v_2{}^\gamma}{1-\gamma}(v_3{}^{1-\gamma}-v_2{}^{1-\gamma})$; $-p_3v_3\log\frac{v_4}{v_3}$; $-\frac{p_4v_4{}^\gamma}{1-\gamma}(v_1{}^{1-\gamma}-v_4{}^{1-\gamma})$.

29. 97·25 units.

33. $y=x+\frac{x^2}{2}+\frac{x^3}{6}+*-\frac{x^5}{40}\dots$

35. $-\left[\frac{z^{2n+1}}{2n+1}+n(1-c)\frac{z^{2n}}{2n}+\frac{n(n-1)}{1.2}(1-c)^2\frac{z^{2n-1}}{2n-1}+\dots+(1-c)^n\frac{z^{n+1}}{n+1}\right]$,
where $z=\frac{1-x}{x}$.

37. $\frac{\sin 3\theta}{3}-\frac{\sin 2\theta}{2}$.

38. $f(x)\equiv 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots\equiv\cos x$. $F(x)\equiv\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\equiv\sin x$.

42. $\left[-16\frac{\cos^5\theta}{5}-8\frac{\cos^4\theta}{4}-12\frac{\cos^3\theta}{3}+4\frac{\cos^2\theta}{2}+\cos\theta\right]$.

CHAPTER III.

Page 75

1. (i) $\log(1+x^3)$, (ii) $\tan^{-1}\frac{1}{7}$, (iii) $\frac{\pi}{3}$, (iv) $\tan^{-1}\left(\frac{e-1}{e+1}\right)$,
(v) $\tan^{-1}\left(\frac{e^x}{e^x+1}\right)$, (vi) $\frac{1}{3}\tan^3 x$, (vii) $\frac{1}{m}\tanh mx$.

2. (i) $\frac{\pi a^2}{4}$, (ii) $\frac{\pi a^2}{2}$.

3. (i) $\frac{a^3}{3}$, (ii) $\frac{\pi a^4}{16}$.

4. $\sin^{-1}\frac{1}{c}\left(ax+\frac{b}{x}\right)$.

5. $\frac{1}{2n+1}\frac{1}{a^2}\left(\frac{x}{\sqrt{x^2+a^2}}\right)^{2n+1}$.

6. $5\tan^{\frac{1}{5}}\theta$.

7. $\frac{2}{3}\tan^{\frac{3}{2}}x$.

8. (i) $\frac{1}{2}\sec^{-1}x^2$, (ii) $-\frac{1}{2}\operatorname{sech}^{-1}x^2$, (iii) $-\frac{1}{2}\operatorname{cosech}^{-1}x^2$.

9. (i) $e^{x+\frac{1}{x}}$, (ii) $\tan^{-1}\left(ax+\frac{b}{x}\right)$, (iii) $\frac{\left(ax+\frac{b}{x}\right)^{n+1}}{n+1}$.
(iv) $\frac{1}{bc-ae}\sin\frac{a+bx}{c+ex}$, (v) $\frac{1}{a}e^{a\tan^{-1}x}$, (vi) $\frac{1}{a}e^{a\sin^{-1}x}$,
(vii) $\frac{\log(a^2\cos^2 x+b^2\sin^2 x)}{2(b^2-a^2)}$.

10. (i) $\phi(x)\,\psi(x)$, (ii) $\frac{\psi(x)}{\phi(x)}$, (iii) $\tan^{-1}\phi(x)$,
(iv) $e^{\phi(x)}$, (v) $e^{-\psi(x)}\log\phi(x)$.

Page 98.

1. $\frac{1}{6}\log\frac{3+x}{3-x}$, $\frac{1}{12}\log\frac{3+2x}{3-2x}$, $\frac{1}{4}\log\frac{x-2}{x+2}$, $\frac{1}{12}\log\frac{3x-2}{3x+2}$,
$\frac{x}{2}\sqrt{16-9x^2}+\frac{8}{3}\sin^{-1}\frac{3x}{4}$, $\frac{x}{2}\sqrt{3x^2-5}-\frac{5}{6}\sqrt{3}\cosh^{-1}\frac{x\sqrt{3}}{\sqrt{5}}$,
$\frac{x}{2}\sqrt{3x^2+2}+\frac{\sqrt{3}}{3}\sinh^{-1}\left(\frac{x\sqrt{3}}{2}\right)$.

2. $2\cosh^{-1}\sqrt{\frac{x}{4}},\ \ 2\sin^{-1}\sqrt{\frac{x}{4}},\ \ 2\sinh^{-1}\sqrt{\frac{x}{4}},\ \ \sin^{-1}\frac{x-1}{\sqrt{3}},$

$\sinh^{-1}(x-1),\ \ \frac{x+a}{2}\sqrt{x^2+2ax}-\frac{a^2}{2}\cosh^{-1}\frac{x+a}{a}.$

3. $-\sqrt{9-x^2},\ \ \sqrt{x^2-9},\ \ -\frac{1}{4}\sqrt{9-4x^2},\ \ \frac{1}{2}(\sin^{-1}x-x\sqrt{1-x^2}),$
$\frac{x\sqrt{1+x^2}}{2}-\frac{1}{2}\sinh^{-1}x.$

4. $\frac{1}{3}(x^2+a^2)^{\frac{3}{2}},\ \ \frac{1}{3}(x^2+a^2)^{\frac{3}{2}}+\frac{b}{2}\left[x(x^2+a^2)^{\frac{1}{2}}+a^2\sinh^{-1}\frac{x}{a}\right],$

$a\sqrt{x^2+c^2}+b\sinh^{-1}\frac{x}{c}.$

5. $\frac{1}{n+2}(x^2+a^2)^{\frac{n+2}{2}},\ \frac{1}{n+2}(x^2+2ax+b)^{\frac{n+2}{2}},\ \frac{1}{n+2}(ax^2-2bx+c)^{\frac{n+2}{2}}.$

6. $\frac{7}{2}\sin^{-1}x-\frac{x+4}{2}\sqrt{1-x^2},\ \ \frac{5}{2}\sinh^{-1}x+\frac{x+4}{2}\sqrt{x^2+1},$

$\frac{15}{8}\sinh^{-1}\frac{2x+1}{\sqrt{3}}+\frac{2x+5}{4}\sqrt{x^2+x+1}\,;$

$\frac{2x+4a-3c}{4}\sqrt{x^2+cx+d}+\frac{1}{8}(8b-4d-4ac+3c^2)\sinh^{-1}\frac{2x+c}{\sqrt{4d-c^2}},$
if $c^2<4d$, with a similar result if $c^2>4d$.

7. $\frac{x+2}{2}\sqrt{x^2+4x+5}+\frac{1}{2}\sinh^{-1}(x+2),\ \frac{x-2}{2}\sqrt{-x^2+4x+5}+\frac{9}{2}\sin^{-1}\frac{x-2}{3},$

$\frac{2x+1}{4}\sqrt{4x^2+4x+5}+\sinh^{-1}\frac{2x+1}{2},$

$\frac{2x-1}{4}\sqrt{-4x^2+4x+5}+\frac{3}{2}\sin^{-1}\frac{2x-1}{\sqrt{6}}.$

8. $\sqrt{x^2-a^2}+a\cosh^{-1}\frac{x}{a},\ \ a\sin^{-1}\frac{x}{a}-\sqrt{a^2-x^2},\ \ \frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{x+2a}{2}\sqrt{a^2-x^2},$

$(x+a+b)\sqrt{x^2-b^2}+\frac{b}{2}(2a+b)\cosh^{-1}\frac{x}{b},\ \ \frac{x+4a}{2}\sqrt{x^2-a^2}+\frac{3a^2}{2}\cosh^{-1}\frac{x}{a}.$

9. $\frac{1}{n}\log\tan\frac{nx}{2},\ \frac{1}{2}\log\tan\left(x+\frac{b}{2}\right),\ \frac{1}{3}\log\tan\left(\frac{3x}{2}+\frac{\pi}{4}\right),$

$\frac{1}{2}\log\tan\left(x+\frac{\pi}{4}\right),\ \frac{1}{2}\log\tan x.$

10. $\frac{1}{\sqrt{a^2+b^2}}\log\tan\frac{1}{2}\left(x+\tan^{-1}\frac{b}{a}\right),\ \ \frac{1}{2\sqrt{2}}\log\tan\left(x+\frac{\pi}{8}\right),$

$\frac{ac+bd}{c^2+d^2}\,x+\frac{bc-ad}{c^2+d^2}\log(c\sin x+d\cos x).$

13. $\log\{\operatorname{cosec}\theta\,(1-\sqrt{1-\sin^{2n}\theta})^{\frac{1}{n}}\}.$

Page 99.

2. $b^2\sin^{-1}\frac{x_1}{a},\ \pi b^2.$

3. $\sqrt{e^{2x}+ae^x}+a\log(\sqrt{e^x+a}+\sqrt{e^x}).$

4. (i) $\sin^{-1}\dfrac{2x+3}{\sqrt{13}}$; (ii) $\dfrac{1}{\sqrt{6}}\cos^{-1}\dfrac{12-x}{5x}$;

(iii) $\dfrac{1}{2}\sqrt{2x^2+3x+4}+\dfrac{1}{4\sqrt{2}}\sinh^{-1}\dfrac{4x+3}{\sqrt{23}}$;

(iv) $\sqrt{x^2+2x-1}-2\cosh^{-1}\dfrac{x+1}{\sqrt{2}}$; (v) $3\sqrt{x^2+2x+5}+\sinh^{-1}\dfrac{x+1}{2}$

7. (i) $\dfrac{1}{3}\log\dfrac{\sqrt{1+x^6}-1}{x^3}$; (ii) $\sqrt{\dfrac{x-1}{x+1}}$.

9. $\sqrt{e^{2x}+e^x+1}+\frac{1}{2}\sinh^{-1}\dfrac{2e^x+1}{\sqrt{3}}-\sinh^{-1}\dfrac{2e^{-x}+1}{\sqrt{3}}$.

10. Mass $=\dfrac{4\pi k}{n+3}a^{n+3}$, where density $=kr^n$ and a is the radius.

(i) Mass $=4\pi ak$; (ii) $2\pi^2 k$.

11. $\dfrac{ap^3}{12}$, a being BC and p the perpendicular from A upon BC.

13. $\log x=\pm\dfrac{2}{3b^2}\sqrt{a^2+b^2y^3}+\text{const.}$

15. (i) $-\dfrac{1}{\sqrt{-a}}\sin^{-1}\dfrac{\sqrt{-aR}}{\sqrt{b^2-ac}}$ ($a-^{\text{ve}}$), $-\dfrac{1}{\sqrt{a}}\sinh^{-1}\dfrac{\sqrt{aR}}{\sqrt{b^2-ac}}$ ($b^2>ac$, $a+^{\text{ve}}$),

$-\dfrac{1}{\sqrt{a}}\cosh^{-1}\dfrac{\sqrt{aR}}{\sqrt{ac-b^2}}$ ($b^2<ac$, $a+^{\text{ve}}$),

where $R\equiv a\cos^2\theta+2b\cos\theta+c$;

(ii) $\dfrac{1}{\sqrt{-a}}\sin^{-1}\dfrac{\sqrt{-aR}}{\sqrt{b^2-ac}}$ ($a-^{\text{ve}}$), $\dfrac{1}{\sqrt{a}}\sinh^{-1}\dfrac{\sqrt{aR}}{\sqrt{b^2-ac}}$ ($b^2>ac$, $a+^{\text{ve}}$),

$\dfrac{1}{\sqrt{a}}\cosh^{-1}\dfrac{\sqrt{aR}}{\sqrt{ac-b^2}}$ ($b^2<ac$, $a+^{\text{ve}}$),

where $R\equiv a\sin^2\theta+2b\sin\theta+c$;

(iii) $\dfrac{1}{\sqrt{-c}}\sin^{-1}\dfrac{\sqrt{-cR}}{\sqrt{b^2-ac}}$ ($c-^{\text{ve}}$), $\dfrac{1}{\sqrt{c}}\sinh^{-1}\dfrac{\sqrt{cR}}{\sqrt{b^2-ac}}$ ($b^2>ac$, $a+^{\text{ve}}$),

$\dfrac{1}{\sqrt{c}}\cosh^{-1}\dfrac{\sqrt{cR}}{\sqrt{ac-b^2}}$ ($b^2<ac$, $c+^{\text{ve}}$),

where $R\equiv c\tan^2\theta+2b\tan\theta+a$;

(iv) $-\dfrac{1}{\sqrt{-a}}\sin^{-1}\dfrac{\sqrt{-aR}}{\sqrt{b^2-ac}}$ ($a-^{\text{ve}}$), $-\dfrac{1}{\sqrt{a}}\sinh^{-1}\dfrac{\sqrt{aR}}{\sqrt{b^2-ac}}$ ($b^2>ac$, $a+^{\text{ve}}$),

$-\dfrac{1}{\sqrt{a}}\cosh^{-1}\dfrac{\sqrt{aR}}{\sqrt{ac-b^2}}$ ($b^2<ac$, $a+^{\text{ve}}$),

where $R\equiv a\cot^2\theta+2b\cot\theta+c$;

(v) $-\dfrac{1}{\sqrt{a+c}}\sinh^{-1}\left(\sqrt{\dfrac{a+c}{b+c}}\cot\theta\right)$,

if $\dfrac{b+c}{a+c}$ be $+^{\text{ve}}$, and a modification (Art. 77) if $\dfrac{b+c}{a+c}$ be $-^{\text{ve}}$.

16. (i) $\frac{a^4}{8}\left[3\sin^{-1}\frac{x}{a}-\frac{x}{a^4}(2x^2+3a^2)\sqrt{a^2-x^2}\right]$;

(ii) $\frac{\theta}{b}+\frac{1}{b}\sqrt{\frac{a}{a+bc^2}}\tan^{-1}\left(\sqrt{\frac{a}{a+bc^2}}\cot\theta\right)$, where $\theta=\sin^{-1}\frac{x}{c}$,

provided $\frac{a}{a+bc^2}$ be positive, with a modification (Art. 89, 17 and 18) if negative.

17. (i) 48; (ii) $\frac{2b}{3a^2}(3ac-2b^2)$; (iii) $\frac{3bc}{a}$.

22. $\frac{2}{\sqrt{a-c}}\tan^{-1}\sqrt{\frac{c+x}{a-c}}\ (a>c)$; $\frac{1}{\sqrt{c-a}}\log\frac{\sqrt{c+x}-\sqrt{c-a}}{\sqrt{c+x}+\sqrt{c-a}}\ (a<c)$;

$-\frac{d}{da}\left\{\frac{2}{\sqrt{a-c}}\tan^{-1}\sqrt{\frac{c+x}{a-c}}\right\}$; $-2\frac{d}{dc}\left\{\frac{2}{\sqrt{a-c}}\tan^{-1}\sqrt{\frac{c+x}{a-c}}\right\}$.

23. (i) $a>c$, $\frac{1}{2a\sqrt{a^2-c^2}}\log\frac{a\sin\phi-\sqrt{a^2-c^2}}{a\sin\phi+\sqrt{a^2-c^2}}$, where $\phi=\cos^{-1}\frac{c}{x}$;

(ii) $a<c$, $\frac{1}{a\sqrt{c^2-a^2}}\tan^{-1}\frac{a\sin\phi}{\sqrt{c^2-a^2}}$.

26. (i) $\frac{1}{\sqrt{2}}\sinh^{-1}\frac{x\sqrt{2}}{1-x^2}$; (ii) $\frac{1}{\sqrt{2}}\sin^{-1}\frac{x\sqrt{2}}{1+x^2}$. **30.** $\frac{a}{\sin\alpha\cos\alpha}$.

CHAPTER IV.

Page 113.

1. $\frac{e^{3x}}{9}(3x-1)$, $\frac{e^{ax}}{a^3}(a^2x^2-2ax+2)$,

$-e^{-x}(x^5+5x^4+5.4x^3+5.4.3x^2+5.4.3.2x+5.4.3.2.1)$,

$x\sinh x-\cosh x$, $(x^2+2)\cosh x-2x\sinh x$.

2. $x\sin x+\cos x$, $\left(\frac{x^5}{2}-\frac{5.4x^3}{2^3}+\frac{5.4.3.2x}{2^5}\right)\sin 2x$

$$+\left(\frac{5x^4}{2^2}-\frac{5.4.3x^2}{2^4}+\frac{5.4.3.2.1}{2^6}\right)\cos 2x.$$

$\frac{x^3}{6}+\frac{1}{8}(2x^2-1)\sin 2x+\frac{x}{4}\cos 2x$,

$\frac{1}{2}\left[x^2\left(\frac{\cos 2x}{2}-\frac{\cos 4x}{4}\right)-x\left(\frac{\sin 2x}{2}-\frac{\sin 4x}{8}\right)-\left(\frac{\cos 2x}{4}-\frac{\cos 4x}{32}\right)\right]$

$-\frac{x}{8}\left(\frac{\cos 2x}{1}+\frac{\cos 4x}{2}-\frac{\cos 6x}{3}\right)+\frac{1}{16}\left(\frac{\sin 2x}{1^2}+\frac{\sin 4x}{2^2}-\frac{\sin 6x}{3^2}\right)$.

3. $\frac{1}{\sqrt{5}}e^x\sin(2x-\tan^{-1}2)$, $\frac{e^x}{2}-\frac{e^x}{2\sqrt{5}}\cos(2x-\tan^{-1}2)$,

$\frac{1}{4}\frac{e^{3x}}{\sqrt{13}}\sin(2x-\tan^{-1}\tfrac{2}{3})-\frac{1}{40}e^{3x}\sin(4x-\tan^{-1}\tfrac{4}{3})$,

$\frac{e^{-5x}}{8}\left[+\tfrac{1}{5}+\frac{1}{\sqrt{29}}\cos(2x+\tan^{-1}\tfrac{2}{5})+\frac{1}{\sqrt{41}}\cos(4x+\tan^{-1}\tfrac{4}{5})\right.$

$$\left.-\frac{1}{\sqrt{61}}\cos(6x+\tan^{-1}\tfrac{6}{5})\right].$$

4. $\frac{x^4}{4}\log x-\frac{x^4}{16}$; $\frac{x^{n+1}}{n+1}\log x-\frac{x^{n+1}}{(n+1)^2}$;

$\frac{x^{n+1}}{n+1}\left[(\log x)^2-\frac{2}{n+1}\log x+\frac{2}{(n+1)^2}\right]$;

$\frac{x^{n+1}}{n+1}\left[(\log x)^3-\frac{3}{n+1}(\log x)^2+\frac{6}{n+1}(\log x)-\frac{6}{(n+1)^3}\right]$.

5. $$\frac{e^{ax}}{4}\left\{\frac{\sin\left\{(q+r-p)x-\tan^{-1}\frac{q+r-p}{a}\right\}}{[(q+r-p)^2+a^2]^{\frac{1}{2}}}+\text{two similar terms}-\frac{\sin\left\{(p+q+r)x-\tan^{-1}\frac{p+q+r}{a}\right\}}{\sqrt{(p+q+r)^2+a^2}}\right\};$$

$$\frac{e^{ax}}{4}\left\{\frac{\cos\left\{(q+r-p)x-\tan^{-1}\frac{q+r-p}{a}\right\}}{\sqrt{(q+r-p)^2+a^2}}+\text{etc.}-\text{etc.}-\text{etc.}\right\}.$$

6. $8\sin px\sin qx\cos^2 rx=2\cos(p-q)x+\cos(p-q+2r)x$
$+\cos(p-q-2r)x-2\cos(p+q)x-\cos(p+q+2r)x-\cos(p+q-2r)x.$

Then apply rule for $\int e^{ax}\cos Nx\,dx$ to each term.

$8\cos px\cos qx\cos^2(p+q)x=2\cos(p+q)x+2\cos(p-q)x+\cos(p+q)x$
$+\cos 3(p+q)x+\cos(3p+q)x+\cos(3q+p)x=\sum A\cos Nx$, say.

Then $$\text{Integral}=\sum A\,\frac{e^{ax}\cos\left(Nx-\tan^{-1}\frac{N}{a}\right)}{\sqrt{a^2+N^2}}.$$

7. π; $\frac{1}{4}(\pi^2-8)$; $-\frac{\pi}{4}$.

8. $x\sin^{-1}x+\sqrt{1-x^2}$; $\frac{2x^2-1}{4}\sin^{-1}x-\frac{1}{4}x\sqrt{1-x^2}$;

$\frac{8x^4-3}{32}\sin^{-1}x+\frac{x(2x^2+3)}{32}\sqrt{1-x^2}$; $\frac{x^2}{2}-\frac{x}{2}+\frac{1}{2}\tan^{-1}x$.

Page 114.

1. $e^x(x^6-6x^5+6.5x^4-6.5.4x^3+6.5.4.3x^2$
$-6.5.4.3.2x+6.5.4.3.2.1)$,

$(x^5+5.4x^3+5.4.3.2x)\cosh x-(5x^4+5.4.3x^2+5.4.3.2.1)\sinh x$,

$\frac{x^6}{12}+\frac{\sinh 2x}{2}\left(\frac{x^5}{2}+\frac{5.4x^3}{2^3}+\frac{5.4.3.2x}{2^5}\right)$
$-\frac{\cosh 2x}{2}\left(\frac{5x^4}{2^2}+\frac{5.4.3x^2}{2^4}+\frac{5.4.3.2.1}{2^6}\right)$.

2. $\frac{3}{4}(\pi^2-8)$; $\frac{\pi^4}{128}+\frac{3\pi^2}{32}-\frac{3}{8}$; $\frac{\pi^4}{2^6}-\frac{3\pi^2}{2^4}+\frac{3}{4}$.

3. $\pi^5-20\pi^3+120\pi$; $\frac{\pi^2}{24}(2\pi^4+15\pi^2-45)$; $-e-8e^{-1}+6$.

4. $\frac{\pi^4}{128}(a^2+b^2)-3\frac{a^2-b^2}{32}(\pi^2-4)$; $\frac{81}{4}\log 3-5$; $\frac{\pi-2}{4}$.

5. $\frac{1}{10}(e^{\frac{\pi}{2}}+2)$; $\frac{\pi}{96}$.

PAGE 130.

1. $(2x-\sin 2x)/4$; $(\cos 3x-9\cos x)/12$ or $-\cos x+\frac{\cos^3 x}{3}$

$(12x-8\sin 2x+\sin 4x)/32$;

$\frac{1}{2^4}\left(-\frac{\cos 5x}{5}+\frac{5}{3}\cos 3x-10\cos x\right)$ or $-\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x$;

$\frac{1}{2^7}\left(\frac{\sin 8x}{8}-\frac{4}{3}\sin 6x+7\sin 4x-28\sin 2x+35x\right)$;

$\frac{1}{2^8}\left(-\frac{\cos 9x}{9}+9\frac{\cos 7x}{7}-36\frac{\cos 5x}{5}+84\frac{\cos 3x}{3}-126\cos x\right)$

or $-\cos x+4\frac{\cos^3 x}{3}-6\frac{\cos^5 x}{5}+4\frac{\cos^7 x}{7}-\frac{\cos^9 x}{9}$;

$\frac{(-1)^n}{2^{2n-1}}\left[\frac{\sin 2nx}{2n}-{}^{2n}C_1\frac{\sin(2n-2)x}{2n-2}+\ldots+\frac{(-1)^n}{2}{}^{2n}C_n x\right]$;

$\frac{(-1)^{n+1}}{2^{2n}}\left[\frac{\cos(2n+1)x}{2n+1}-{}^{2n+1}C_1\frac{\cos(2n-1)x}{2n-1}+\ldots+(-1)^n\,{}^{2n+1}C_n\cos x\right]$

or $-\cos x+{}^nC_1\frac{\cos^3 x}{3}-{}^nC_2\frac{\cos^5 x}{5}+\ldots$.

2. $\frac{1}{8}\left(x-\frac{\sin 4x}{4}\right)$; $\frac{\sin^4 x}{4}-\frac{\sin^6 x}{6}$; $\frac{1}{128}(3x-\sin 4x+\frac{1}{8}\sin 8x)$;

$-\frac{\cos^7 x}{7}+\frac{\cos^9 x}{9}$; $\frac{\sin^7 x}{7}-\frac{\sin^9 x}{9}$;

$-\frac{1}{2^9}\left[\frac{\sin 10x}{10}-\frac{\sin 8x}{4}-\frac{\sin 6x}{2}+2\sin 4x+\sin 2x-6x\right]$.

3. $\frac{1}{3}\tan^3 x$; $-\frac{1}{3}\cot^3 x$; $\tan x-\cot x$; $-\frac{1}{3\tan^3 x}-\frac{3}{\tan x}+3x+\frac{\tan^3 x}{3}$.

4. $(\pi-2)/8$; $43\sqrt{2}/120$; $(15\pi+44)/192$.

5. $-\frac{1}{4}\left[\frac{2\cos ax}{a}+\frac{\cos(a+2b)x}{a+2b}+\frac{\cos(a-2b)x}{a-2b}\right]$;

$\frac{3}{2}\sin^2 x-\frac{7}{4}\sin^4 x+\frac{2}{3}\sin^6 x$;

$-\frac{1}{4}\left[\frac{2\cos nx}{n}+\frac{\cos(n+2)x}{n+2}+\frac{\cos(n-2)x}{n-2}\right]$.

PAGE 131.

2. (i) $x\cos^{-1}x-\sqrt{1-x^2}$; (ii) $x\sec^{-1}x-\log(x+\sqrt{x^2-1})$;

(iii) $\frac{x^4-1}{4}\tan^{-1}x+\frac{x}{4}-\frac{x^3}{12}$; (iv) $x\tan x+\log\cos x$;

(v) $x\sec x-\log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$;

(vi) $\frac{c^2(ax+b)^2-(bc-ad)^2}{2ac^2}\log(cx+d)-\frac{a}{4c^2}(cx+d)^2-\frac{1}{c}(bc-ad)x$;

(vii) $x\tan^{-1}\sqrt{1-x^2}-\sin^{-1}x+\sqrt{2}\tan^{-1}\dfrac{x}{\sqrt{2}\sqrt{1-x^2}}$;

(viii) $\left(\dfrac{x^3}{3}-x\right)\tan^{-1}x+\frac{1}{2}(\tan^{-1}x)^2-\frac{1}{6}x^2+\frac{2}{3}\log(1+x^2)$;

(ix) $(a+x)\tan^{-1}\sqrt{\dfrac{x}{a}}-\sqrt{ax}$; (x) $\dfrac{1}{4}(x^2-2a^2)\cos^{-1}\dfrac{x}{2a}-\dfrac{1}{8}x\sqrt{4a^2-x^2}$;

(xi) $(2a+x)\tan^{-1}\sqrt{\dfrac{x}{2a}}-\sqrt{2ax}$; (xii) $\dfrac{x^{n+1}}{n+1}\left[\log x-\dfrac{1}{n+1}\right]$.

3. (i) $\dfrac{e^{a\sin^{-1}x}}{\sqrt{a^2+1}}\cos(\sin^{-1}x-\cot^{-1}a)$; (ii) $x-\sqrt{1-x^2}\sin^{-1}x$;

(iii) $\theta(\sec\theta+\cos\theta)-\sin\theta-\log\tan\left(\dfrac{\theta}{2}+\dfrac{\pi}{4}\right)$, where $x=\sin\theta$.

4. (i) $\dfrac{1}{m}e^{m\theta}$; (ii) $\dfrac{e^{m\theta}}{\sqrt{1+m^2}}\cos(\theta-\tan^{-1}m)$;

(iii) $\dfrac{e^{m\theta}}{2}\left\{\dfrac{1}{m}+\dfrac{1}{\sqrt{m^2+4}}\cos\left(2\theta-\tan^{-1}\dfrac{2}{m}\right)\right\}$;

(iv) $\dfrac{e^{m\theta}}{4}\left\{\dfrac{3}{\sqrt{m^2+1}}\cos\left(\theta-\tan^{-1}\dfrac{1}{m}\right)+\dfrac{1}{\sqrt{m^2+9}}\cos\left(3\theta-\tan^{-1}\dfrac{3}{m}\right)\right\}$;

(v) $$\frac{e^{m\theta}}{2^{n-2}}\left[\frac{\cos\left\{(n-1)\theta-\tan^{-1}\dfrac{n-1}{m}\right\}}{\sqrt{m^2+(n-1)^2}}+{}^{n-1}C_1\frac{\cos\left\{(n-3)\theta-\tan^{-1}\dfrac{n-3}{m}\right\}}{\sqrt{m^2+(n-3)^2}}+\dots\right],$$

where $\tan\theta=x$.

5. (i) $x\dfrac{e^{bx}}{\sqrt{a^2+b^2}}\cos\left(ax-\tan^{-1}\dfrac{a}{b}\right)-\dfrac{e^{bx}}{a^2+b^2}\cos\left(ax-2\tan^{-1}\dfrac{a}{b}\right)$;

(ii) $$x^2\frac{e^{ax}}{(a^2+b^2)^{\frac{1}{2}}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)-2x\frac{e^{ax}}{a^2+b^2}\sin\left(bx-2\tan^{-1}\frac{b}{a}\right)+2\frac{e^{ax}}{(a^2+b^2)^{\frac{3}{2}}}\sin\left(bx-3\tan^{-1}\frac{b}{a}\right);$$

(iii) $\frac{1}{2}e^x\left[x-1-\dfrac{x}{\sqrt{5}}\cos(2x-\tan^{-1}2)+\frac{1}{5}\cos(2x-2\tan^{-1}2)\right]$.

6. (i) $e^{ax}\dfrac{(a-b)\cos bx+(a+b)\sin bx}{a^2+b^2}$; (ii) $\dfrac{e^{(a+b)x}}{a+b}$;

(iii) $\dfrac{1}{4}\left[\dfrac{e^{(2a+b)x}}{2a+b}+\dfrac{e^{(2a-b)x}}{2a-b}+\dfrac{e^{bx}}{b}+\dfrac{e^{-bx}}{b}\right]$;

(iv) $-\dfrac{\cos bx}{2b}+\dfrac{1}{2}\dfrac{e^{2ax}}{\sqrt{4a^2+b^2}}\sin\left(bx-\tan^{-1}\dfrac{b}{2a}\right)$;

(v) $3^x(P\sin 4x-Q\cos 4x)$, where

$$P=\frac{x^2\cos\phi}{r}-\frac{2x\cos 2\phi}{r^2}+\frac{2\cos 3\phi}{r^3},$$

$$Q=\frac{x^2\sin\phi}{r}-\frac{2x\sin 2\phi}{r^2}+\frac{2\sin 3\phi}{r^3},$$

and $\phi=\tan^{-1}(4/\log 3)$, $r^2=4^2+(\log 3)^2$;

(vi) $\dfrac{x}{\sqrt{b^2+1}}\cos\left(b\log\dfrac{x}{a}-\tan^{-1}b\right)$;

(vii) $\dfrac{1}{2}\left[\dfrac{x}{1+b}\left(\dfrac{x}{a}\right)^b+\dfrac{a}{1-b}\left(\dfrac{a}{x}\right)^{b-1}\right]$; (viii) $\pi\sinh 1$.

7. (i) $\dfrac{e^x}{x+1}$; (ii) $e^x\tan\dfrac{x}{2}$; (iii) $-e^x\cot\dfrac{x}{2}$;

(iv) $\cosh x\tan\dfrac{x}{2}$; (v) $-\log(1+e^{-x})$;

(vi) $\dfrac{2}{n}\sqrt{1+e^{nx}}+\dfrac{1}{n}\log\dfrac{\sqrt{1+e^{nx}}-1}{\sqrt{1+e^{nx}}+1}$; (vii) $\dfrac{x-1}{x+1}e^x$.

8. (i) $x(\log x)^2-2x\log x+2x$;

(ii) $\frac{1}{2}(\log x)^2+\left(\dfrac{x^2}{2}-\dfrac{1}{x}\right)\log x-\left(\dfrac{x^2}{4}+\dfrac{1}{x}\right)$;

(iii) $-\dfrac{1}{x}\tan^{-1}x+\log x-\log\sqrt{1+x^2}$;

(iv) $x\log(x+\sqrt{a^2+x^2})-\sqrt{a^2+x^2}$;

(v) $\dfrac{2x^2+a^2}{4}\log(x+\sqrt{x^2+a^2})-\dfrac{x}{4}\sqrt{x^2+a^2}$;

(vi) $\dfrac{2x^2+3ax+2a^2}{6}\sqrt{a^2+x^2}+\dfrac{a^3}{2}\sinh^{-1}\dfrac{x}{a}$;

(vii) $\frac{2}{105}(x+a)^{\frac{3}{2}}(15x^2-12ax+43a^2)$;

(viii) $$e^{ax}\left[\frac{x^2}{(b^2+c^2)^{\frac{1}{2}}}\sin\left(bx+c-\tan^{-1}\frac{b}{a}\right)-\frac{2x}{b^2+c^2}\sin\left(bx+c-2\tan^{-1}\frac{b}{a}\right)\right.$$
$$\left.+\frac{2}{(b^2+c^2)^{\frac{3}{2}}}\sin\left(bx+c-3\tan^{-1}\frac{b}{a}\right)\right];$$

(ix) $$-9\left[\tfrac{1}{10}\cos^{\frac{10}{3}}\theta-\tfrac{5}{16}\cos^{\frac{16}{3}}\theta+\tfrac{10}{22}\cos^{\frac{22}{3}}\theta-\tfrac{10}{28}\cos^{\frac{28}{3}}\theta\right.$$
$$\left.+\tfrac{5}{34}\cos^{\frac{34}{3}}\theta-\tfrac{1}{40}\cos^{\frac{40}{3}}\theta\right],$$
where $\sin\theta=x^{\frac{1}{3}}$.

9. (i) $$\frac{e^{ax}}{2}\left[\frac{x\cos\left\{(b-c)x-\tan^{-1}\frac{b-c}{a}\right\}}{\sqrt{(b-c)^2+a^2}}-\frac{\cos\left\{(b-c)x-2\tan^{-1}\frac{b-c}{a}\right\}}{(b-c)^2+a^2}\right.$$
$$\left.-x\frac{\cos\left\{(b+c)x-\tan^{-1}\frac{b+c}{a}\right\}}{\sqrt{(b+c)^2+a^2}}+\frac{\cos\left\{(b+c)x-2\tan^{-1}\frac{b+c}{a}\right\}}{(b+c)^2+a^2}\right];$$

(ii) $$\frac{e^{ax}}{4}\left[\frac{2x}{\sqrt{a^2+b^2}}\sin\left(bx-\tan^{-1}\frac{b}{a}\right)-\frac{2}{a^2+b^2}\sin\left(bx-2\tan^{-1}\frac{b}{a}\right)\right.$$
$$-\frac{x}{\sqrt{a^2+(b+2c)^2}}\sin\left\{(b+2c)x-\tan^{-1}\frac{b+2c}{a}\right\}$$
$$+\frac{1}{a^2+(b+2c)^2}\sin\left\{(b+2c)x-2\tan^{-1}\frac{b+2c}{a}\right\}$$
$$-\frac{x}{\sqrt{a^2+(b-2c)^2}}\sin\left\{(b-2c)x-\tan^{-1}\frac{b-2c}{a}\right\}$$
$$\left.+\frac{1}{a^2+(b-2c)^2}\sin\left\{(b-2c)x-2\tan^{-1}\frac{b-2c}{a}\right\}\right].$$

12. $\dfrac{x^3}{3}\log(1-x^2)+\dfrac{1}{3}\log\dfrac{1+x}{1-x}-\dfrac{2}{3}\left(x+\dfrac{x^3}{3}\right)$.

13. $-\frac{5}{2}\cot^{\frac{2}{5}}\theta$; $-\frac{5}{2}\cos^{\frac{2}{5}}\theta$. 14. $uv^{(n-1)}-u'v^{(n-2)}+\ldots+(-1)^{n-1}u^{(n-1)}v$.

15. $\begin{vmatrix} u'', & v'', & w'' \\ u', & v', & w' \\ 1, & 1, & 1 \end{vmatrix}$. 20. $\cdot 78343$. 22. $\dfrac{(x^2+1)^2}{8}\tan^{-1}x-\dfrac{5x^3+3x}{24}$.

27. $\displaystyle\int_0^1 \frac{1-\sqrt{1-x}}{x\sqrt{1-x}}\,dx=2\int_0^{\frac{\pi}{2}}\tan\frac{\theta}{2}\,d\theta=2\log 2$. 29. $\dfrac{a_1a_2}{2}T\cos\dfrac{2\pi\lambda}{T}$.

33. $2\sin\dfrac{4n-1}{2}\theta\cos^{\frac{3}{2}}\theta$. 34. $n^2\pi a^2$.

35. $2^{n+1}a^n\dfrac{2n-1}{2n}\,\dfrac{2n-3}{2n-2}\cdots\dfrac{1}{2}\cdot\dfrac{\pi}{2}$. 39. $A=\dfrac{518}{225}a,\quad V=\dfrac{\pi^2 a}{4}A$.

CHAPTER V.

Page 143.

1. $\frac{1}{2}\log(x^2+2x+3)-\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{x+1}{\sqrt{2}}$. 2. $\log(x+1)+\dfrac{1}{x+1}$.

3. $\frac{1}{2}\log(x^2+4x+5)-\tan^{-1}(x+2)$. 4. $-\log(3-x)$.

5. $x-2\log(x^2+2x+2)+3\tan^{-1}(x+1)$.

6. $2x-\frac{9}{2}\log(x^2+6x+10)+11\tan^{-1}(x+3)$.

7. $\dfrac{1}{ad-bc}\tan^{-1}\dfrac{(a^2+c^2)x+(ab+cd)}{ad-bc}$.

8. $\dfrac{1}{2(bc-ad)}\log\dfrac{(a+c)x+(b+d)}{(a-c)x+(b-d)}$.

9. $\dfrac{1}{2(ad-bc)}\tan^{-1}\dfrac{(a^2+c^2)x^2+(ab+cd)}{ad-bc}$.

10. $\dfrac{1}{2\sqrt{(ad-bc)^2+(cf-de)^2+(eb-af)^2}}$

$$\times\tan^{-1}\frac{(a^2+c^2+e^2)x^2+(ab+cd+ef)}{\sqrt{(ad-bc)^2+(cf-de)^2+(eb-af)^2}}.$$

11. $\dfrac{1}{2(ad-bc)}\log\dfrac{ax^2+b}{cx^2+d}$ 12. $\frac{1}{2}\log(e^{2x}+2e^x+3)-\dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{e^x+1}{\sqrt{2}}$.

Page 161.

1. (i) $\log\dfrac{\sqrt{x^2-1}}{x}$; (ii) $\frac{1}{8}\log\dfrac{(x-1)(x-5)}{(x-3)^2}$;

(iii) $\frac{1}{3}\log\{x(3-x^2)^4\}$; (iv) $\frac{1}{12}\log\left\{\dfrac{(x+1)^2}{(x-1)^6}\cdot\dfrac{(x-2)^7}{(x+2)^3}\right\}$;

(v) $\frac{1}{7}[-\frac{1}{3}\log(x-3)+\frac{10}{3}\log(x+3)+\frac{3}{4}\log(x-4)-\frac{15}{4}\log(x+4)]$;

(vi) $x+\Sigma\dfrac{(a_1-a)(a_1-b)(a_1-c)}{(a_1-b_1)(a_1-c_1)}\log(x-a_1)$,

where Σ refers to a cyclic interchange of the letters a_1, b_1, c_1;

(vii) $\frac{1}{2}\Sigma\frac{\left\{\begin{array}{l}(a_1-a)(a_1-b)(a_1-c)\log(x-a_1)\\ \quad+(a_1+a)(a_1+b)(a_1+c)\log(x+a_1)\end{array}\right\}}{a_1(a_1^2-b_1^2)(a_1^2-c_1^2)}$,

where Σ refers to a cyclic interchange of a_1, b_1, c_1;

(viii) $\frac{1}{10}\log\{(x-5)^3(x+15)^7\}$; (ix) $\frac{1}{3}\log\{(x-7)(x+17)^2\}$;

(x) $\frac{1}{6}\log\left\{\frac{(x-7)^2(x-13)^7}{(x-11)^9}\right\}$.

2. (i) $-\frac{1}{4}\frac{1}{(x-1)^2}+\frac{1}{4}\frac{1}{x-1}+\frac{1}{8}\log(x-1)-\frac{1}{8}\log(x+1)$;

(ii) $-\frac{1}{24}\frac{x(x^2+3)}{(x^2-1)^3}+\frac{1}{4}\frac{x}{(x^2-1)^2}-\frac{5}{16}\frac{x}{x^2-1}-\frac{5}{32}\log\left(\frac{x-1}{x+1}\right)$;

(iii) $-\frac{1}{3x^3}-\frac{5}{2x^2}-\frac{14}{x}+30\log x-\frac{2}{3(x-1)^3}+\frac{7}{2(x-1)^2}$

$-\frac{16}{x-1}-30\log(x-1)$;

(iv) $-\frac{1}{ax}+\frac{b}{a^2}\log\frac{a+bx}{x}$; (v) $-\frac{1}{x-3}-\frac{1}{x-4}+2\log\frac{x-3}{x-4}$;

(vi) $x-\frac{a^3}{a-b}\frac{1}{x-a}+\frac{(2a-3b)a^2}{(a-b)^2}\log(x-a)+\frac{b^3}{(a-b)^2}\log(x-b)$;

(vii) $\frac{1}{x-2}+\log\frac{(x-3)^3}{(x-2)^2}$.

3. (i) $\frac{1}{b^2-a^2}\left(\frac{1}{a}\tan^{-1}\frac{x}{a}-\frac{1}{b}\tan^{-1}\frac{x}{b}\right)$;

(ii) $x+\frac{1}{d^2-c^2}\left[\frac{(a^2-c^2)(b^2-c^2)}{c}\tan^{-1}\frac{x}{c}-\frac{(a^2-d^2)(b^2-d^2)}{d}\tan^{-1}\frac{x}{d}\right]$;

(iii) $\frac{x^3}{3}+(a^2-c^2)x-c(a^2-c^2)\tan^{-1}\frac{x}{c}$;

(iv) $\tan^{-1}x-\frac{1}{\sqrt{2}}\tan^{-1}x\sqrt{2}$;

(v) $\frac{ad-bc}{ed-fc}\frac{1}{\sqrt{cd}}\tan^{-1}\left(x\sqrt{\frac{c}{d}}\right)+\frac{af-be}{fc-ed}\frac{1}{\sqrt{ef}}\tan^{-1}\left(x\sqrt{\frac{e}{f}}\right)$;

(vi) $-\frac{b}{dfhx}+\Sigma\frac{(ad-bc)c^2}{(ed-fc)(gd-hc)}\frac{1}{\sqrt{cd}}\tan^{-1}\left(x\sqrt{\frac{c}{d}}\right)$.

4. (i) $\log\frac{x}{\sqrt{x^2+1}}$; (ii) $\frac{3}{4}\log(x^2-1)+\frac{1}{4}\log(x^2+1)-2\log x$;

(iii) $-\frac{1}{6}\log x+\frac{1}{4}\log(x^2-1)-\frac{1}{4}\log(x^2-2)+\frac{1}{12}\log(x^2-3)$;

(iv) $\frac{1}{4b\sqrt{b^2+4ac}}\log\frac{2a^2x^2+2ac+b^2-b\sqrt{b^2+4ac}}{2a^2x^2+2ac+b^2+b\sqrt{b^2+4ac}}$ $(b^2+4ac>0)$.

5. (i) $\frac{1}{4}\log\frac{x^2+x+1}{x^2-x+1}+\frac{1}{2\sqrt{3}}\tan^{-1}\frac{x\sqrt{3}}{1-x^2}$;

(ii) $\sqrt{3}\tan^{-1}\frac{2x-1}{\sqrt{3}}-\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$

or $\frac{1}{\sqrt{3}}\tan^{-1}\frac{x\sqrt{3}}{1-x^2}-\frac{2}{\sqrt{3}}\tan^{-1}\frac{\sqrt{3}}{2x^2+1}$, which is the same thing;

(iii) $\frac{1}{\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2}$; (iv) $\tan^{-1}\frac{x}{1-x^2}$; (v) $\frac{1}{\sqrt{3}}\tan^{-1}\frac{ax\sqrt{3}}{a^2-x^2}$;

(vi) $\frac{1}{2a}\log\frac{x^2-ax+a^2}{x^2+ax+a^2}$;

(vii) $\frac{4}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}-\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$

or $\frac{1}{\sqrt{3}}\tan^{-1}\frac{x\sqrt{3}}{1-x^2}-\sqrt{3}\tan^{-1}\frac{\sqrt{3}}{2x^2+1}$;

(viii) $\frac{1}{4\sqrt{2}}\log\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\frac{1}{2\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2}$.

6. (i) $\frac{1}{2}\log(x-2)-\frac{1}{x-2}-\frac{1}{4}\log(x^2-2x+4)-\frac{1}{2\sqrt{3}}\tan^{-1}\frac{x-1}{\sqrt{3}}$;

(ii) $\frac{2}{3}\log(1+x)-\frac{1}{3}\log(1+2x+4x^2)-\frac{1}{3}\frac{1}{1+x}+\frac{2}{3\sqrt{3}}\tan^{-1}\frac{4x+1}{\sqrt{3}}$;

(iii) $x+\frac{18}{25}\log(x-1)+\frac{16}{25}\log(x^2+4)-\frac{1}{5}\frac{1}{x-1}-\frac{24}{25}\tan^{-1}\frac{x}{2}$;

(iv) $\frac{1}{4}\log\frac{(x+1)^2}{x^2+1}-\frac{1}{2}\frac{1}{x+1}$ (v) $\frac{1}{4}\log\frac{x^2+1}{(x-1)^2}-\frac{1}{2}\frac{1}{x-1}$;

(vi) $\log\frac{x}{x-1}-\frac{1}{2}\frac{1}{x-1}+\frac{1}{2}\tan^{-1}x$. (vii) $\frac{1}{a^4}\log\frac{\sqrt{a^2+x^2}}{x}-\frac{1}{2a^2x^2}$;

(viii) $-\frac{1}{2a^2b^2x^2}-\frac{a^2+b^2}{a^4b^4}\log x$

$-\frac{1}{2(a^2-b^2)}\left\{\frac{1}{a^4}\log(a^2+x^2)-\frac{1}{b^4}\log(b^2+x^2)\right\}$;

(ix) $-\frac{1}{6(x-1)^2}+\frac{1}{3(x-1)}+\frac{2}{9}\log(x-1)-\frac{1}{9}\log(x^2+x+1)$;

(x) $-\frac{1}{28}\frac{1}{2x-3}-\frac{3}{196}\log(2x-3)+\frac{3}{392}\log(4x^2+5)+\frac{1}{98\sqrt{5}}\tan^{-1}\frac{2x}{5}$.

7. (i) $\log\frac{x}{\sqrt{x^2+1}}+\frac{1}{2}\frac{1}{x^2+1}$;

(ii) $-\frac{1}{2}\log(x-1)-\frac{1}{4}\frac{1}{x-1}+\frac{1}{4}\log(x^2+1)+\frac{1}{4}\tan^{-1}x-\frac{1}{4}\frac{1}{x^2+1}$;

(iii) $\frac{1}{2}\tan^{-1}x+\frac{1}{2}\frac{x-1}{1+x^2}$;

(iv) $\frac{c^2+3ab}{8c^5}\tan^{-1}\frac{x}{c}+\frac{ab}{2c^4}\frac{x}{c^2+x^2}+\frac{ab-c^2}{8c^4}\frac{x(c^2-x^2)}{(c^2+x^2)^2}-\frac{a+b}{4}\frac{1}{(c^2+x^2)^2}$.

8. $\frac{1}{2\sqrt{2}}\{\pi+2\log(\sqrt{2}-1)\}$; $\frac{1}{2\sqrt{2}}\{\pi+2\log(\sqrt{2}+1)\}$.

9. (i) $\frac{\pi}{2}$; (ii) $\frac{\pi}{2\sqrt{3}}$. **10.** $\log\frac{4}{3}$.

14. (i) $4\log(2x-1)-\log(x+2)-\frac{3}{2}\log(x^2+1)-4\tan^{-1}x$;

(ii) $x-2\log x+\frac{3}{4}\log(x-1)+\frac{1}{4}\log(x+1)+\frac{1}{2}\log(x^2+1)-\frac{1}{2}\tan^{-1}x$;

(iii) $\frac{1}{2}\frac{1}{2\sin\frac{a}{2}}\tan^{-1}\frac{2x\sin\frac{a}{2}}{1-x^2}$;

(iv) $$\frac{1}{5}\Big[\log(x+1)-\cos\frac{\pi}{5}\log\left(x^2-2ax\cos\frac{\pi}{5}+a^2\right) -\cos\frac{3\pi}{5}\log\left(x^2-2ax\cos\frac{3\pi}{5}+a^2\right)+2\sin\frac{\pi}{5}\tan^{-1}\frac{x-a\cos\frac{\pi}{5}}{a\sin\frac{\pi}{5}} +2\sin\frac{3\pi}{5}\tan^{-1}\frac{x-a\cos\frac{3\pi}{5}}{a\sin\frac{3\pi}{5}}\Big];$$

(v) $\frac{\pi}{2a}$.

17. (i) $$\frac{1}{9x}+\frac{1}{8}\log\frac{x-1}{x+1}-\frac{5}{72}\sqrt{\frac{5}{3}}\log\frac{x\sqrt{5}-\sqrt{3}}{x\sqrt{5}+\sqrt{3}} +\frac{5\sqrt{5}}{36\sqrt{3}}\Big[\cot\theta\operatorname{cosec}\theta-\log\cot\frac{\theta}{2}\Big],\ \text{where}\ \theta=\sec^{-1}\frac{x\sqrt{5}}{\sqrt{3}};$$

(ii) $$\frac{1}{27x^3}+\frac{5}{27x^2}+\frac{28}{27x}-\frac{590}{243}\log x+\frac{5^5}{2^4.3^4}\frac{1}{5x-3}+\frac{5^5.23}{2^7.3^5}\log(5x-3) +\frac{1}{8}\log(x-1)-\frac{1}{128}\log(x+1);$$

(iii) $(2\sqrt{2}-\sqrt{3}-1)\frac{\pi}{2}$.

19. $-\frac{1}{3}\log(x+1)+\frac{1}{6}\log(x^2-x+1)+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}$.

20. $-\tan^{-1}\frac{1}{2}(\sqrt{\tan x}+\sqrt{\cot x})$.

21. (i) $\left(x-\frac{2}{a^2+b^2}\right)\tan^{-1}\sqrt{\frac{a^2x-1}{b^2x-1}}+\frac{1}{ab}\frac{a^2-b^2}{a^2+b^2}\log\{a\sqrt{b^2x-1}+b\sqrt{a^2x-1}\}$;

(ii) $\frac{c}{4b}\log\left\{\left(\frac{z+\rho_1}{z-\rho_1}\right)^{\frac{1}{\rho_1}}\left(\frac{z-\rho_2}{z+\rho_2}\right)^{\frac{1}{\rho_2}}\right\}+\frac{cz}{(z^2-\rho_1{}^2)(z^2-\rho_2{}^2)}$,

where $(z^2-a^2)^2=b^2+\frac{c}{x}$, $a^2+b=\rho_1{}^2$, $a^2-b=\rho_2{}^2$.

22. $\frac{2\sqrt{3}}{\sqrt{a}}\tan^{-1}\frac{2\sqrt{x}+\sqrt{a}}{\sqrt{3a}}-\frac{2}{\sqrt{3a}}\tan^{-1}\frac{2\sqrt{x}-\sqrt{a}}{\sqrt{3a}}$. **23.** $\frac{-x}{(x^3+3x+1)^2}$.

24. $\left[\frac{1}{4}\frac{\sin x}{\cos^4 x}-\frac{5}{8}\frac{\sin x}{\cos^2 x}+\frac{3}{16}\log\frac{1+\sin x}{1-\sin x}\right]_0^{\frac{\pi}{4}}=\frac{3}{16}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}-\frac{\sqrt{2}}{8}$.

25. $$-\frac{1}{(n-1)a}\frac{1}{(x-a)^{n-1}}+\frac{1}{(n-2)a^2}\frac{1}{(x-a)^{n-2}}-\frac{1}{(n-3)a^3}\frac{1}{(x-a)^{n-3}}+\dots +\frac{(-1)^{n-1}}{a^{n-1}}\frac{1}{x-a}+\frac{(-1)^{n-1}}{a^n}\log(x-a)+\frac{(-1)^n}{a^n}\log x.$$

26. If n be even, $=2m$,

$$x+{}^mC_1(a-b)\log(x-a)-{}^mC_2\frac{(a-b)^2}{x-a}-\frac{{}^mC_3}{2}\frac{(a-b)^3}{(x-a)^2}-\dots$$
$$-\frac{{}^mC_m}{m-1}\frac{(a-b)^m}{(x-a)^{m-1}}.$$

If n be odd, $=2m+1$,

$$2(b-a)\left[\frac{1}{2m-1}\left(\frac{x-b}{x-a}\right)^{\frac{2m-1}{2}}+\frac{2}{2m-3}\left(\frac{x-b}{x-a}\right)^{\frac{2m-3}{2}}+\frac{3}{2m-5}\left(\frac{x-b}{x-a}\right)^{\frac{2m-5}{2}}+\dots\right.$$
$$\left.+\frac{m}{1}\left(\frac{x-b}{x-a}\right)^{\frac{1}{2}}+\frac{1}{2}\frac{(x-a)^{\frac{1}{2}}(x-b)^{\frac{1}{2}}}{b-a}-\frac{2m+1}{2}\tanh^{-1}\left(\frac{x-b}{x-a}\right)^{\frac{1}{2}}\right].$$

27. $\log\dfrac{e^x(e^x+1)}{(2e^x+1)^2}$. **28.** $\dfrac{x^2-1}{4}\log\dfrac{1+x}{1-x}+\dfrac{1}{2}x$.

30. $\dfrac{x^3}{3}\log(1-x^2)-\dfrac{2x}{3}-\dfrac{2x^3}{9}+\dfrac{1}{3}\log\dfrac{1+x}{1-x}$.

45. Let $A\equiv a\alpha^2+b\alpha+c$, $B=a\beta^2+b\beta+c$, $C=a\gamma^2+b\gamma+c$,

$$P=-\frac{A^2}{(\alpha-\beta)^2(\alpha-\gamma)^2},\quad P'=\frac{2A}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}\left\{\frac{B}{(\alpha-\beta)^2}+\frac{C}{(\alpha-\gamma)^2}\right\},$$

and Q, Q'; R, R' similar expressions obtained by a cyclic interchange of letters,

$$I=\frac{P}{x-\alpha}+\frac{Q}{x-\beta}+\frac{R}{x-\gamma}+P'\log(x-\alpha)+Q'\log(x-\beta)+R'\log(x-\gamma).$$

CHAPTER VI.

Page 200.

1. (i) $[(ac+be)\theta+(bc-ae)\log(c\sin\theta+e\cos\theta)]/(c^2+e^2)$;

(ii) $\dfrac{1}{\sqrt{2}}\log\tan\left(\dfrac{\theta}{2}+\dfrac{3\pi}{8}\right)$;

(iii) $aK-\dfrac{\beta}{b}\log(a+b\cos\theta)$, where

$$K=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\quad(a>b)$$

or $$=\frac{2}{\sqrt{b^2-a^2}}\tanh^{-1}\sqrt{\frac{b-a}{b+a}}\tan\frac{\theta}{2}\quad(a<b);$$

(iv) $\dfrac{1}{\sin\alpha}\cosh^{-1}\dfrac{1+\cos\alpha\cos x}{\cos\alpha+\cos x}=\dfrac{2}{\sin\alpha}\tanh^{-1}\left(\tan\dfrac{\alpha}{2}\tan\dfrac{x}{2}\right)$;

(v) $\dfrac{1}{\sqrt{a^2+b^2}}\log\tan\dfrac{1}{2}\left(x+\tan^{-1}\dfrac{a}{b}\right)$; (vi) $\log(\cos\theta+\sin\theta)$;

(vii) $\dfrac{1}{4}\log\dfrac{1+\sin\theta}{1-\sin\theta}+\dfrac{1}{2}\dfrac{1}{1+\sin\theta}=\dfrac{1}{2}\left[\log(\sec\theta+\tan\theta)+\dfrac{1}{1+\sin\theta}\right]$;

(viii) $\cosh^{-1}\dfrac{3\cos(x-\tan^{-1}3)-\sqrt{10}}{3-\sqrt{10}\cos(x-\tan^{-1}3)}$;

(ix) $\dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{1}{\sqrt{3}}\tan\left(\dfrac{x}{2}-\dfrac{\pi}{8}\right)$;

(x) $[ax+b\log(a\cos x+b\sin x)]/(a^2+b^2)$.

2. (i) $\frac{2}{3}\log 2$; (ii) $\frac{\pi}{\sqrt{a^2-c^2}}\ (a>c)$; (iii) $\frac{\pi}{3\sqrt{3}}$; (iv) $\frac{\pi}{1-a^2}$.

3. $x\cos\alpha+\sin\alpha\cosh^{-1}\frac{1+\cos\alpha\cos x}{\cos\alpha+\cos x}$.

4. (i) $\frac{1}{\alpha\sqrt{\alpha^2-\beta^2}}\tan^{-1}\left(\frac{\alpha}{\sqrt{\alpha^2-\beta^2}}\tan x\right)$;

(ii) $\frac{1}{2}\left[\cosh^{-1}\frac{1}{\sqrt{2}}\frac{2-\cos x-\sin x}{1-\cos x-\sin x}+\frac{1}{\sqrt{7}}\cos^{-1}\frac{1}{\sqrt{2}}\frac{2+3\cos x+3\sin x}{3+\cos x+\sin x}\right]$;

(iii) $\frac{1}{3}\log\frac{\sin x(1+\cos x)}{(1+2\cos x)^2}$.

5. $\frac{1}{4}\tanh x$.

6. (i) $\frac{5}{9}\frac{\sin x}{4+5\cos x}-\frac{4}{27}\cosh^{-1}\frac{5+4\cos x}{4+5\cos x}$;

(ii) $\frac{a}{a^2-b^2}\int\frac{dx}{a+b\cos x}-\frac{b}{a^2-b^2}\frac{\sin x}{a+b\cos x}=$ etc., by Art. 173;

(iii) $\frac{1}{a^2+b^2}\tan\left(\theta-\tan^{-1}\frac{b}{a}\right)$;

(iv) $I=\int\left[\frac{dx}{a+\sqrt{b^2+c^2}\cos(x-\gamma)}\right]^2$, where $\tan\gamma=\frac{c}{b}$, and then use (ii).

7. (iii) $\frac{\pi}{2\sin^4\alpha\cos\alpha}\{(1+\cos\alpha)^2-\sin\alpha\}$;

(iv) $\frac{\pi}{6\sin^6\alpha\cos\alpha}\{2(1+\cos\alpha)^3-\sin\alpha(2+\cos^2\alpha)\}$.

8. $\sin\theta\cos\theta\log(1+\tan\theta)-\frac{\theta}{2}+\frac{1}{2}\log\sin\left(\theta+\frac{\pi}{4}\right)$.

9. (i) $\alpha/2\sin\alpha$; (ii) $\tanh^{-1}\left(\tan\frac{\alpha}{2}\right)\Big/\sin\alpha$.

10. (i) $\pi/2ab$; (ii) $\pi/12$; (iii) $\frac{\pi}{2}\left(\frac{a-b}{c-d}+\frac{bc-ad}{c-d}\frac{1}{\sqrt{cd}}\right)$;

(iv) $\pi(\alpha^2+\beta^2)/4\alpha^3\beta^3$; (v) $\pi/4$.

11. $\frac{\pi}{2}\frac{2a^2+b^2}{(a^2-b^2)^{\frac{5}{2}}}$. **12.** $\frac{\pi}{2}\frac{2+3e^2}{(1-e^2)^{\frac{7}{2}}}$. **13.** $\frac{2}{\sqrt{4bc-a^2}}\tan^{-1}\frac{2be^x+a}{\sqrt{4bc-a^2}}$.

16. (i) $2\sqrt{\tan x}$;

(ii) $I=\frac{a}{a^2-b^2}\int\frac{dx}{a+b\cos x}-\frac{b}{a^2-b^2}\frac{\sin x}{a+b\cos x}=$ etc. (Art. 173);

(iii) $\frac{1}{a^2+b^2}\tan\left(\theta-\tan^{-1}\frac{b}{a}\right)=\frac{1}{a^2+b^2}\frac{a\sin\theta-b\cos\theta}{a\cos\theta+b\sin\theta}$.

17. (i) $\frac{3}{68}\tan^{-1}\left(\frac{1}{2}\tan\frac{\theta}{2}\right)-\frac{5}{68}\tanh^{-1}\left(2\tan\frac{\theta}{2}\right)$; (ii) π;

(iii) $\theta+\frac{1}{\sqrt{3}}\log\frac{\tan\theta-\sqrt{3}}{\tan\theta+\sqrt{3}}$; (iv) $-\frac{2}{b^2}\left\{\log(a+b\cos x)+\frac{a}{a+b\cos x}\right\}$.

18. (i) $\dfrac{\sin 2\theta}{2}\log\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}-\dfrac{1}{2}\log\sec 2\theta$;

(ii) $-\cosh^{-1}(\cos\theta+\sin\theta)$; (iii) $\operatorname{cosec}^{-1}\left(2\cos^2\dfrac{\theta}{2}\right)$;

19. $\cos^{-1}\left(\dfrac{\sin x}{2}\right)+2\sqrt{3}\tanh^{-1}\left[\sqrt{3}\tan\dfrac{1}{2}\left\{\cos^{-1}\left(\dfrac{\sin x}{2}\right)\right\}\right]$;

20. $\operatorname{cosec}^{-1}(1+\sin 2\theta)$. 21. $\sec^{-1}(\cos\theta+\sec\theta)$.

22. (i) $-2\sqrt{1-\sin x}$; (ii) $-2\sqrt{1-\sin x}-\sqrt{2}\log\tan\left(\dfrac{x}{4}+\dfrac{\pi}{8}\right)$;

(iii) $\dfrac{1}{\sqrt{b-a}}\cos^{-1}\left[\sqrt{\dfrac{b-a}{b}}\cos x\right]$.

23. $\cosh x\cot\dfrac{x}{2}$.

24. $\dfrac{\sin x-x\cos x}{\cos x+x\sin x}$. 25. $\log\log\tan x$.

26. (i) $2x\tan^{-1}x-\log(1+x^2)$; (ii) $3x\tan^{-1}x-\frac{3}{2}\log(1+x^2)$;
(iii) $\frac{1}{2}x\tan^{-1}x-\frac{1}{4}\log(1+x^2)$.

27. $\dfrac{1}{2}\log\dfrac{1-\sin\theta}{1+\sin\theta}-\dfrac{1}{\sqrt{2}}\log\dfrac{1-\sqrt{2}\sin\theta}{1+\sqrt{2}\sin\theta}$, where $x=\tan\theta$.

28. $\dfrac{1}{2}\log\tan\left(\dfrac{x}{2}+\dfrac{\pi}{4}\right)$, $\dfrac{1}{2\sqrt{3}}\log\dfrac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}$,

$\dfrac{1}{8}\log\dfrac{1-\sin x}{1+\sin x}-\dfrac{1}{4\sqrt{2}}\log\dfrac{1-\sqrt{2}\sin x}{1+\sqrt{2}\sin x}$.

29. (i) $\dfrac{1}{\sin 2\alpha}\log\dfrac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}$; (ii) $\dfrac{1}{2\sin\alpha}\log\dfrac{\sin\theta-\sin\alpha}{\sin\theta+\sin\alpha}$.

30. (i) $\dfrac{8a^2+4ab-b^2}{ab^2(a+b)}-\dfrac{8a}{b^3}\log\dfrac{a+b}{a}$;

(ii) $\dfrac{4a^2-2ab-b^2}{2b^3}-\dfrac{a^2-b^2}{b^3}\log\dfrac{a+b}{a}$; (iii) $\dfrac{b-2a}{ab^2}+\dfrac{2a}{b^3}\log\dfrac{a+b}{a}$;

(iv) $\dfrac{1}{b^3}\left[\dfrac{4}{n-3}\left\{\dfrac{1}{a^{n-3}}-\dfrac{1}{(a+b)^{n-3}}\right\}-\dfrac{8a}{n-2}\left\{\dfrac{1}{a^{n-2}}-\dfrac{1}{(a+b)^{n-2}}\right\}\right.$

$$\left.+\frac{4a^2-b^2}{n-1}\left\{\frac{1}{a^{n-1}}-\frac{1}{(a+b)^{n-1}}\right\}\right],$$

unless $n=1$, 2 or 3, when a logarithmic term occurs from one of the integrations.

32. $-x+\cot(\alpha-\beta)\log\dfrac{\sin(x-\alpha)}{\sin(x-\beta)}$.

42. $\dfrac{2}{1-ab}\left[\dfrac{1}{\sqrt{1-a^2}}\tan^{-1}\sqrt{\dfrac{1+a}{1-a}}\tan\dfrac{x}{2}-\dfrac{b}{\sqrt{b^2-1}}\tan^{-1}\sqrt{\dfrac{b+1}{b-1}}\tan\dfrac{x}{2}\right]$.

43. (i) $\frac{1}{2}e^x\{x\sin x+(x-1)\cos x\}$;

(ii) $(3x+2x^3)/3(1+x^2)^{\frac{3}{2}}$;

(iii) $\frac{\sqrt{3}}{18}(\sin 4\theta-4\sin 2\theta-12\cos^4\theta)$, where $\tan\theta=(2x+1)/\sqrt{3}$.

44. $\Sigma\frac{\cos^3 a}{\sin(a-b)\sin(a-c)}\log\sin(x-a)-x\Sigma\frac{\sin a\cos^2 a}{\sin(a-b)\sin(a-c)}$.

46. (iii) Put $x+a\log x=xy$.

48. (i) $\frac{\cos x-\sin x}{(x-1)\cos x-(x+1)\sin x}$. (ii) $\frac{1}{2}\frac{(x+1)\cos x+(x-1)\sin x}{(x-1)\cos x-(x+1)\sin x}$.

CHAPTER VII.

PAGE 221.

6. (i) $\frac{x^5}{5b}-\frac{ax^2}{2b^2}+\frac{a^2}{b^2}I_1$,

where $I_1=\frac{1}{3bk}\left[\log\frac{\sqrt{x^2-kx+k^2}}{x+k}+\sqrt{3}\tan^{-1}\frac{2x-k}{k\sqrt{3}}\right]$ and $k^3=\frac{a}{b}$;

(ii) $-\frac{x}{3b(a+bx^3)}+\frac{1}{3b}I$,

where $I=\frac{1}{3bk^2}\left[\log\frac{x+k}{\sqrt{x^2-kx+k^2}}+\sqrt{3}\tan^{-1}\frac{2x-k}{k\sqrt{3}}\right]$;

(iii) $-\frac{1}{2ax^2}-\frac{b}{a}I$.

7. (i) $\frac{x}{12a(a+bx^4)^3}+\frac{11}{12a}\left[\frac{x}{8a(a+bx^4)^2}+\frac{7}{8}\left\{\frac{x}{4a(a+bx^4)}+\frac{3}{4}I_0\right\}\right]$,

where $I_0=\int\frac{dx}{a+bx^4}$; and if a, b be of like sign and $k^4=\frac{a}{b}$,

$$I_0=\frac{1}{2k^3}\frac{1}{b\sqrt{2}}\left[\tanh^{-1}\frac{kx\sqrt{2}}{k^2+x^2}+\tan^{-1}\frac{kx\sqrt{2}}{k^2-x^2}\right];$$

or if of unlike sign and $k'^4=-\frac{a}{b}$,

$$I_0=-\frac{1}{2bk'^3}\left(\tanh^{-1}\frac{k'}{x}+\tan^{-1}\frac{x}{k'}\right);$$

(ii) $-\frac{1}{a^3x}-\frac{13b}{32a^3}\frac{x^3}{a+bx^4}-\frac{b}{8a^2}\frac{x^3}{(a+bx^4)^2}-\frac{45}{32}\frac{b}{a^3}J_1$,

and $J_1=\frac{1}{2bk\sqrt{2}}\left[-\tanh^{-1}\frac{kx\sqrt{2}}{k^2+x^2}+\tan^{-1}\frac{kx\sqrt{2}}{k^2-x^2}\right]$, if $\frac{a}{b}$ be $+^{\text{ve}}=k^4$,

or $=\frac{1}{2bk'}\left[-\tanh^{-1}\frac{k'}{x}+\tan^{-1}\frac{x}{k'}\right]$, if $\frac{a}{b}$ be $-^{\text{ve}}=-k'^4$.

PAGE 267.

4. If $I_{m,n}$ denote the given integral,

$$I_{m,n}=\frac{x^{m-1}(1+x^2)^{\frac{n}{2}+1}}{m+n+1}-\frac{m-1}{m+n+1}I_{m-2,n},$$

$$I_{5,7}=(1+x^2)^{\frac{9}{2}}\left\{\frac{x^4}{13}-\frac{4x^2}{13.11}+\frac{4.2}{13.11.9}\right\}.$$

6. With a similar notation,

(a) $I_n=\dfrac{x}{(n-2)a^2(a^2+x^2)^{\frac{n-2}{2}}}+\dfrac{n-3}{n-2}\dfrac{1}{a^2}I_{n-1}$;

(b) $I_{n,p}=\dfrac{x^n(a+bx)^{p+\frac{3}{2}}}{(p+n+\frac{3}{2})b}-\dfrac{an}{(p+n+\frac{3}{2})}I_{n-1,p}$;

(c) $mI_m=x^{m-1}(a^2+x^2)^{\frac{1}{2}}-(m-1)a^2I_{m-2}$;

(d) $(m-n+1)I_{m,n}=\dfrac{x^{m-2}}{(a^3+x^3)^{\frac{n}{3}-1}}-(m-2)a^3I_{m-3,n}$;

(e) $mI_m=x^{m-2}(x^3-1)^{\frac{2}{3}}+(m-2)I_{m-3}$;

(f) $I_{n,p}=\dfrac{x^{2n-1}(x^2+a^2)^{p+\frac{3}{2}}}{2n+2p+2}-\dfrac{(2n-1)a^2}{2n+2p+2}I_{n-1,p}$,

$$(x^3-1)^{\frac{2}{3}}\left(\frac{x^6}{8}+\frac{6x^3}{8.5}+\frac{6.3}{8.5.2}\right).$$

7. $x^{2n}(1-x^2)^{\frac{1}{2}}=2nI_{2n-1}-(2n+1)I_{2n+1}$, where the integral $\equiv I_{2n+1}$.

8. $I_n=\dfrac{2}{2n+1}x^n\sqrt{x-1}+\dfrac{2n}{2n+1}I_{n-1}$.

11. $I_n=e^{ax}\cos^{n-1}x\dfrac{a\cos x+n\sin x}{a^2+n^2}+\dfrac{n(n-1)}{a^2+n^2}I_{n-2}$,

$$I_4=\frac{e^{ax}}{a^2+4^2}\Big[\cos^3x(a\cos x+4\sin x)+\frac{4.3}{a^2+2^2}\left\{\cos x(a\cos x+2\sin x)+2.1.\frac{1}{a}\right\}\Big].$$

12. (1) $I_n=-x^n\cos x+nx^{n-1}\sin x-n(n-1)I_{n-2}$;

(2) $I_n=e^{ax}\sin^{n-1}x\dfrac{a\sin x-n\cos x}{n^2+a^2}+\dfrac{n(n-1)}{n^2+a^2}I_{n-2}$,

$$I_n=-\sin^{n-1}x\frac{a\sin x\sin ax+n\cos x\cos ax}{n^2-a^2}+\frac{n(n-1)}{n^2-a^2}I_{n-2}.$$

16. (m even) $\dfrac{m(m-1)(m-2)(m-3)\dots 2.1}{(n^2+m^2)\{n^2+(m-2)^2\}\dots(n^2+2^2)}\dfrac{2\sinh\frac{n\pi}{2}}{n}$;

(m odd) $\dfrac{m(m-1)(m-2)(m-3)\dots 3.2}{(n^2+m^2)\{n^2+(m-2)^2\}\dots(n^2+3^2)}\dfrac{2\cosh\frac{n\pi}{2}}{n^2+1^2}$.

18. $$\frac{1}{3m}+\frac{m}{3m(3m-2)}+\frac{m(m-1)}{3m(3m-2)(3m-4)}+\ldots$$
$$+\frac{m(m-1)\ldots 2}{3m(3m-2)\ldots(m+2)}+\frac{m(m-1)\ldots 1}{3m(3m-2)\ldots(m+2)}\cdot\frac{1}{m}\left(1-\cos\frac{m\pi}{2}\right).$$

34. If $m^2\equiv\frac{b+\sqrt{b^2-4ac}}{2c}$, $n^2\equiv\frac{b-\sqrt{b^2-4ac}}{2c}$, and $b^2>4ac$,
$$\int\frac{dx}{a+bx^2+cx^4}=\frac{1}{\sqrt{b^2-4ac}}\left[\frac{1}{n}\tan^{-1}\frac{x}{n}-\frac{1}{m}\tan^{-1}\frac{x}{m}\right];$$
or $$=\frac{1}{4ck^3}\left[\sec\phi\tanh^{-1}\frac{2kx\cos\phi}{k^2+x^2}+\operatorname{cosec}\phi\tan^{-1}\frac{2kx\sin\phi}{k^2-x^2}\right],$$
where $a=ck^4$;

and $\cos 2\phi=-\frac{b}{2\sqrt{ac}}$, where $b^2<4ac$.

If $b^2=4ac$, the integral $=\frac{x}{bx^2+2a}+\frac{1}{\sqrt{2ab}}\tan^{-1}x\sqrt{\frac{b}{a}}$,
$$\int\frac{x^2\,dx}{a+bx^2+cx^4}=\frac{1}{\sqrt{b^2-4ac}}\left(m\tan^{-1}\frac{x}{m}-n\tan^{-1}\frac{x}{n}\right),\text{ if } b^2>4ac,$$
$$=\frac{1}{4kc}\left(-\sec\phi\tanh^{-1}\frac{2kx\cos\phi}{k^2+x^2}+\operatorname{cosec}\phi\tan^{-1}\frac{2kx\sin\phi}{k^2-x^2}\right),$$
if $b^2<4ac$,
$$=\frac{2a}{b}\left(\frac{1}{\sqrt{2ab}}\tan^{-1}x\sqrt{\frac{b}{2a}}-\frac{x}{2a+bx^2}\right),\text{ if } b^2=4ac.$$
[Bertrand, *I. C.*, p. 36.]

36. (α) $I_n=I_{n-2}-\frac{\tanh^{n-1}x}{n-1}$;

(β) $I_n=-\frac{(n-2)x\cos x+\sin x}{(n-1)(n-2)\sin^{n-1}x}+\frac{n-2}{n-1}I_{n-2}$;

(γ) $\frac{be^x-ce^{-x}}{(a+be^x+ce^{-x})^n}=-(n-2)I_{n-2}+(2n-3)a\,I_{n-1}+(n-1)(4bc-a^2)\,I_n$.

40. $\frac{b+cx}{(a+2bx+cx^2)^{n-1}}=-2(n-1)(b^2-ac)\,I_n-(2n-3)c\,I_{n-1}$.

43. $\frac{\pi}{2^4}(a+b)(5a^2-2ab+5b^2)$.

44. $I_n-2I_{n-1}+I_{n-2}=-\frac{2}{n-1}\sin 2(n-1)x$,
$$n(2x-\pi)+\cot x+2\left[(n-1)\sin 2x+(n-2)\frac{\sin 4x}{2}+\ldots+\frac{\sin 2(n-1)x}{n-1}\right].$$

49. See Art. 202.

CHAPTER VIII.

Page 286.

1. (i) $\log\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}$; (ii) $\frac{1}{\sqrt{3}}\log\frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}$;

(iii) $2\sqrt{x+2}+\frac{2}{\sqrt{3}}\log\frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}$; (iv) $\frac{2}{3}(x-1)^{\frac{3}{2}}+2\sqrt{3}\tan^{-1}\sqrt{\frac{x-1}{3}}$.

2. (i) $\frac{1}{\sqrt{2}}\left(\tanh^{-1}\frac{\sqrt{2x}}{1+x}+\tan^{-1}\frac{\sqrt{2x}}{1-x}\right)$;

(ii) $\frac{1}{\sqrt{2}}\left(\tanh^{-1}\frac{\sqrt{2}\sqrt{x+1}}{x+2}+\tan^{-1}\frac{\sqrt{2}\sqrt{x+1}}{-x}\right)$;

(iii) $-\sqrt{2}\tanh^{-1}\sqrt{2}\frac{\sqrt{x+1}}{x+2}$;

(iv) $2\sqrt{x+1}+\frac{3}{\sqrt{2}}\tanh^{-1}\sqrt{2}\frac{\sqrt{x+1}}{x+2}-\frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{2(x+1)}}{-x}$.

3. (i) $-\operatorname{cosech}^{-1}x$; (ii) $-\frac{1}{\sqrt{2}}\sinh^{-1}\frac{1-x}{1+x}$;

(iii) $\sinh^{-1}x+\frac{1}{\sqrt{2}}\sinh^{-1}\frac{1-x}{1+x}$;

(iv) $\sqrt{x^2+2x+3}-\sinh^{-1}\frac{x+1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\sinh^{-1}\frac{\sqrt{2}}{x+1}$.

5. $\log\frac{\sqrt{2\cot\theta+3}-1}{\sqrt{2\cot\theta+3}+1}-\frac{1}{\sqrt{3}}\log\frac{\sqrt{2\cot\theta+3}-\sqrt{3}}{\sqrt{2\cot\theta+3}+\sqrt{3}}$.

6. $$\frac{2^{\frac{1}{4}}}{ab}\left[\sqrt{\frac{a}{a+b}}\coth^{-1}\left\{\sqrt{\frac{a}{a+b}(\cot\theta+1)}\right\}-\sqrt{\frac{a}{a-b}}\coth^{-1}\left\{\sqrt{\frac{a}{a-b}(\cot\theta+1)}\right\}+\sqrt{\frac{b}{a+b}}\tanh^{-1}\left\{\sqrt{\frac{b}{a+b}(\tan\theta+1)}\right\}+\sqrt{\frac{b}{a-b}}\tan^{-1}\left\{\sqrt{\frac{b}{a-b}(\tan\theta+1)}\right\}\right].$$

7. $\sinh^{-1}\left(\frac{1}{\sqrt{3}}\sec 2\theta\right)$.

8. $\sqrt{x^2+1}\left[\frac{x^4}{5}+\frac{x^3}{4}+\frac{x^2}{15}+\frac{9x}{8}+\frac{43}{15}\right]+\frac{15}{8}\sinh^{-1}x-2\sqrt{2}\sinh^{-1}\frac{x+1}{x-1}$.

9. (i) $\sin^{-1}\frac{2x-a-b}{a-b}$;

(ii) $\frac{1}{\sqrt{a-b}}\log\frac{\sqrt{a-b}+\sqrt{x-b}}{\sqrt{a-b}-\sqrt{x-b}}$ $(a>b)$, $-\frac{2}{\sqrt{b-a}}\tan^{-1}\frac{\sqrt{x-b}}{\sqrt{b-a}}$ $(a<b)$;

(iii) $\frac{1}{\sqrt{a-b}}\log\frac{\sqrt{a-b}-\sqrt{a-x}}{\sqrt{a-b}+\sqrt{a-x}}$ $(b<a)$, $\frac{2}{\sqrt{b-a}}\tan^{-1}\sqrt{\frac{a-x}{b-a}}$ $(b>a)$;

(iv) (*a*) $-\frac{1}{a\sqrt{2}}\sinh^{-1}\frac{a-x}{a+x}$; (*b*) $\frac{1}{a\sqrt{2}}\sinh^{-1}\frac{a+x}{a-x}$;

(*c*) $-\frac{1}{a}\sqrt{\frac{a-x}{a+x}}$; (*d*) $\frac{1}{a}\sqrt{\frac{a+x}{a-x}}$.

10. $-\frac{1}{\sqrt{ab}}\cosh^{-1}\frac{2ab-(a+b)x}{(a-b)x}$.

12. $\cosh^{-1}\frac{x+p}{\sqrt{p^2-q}}+\frac{\alpha-\beta}{\sqrt{\beta^2+2p\beta+q}}\cosh^{-1}\frac{(\beta+p)x+p\beta+q}{\sqrt{p^2-q}(\alpha-\beta)}$, if $p^2>q$, with a modification if $p^2<q$.

13. (i) $-\sinh^{-1}\frac{2+x}{x\sqrt{3}}$; (ii) $\frac{1}{a}\sec^{-1}\frac{x}{a}$; (iii) $\frac{1}{\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{\sqrt{1-x^2}}$.

14. (i) $\sqrt{\frac{1+x}{1-x}}$; (ii) $\frac{1}{\sqrt{3}}\tanh^{-1}\sqrt{\frac{1+4x}{3}}$.

15. (i) $\frac{2}{\sqrt{\mu-\lambda}}\tan^{-1}\sqrt{\frac{x-\mu}{\mu-\lambda}}$ $(\lambda<\mu)$, $-\frac{2}{\sqrt{\lambda-\mu}}\coth^{-1}\sqrt{\frac{x-\mu}{\lambda-\mu}}$ $(\lambda>\mu)$;

(ii) $\frac{x^2}{2}-\frac{\lambda^2}{2}\log(x^2+\lambda^2)$; (iii) $\frac{1}{2}\log\frac{(x+1)(x+3)}{(x+2)^2}$.

PAGE 314.

1. (i) $2\tan^{-1}\sqrt{x}$ (ii) $2\tan^{-1}\sqrt{1+2x}$;

(iii) $-\frac{1}{\sqrt{2}}\cosh^{-1}\frac{4-3x}{x}$; (iv) $-\sinh^{-1}\frac{1}{\sqrt{3}}\frac{1-x}{1+x}$;

(v) $\sqrt{x^2+x+1}-\frac{1}{2}\sinh^{-1}\frac{2x+1}{\sqrt{3}}-\sinh^{-1}\frac{1}{\sqrt{3}}\frac{1-x}{1+x}$; (vi) $\frac{x\sqrt{x-1}}{\sqrt{x+1}}$;

(vii) $-\frac{2}{na^{\frac{n}{2}}}\sinh^{-1}\left(\frac{a}{x}\right)^{\frac{n}{2}}$; (viii) $2\operatorname{cosec}^{-1}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)$.

2. (i) $-\frac{1}{\sqrt{3}}\cosh^{-1}\left\{-\frac{2x+1}{x+2}\right\}$;

(ii) $\sqrt{x^2-1}-2\cosh^{-1}x+\sqrt{3}\cosh^{-1}\left\{-\frac{2x+1}{x+2}\right\}$.

6. (i) $\frac{10}{19}\cosh^{-1}\frac{4}{\sqrt{3}}\sqrt{\frac{4x^2-2x+1}{5x^2+8x}}-\frac{9}{19}\sinh^{-1}\sqrt{\frac{4x^2-2x+1}{5x^2+8x}}$;

(ii) $\frac{1}{\sqrt{(b^2-a^2)(c^2-b^2)}}\cos^{-1}\sqrt{\frac{b^2-a^2}{c^2-a^2}}\sqrt{\frac{x^2+2ax+c^2}{x^2+2ax+b^2}}$ $(a<b<c)$,

with similar results for other cases.

7. (i) $\frac{1}{2ab}\sin^{-1}\frac{(a^2+b^2)x^2-(a^4+b^4)}{(a^2-b^2)(a^2+b^2-x^2)}$;

(ii) $\frac{1}{\sqrt{a^2-c^2}}\sin^{-1}\sqrt{\frac{x^2+c^2}{x^2+a^2}}+\frac{b}{a\sqrt{a^2-c^2}}\cosh^{-1}\frac{a}{c}\sqrt{\frac{x^2+c^2}{x^2+a^2}}$;

(iii) $-\frac{1}{\sqrt{a+c}}\sinh^{-1}\left\{\sqrt{\frac{a+c}{b+c}}\cot\theta\right\}$.

8. (i) $$\frac{1}{\sqrt{(\cos\alpha-\cos\beta)(\cos\alpha-\cos\gamma)}}$$
$$\times\cosh^{-1}\left(\frac{\dfrac{2}{\cos x+\cos\alpha}-\dfrac{1}{\cos\alpha-\cos\beta}-\dfrac{1}{\cos\alpha-\cos\gamma}}{\dfrac{1}{\cos\alpha-\cos\beta}-\dfrac{1}{\cos\alpha-\cos\gamma}}\right)$$

for the case $\cos\alpha > \cos\beta$ or $\cos\gamma$, with modifications for other cases;

(ii) $$-\frac{1}{\sqrt{\sin(\alpha-\beta)\sin(\alpha-\gamma)}}$$
$$\times\cosh^{-1}\left(\frac{\dfrac{2}{\tan x-\cot\alpha}+\dfrac{1}{\cot\alpha-\cot\beta}+\dfrac{1}{\cot\alpha-\cot\gamma}}{\dfrac{1}{\cot\beta-\cot\alpha}-\dfrac{1}{\cot\gamma-\cot\alpha}}\right).$$

9. $$\frac{1}{a}\sqrt{\frac{x^2+ax+a^2}{x^2-ax+a^2}}.$$

10. (i) $$-\frac{1}{2\sqrt{5}}\left[3\sqrt{2}\sin^{-1}\sqrt{\frac{1}{3}\,\frac{-x^2+10x-13}{3x^2-10x+9}}\right.$$
$$\left.+5\sinh^{-1}\sqrt{\frac{1}{2}\,\frac{-x^2+10x-13}{3x^2-10x+9}}\right];$$

(ii) $$\frac{1}{\sqrt{6}}\cosh^{-1}\frac{17-5x}{x-1}-\frac{1}{\sqrt{2}}\cosh^{-1}\frac{10-3x}{(x-2)};$$

(iii) $$\frac{10}{3}\sinh^{-1}\frac{1}{x-1}-\frac{13}{3\sqrt{10}}\sinh^{-1}\frac{3x-2}{x-4};$$

(iv) $$-\frac{b-a}{(b-c)(b-d)}\,\frac{2}{\sqrt{b-e}}\sinh^{-1}\sqrt{\frac{b-e}{x-b}}$$
$$-\frac{c-a}{(c-b)(c-d)}\,\frac{2}{\sqrt{c-a}}\sinh^{-1}\sqrt{\frac{c-e}{x-c}}$$
$$-\frac{d-a}{(d-b)(d-c)}\,\frac{2}{\sqrt{d-e}}\sinh^{-1}\sqrt{\frac{d-e}{x-d}};$$

(v) $$-\cosh^{-1}\sqrt{\frac{x^2+x+2}{x^2+x+1}}+\frac{5}{\sqrt{3}}\cos^{-1}\sqrt{\frac{3}{7}\cdot\frac{x^2+x+2}{x^2+x+1}}.$$

11. (i) $\dfrac{\sqrt{x^4+x^2+1}}{x}$; (ii) $\cosh^{-1}\left(x+\dfrac{1}{x}\right)$.

13. (i) $\dfrac{1}{2}\sin\theta-\dfrac{1}{\sqrt{3}}\tanh^{-1}\left(\dfrac{1}{\sqrt{3}}\tan\dfrac{\theta}{2}\right)$, where $\cos\theta=x^2$;

(ii) $\tan^{-1}\{x(\sqrt{1+x^4}+x^2)^{\frac{1}{2}}\}$; (iii) $\dfrac{2}{\sqrt{5}}\cosh^{-1}\sqrt{5\cdot\dfrac{x^2+ax}{x^2+ax-a^2}}$.

14. $$\frac{1}{(a^2+1)}\left\{\frac{1}{\sqrt{b^2+1}}\sin^{-1}\left(\frac{\sqrt{b^2+1}}{b}\sin x\right)\right.$$
$$\left.+\frac{1}{a\sqrt{b^2-a^2}}\sinh^{-1}\sqrt{\frac{b^2-a^2}{b^2}\,\frac{\tan^2 x}{a^2-\tan^2 x}}\right\},$$

if $b^2>a^2$, with other forms for other cases.

18. $-\frac{\sqrt{2}}{18}\left[4\cos^{-1}\sqrt{\frac{1}{3}\,\frac{4x^2-26x+49}{2x^2-10x+17}}+7\cosh^{-1}\sqrt{\frac{4x^2-26x+49}{2x^2-10x+17}}\right].$

20. $\frac{1}{ab}\tan^{-1}\frac{a}{b}\frac{x}{\sqrt{a^2+b^2+x^2}}.$

21. (i) $\sec^{-1}(\cos x+\sec x)$; (ii) $\frac{\pi}{a\sqrt{a^2+c^2}}.$

25. If $s_1=s_2$, $-\frac{1}{\sqrt{s_1-s_3}}\sinh^{-1}\sqrt{\frac{s_1-s_3}{s-s_1}}.$

If $s_2=s_3$, $\frac{1}{\sqrt{s_1-s_3}}\cos^{-1}\sqrt{\frac{s_1-s_3}{s-s_3}}.$

30. $\frac{1}{\sqrt{2}}\sin^{-1}\frac{x\sqrt{2}}{x^2+1}.$

31. (i) $\frac{1}{2\sqrt{2}}\log\frac{\sqrt{1+x^4}+x\sqrt{2}}{1-x^2}+\frac{1}{2\sqrt{2}}\sin^{-1}\frac{x\sqrt{2}}{1+x^2}$;

(ii) $\frac{1}{4\sqrt{2}}\log\frac{\sqrt{1+x^4}+x\sqrt{2}}{1-x^2}-\frac{1}{4\sqrt{2}}\sin^{-1}\frac{x\sqrt{2}}{1+x^2}.$

34. (ii) $e^x\sqrt{\frac{1+x^n}{1-x^n}}.$

35. (i) $\sin\theta-\frac{1}{3}\theta-\frac{4}{3\sqrt{5}}\log\frac{\sqrt{5}+\tan\frac{\theta}{2}}{\sqrt{5}-\tan\frac{\theta}{2}}$, where $x=\cos\theta$;

(ii) $-\frac{1}{4}[\tan\theta-2\log\tan\theta+\frac{3}{4}\log(\tan\theta-1)+\frac{1}{4}\log(\tan\theta+1)$
$+\frac{1}{2}\log(\tan^2\theta+1)-\frac{1}{2}\theta]$.

41. (i) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\sin^2\theta\right)$; (ii) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\frac{z^2+1}{z^2+z+1}\right).$

45. (i) $\log\sqrt{\frac{x^3-x+1}{x^3+x+1}}$; (ii) $\frac{2}{\sqrt{3a}}\left\{3\tan^{-1}\frac{2\sqrt{x}+\sqrt{a}}{\sqrt{3a}}-\tan^{-1}\frac{2\sqrt{x}-\sqrt{a}}{\sqrt{3a}}\right\}$

52. (i) $\frac{1}{2}\,\frac{x^2}{b^4-x^4}\sin^{-1}\frac{x^2}{b^2}-\frac{1}{2}\,\frac{1}{(b^4-x^4)^{\frac{1}{2}}}$;

(ii) $\frac{1}{2\sqrt{2}}\log\tan\left(\theta+\frac{\pi}{4}\right)+\frac{\theta}{\sqrt{2}}$, where $\sin\phi=\sqrt{2}\sin\theta$.

CHAPTER IX.

Page 326.

1. (i) $\log_e 2$; (ii) $\frac{\pi}{4}$; (iii) $\frac{\pi}{2}$; (iv) $\frac{(2k-1)(2k-3)\dots1}{2k(2k-2)\dots2}.$

3. 2; 4. $\sqrt{2}/a$; 5. $1/\sqrt{2}$.

Page 353.

1. (i) $\frac{\cos x}{x\cos x-\sin x}$; (ii) $\frac{1}{x(1-\log x)}.$

3. (i) $2(n-1)(ac-b^2)\int\frac{dx}{X^n}=\frac{b+cx}{X^{n-1}}+(2n-3)c\int\frac{dx}{X^{n-1}}.$

(ii) $\int \cos mx \sin^4 x\, dx = \frac{\cos^2 mx}{m^2-4^2}\frac{d}{dx}\frac{\sin^4 x}{\cos mx}$

$$-\frac{4\,.\,3}{(m^2-4^2)(m^2-2^2)}\cos^2 mx\frac{d}{dx}\frac{\sin^2 x}{\cos mx}+\frac{4\,.\,3\,.\,2\,.\,1}{(m^2-4^2)(m^2-2^2)}\frac{\sin mx}{m}.$$

4. $\frac{1}{\sqrt{2}}\cosh^{-1}\sqrt{2\frac{3x^2-10x+9}{5x^2-16x+14}}-\frac{1}{\sqrt{3}}\cos^{-1}\sqrt{\frac{3}{2}\cdot\frac{3x^2-10x+9}{5x^2-16x+14}}.$

6. $a\frac{\sqrt{p^2n^2+q^2}}{\sqrt{P^2n^2+Q^2}}\cos\left(nt+e+\tan^{-1}\frac{pn}{q}-\tan^{-1}\frac{Pn}{Q}\right);$

$$a\frac{(PP'+QQ'n^2)\sin(nx+\alpha)+(P'Q-Q'P)n\cos(nx+\alpha)}{P'^2+Q'^2n^2},$$

where $\left.\begin{matrix}P=\alpha-\gamma n^2+\dots,\\ Q=\beta-\delta n^2+\dots,\end{matrix}\right\}$

and P', Q' are the corresponding expressions, with Capitals instead of Greek letters.

8. $\frac{1}{e}$. 9. $\frac{3}{2}$, $\frac{4}{e}$. 12. 2. 13. 1.

15. $\frac{1}{x}\log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)$. If $\pi > x > \frac{\pi}{2}$,

Principal Value $=\frac{1}{x}\log\left\{-\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right\}=\frac{1}{x}\log\tan\left(\frac{3\pi}{4}-\frac{x}{2}\right)$.

16. $2-\log 2-\pi$. 32. $\frac{n\pi}{2ab}$.

41. Principal Value $=\frac{1}{2c}\log\left(\frac{b+c}{b-c}\cdot\frac{c-a}{c+a}\right)$. [See Art. 347 (c).]

47. (i) $\frac{e^x}{x+1}$; (ii) $e^x\frac{x-1}{x+1}$; (iii) $e^x\sqrt{\frac{1+x}{1-x}}$;

(iv) $\frac{1}{2}\left[\log\frac{e^x}{e^x-1}-\frac{1}{e^x-1}\left\{1+\frac{x}{(e^x-1)}\right\}\right]$;

(v) $\frac{\sin x}{\cos x+x\sin x}$; (vi) $\log\left(\frac{\log\tan e^2}{\log\tan e}\right)$;

(vii) $-2\sqrt{1-x}\log(1+x^2)+8\sqrt{1-x}$

$$-4\left\{R\tanh^{-1}\frac{2R\sqrt{1-x}}{2R^2-x}+S\tan^{-1}\frac{2S\sqrt{1-x}}{2S^2+x}\right\},$$

where $R^2=\frac{\sqrt{2}+1}{2}$, $S^2=\frac{\sqrt{2}-1}{2}$.

CHAPTER X.

Page 377.

12. The integrand becomes ∞ at the limit $\theta=\alpha$, but remains real and finite from $\theta=0$ to $\theta=\alpha$, and the rule of differentiation is not established for this case. But putting $\sin\frac{\theta}{2}=\sin\frac{\alpha}{2}\sin\frac{\phi}{2}$, the difficulty disappears.

14. $y = Ax^{\frac{2\lambda-1}{1-\lambda}}$, where $\dfrac{\int_0^x \xi\eta\, d\xi}{\int_0^x \eta\, d\xi} = \lambda x$.

16. $y = Ax^{\frac{2-n}{2(n-1)}}$, the height of the centroid being $\dfrac{1}{n}$ of the height of the segment.

17. A straight line through the origin.

19. The density at each point varies inversely as the square of the abscissa.

20. $y = (Ax+B)^k$, A, B, k being constants. **21.** If $F(x) = A/\sqrt{x}$.

38. The first $= \dfrac{\pi}{4}$. The second $= -\dfrac{\pi}{4}$. The rule for the reversal of the order of integration is not established when the subject of integration becomes infinite at any point of the range of integration. For $a = 0$, $\int_0^1 \dfrac{a^2 - x^2}{(a^2+x^2)^2}\, dx$ is infinite.

39. The case reduces to $\int_0^\infty e^{-x^2}\cos 2\beta x\, dx = e^{-\beta^2}\int_0^\infty e^{-x^2}\, dx$.

CHAPTER XII.

Page 415.

1. $\frac{8}{3}a^2$.

2. (*a*) $c^2 \sinh \dfrac{h}{c}$; (*b*) $e^h - 1$; (*c*) $h(\log h - 1) + 1$, $(h > 1)$;

(*d*) $\dfrac{\pi ab}{4} - \dfrac{b^2}{2a}\sqrt{a^2-b^2} - \dfrac{ab}{2}\cos^{-1}\dfrac{b}{a}$;

(*e*) (i) $k^2 \log \dfrac{b}{a}$, (ii) $k^2 \sin\omega \log \dfrac{b}{a}$; (*f*) $\frac{1}{2}(e^{h^2} - 1)$.

3. (1) $\frac{16}{3}a^2$; (2) $\frac{16}{3}ab$. Area bisected in either case.

4. (1) $\dfrac{\pi ab}{2} \pm \dfrac{a}{b}\left(c\sqrt{b^2-c^2} + b^2\sin^{-1}\dfrac{c}{b}\right)$;

(2) If $A_1 = \dfrac{a}{2b}\left[c\sqrt{b^2-c^2} + b^2\sin^{-1}\dfrac{c}{b}\right]$, $A_2 = \dfrac{b}{2a}\left[d\sqrt{a^2-d^2} + a^2\sin^{-1}\dfrac{d}{a}\right]$,

the four regions are

$$\frac{\pi ab}{4} - A_1 - A_2 + cd,$$
$$\frac{\pi ab}{4} + A_1 - A_2 - cd,$$
$$\frac{\pi ab}{4} - A_1 + A_2 - cd,$$
$$\frac{\pi ab}{4} + A_1 + A_2 + cd.$$

5. $4a^2$. **6.** $3\pi a^2$. **7.** $\dfrac{a^2}{2}(4-\pi)$. **11.** $\dfrac{352}{15}a^2\sqrt{2}$.

13. (i) $\frac{8a^2}{15}$; (ii) $\frac{4}{3}a^2$. **16.** $\frac{\pi}{4}+\frac{1}{2}\log 2-\frac{1}{2}$. **17.** (i) $\frac{3\pi a^2}{4}$.

19. $a^2\left(\frac{16}{3}+4\sqrt{3}-\frac{\pi}{2}\right)$. **21.** $c^2\left(\frac{\sqrt{3}}{2}+\frac{\pi}{3}\mp\frac{\pi}{2}\right)$. **24.** $\frac{2}{3}$.

PAGE 428.

1. $(a^2-b^2)\tan^{-1}\frac{a}{b}+ab$. **2.** $\frac{\pi a^2}{16}, \frac{\pi a^2}{2}$. **3.** $\frac{\pi a^2}{20}, \frac{\pi a^2}{4}$.

4. $\frac{\pi a^2}{4n}$; n even, $\frac{\pi a^2}{2}$; n odd, $\frac{\pi a^2}{4}$. **5.** $\frac{a^2}{4}\tan\alpha\, e^{2\beta\cot\alpha}(e^{2\gamma\cot\alpha}-1)$.

6. $\frac{a^2}{6}\left(\frac{1}{\alpha^3}-\frac{1}{\beta^3}\right)$. **7.** $\frac{a^2}{2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)$. **8.** $\frac{3}{2}\pi a^2$.

9. (i) $\pi\left(a^2+\frac{1}{2}b^2\right)$; (ii) $A_o=\frac{2a^2+b^2}{2}\cos^{-1}\left(\frac{-a}{b}\right)+\frac{3a}{2}\sqrt{b^2-a^2}$,

$$A_i=\frac{2a^2+b^2}{2}\cos^{-1}\left(\frac{a}{b}\right)-\frac{3a}{2}\sqrt{b^2-a^2}.$$

10. $\frac{q^2}{3}(10\pi+9\sqrt{3})$. **12.** $\frac{3a^2}{2}$.

14. $\frac{a^2}{4}\log\left(\frac{1+\sqrt{\sin\alpha}}{1-\sqrt{\sin\alpha}}\,\frac{1-\sqrt{\sin\beta}}{1+\sqrt{\sin\beta}}\right)-\frac{a^2}{2}[\tan^{-1}\sqrt{\sin\alpha}-\tan^{-1}\sqrt{\sin\beta}]$.

15. Area of lozenge $=\frac{a^2}{16}(16-9\sqrt{3})$.

17. $\frac{5}{4}\pi a^2$. **19.** $\frac{5}{2}\pi a^2$. **20.** $\frac{\pi a^2}{16}\left(\frac{\pi^2}{6}-1\right)$.

PAGE 429.

1. $\left(\frac{\sqrt{3}}{2}+\frac{\pi}{12}-1\right)a^2$, $\left(\frac{\sqrt{3}}{2}+\frac{25\pi}{12}+1\right)a^2$, $\frac{8a^2}{15}\sqrt{\frac{a}{b}}$. **2.** $(\pi-2)a^2$.

3. $\left\{2\log(\sqrt{2}+1)-\frac{11\sqrt{2}}{24}\right\}a^2$. **4.** a^2, $\pi a^2\sqrt{2}$.

7. $\frac{b^2-a^2}{2}\log\frac{b-a}{b+a}+(b^2+a^2)\cot^{-1}\frac{b}{a}$.

8. $3\pi a^2$. **17.** $16\pi a^2/3\sqrt{3}$. **18.** $\pi a^2/2$.

19. $\frac{l^2}{2(1-e^2)^{\frac{3}{2}}}\left[\cos^{-1}\frac{e+\cos\theta}{1+e\cos\theta}-e\sqrt{1-e^2}\frac{\sin\theta}{1+e\cos\theta}\right]_{-\alpha}^{\pi-\alpha}$.

21. $$\frac{a^2}{4b^3}[2b^2\{(\alpha+2\pi)^2e^{2b\alpha}-(\beta+2\pi)^2e^{2b\beta}\}-2b\{(\alpha+2\pi)e^{2b\alpha}-(\beta+2\pi)e^{2b\beta}\}+(e^{2b\alpha}-e^{2b\beta})]e^{2b\pi}$$
$$-\frac{a^2}{4b^3}[2b^2(\alpha^2e^{2b\alpha}-\beta^2e^{2b\beta})-2b(\alpha e^{2b\alpha}-\beta e^{2b\beta})+(e^{2b\alpha}-e^{2b\beta})].$$

22. $\frac{\pi a^2}{16}\left(\frac{\pi^2}{6}-1\right)$ 23. $\frac{\pi a^2}{2}\sqrt{2}$. 24. 2 : 1.

25. $\frac{19}{12}\sqrt{7}-\frac{7}{12}+\frac{5}{4}\log\frac{5+2\sqrt{7}}{3}$. 26. $\frac{2\pi}{\sqrt{3}}$. 27. (i) $\frac{3\pi a^2}{8}$.

30. $(\pi+2)a^2$. 31. $\pi a^2 - a^2\cos^{-1}\frac{b^2}{a^2}+b^2\cosh^{-1}\frac{a^2}{b^2}$. 33. $a^2\left(1-\frac{\pi}{4}\right)$.

35. $A=\sqrt{R^2-b^2}-b\cos^{-1}\frac{b}{R}$, where $R^2=(p-a)^2+q^2$.

43. $$\frac{a_1b_1}{2}\tan^{-1}\frac{a_1b_1\sin(\theta_2-\theta_1)}{a_1{}^2\sin\theta_1\sin\theta_2+b_1{}^2\cos\theta_1\cos\theta_2}$$
$$-\frac{ab}{2}\tan^{-1}\frac{ab\sin(\theta_2-\theta_1)}{a^2\sin\theta_1\sin\theta_2+b^2\cos\theta_1\cos\theta_2},$$ where $a_1{}^2-a^2=b_1{}^2-b^2=\lambda$.

52. πa^2. 53. $v_1+\frac{c}{v_1{}^2}\left[\frac{1}{2}\log 3-\frac{\pi b}{6\sqrt{3}}\right]$. 54. $\frac{ab}{2}\sinh c\,[\sinh 2c+c]$.

55. $\pi c(\sqrt{a}-\sqrt{b})^2$. 56. At the cusps.

57. $\left\{\begin{array}{l}\text{Area of loop of first } =\frac{\pi a^2}{2} =157 \text{ sq. cm., about,}\\ \text{Area of loop of second} =\frac{\pi a^2}{2}\sqrt{2}=222 \text{ sq. cm., about}\end{array}\right\}(a=10).$

58. $(\pi+1)a^2$.

CHAPTER XIII.

Page 466.

1. Double the area swept out by the portion of the tangent intercepted between the original curve and the first positive pedal.

3. $\frac{\pi ab}{4}-\frac{b}{2}\sqrt{a^2-b^2}$.

4. $\frac{3a^4+2a^2b^2+3b^4}{16ab}\tan^{-1}\frac{b}{a}-\frac{(3a^2+b^2)(a^2+3b^2)(a^2-b^2)}{16(a^2+b^2)^2}$.

7. $\pi a(a-b)$. 13. $\frac{\pi^3a^2}{24}+\frac{\pi}{8}\{(h-a)^2+a^2\}$, and is least if $h=a$.

14. $x^2y^2=(a^2-y^2)(y^2-b^2)$. 20. πc^2, c being the constant.

25. $\left[a^2\theta+\frac{a^4}{2c^2}\tan\theta\right]_{\theta_1}^{\theta_2}$, where c is the diameter of the circle.

31. The vertex. 34. A circle of radius a; πa^2.

CHAPTER XIV.

Page 478.

1. (i) $\frac{\mu a^4}{8}$. Density $=\mu xy$; (ii) $\bar{x}=\bar{y}=\frac{8}{15}a$; (iii) $B=\frac{1}{3}Ma^2$.

2. (i) $\mu\dfrac{2^{q+2}a^{p+q+2}}{(q+1)(2p+q+3)}$;

(ii) $\bar{x}=\dfrac{2p+q+3}{2p+q+5}a$; $\bar{y}=2\dfrac{q+1}{q+2}\cdot\dfrac{2p+q+3}{2p+q+4}a$; (iii) $B=\dfrac{2p+q+3}{2p+q+7}Ma^2$.

3. (i) $\bar{x}=\dfrac{n+1}{n+2}l$, ($l$ = length); (ii) $\dfrac{n+1}{n+3}Ml^2$.

(iii) $\dfrac{2}{(n+2)(n+3)}Ml^2$; (iv) $\dfrac{1}{4}\dfrac{n^2+n+2}{(n+2)(n+3)}Ml^2$.

4. (i) $\bar{x}=\frac{4}{5}a$, $\bar{y}=\dfrac{3}{5}\dfrac{2+m^2}{3+m^2}ma$; (ii) $B=\frac{2}{3}Ma^2$.

5. (i) $\bar{x}=\dfrac{a}{5}\dfrac{15\pi-44}{3\pi-8}$, $\bar{y}=\dfrac{a}{3\pi-8}$; (ii) $\bar{x}=\frac{9}{5}a^{\frac{1}{3}}b^{\frac{2}{3}}$, $\bar{y}=\frac{9}{5}a^{\frac{2}{3}}b^{\frac{1}{3}}$;

(iii) $\bar{x}=\frac{2}{5}a$; $\bar{y}=a$.

6. (i) Moment of Inertia about base $=\dfrac{Mh^2}{6}$, h being the perpendicular from the vertex to the base;

(ii) $\dfrac{\Delta}{3}(AL^2+AM^2+AN^2)$, where A is the angular point and L, M, N the mid-points of the sides.

Page 484.

1. (α) $\bar{x}=\dfrac{2}{3}\dfrac{a\sin\alpha}{\alpha}$, $\bar{y}=0$, } 2α being the angle of the sector, and a the radius;

(β) $\bar{x}=\dfrac{n+2}{n+3}\dfrac{a\sin\alpha}{\alpha}$, $\bar{y}=0$.

2. $\bar{x}=\dfrac{n+2}{n+4}a$, $\bar{y}=0$, a being the diameter;

(i) $\dfrac{(n+2)(n+3)(n+5)}{(n+4)^2(n+6)}Ma^2$; (ii) $\dfrac{(n+2)(n+3)}{(n+4)^2(n+6)}Ma^2$;

(iii) $\dfrac{(n+2)(n+3)}{(n+4)^2}Ma^2$.

3. (b) If (p_1, q_1), (p_2, q_2), (p_3, q_3) be the coordinates of A, B, C, viz.

$$p_1=-\frac{c_2-c_3}{m_2-m_3},\qquad q_1=\frac{m_2c_3-m_3c_2}{m_2-m_3},\ \text{etc.},$$

$$A=\frac{M}{12}\Sigma(q_2+q_3)^2,\quad B=\frac{M}{12}\Sigma(p_2+p_3)^2.$$

4. $\bar{x}=\dfrac{p+1}{p+2}\dfrac{a_2^{p+2}-a_1^{p+2}}{a_2^{p+1}-a_1^{p+1}}$, $\bar{y}=\dfrac{q+1}{q+2}\dfrac{b_2^{p+2}-b_1^{p+2}}{b_2^{p+1}-b_1^{p+1}}$,

$A=\dfrac{q+1}{q+3}M\dfrac{b_2^{q+3}-b_1^{q+3}}{b_2^{q+1}-b_1^{q+1}}$, $B=\dfrac{p+1}{p+3}M\dfrac{a_2^{p+3}-a_1^{p+3}}{a_2^{p+1}-a_1^{p+1}}$.

7. Area $=\dfrac{a^2}{6}(2\pi+3\sqrt{3})$;

(1) $\bar{x}=\dfrac{3a\sqrt{3}}{2(3\sqrt{3}-\pi)}$, $\bar{y}=0$; (2) $Ma^2\dfrac{9\sqrt{3}-\pi}{9\sqrt{3}-3\pi}$.

8. (i) $A=\frac{2^4 \cdot 3^2}{35} Ma^{\frac{4}{3}}b^{\frac{2}{3}}$, $B=\frac{2^4 \cdot 3^2}{35} a^{\frac{2}{3}}b^{\frac{4}{3}}$;

(ii) $C=\frac{2^4 \cdot 3^2}{35} Ma^{\frac{2}{3}}b^{\frac{2}{3}}(a^{\frac{2}{3}}+b^{\frac{2}{3}})$.

9. $\bar{x}=x-\frac{c}{s}(y-c)$, $\bar{y}=\frac{1}{4}\left(y+\frac{cx}{s}\right)$.

PAGE 492.

1. (i) $2a^2\left(1-\frac{\pi}{4}\right)$; (ii) $\frac{a^2n^2-b^2m^2}{2m^3n^3}\tan^{-1}\frac{an}{bm}+\frac{ab}{2m^2n^2}$.

2. (i) $7\pi a^2/2^9$; (ii) $7\pi a^2\sqrt{2}/2^{14}$.

7. $\pi\left[c^2+\frac{l^2}{(1-e^2)^{\frac{3}{2}}}\right]$. 9. $15\pi ab/2^7$.

15. (i) ab. (ii) $\pi ab/2$.

17. $11\pi a^2/2^{15}\cdot 3^{12}$. 21. $\bar{x}=8a\sqrt{2}\{\log(\sqrt{2}+1)-\frac{7}{12}\sqrt{2}\}/\pi(4\sqrt{2}-5)$.

25. $2(a^2x^2+b^2y^2)^3=(a^2-b^2)^2(a^2x^2-b^2y^2)^2$. 26. $\pi(a^2+b^2)c^2/2ab$.

CHAPTER XV.

PAGE 521.

7. $\frac{\Sigma[A]\sin 2A}{4\sin A\sin B\sin C}-\pi R^2$, R being the radius of the circumcircle.

CHAPTER XVI.

PAGE 533.

1. $a\left[2\sqrt{\frac{8a-3x}{2a-x}}+\sqrt{3}\cosh^{-1}\frac{3x-7a}{a}\right]_{x_1}^{x_2}$. 2. A cycloid.

4. $\left(x_2^{\frac{2}{3}}+y_2^{\frac{2}{3}}\right)^{\frac{3}{2}}-\left(x_1^{\frac{2}{3}}+y_1^{\frac{2}{3}}\right)^{\frac{3}{2}}$.

PAGE. 538.

1. (i) $a(\theta_2-\theta_1)$; (ii) $\frac{a\sqrt{1+m^2}}{m}(e^{m\theta_2}-e^{m\theta_1})$;

(iii) $2a\left(\cos\frac{\theta_1}{2}-\cos\frac{\theta_2}{2}\right)$;

(iv) $2a\left\{\left(\tan\frac{\theta_2}{2}-\tan\frac{\theta_1}{2}\right)+\frac{1}{3}\left(\tan^3\frac{\theta_2}{2}-\tan^3\frac{\theta_1}{2}\right)\right\}$;

(v) $a\left[\frac{\sqrt{1+3\cos^2\theta}}{\cos\theta}-\frac{\sqrt{3}}{2}\cosh^{-1}(1+6\cos^2\theta)\right]$ (cf. Ex. 1, p. 533);

(vi) $\frac{a}{18}\left[(4+9\tan^2\theta)^{\frac{3}{2}}\right]_{\theta_1}^{\theta_2}$.

PAGE 541.

1. (i) A circle; (ii) A catenary; (iii) An involute of a circle; (iv) The tractrix; (v) An equiangular spiral; (vi) A cycloid;

(vii) $\theta+2\sin^{-1}\sqrt{\frac{r}{2a}}+2\sqrt{\frac{2a-r}{r}}=\text{const.}$

PAGE 546.

2. $\frac{8a}{3}$.

PAGE 570.

2. $4a/\sqrt{3}$.

5. (i) − the area;

(ii) the area;

(iii) 0 or 2π, according as the origin lies within or without the area, there being one convolution about the pole; or if there be n convolutions, $2n\pi$.

10. Equiangular spirals. 12. $5a$. 13. Involute of a circle.

15. $2a[3\sqrt{3}+3\sqrt{2}+\log(\sqrt{2}+1)]$, $4a$ being the latus rectum.

17. Epicycloid. $2\frac{c^2-a^2}{a}$. 19. $4a$.

25. $\bar{x}=a\frac{\sqrt{2}}{3}(B+C)/A,\ \bar{y}=a\frac{\sqrt{2}}{3}(B-C)/A,$

where $A=\Big[\tan\psi-\psi\Big]_{\psi_1}^{\psi_2},\ B=\Big[\sec\psi+\cos\psi\Big]_{\psi_1}^{\psi_2},$

$$C=\left[\frac{\sin^3\psi}{\cos^2\psi}-\frac{3}{2}\frac{\sin\psi}{\cos^2\psi}-\frac{3}{2}\log\tan\left(\frac{\pi}{4}+\frac{\psi}{2}\right)\right]_{\psi_1}^{\psi_2},\ \text{and}\ \psi=\frac{\pi}{4}-\theta.$$

[*E. T.*]

27. $s=\dfrac{2r\cos\frac{a}{2}}{\cos a}$. 28. $s=\frac{\pi}{16}\frac{1}{a^{\frac{5}{2}}b^{\frac{5}{2}}}\{3a^2+2ab+3b^2\}$.

29. Area $=\pi(a^2+2b^2)$. 30. $\frac{1}{2}\pi a^2$.

31. $s=\dfrac{1-m^2}{n}\displaystyle\int\frac{\sin\phi\,d\phi}{(\sin^2\phi+m^2\cos^2\phi)^{\frac{2n-1}{m}}}$.

39. $s=2a(\sec^3\psi-1)$. If $c=0$, the involute is $y^2=4a(x+2a)$.

40. $\dfrac{a^3-b^3}{a^2-b^2}$.

CHAPTER XVII.

PAGE 600.

2. $A=\frac{1}{\sqrt{2}}F_1,\ \left(\text{mod.}\ \frac{1}{\sqrt{2}}\right)=1{\cdot}31102\ldots$ square units.

PAGE 636.

2. With notation in *Diff. Calc.*, Art. 458,
$$\begin{cases} b=a, & A=2a^2, \\ b>a, & A=2b^2E_1, \text{ mod. } \dfrac{a^2}{b^2}. \\ b<a, & A=2a^2\left[E_1-\dfrac{a^4-b^4}{a^4}F_1\right], \text{ mod. } \dfrac{b^2}{a^2}. \end{cases}$$

10. $$x=\frac{(y+\sqrt{y^2-4a^6})^{\frac{1}{3}}+(y-\sqrt{y^2-4a^6})^{\frac{1}{3}}}{2^{\frac{1}{3}}},$$

$$x=11 \text{ or } -4\pm 3\sqrt{-3}.$$

24. (i) $\tanh^{-1}\dfrac{\sqrt{x^4+2x^3-3x^2-4x+3}}{x^2+x-2}$;

(ii) $\dfrac{1}{3}\tanh^{-1}\dfrac{(x+2)\sqrt{R}}{x^3+3x^2-2-\dfrac{a}{2}}$, where $R=x^4+2x^3-3x^2-ax+a$;

(iii) $2\tanh^{-1}\dfrac{x}{x+3}\sqrt{\dfrac{x+2}{x+1}}$; (iv) $\tanh^{-1}\dfrac{x}{x+1}\sqrt{\dfrac{1+6x+4x^2}{1-2x+4x^2}}$;

(v) $\cosh^{-1}\dfrac{x^2+ax+a^2}{a\sqrt{2}}$; (vi) $\tanh^{-1}\dfrac{x}{x^3+1}\sqrt{x^4+1}$;

(vii) $2\tanh^{-1}(x+1)\sqrt{x}$; (viii) $\tanh^{-1}x\sqrt{x^4+1}$;

(ix) $2\tanh^{-1}\dfrac{x+b}{x+a}\sqrt{\dfrac{x^2+a^2}{x^2+b^2}}$; (x) $\tanh^{-1}x\sqrt{\dfrac{x+1}{x-1}}$;

(xi) $\tanh^{-1}\dfrac{x}{\sqrt{x^4-1}}$; (xii) $\tanh^{-1}\dfrac{x\sqrt{1+x^4}}{1+x}$.

CHAPTER XVIII.

PAGE 669.

4. $$s=2ak^2\int\frac{\sqrt{1+m^2}\,dm}{(am^2-h)^2+4a^2m^2},$$

$y^2=4ax$ being the parabola, k^2 the const. of inversion, and $(h, 0)$ the pole.

10. $\dfrac{1}{r-1}\dfrac{1}{ca^{r-1}}$.

12. $$I=\frac{2}{\sqrt{(1+\cos v)(\cosh u-\cos v)}}\sin^{-1}\sqrt{\frac{1+\cos v}{1+\cosh u}\cdot\frac{x^2-2x\cosh u+1}{x^2-2x\cos v+1}}.$$

$$I_{e^u}^{\infty}=\frac{\sqrt{2}}{\cos\frac{v}{2}(\cosh u-\cos v)^{\frac{1}{2}}}\sin^{-1}\left(\frac{\cos\frac{v}{2}}{\cosh\frac{u}{2}}\right).$$

14. $F_1(x-\sqrt{x^2-1})+\log F_2(x-\sqrt{x^2-1})$ 15. $\dfrac{3a^2}{2}$.

17. (i) $\dfrac{1}{a-b}\sin^{-1}\dfrac{(x-a)^a}{(x-b)^b}$; (ii) $\dfrac{1}{a^q}\tan^{-1}\left(\dfrac{x^p}{a^q}+\dfrac{a^p}{x^q}\right)$.

19. $\alpha=\frac{\pi}{2}-\frac{1}{2}\cos^{-1}e.$ **24.** $x\cos a+\sin a\log\sin(x-a).$

25. (i) $I=\frac{7}{25}\log(x+1)-\frac{1}{5}\frac{1}{x+1}-\frac{7}{50}\log(x^2+4)+\frac{1}{25}\tan^{-1}\frac{x}{2},$

$[I]_0^\infty=(\pi+14\log 2+10)/50;$

(ii) $I=\frac{1}{2^5}\left\{\frac{\sin 6\theta}{6}+\frac{3\sin 4\theta}{2}+\frac{15\sin 2\theta}{2}+10\theta\right\}$, where $\theta=\tan^{-1}x$,

$[I]_0^\infty=\frac{5\pi}{32};$

(iii) $I=\frac{3}{16}\left\{\frac{\sin x}{5-3\cos x}+\frac{5}{6}\tan^{-1}2\tan\frac{x}{2}\right\},$

$[I]_0^\pi=\frac{5\pi}{64}.$

CHAPTER XIX.

PAGE 723.

3. $\frac{1}{2}(r_2{}^2-r_1{}^2)\tan\alpha\sin^2\alpha\quad(r_1=OP_1,\ r_2=OP_2).$

6. Evolute of roulette of the cusp is a four-cusped hypocycloid. Intrinsic equation of envelope of axis with notation of Ex. 2, Art. 670, is

$$s=a\sin^2\frac{\chi}{3}\left(5+7\cos^2\frac{\chi}{3}\right).$$

20. See Art. 657.

25. The rolling of a catenary upon a straight line.

30. $s=a\psi-3a\sin\left(\frac{\pi}{4}+\frac{\psi}{2}\right)+\text{const.}$

CHAPTER XX.

PAGE 772.

6. $\text{Arc}=\frac{a}{\sqrt{2}}\left[\frac{1}{R}\log\frac{z^2-Rz+\sqrt{2}}{z^2+Rz+\sqrt{2}}-R\tan^{-1}\frac{2z}{R(\sqrt{2}-z^2)}\right]_{\theta_1}^{\theta_2},$

where $R^2=2(\sqrt{2}+1)$, $z=\cos\frac{\theta}{2}$, and θ is the azimuthal angle of a point on the curve.

CHAPTER XXI.

PAGE 790.

2. $\pi^2a^3.$ **3.** $\frac{8\sqrt{2}}{15}\pi a^3.$ **5.** $\frac{2}{3}\pi a^3(3\log 2-2).$

6. $\left.\begin{pmatrix}x=a\cos\theta\\ y=b\sin\theta\end{pmatrix}\right\}.$ For surface from $\theta=\theta_1$ to $\theta=\theta_2$, revolution about the y-axis,

$$S=\pi a\left[\sin\theta\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}+\frac{1-e^2}{e}a\log\left\{ae\sin\theta+\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}\right\}\right]_{\theta_1}^{\theta_2}.$$

8. $\frac{\pi a^3}{12}$. 10. About axis, $\frac{8}{3}\pi a^2(3\pi-4)$; about base, $\frac{64}{3}\pi a^2$.

11. $\frac{\pi^2 a^3}{4\sqrt{2}}$. 14. $\frac{\pi^2 a^3}{2}$. 16. $\frac{4\pi n^3 a^3 \sin\frac{\pi}{n}}{(n^2-1)(9n^2-1)}$. 22. A circular cylinder.

27. $\frac{\pi}{3c^3}(\sqrt{a+c}-\sqrt{a-c})\{a(c-2a)\sqrt{a+c}+(2a^2+ac+2c^2)\sqrt{a-c}\}$.

29. $\frac{2\pi}{3}\{(1+2h)^{\frac{3}{2}}-1\}$.

CHAPTER XXII.

Page 862.

1. In each case $V=\frac{h}{3}(A+\sqrt{AB}+B)$, where h = height of frustum and A, B the areas of the ends.

2. $\frac{1}{3}Ea^3$, a being the radius of the sphere and E the spherical excess.

8. $\int\frac{dS}{p^3}=\frac{4}{9}\pi abc\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)^2$. 9. $\frac{4}{3}\pi abc$.

21. $\frac{d_1 d_2 d_3}{\Delta}$, where $\Delta=\begin{vmatrix} a, & b, & c \\ a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{vmatrix}$.

29. $\frac{\pi}{2}(x_2-x_1)\left\{\alpha^2+\alpha'^2+(\beta+\beta')\frac{x_2+x_1}{2}\right\}$.

31. $\frac{1}{4}\frac{1+\gamma}{1-\gamma}\left(\alpha_1^{\frac{2}{1+\gamma}}-\alpha_2^{\frac{2}{1+\gamma}}\right)\left\{(4\beta_1)^{\frac{1-\gamma}{1+\gamma}}-(4\beta_2)^{\frac{1-\gamma}{1+\gamma}}\right\}(\tan^{-1}b_1-\tan^{-1}b_2)$.

39. $\frac{4\pi bc}{a^2}(a\cosh a-\sinh a)$, $\frac{4\pi abc}{(a^2+b^2)^{\frac{3}{2}}}(\sqrt{a^2+b^2}\cosh\sqrt{a^2+b^2}-\sinh\sqrt{a^2+b^2})$,

$\frac{4\pi abc}{(a^2+b^2+c^2)^{\frac{3}{2}}}(\sqrt{a^2+b^2+c^2}\cosh\sqrt{a^2+b^2+c^2}-\sinh\sqrt{a^2+b^2+c^2})$.

43. Envelope $y=\pm x$, $y^4(x^4-a^4)+a^4x^4=0$.

50. $\frac{\pi}{2}-1$. 51. $\frac{\pi}{(p-1)a^{2(p-1)}}$.