56.

ON THE MULTISECTION OF THE ROOTS OF UNITY.

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IF p be a prime number, e a divisor of $p-1$, and the e periods into which the primitive pth roots of unity may be distributed are the roots of $\eta^e + B\eta^{e-1} + \ldots$, I call this last written function (say E), the e-period function to p. Every divisor of such function, it is well known, if not p itself or an eth power residue of p, must be a divisor of the discriminant of E.

Every divisor q of the discriminant is necessarily a divisor of E but may or may not be, according to circumstances, an eth power residue to p; if it is not, then q may be called an exceptional divisor of the period-function.

When $e = 2$ the discriminant is p itself so that (as is well known) there are no exceptional factors to the two-period function. When $e = 3$, it may be shown that every factor of the discriminant is necessarily a cubic residue of p.

This may be proved by the Law of Reciprocity for cubic residues, although obtained in quite a different manner. It follows that the three-period function has no exceptional divisor.

When $e = 4$ it is better to distinguish between the two cases of $p = 8i + 1$ and $p = 8i + 5$.

In the former case 2 is not necessarily a biquadratic but may be only a quadratic residue of p, although a divisor of the 4-period function, and consequently 2 may be an exceptional divisor. When $p = 8i + 5$, if $p = f^2 + 4\gamma^2$, every divisor of γ is necessarily a divisor of the function inasmuch as γ is contained in the discriminant, but whilst divisors of γ of the form $4i + 1$ are biquadratic, those of the form $4i - 1$ will be only quadratic and not biquadratic residues of p. The results for $e = 4$ so far as yet stated may be proved by the law of reciprocity for biquadratic residues.

But when $p = 8i + 5$, or in other words, when $p = f^2 + 4\gamma^2$ where γ is odd, it may be shown that $\frac{3p^2 + f^2}{16}$ is also that factor of the discriminant which is represented by $(\eta_0 - \eta_1)(\eta_1 - \eta_2)(\eta_2 - \eta_3)(\eta_3 - \eta_0)$, ($\eta_0, \eta_1, \eta_2, \eta_3$ being the four periods taken in natural order), and it is capable of proof that every divisor of this chain of products cannot but be a biquadratic residue to p*, or in other words, every divisor of $\frac{f^2 + 4\gamma^2}{4}$ is a biquadratic residue of $f^2 + 4\gamma^2$ when this last quantity is a prime number. This theorem, deduced from the method applied to the divisors of period-functions, does not appear to be referable to any known theorem concerning biquadratic residues. Professor Sylvester finally stated that he had under consideration the question of the existence or otherwise of exceptional factors to the e-period function in the general case of e being a prime number.

* For suppose q, a prime-number divisor of the "chain-product," to be not a biquadratic residue of p; then if q is a quadratic residue of p, it may be shown that q must be also a divisor of $(\eta_0 - \eta_2)^2 (\eta_1 - \eta_3)^2$ and therefore of $p\gamma^2$, which is impossible because γ is prime to f and p, and if q is a non-quadratic residue of p, it may be shown that all four roots of the congruence, which expresses that the 4-period function contains q, must be equal to one another, which admits of easy disproof. Hence q cannot but be a biquadratic residue of p.