59.

TABLES OF THE GENERATING FUNCTIONS AND GROUNDFORMS OF THE BINARY DUODECIMIC, WITH SOME GENERAL REMARKS, AND TABLES OF THE IRREDUCIBLE SYZYGIES OF CERTAIN QUANTICS*.

[*American Journal of Mathematics*, IV. (1881), pp. 41—61.]

Generating Function for differentiants,

Denominator:

$$(1-a)(1-a^2)^2(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^8)(1-a^9)(1-a^{10})(1-a^{11}).$$

Numerator:

$$\begin{aligned}&1+4a^2+17a^3+49a^4+125a^5+285a^6+594a^7+1143a^8+2063a^9\\&+3517a^{10}+5693a^{11}+8817a^{12}+13104a^{13}+18769a^{14}+25979a^{15}\\&+34830a^{16}+45317a^{17}+57327a^{18}+70595a^{19}+84730a^{20}+99214a^{21}\\&+113430a^{22}+126698a^{23}+138345a^{24}+147722a^{25}+154297a^{26}\\&+157689a^{27}+157689a^{28}+154297a^{29}+147722a^{30}+138345a^{31}\\&+126698a^{32}+113430a^{33}+99214a^{34}+84730a^{35}+70595a^{36}+57327a^{37}\\&+45317a^{38}+34830a^{39}+25979a^{40}+18769a^{41}+13104a^{42}+8817a^{43}\\&+5693a^{44}+3517a^{45}+2063a^{46}+1143a^{47}+594a^{48}+285a^{49}+125a^{50}\\&+49a^{51}+17a^{52}+4a^{53}+a^{55}.\end{aligned}$$

Generating Function for covariants, reduced form,

Denominator:

$$(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^8)(1-a^9)(1-a^{10})(1-a^{11})(1-ax^2)(1-ax^4)(1-ax^6)(1-ax^8)(1-ax^{10})(1-ax^{12}).$$

* The tables of the duodecimic have been calculated by Mr F. Franklin in accordance with Professor Sylvester's second method (see this *Journal*†, Vol. III. p. 146), in pursuance of a grant made by the British Association for the Advancement of Science. The corresponding tables for the binary quantics of the first ten orders are given in this *Journal*, Vol. II. p. 223 [p. 283, above]; those for systems of quantics of the first four orders, taken two and two together, are given at page 293 of the same volume [p. 392, above].

[† *On the Calculation of the Generating Functions and Tables of Groundforms for Binary Quantics*, By F. Franklin, *American Journal of Mathematics*, III. (1880), pp. 128—153.]

Numerator:*

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}	x^{36}	x^{38}	x^{40}
a^0	1																				
a^1		$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$															
a^2			1	1	2	2	3	2	2	1	1										
a^3			1		1	$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{2}$	$\overline{1}$	$\overline{1}$						
a^4	1		3	1	3		1			1	1	2	2	3	2	2	1	1			
a^5	1	1	4	2	2	$\overline{1}$	$\overline{2}$	$\overline{2}$	$\overline{3}$	$\overline{2}$	$\overline{2}$		$\overline{1}$		$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	
a^6	3	3	7	4	4		$\overline{1}$	$\overline{2}$	$\overline{3}$			3	1	2	1	1					1
a^7	4	7	10	7	4	$\overline{1}$	$\overline{6}$	$\overline{7}$	$\overline{8}$	$\overline{4}$	$\overline{3}$	1	$\overline{1}$	1		$\overline{1}$	$\overline{1}$	$\overline{1}$			
a^8	7	12	17	10	6	$\overline{2}$	$\overline{11}$	$\overline{13}$	$\overline{12}$	$\overline{5}$	$\overline{2}$	5	3	6	2	1			1		
a^9	9	23	25	18	7	$\overline{6}$	$\overline{21}$	$\overline{24}$	$\overline{22}$	$\overline{10}$	$\overline{3}$	7	7	8	2	$\overline{1}$	$\overline{1}$	$\overline{2}$		$\overline{1}$	—
a^{10}	17	36	39	25	5	$\overline{15}$	$\overline{39}$	$\overline{45}$	$\overline{37}$	$\overline{16}$	$\overline{1}$	16	16	15	5	1	$\overline{2}$	$\overline{3}$		$\overline{1}$	1
a^{11}	21	56	53	32	$\overline{1}$	$\overline{32}$	$\overline{67}$	$\overline{72}$	$\overline{54}$	$\overline{19}$	9	31	34	24	9	$\overline{1}$	$\overline{6}$	$\overline{7}$	$\overline{2}$	$\overline{1}$	1
a^{12}	36	81	76	44	$\overline{7}$	$\overline{52}$	$\overline{100}$	$\overline{108}$	$\overline{72}$	$\overline{17}$	24	56	56	35	11	$\overline{5}$	$\overline{14}$	$\overline{13}$	$\overline{4}$		3
a^{13}	45	112	97	51	$\overline{27}$	$\overline{93}$	$\overline{158}$	$\overline{162}$	$\overline{101}$	$\overline{21}$	45	85	87	47	11	$\overline{14}$	$\overline{24}$	$\overline{22}$	$\overline{6}$	3	4
a^{14}	65	151	133	63	$\overline{45}$	$\overline{134}$	$\overline{216}$	$\overline{218}$	$\overline{120}$	$\overline{3}$	90	143	136	69	15	$\overline{22}$	$\overline{40}$	$\overline{32}$	$\overline{8}$	7	7
a^{15}	81	199	163	66	$\overline{83}$	$\overline{206}$	$\overline{309}$	$\overline{303}$	$\overline{157}$	5	139	199	187	83	4	$\overline{48}$	$\overline{66}$	$\overline{51}$	$\overline{14}$	13	9
a^{16}	110	251	206	69	$\overline{124}$	$\overline{282}$	$\overline{404}$	$\overline{386}$	$\overline{173}$	52	226	301	267	114		$\overline{69}$	$\overline{97}$	$\overline{68}$	$\overline{16}$	24	17
a^{17}	131	309	241	59	$\overline{188}$	$\overline{389}$	$\overline{532}$	$\overline{489}$	$\overline{196}$	97	323	403	345	127	$\overline{29}$	$\overline{120}$	$\overline{146}$	$\overline{100}$	$\overline{23}$	37	21
a^{18}	168	370	288	51	$\overline{253}$	$\overline{495}$	$\overline{653}$	$\overline{580}$	$\overline{193}$	188	460	550	446	150	$\overline{58}$	$\overline{169}$	$\overline{202}$	$\overline{127}$	$\overline{23}$	57	36
a^{19}	193	433	318	22	$\overline{347}$	$\overline{636}$	$\overline{808}$	$\overline{692}$	$\overline{196}$	274	604	691	539	149	$\overline{116}$	$\overline{251}$	$\overline{274}$	$\overline{168}$	$\overline{26}$	83	45
a^{20}	232	493	359	$\overline{6}$	$\overline{436}$	$\overline{759}$	$\overline{939}$	$\overline{770}$	$\overline{153}$	421	797	883	655	160	$\overline{176}$	$\overline{329}$	$\overline{352}$	$\overline{204}$	$\overline{19}$	115	65
a^{21}	256	551	377	$\overline{54}$	$\overline{546}$	$\overline{912}$	$\overline{1093}$	$\overline{861}$	$\overline{119}$	554	980	1045	742	128	$\overline{277}$	$\overline{448}$	$\overline{450}$	$\overline{254}$	$\overline{18}$	154	81
a^{22}	293	598	402	$\overline{97}$	$\overline{648}$	$\overline{1035}$	$\overline{1209}$	$\overline{909}$	$\overline{35}$	746	1201	1252	844	110	$\overline{372}$	$\overline{553}$	$\overline{543}$	$\overline{292}$	1	201	110
a^{23}	307	638	402	$\overline{161}$	$\overline{762}$	$\overline{1174}$	$\overline{1335}$	$\overline{956}$	43	914	1402	1410	907	45	$\overline{510}$	$\overline{697}$	$\overline{654}$	$\overline{343}$	14	251	131
a^{24}	336	667	407	$\overline{210}$	$\overline{852}$	$\overline{1266}$	$\overline{1404}$	$\overline{948}$	168	1131	1619	1594	972	$\overline{14}$	$\overline{637}$	$\overline{821}$	$\overline{752}$	$\overline{375}$	44	308	168
a^{25}	339	679	381	$\overline{280}$	$\overline{953}$	$\overline{1371}$	$\overline{1480}$	$\overline{944}$	274	1296	1791	1706	983	$\overline{119}$	$\overline{800}$	$\overline{972}$	$\overline{855}$	$\overline{417}$	69	369	193
a^{26}	351	678	367	$\overline{326}$	$\overline{1012}$	$\overline{1405}$	$\overline{1477}$	$\overline{867}$	430	1508	1972	1836	997	$\overline{209}$	$\overline{934}$	$\overline{1086}$	$\overline{937}$	$\overline{434}$	114	430	232
a^{27}	339	664	323	$\overline{382}$	$\overline{1070}$	$\overline{1446}$	$\overline{1479}$	$\overline{810}$	542	1632	2069	1860	937	$\overline{351}$	$\overline{1100}$	$\overline{1225}$	$\overline{1017}$	$\overline{457}$	148	490	256
a^{28}	336	635	291	$\overline{410}$	$\overline{1086}$	$\overline{1418}$	$\overline{1410}$	$\overline{692}$	688	1782	2164	1900	893	$\overline{453}$	$\overline{1211}$	$\overline{1301}$	$\overline{1056}$	$\overline{446}$	203	547	293
a^{29}	307	595	239	$\overline{445}$	$\overline{1093}$	$\overline{1393}$	$\overline{1342}$	$\overline{595}$	784	1837	2172	1837	784	$\overline{595}$	$\overline{1342}$	$\overline{1393}$	$\overline{1093}$	$\overline{445}$	239	595	307

* In the tabulated numerators, the *minus* sign is placed *over* the number which it affects.

Numerator—(*Continued.*)

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}	x^{36}	x^{38}	x^{40}
a^{30}	293	547	203	$\overline{446}$	$\overline{1056}$	$\overline{1301}$	$\overline{1211}$	$\overline{453}$	893	1900	2164	1782	688	$\overline{692}$	$\overline{1410}$	$\overline{1418}$	$\overline{1086}$	$\overline{410}$	291	635	336
a^{31}	256	490	148	$\overline{457}$	$\overline{1017}$	$\overline{1225}$	$\overline{1100}$	$\overline{351}$	937	1860	2069	1632	542	$\overline{810}$	$\overline{1479}$	$\overline{1446}$	$\overline{1070}$	$\overline{382}$	323	664	339
a^{32}	232	430	114	$\overline{434}$	$\overline{937}$	$\overline{1086}$	$\overline{934}$	$\overline{209}$	997	1836	1972	1508	430	$\overline{867}$	$\overline{1477}$	$\overline{1405}$	$\overline{1012}$	$\overline{326}$	367	678	351
a^{33}	193	369	69	$\overline{417}$	$\overline{855}$	$\overline{972}$	$\overline{800}$	$\overline{119}$	983	1706	1791	1296	274	$\overline{944}$	$\overline{1480}$	$\overline{1371}$	$\overline{953}$	$\overline{280}$	381	679	339
a^{34}	168	308	44	$\overline{375}$	$\overline{752}$	$\overline{821}$	$\overline{637}$	$\overline{14}$	972	1594	1619	1131	168	$\overline{948}$	$\overline{1404}$	$\overline{1266}$	$\overline{852}$	$\overline{210}$	407	667	336
a^{35}	131	251	14	$\overline{343}$	$\overline{654}$	$\overline{697}$	$\overline{510}$	45	907	1410	1402	914	43	$\overline{956}$	$\overline{1335}$	$\overline{1174}$	$\overline{762}$	$\overline{161}$	402	638	307
a^{36}	110	201	1	$\overline{292}$	$\overline{543}$	$\overline{553}$	$\overline{372}$	110	844	1252	1201	746	$\overline{35}$	$\overline{909}$	$\overline{1209}$	$\overline{1035}$	$\overline{648}$	$\overline{97}$	402	598	293
a^{37}	81	154	$\overline{18}$	$\overline{254}$	$\overline{450}$	$\overline{448}$	$\overline{277}$	128	742	1045	980	554	$\overline{119}$	$\overline{861}$	$\overline{1093}$	$\overline{912}$	$\overline{546}$	$\overline{54}$	377	551	256
a^{38}	65	115	$\overline{19}$	$\overline{204}$	$\overline{352}$	$\overline{329}$	$\overline{176}$	160	655	883	797	421	$\overline{153}$	$\overline{770}$	$\overline{939}$	$\overline{759}$	$\overline{436}$	$\overline{6}$	359	493	232
a^{39}	45	83	$\overline{26}$	$\overline{168}$	$\overline{274}$	$\overline{251}$	$\overline{116}$	149	539	691	604	274	$\overline{196}$	$\overline{692}$	$\overline{808}$	$\overline{636}$	$\overline{347}$	22	318	433	193
a^{40}	36	57	$\overline{23}$	$\overline{127}$	$\overline{202}$	$\overline{169}$	$\overline{58}$	150	446	550	460	188	$\overline{193}$	$\overline{580}$	$\overline{658}$	$\overline{495}$	$\overline{253}$	51	288	370	168
a^{41}	21	37	$\overline{23}$	$\overline{100}$	$\overline{146}$	$\overline{120}$	$\overline{29}$	127	345	403	323	97	$\overline{196}$	$\overline{489}$	$\overline{532}$	$\overline{389}$	$\overline{188}$	59	241	309	131
a^{42}	17	24	$\overline{16}$	$\overline{68}$	$\overline{97}$	$\overline{69}$		114	267	301	226	52	$\overline{173}$	$\overline{386}$	$\overline{404}$	$\overline{282}$	$\overline{124}$	69	206	251	110
a^{43}	9	13	$\overline{14}$	$\overline{51}$	$\overline{66}$	$\overline{48}$	4	83	187	190	139	5	$\overline{157}$	$\overline{303}$	$\overline{309}$	$\overline{206}$	$\overline{83}$	66	163	199	81
a^{44}	7	7	$\overline{8}$	$\overline{32}$	$\overline{40}$	$\overline{22}$	15	69	136	143	90	$\overline{3}$	$\overline{120}$	$\overline{218}$	$\overline{216}$	$\overline{134}$	$\overline{45}$	63	133	151	65
a^{45}	4	3	$\overline{6}$	$\overline{22}$	$\overline{24}$	$\overline{14}$	11	47	87	85	45	$\overline{21}$	$\overline{101}$	$\overline{162}$	$\overline{158}$	$\overline{93}$	$\overline{27}$	51	97	112	45
a^{46}	3		$\overline{4}$	$\overline{13}$	$\overline{14}$	$\overline{5}$	11	35	56	56	24	$\overline{17}$	$\overline{72}$	$\overline{108}$	$\overline{100}$	$\overline{52}$	$\overline{7}$	44	76	81	36
a^{47}	1	$\overline{1}$	$\overline{2}$	$\overline{7}$	$\overline{6}$	$\overline{1}$	9	24	34	31	9	$\overline{19}$	$\overline{54}$	$\overline{72}$	$\overline{67}$	$\overline{32}$	$\overline{1}$	32	53	56	21
a^{48}	1	$\overline{1}$		$\overline{3}$	$\overline{2}$	1	5	15	16	16	$\overline{1}$	$\overline{16}$	$\overline{37}$	$\overline{45}$	$\overline{39}$	$\overline{15}$	5	25	39	36	17
a^{49}		$\overline{1}$		$\overline{2}$	$\overline{1}$	$\overline{1}$	2	8	7	7	$\overline{3}$	$\overline{10}$	$\overline{22}$	$\overline{24}$	$\overline{21}$	$\overline{6}$	7	18	25	23	9
a^{50}			1			1	2	6	3	5	$\overline{2}$	5	12	$\overline{13}$	$\overline{11}$	$\overline{2}$	6	10	17	12	7
a^{51}				$\overline{1}$	$\overline{1}$	$\overline{1}$		1	$\overline{1}$	1	$\overline{3}$	$\overline{4}$	$\overline{8}$	$\overline{7}$	$\overline{6}$	$\overline{1}$	4	7	10	7	4
a^{52}	1					1	1	2	1	3			$\overline{3}$	$\overline{2}$	$\overline{1}$		4	4	7	3	3
a^{53}		$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$		$\overline{1}$		$\overline{2}$	2	$\overline{3}$	$\overline{2}$	$\overline{2}$	$\overline{1}$	2	2	4	1	1
a^{54}				1	1	2	2	3	2	2	1	1			1		3	1	3		1
a^{55}							$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{2}$	$\overline{1}$	$\overline{1}$	1		1		
a^{56}											1	1	2	2	3	2	2	1	1		
a^{57}																$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	$\overline{1}$	
a^{58}																					1

Generating Function for covariants, representative form,

Denominator:

$$(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^8)(1-a^9)(1-a^{10})$$
$$(1-a^{11})(1-a^2x^4)(1-a^2x^8)(1-a^2x^{12})(1-a^2x^{16})(1-a^2x^{20})(1-ax^{12}).$$

Numerator:

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}
a^0	1																	
a^3			1	1	2	1	2	2	1	2	1	1	1	1		1		
a^4	1		3	2	4	3	4	4	3	4	2	3	1	2	1	1		1
a^5	1	2	5	6	7	8	6	9	5	6	3	4	1	1		−1		−2
a^6	3	4	9	11	13	15	13	15	9	11	5	6	2	1		−1		−3
a^7	4	10	16	21	23	27	23	24	18	15	10	6	3	−2	1	−5		−4
a^8	7	16	28	34	40	46	40	37	27	22	12	6		−5	−5	−10	−5	−7
a^9	9	30	44	58	64	71	64	55	39	27	13	−3	−8	−18	−16	−22	−14	−13
a^{10}	17	45	71	89	99	110	97	77	51	29	5	−15	−30	−40	−39	−42	−30	−23
a^{11}	21	73	106	133	148	156	137	101	62	20	−10	−50	−65	−83	−76	−74	−54	−38
a^{12}	36	102	153	191	208	218	187	123	61	1	−53	−102	−129	−146	−136	−120	−89	−59
a^{13}	45	148	214	265	287	288	240	143	55	−44	−113	−190	−220	−239	−218	−181	−131	−75
a^{14}	65	196	290	353	377	373	299	152	21	−114	−218	−309	−352	−363	−327	−258	−178	−89
a^{15}	81	264	379	460	486	460	357	147	−33	−226	−357	−483	−524	−528	−467	−344	−227	−90
a^{16}	110	332	486	577	601	558	408	118	−129	−376	−558	−694	−750	−725	−627	−439	−266	−68
a^{17}	131	419	602	707	728	647	442	61	−261	−587	−805	−972	−1018	−960	−810	−529	−286	−12
a^{18}	168	501	728	842	848	734	457	−30	−452	−840	−1122	−1287	−1330	−1216	−996	−604	−266	91
a^{19}	193	601	856	979	970	800	443	−160	−677	−1160	−1476	−1654	−1664	−1489	−1171	−644	−194	265
a^{20}	232	686	985	1106	1068	854	397	−331	−964	−1516	−1887	−2038	−2017	−1751	−1318	−636	−48	507
a^{21}	256	783	1102	1223	1158	867	318	−541	−1279	−1920	−2310	−2451	−2356	−1995	−1422	−556	171	842
a^{22}	293	854	1209	1319	1207	865	203	−785	−1635	−2332	−2758	−2831	−2673	−2183	−1451	−403	491	1259
a^{23}	307	931	1293	1388	1241	814	54	−1050	−1993	−2763	−3171	−3200	−2930	−2310	−1408	−151	893	1768
a^{24}	336	974	1352	1430	1225	744	−118	−1330	−2364	−3155	−3567	−3490	−3115	−2352	−1263	182	1393	2335
a^{25}	339	1015	1384	1437	1192	631	−310	−1607	−2691	−3525	−3876	−3719	−3204	−2299	−1032	614	1953	2967
a^{26}	351	1017	1385	1409	1110	507	−511	−1867	−2998	−3812	−4123	−3834	−3192	−2144	−698	1105	2575	3605
a^{27}	339	1015	1352	1353	1018	350	−704	−2095	−3224	−4033	−4249	−3860	−3067	−1893	−301	1661	3199	4244
a^{28}	336	974	1294	1267	887	203	−883	−2274	−3389	−4139	−4281	−3752	−2839	−1550	175	2223	3824	4823
a^{29}	307	931	1210	1156	762	38	−1032	−2399	−3452	−4162	−4183	−3558	−2518	−1143	658	2788	4374	5329
a^{30}	293	854	1105	1031	611	−100	−1146	−2456	−3446	−4059	−3998	−3251	−2127	−692	1158	3290	4850	5697
a^{31}	256	783	988	893	482	−236	−1219	−2447	−3333	−3876	−3701	−2881	−1690	−226	1612	3730	5188	5942

Numerator—(*Continued.*)

	x^{36}	x^{38}	x^{40}	x^{42}	x^{44}	x^{46}	x^{48}	x^{50}	x^{52}	x^{54}	x^{56}	x^{58}	x^{60}	x^{62}	x^{64}	x^{66}	x^{68}	x^{70}
a^0																		
a^3																		
a^4																		
a^5		2		1		1												
a^6		2		1		1												
a^7		1		1		2		2		1		1						
a^8	2	2	1			2		1		1								
a^9	7	4	3	2		3		1				1		1		1		
a^{10}	13	8	2	2	3	5	2	3	1			1						
a^{11}	22	6		10	7	11	6	7	2	1		1		1				1
a^{12}	28	3	11	20	20	22	15	11	4	1		1	1	1				
a^{13}	32	15	31	45	40	43	29	18	7		1	5	2	3				
a^{14}	18	46	75	85	78	72	51	27	9	1	5	9	7	4	1			
a^{15}	8	106	139	149	130	115	79	37	9	10	14	19	12	8	2			1
a^{16}	73	199	247	237	209	169	116	46	2	23	31	34	24	13	4	1	2	1
a^{17}	177	344	392	367	311	243	161	51	11	50	58	55	37	21	5	3	4	3
a^{18}	348	540	602	526	445	326	206	48	43	90	100	88	63	28	7	8	10	4
a^{19}	582	815	861	740	601	427	250	29	94	155	163	132	89	42	7	14	16	7
a^{20}	906	1152	1192	980	780	525	279	12	179	244	250	193	132	51	5	27	30	9
a^{21}	1298	1574	1564	1264	962	621	287	91	298	375	365	271	174	69	3	42	45	17
a^{22}	1788	2048	1998	1553	1149	692	260	207	468	537	510	366	232	75	15	70	73	21
a^{23}	2334	2588	2443	1862	1312	734	191	378	680	748	684	476	283	88	40	105	102	36
a^{24}	2947	3140	2910	2135	1441	723	57	600	954	993	890	597	348	82	74	154	148	45
a^{25}	3568	3714	3338	2388	1510	656	134	886	1269	1285	1116	730	394	78	123	216	196	65
a^{26}	4201	4240	3733	2561	1511	513	399	1221	1639	1596	1360	860	444	48	189	293	264	81
a^{27}	4772	4721	4029	2667	1422	298	723	1617	2029	1942	1605	987	466	14	272	333	332	110
a^{28}	5285	5088	4239	2658	1250	2	1115	2036	2454	2276	1844	1098	480	53	370	490	419	131
a^{29}	5665	5349	4301	2561	981	359	1538	2489	2858	2612	2062	1183	454	124	488	606	501	168
a^{30}	5920	5442	4250	2331	634	780	2003	2925	3257	2905	2242	1237	417	233	615	731	601	193
a^{31}	6003	5403	4043	2023	225	1231	2446	3347	3592	3161	2374	1251	330	340	753	860	686	232

Numerator—(*Continued.*)

	x^0	x^2	x^4	x^6	x^8	x^{10}	x^{12}	x^{14}	x^{16}	x^{18}	x^{20}	x^{22}	x^{24}	x^{26}	x^{28}	x^{30}	x^{32}	x^{34}
a^{32}	232	686	860	753	340	$\bar{330}$	$\bar{1251}$	$\bar{2374}$	$\bar{3161}$	$\bar{3592}$	$\bar{3347}$	$\bar{2446}$	$\bar{1231}$	225	2023	4043	5403	6003
a^{33}	193	601	731	615	233	$\bar{417}$	$\bar{1237}$	$\bar{2242}$	$\bar{2905}$	$\bar{3257}$	$\bar{2925}$	$\bar{2003}$	$\bar{780}$	634	2331	4250	5442	5920
a^{34}	168	501	606	488	124	$\bar{454}$	$\bar{1183}$	$\bar{2062}$	$\bar{2612}$	$\bar{2858}$	$\bar{2489}$	$\bar{1538}$	$\bar{359}$	981	2561	4301	5349	5665
a^{35}	131	419	490	370	53	$\bar{480}$	$\bar{1098}$	$\bar{1844}$	$\bar{2276}$	$\bar{2454}$	$\bar{2036}$	$\bar{1115}$	2	1250	2658	4239	5088	5285
a^{36}	110	332	383	272	$\bar{14}$	$\bar{466}$	$\bar{987}$	$\bar{1605}$	$\bar{1942}$	$\bar{2029}$	$\bar{1617}$	$\bar{723}$	298	1422	2667	4029	4721	4772
a^{37}	81	264	293	189	$\bar{48}$	$\bar{444}$	$\bar{860}$	$\bar{1360}$	$\bar{1596}$	$\bar{1639}$	$\bar{1221}$	$\bar{399}$	513	1511	2561	3733	4240	4201
a^{38}	65	196	216	123	$\bar{78}$	$\bar{394}$	$\bar{730}$	$\bar{1116}$	$\bar{1285}$	$\bar{1269}$	$\bar{886}$	$\bar{134}$	656	1510	2388	3338	3714	3568
a^{39}	45	148	154	74	$\bar{82}$	$\bar{348}$	$\bar{597}$	$\bar{890}$	$\bar{993}$	$\bar{954}$	$\bar{600}$	57	723	1441	2135	2910	3140	2947
a^{40}	36	102	105	40	$\bar{88}$	$\bar{283}$	$\bar{476}$	$\bar{684}$	$\bar{748}$	$\bar{680}$	$\bar{378}$	191	734	1312	1862	2443	2588	2334
a^{41}	21	73	70	15	$\bar{75}$	$\bar{232}$	$\bar{366}$	$\bar{510}$	$\bar{537}$	$\bar{468}$	$\bar{207}$	260	692	1149	1553	1998	2048	1788
a^{42}	17	45	42	3	$\bar{69}$	$\bar{174}$	$\bar{271}$	$\bar{365}$	$\bar{375}$	$\bar{298}$	$\bar{91}$	287	621	962	1264	1564	1574	1298
a^{43}	9	30	27	$\bar{5}$	$\bar{51}$	$\bar{132}$	$\bar{193}$	$\bar{250}$	$\bar{244}$	$\bar{179}$	$\bar{12}$	279	525	780	980	1192	1152	906
a^{44}	7	16	14	$\bar{7}$	$\bar{42}$	$\bar{89}$	$\bar{132}$	$\bar{163}$	$\bar{155}$	$\bar{94}$	29	250	427	601	740	861	815	582
a^{45}	4	10	8	$\bar{7}$	$\bar{28}$	$\bar{63}$	$\bar{88}$	$\bar{100}$	$\bar{90}$	$\bar{43}$	48	206	326	445	526	602	540	348
a^{46}	3	4	3	$\bar{5}$	$\bar{21}$	$\bar{37}$	$\bar{55}$	$\bar{58}$	$\bar{50}$	$\bar{11}$	51	161	243	311	367	392	344	177
a^{47}	1	2	1	$\bar{4}$	$\bar{13}$	$\bar{24}$	$\bar{34}$	$\bar{31}$	$\bar{23}$	2	46	116	169	209	237	247	199	73
a^{48}	1			$\bar{2}$	$\bar{8}$	$\bar{12}$	$\bar{19}$	$\bar{14}$	$\bar{10}$	9	37	79	115	130	149	139	106	8
a^{49}				$\bar{1}$	$\bar{4}$	$\bar{7}$	$\bar{9}$	$\bar{5}$	$\bar{1}$	9	27	51	72	78	85	75	46	$\bar{18}$
a^{50}					$\bar{3}$	$\bar{2}$	$\bar{5}$	$\bar{1}$		7	18	29	43	40	45	31	15	$\bar{32}$
a^{51}					$\bar{1}$	$\bar{1}$	$\bar{1}$		1	4	11	15	22	20	20	11	$\bar{3}$	$\bar{28}$
a^{52}	1				$\bar{1}$		$\bar{1}$		1	2	7	6	11	7	10		$\bar{6}$	$\bar{22}$
a^{53}							$\bar{1}$			1	3	2	5	3	2	$\bar{2}$	$\bar{8}$	$\bar{13}$
a^{54}			$\bar{1}$		$\bar{1}$		$\bar{1}$				1		3		2	$\bar{3}$	$\bar{4}$	$\bar{7}$
a^{55}									1		1		2			$\bar{1}$	$\bar{2}$	$\bar{2}$
a^{56}							1		1		2		2		1		$\bar{1}$	
a^{57}													$\bar{1}$		$\bar{1}$		$\bar{2}$	
a^{58}													$\bar{1}$		$\bar{1}$		$\bar{2}$	
a^{59}																		
a^{60}																		
a^{63}																		

Numerator—(*Continued.*)

x^{36}	x^{38}	x^{40}	x^{42}	x^{44}	x^{46}	x^{48}	x^{50}	x^{52}	x^{54}	x^{56}	x^{58}	x^{60}	x^{62}	x^{64}	x^{66}	x^{68}	x^{70}	
5942	5188	3730	1612	226	1690	2881	3701	3876	3333	2447	1219	236	482	893	988	783	256	a^{32}
5697	4850	3290	1158	692	2127	3251	3998	4059	3446	2456	1146	100	611	1031	1105	854	293	a^{33}
5329	4374	2788	658	1143	2518	3558	4183	4162	3452	2399	1032	38	762	1156	1210	931	307	a^{34}
4823	3824	2223	175	1550	2839	3752	4281	4139	3389	2274	883	203	887	1267	1294	974	336	a^{35}
4244	3199	1661	301	1893	3067	3860	4249	4033	3224	2095	704	350	1018	1353	1352	1015	339	a^{36}
3605	2575	1105	698	2144	3192	3834	4123	3812	2998	1867	511	507	1110	1409	1385	1017	351	a^{37}
2967	1953	614	1032	2299	3204	3719	3876	3525	2691	1607	310	631	1192	1437	1384	1015	339	a^{38}
2335	1393	182	1263	2352	3115	3490	3567	3155	2364	1330	118	744	1225	1430	1352	974	336	a^{39}
1768	893	151	1408	2310	2930	3200	3171	2763	1993	1050	54	814	1241	1388	1293	931	307	a^{40}
1259	491	403	1451	2183	2673	2831	2758	2332	1635	785	203	865	1207	1319	1209	854	293	a^{41}
842	171	556	1422	1995	2356	2451	2310	1920	1279	541	318	867	1158	1223	1102	783	256	a^{42}
507	48	636	1318	1751	2017	2038	1887	1516	964	331	397	854	1068	1106	985	686	232	a^{43}
265	194	644	1171	1489	1664	1654	1476	1160	677	160	443	800	970	979	856	601	193	a^{44}
91	266	604	996	1216	1330	1287	1122	840	452	30	457	734	848	842	728	501	168	a^{45}
12	286	529	810	960	1018	972	805	587	261	61	442	647	728	707	602	419	131	a^{46}
68	266	439	627	725	750	694	558	376	129	118	408	558	601	577	486	332	110	a^{47}
90	227	344	467	528	524	483	357	226	33	147	357	460	486	460	379	264	81	a^{48}
89	178	258	327	363	352	309	218	114	21	152	299	373	377	353	290	196	65	a^{49}
75	131	181	218	239	220	190	113	44	55	143	240	288	287	265	214	148	45	a^{50}
59	89	120	136	146	129	102	53	1	61	123	187	218	208	191	153	102	36	a^{51}
38	54	74	76	83	65	50	10	20	62	101	137	156	148	133	106	73	21	a^{52}
23	30	42	39	40	30	15	5	29	51	77	97	110	99	89	71	45	17	a^{53}
13	14	22	16	18	8	3	13	27	39	55	64	71	64	58	44	30	9	a^{54}
7	5	10	5	5		6	12	22	27	37	40	46	40	34	28	16	7	a^{55}
4		5	1	2	3	6	10	15	18	24	23	27	23	21	16	10	4	a^{56}
3		1		1	2	6	5	11	9	15	13	15	13	11	9	4	3	a^{57}
2		1		1	1	4	3	6	5	9	6	8	7	6	5	2	1	a^{58}
1		1	1	2	1	3	2	4	3	4	4	3	4	2	3		1	a^{59}
		1		1	1	1	1	2	1	2	2	1	2	1	1			a^{60}
																	1	a^{63}

Table of Groundforms.

Degree in the Coefficients. \ Order in the Variables.	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	34
1							1										
2	1		1		1		1		1		1						
3	1		1	1	2	1	2	2	1	2	1	1	1	1		1	
4	2		3	2	4	3	4	4	3	4	2	3	1	2	1	1	1
5	2	2	5	6	7	8	6	9	5	6	3	4	1	1			
6	4	4	9	11	12	14	10	12	3	5							
7	5	10	15	20	18	21	9	8									
8	7	16	24	29	21	21											
9	9	28	33	37	15												
10	14	39	41	30													
11	15	53	40														
12	19	56	7														
13	18	44															
14	12																

The total number of groundforms (counting in the absolute constant and the quantic itself) is 949.

The manuscript sheets containing the original calculations from which the preceding tables have been constructed (as is the case also with the calculations connected with all the similar tables which have appeared in this journal) are deposited in the iron safe of the Johns Hopkins University, Baltimore, where they can be seen and examined, or copied, by any one interested in the subject. From the manifold independent systematic tests*

* One of these tests depends upon the following property of the generating function, which has been disclosed by observation, and of which the significance is not yet known. On putting $a=1$ in the numerator of the generating function, the coefficients of the various powers of x are integer multiples of the coefficient of x^0. Thus in the case of the duodecimic, the numerator of the reduced form becomes, on putting $a=1$,

$$5663\,(1+2x^2+x^4-x^6-3x^8-4x^{10}-4x^{12}-2x^{14}+2x^{16}+5x^{18}+6x^{20}+5x^{22}+2x^{24}-2x^{26}-4x^{28}-4x^{30}-3x^{32}-x^{34}+x^{36}+2x^{38}+x^{40}).$$

Thus the numerical divisibility of the result of putting $a=1$ furnishes a test for the sums of the *columns*, while the algebraic divisibility of the result of putting $x=1$ (see this *Journal*†, Vol. III. p. 151) tests the sums of the *rows*; and the satisfaction of both tests makes the correctness of the result practically certain.

[† See footnote, above, p. 489.]

to which the work has been subjected, Mr Franklin estimates that the chance is far more than a million to one that the generating functions for the twelfthic as calculated do not contain a single numerical error. The highest order of any ground-covariant to the twelfthic it will be seen is 34, which is the superior limit of order given by M. Camille Jordan's formula for the ground-covariants to a system of an indefinite number of simultaneous binary forms of each of which the order is 12 or less: M. Jordan's "superior limit" in fact in this as in all the other calculated cases, being actually attained by one (and only one) ground-covariant to a single form*. It will also be noticed that for all orders of the primitive which have been calculated, namely, from 3 to 12 (with 11 omitted), the degree of the covariant of highest order is either 3 or 4. Looking at single quantics of the even orders 6, 8, 10, 12, it will be observed that the maximum order of their ground-covariants for any degree (from and after the 4th degree) diminishes, or, to speak more strictly, never increases as the degree increases. As regards quantics of the odd orders 5, 7, 9, the same rule applies for the maximum order of their groundforms of even degrees; and in respect to their groundforms of odd degrees, the maximum order from and after the 3rd degree diminishes or remains stationary as the degree increases. Also (alike for quantics of odd or even order) when (beginning with the 3rd degree) in passing from an odd to the next even or from an even to the next odd degree of the groundforms, an increase in the maximum order takes place, it is only to the extent of a single unit. These facts, which constitute a sort of *law of shrinkage*, assume practical importance when the successive tables of groundforms are compared together, with a view to track the ground-differentiants (or, in Mr Cayley's language, the ground-seminvariants or *sources* of covariants), as the order of the primitive quantic is increased. Some of these ground-sources retain their irreducible character permanently, others only up to a particular limit of order in the primitive. The former may be regarded as the irreducible differentiants to a quantic of an infinite order: such for instance are all the differentiants of the second and third degree. But when we consider differentiants of the 4th degree this is no longer true. Thus we have the well-known example of the discriminant to $(a, b, c, d \between x, y)^3$, namely, $a^2d^2 + 4ac^3 + 4df^3 - 3b^2c^2 - 6abcd$, which is irreducible for this quantic, but for the quantic $(a, b, c, d, e \between x, y)^4$ remains, it is obvious, a differentiant, but no longer a ground-differentiant, being expressible under the form of the difference of two products of lower differentiants, namely, as

$$(ac - b^2)(ae - 4bd + 3c^2) - a \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}$$

* It is also particularly noticeable that the number of the successively positive and negative blocks in the table follows the law observed in the inferior cases, namely, for Quantics of orders 3 and 4 there is a single block, for Quantics of orders 5 and 6 two blocks, for order 8 three blocks, and for orders 9 and 10 four blocks, there being five distinct blocks alternately positive and negative in the instance before us of the Quantic of order 12.

Suppose a differentiant to be the source of a covariant of the deg-order $j \cdot \epsilon$ considered as belonging to the quantic $(a_0, a_1, \ldots a_i \between x, y)^i$; then it is easily seen that it will be the source of a covariant of the deg-order $j \cdot j + \epsilon$ in respect to the quantic $(a_0, a_1, \ldots a_{i+1} \between x, y)^{i+1}$. We can, therefore, in many cases by a mere inspection of successive tables of groundforms eliminate some at least of the transient ground-differentiants: that is, wherever there are K groundforms of deg-order $j \cdot \epsilon$ to a quantic of the order i, but only $K - \Delta$ of the deg-order $j \cdot \epsilon + \lambda j$ to the quantic of the order $i + \lambda$, we know that at least Δ of the sources to the K groundforms, that is, Δ ground-differentiants of degree j and weight $\frac{1}{2}(ij - \epsilon)$ are only transiently irreducible. Thus, for example, the table of groundforms for the quintic exhibits a groundform of deg-order 4 . 4, that is, of deg-weight 4 . 8; but the table of groundforms for the sextic contains no groundform of the same deg-weight, that is, of deg-order 4 . 8. Hence the differentiant of deg-weight 4 . 8, although irreducible when regarded as a function of 6 letters (the number of letters which actually appear in it), is reducible when regarded as a function potentially of 7 or more.

So, again, for a like reason, the ground-differentiants of 5 letters, of deg-orders (in respect to the quintic) 5 . 1 and 5 . 7, that is, of deg-weights 5 . 12, 5 . 9, are only transiently irreducible; and, what is very interesting, it will be seen at a glance (and here the law of shrinkage makes its importance felt) that the sources of all the groundforms to a quintic of a higher order than the 5th are only transitory (or provisional, so to say) ground-differentiants. So in like manner it will be recognized by comparing the tables of groundforms for the seventhic and eighthic, that of the 9 ground-sources of the degree 6 to the former, only two *can be* permanent, namely, one of the weight $\frac{1}{2}(6 \cdot 7 - 2)$ and one of the weight $\frac{1}{2}(6 \cdot 7 - 4)$, that is, of the deg-weights 6 . 20 and 6 . 19 respectively: all the others becoming resoluble when an additional letter is introduced into the quantic. Moreover, as the table for the eighthic contains no groundforms of deg-order 7 . 8, we see from the law of shrinkage that there can be no ground-source to the seventhic of a higher than the 6th degree which is permanently irreducible*.

A systematic weeding out of the transitory ground-sources from the published tables, which cannot in all cases for groundforms of earlier degrees be effected completely without an examination of a more searching kind than that illustrated by the above examples, must be reserved for a future occasion—after I shall have completed, as I hope soon to do, the study of a subject of higher interest and more pressing importance, which has for its object to determine not only the groundforms so called, but also the ground-syzygants, the ground-counter-syzygants, &c., of quantics from their

* For the 6th degree it will at once be seen that there can be no permanent differentiant to the seventhic except one of the 2nd and one of the 4th order.

generating functions by a purely arithmetical process, which I believe to be already substantially in my possession.

As the first fruits of this method, I may state that the invariantive ground-syzygants (or, if the expression is preferred, fundamental syzygies) to the octavian quantic $(x, y)^8$ are 5 in number, and of the degrees 16, 17, 18, 19, 20 respectively in the coefficients. As regards the ground-syzygants (invariantive and covariantive) of the quintic, my method furnishes the same list as that given in Professor Cayley's *Tenth Memoir on Quantics*. Their deg-orders may be found as follows.

By the supernumerary ground-types understand the deg-orders of the ground-covariants exclusive of those represented by the factors which appear in the denominator of the representative generating function*, which are therefore 23 − 6, that is, 17 in number. Let these types be added each to itself and every other, thus giving rise to $\frac{17.18}{2}$ types: out of these sums strike out the types

$$8.4 \quad 9.5 \quad 10.2 \quad 10.4 \quad 11.3 \quad 12.2 \quad 14.4 \quad 16.2$$

and replace them by

$$13.5 \quad 14.6 \quad 15.3 \quad 15.5 \quad 16.4 \quad 17.3 \quad 19.5 \quad 21.3$$

The 153 types thus formed, together with the types, 26 in number, furnished by the negative terms in the numerator to the generating function (see this *Journal*, vol. II. p. 224 [p. 284, above]), 179 in all, will be the deg-orders of the fundamental syzygants. Mr Cayley founds this rule on his theory of the so-called Real Generating Function, which essentially consists in what may be termed the Dialytic Presentation of the Representative G. F. for the Quintic—namely as a sum of 26 pairs, each pair containing one positive and one negative term of the numerator divided by the denominator, so selected for conjunction that the developed expression of each pair shall be seen to be omni-positive by an obvious dialytic process.

The method followed by the eminent author in singling out the fundamental syzygants does not appear (as far as I can make out) to be explicitly stated in his memoir. The dialytic form (supposing, as is probably the case, it always exists for *finite* representative generating functions) is not easy to arrive at: a serious additional obstacle to the use of the dialytic method would arise in the case where (as for the seventhic) the numerator of the representative form becomes an infinite series. The method I employ does not require the use of the dialytic method, nor even of the *representative* form of the G. F., although the practical process is much simplified by the use of the representative form when it has a finite numerator. The result

* In such denominator the number of factors for a Quantic of any odd order $2i-1$ is $3i-3$, and for any even order $2i$ is $3i-2$ (i in each case being supposed greater than unity).

I obtain for the fundamental syzygants of the sextic is as follows: Take the 19 supernumerary ground-types (see* vol. II. p. 225), and add them each to each and to every other, as in the preceding case. Then strike out of the sums so formed the types of the deg-orders 6 . 4, 9 . 6, 8 . 4, 11 . 6, 10 . 4, 7 . 8, 8 . 6, 11 . 4, as well as one of the two sums 13 . 4 obtained from the addition of 5 . 2 and 8 . 2 or of 3 . 2 and 10 . 2 and replace the nine types so omitted by the eight types 12 . 8, 14 . 8, 13 . 6, 15 . 6, 10 . 10, 11 . 8, 14 . 6, 16 . 6. There will thus arise $19 \cdot \frac{20}{2} - 9 + 8$, or 189 types: to these adjoin the 29 types given by the negative terms in the numerator of the Rep. G. F.: the total number of types 189 + 29 or 218 so obtained will be the deg-orders of the complete system of fundamental syzygants to the sextic. The two types of the deg-order 6 . 6 which appear among the supernumerary types, it will of course be understood, are to be treated as distinct types in forming the binary sums. It is just barely possible (but I think very unlikely) that I may have committed some oversight in the table of replacement in the above calculation, and that the true number of ground-syzygies may be $19 \cdot \frac{18}{2} + 29$ or 219 instead of 218†.

I subjoin a brief *aperçu* of the general theory.

A generating function (whatever its subject-matter) developed in a series consists of facients and coefficients, where any facient is a product of a finite set of letters each raised to a certain power. The totality of the exponents expressing these powers may be termed the type of the facient. In the generating functions to be referred to hereinunder, the letters employed are just as many in number as there are quantics in the system to be considered: namely, one letter corresponds to each quantic.

A generating function proper (with reference to the present theory) is defined to be one that is or can be developed into a series of facients whose coefficients and whose types are omni-positive integers, and where each such numerical coefficient is the number of linearly independent invariants whose degrees in the coefficients of the several quantics of the system are identical with the indices of the corresponding letters in the facient to which that numerical coefficient is attached‡. The type of the facient may be also styled the type of the connoted invariants. A binomial expression consisting

[* p. 285, above.]

† Nine binary sums of types are omitted, and are replaced by only eight other combinations. This is analogous to the loss of a unit in counting the irreducible syzygies to the invariants of an eighthic. The *supernumerary* invariants in this case are 3 in number; of degrees 8, 9, 10 respectively. Their binary combinations would give 6, but the true number of irreducible syzygies is only 5.

‡ I speak designedly (for greater facility of expression) of invariants only, which can be done for binary quantics without any loss of generality, inasmuch as covariants may be regarded as invariants of a given system of quantics with a linear quantic superadded.

of unity followed by a facient and separated from it by the negative sign may be termed a *generator**.

A proper generating function to a system of quantics may always by known methods (see this *Journal*, vol. III. p. 133)† be expressed by a fraction whose numerator is a finite series of facients with numerical coefficients and its denominator a finite product of generators.

It may also be expressed (according to a definite process), and in one way only, by a fraction whose numerator and denominator alike consist of a finite or infinite (except in a few trivial cases, an infinite) product of generators‡.

A finite product of generators (or powers of generators) may be termed a generator-group.

For greater uniformity of statement in regard to what follows, let us agree to understand by a syzygant of the grade zero, an irreducible invariant. Then the two infinite products above referred to (whose ratio is algebraically equal to the generating function) may each be resolved into a product (usually infinite) of collect-groups, such that the totality of the types of the 1st, 2nd, ... ith groups of the denominator shall respectively represent the totality of the types of irreducible syzygants of the grades $0, 2, \ldots (2i-2)$ and the totality of the types of the 1st, 2nd, ... ith groups of the numerator the totality of the types of irreducible syzygants of the grades $1, 3, 5, \ldots (2i-1)$, so that each group may be said to be related to or to represent a complete system of irreducible syzygants of a certain grade (invariants being regarded as zero-graded syzygants)—that is to say, as many times as any generator is repeated in a group so many (and no more) irreducible syzygants of that type will there be of the corresponding grade.

Let G be a proper generating function to a system of quantics, $\Gamma_0, \Gamma_1, \Gamma_2 \ldots$ generator-groups such that

$$G = \frac{1 \cdot \Gamma_1 \cdot \Gamma_3 \cdot \Gamma_5 \ldots}{\Gamma_0 \cdot \Gamma_2 \cdot \Gamma_4 \cdot \Gamma_6 \ldots};$$

then, as suggested to me by Mr Franklin, in order that the Γ series may be

* If $a, b, c, \ldots$ are facients, $1 - a^\alpha b^\beta c^\gamma \ldots$ is a *generator*, and $\alpha, \beta, \gamma \ldots$ (taken in a definite order) is its *type*.

[† See above, p. 489, footnote.]

‡ For instance let G be the generating function proper to the invariants of an eighthic.

$$\text{Then } G = \frac{1 + a^8 + a^9 + a^{10} + a^{18}}{(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)}$$
$$= [(1-a^2)(1-a^3)(1-a^4)(1-a^5)(1-a^6)(1-a^7)(1-a^8)(1-a^9)(1-a^{10})]^{-1}$$
$$\cdot (1-a^{16})(1-a^{17})(1-a^{18})(1-a^{19})(1-a^{20})$$
$$\cdot [(1-a^{25})(1-a^{26})(1-a^{27})(1-a^{28})(1-a^{29})]^{-1}$$
$$\cdot (1-a^{33})(1-a^{34})(1-a^{35})^2(1-a^{36})^2(1-a^{37})^2(1-a^{38})(1-a^{39})$$
$$\cdot [(1-a^{41})(1-a^{42})^2(1-a^{43})^3(1-a^{44})^4(1-a^{45})^4(1-a^{46})^4(1-a^{47})^3(1-a^{48})^2]^{-1}$$
$$\cdot \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

representative of complete systems of irreducible syzygants of the successive grades, it is *necessary* that $\frac{1}{\Gamma_0} - G$; $\frac{\Gamma_1}{\Gamma_0} - G$; $\frac{\Gamma_1\Gamma_3}{\Gamma_0} - G$; $\frac{\Gamma_1\Gamma_3}{\Gamma_0\Gamma_2} - G$; ... shall, when developed in series of facients with omni-positive indices, be alternately omni-positive and omni-negative. But the existence of these inequalities, although a *necessary*, is not a *sufficient* condition in order that the Γ's shall be so representative; for example, $\Gamma_0 . \Gamma_2$ and $\Gamma_1 . \Gamma_3$ might evidently be regarded as single groups and the inequalities would still be satisfied; but suppose we further limit the Γ's in succession by the following rule, namely, that on withdrawing any one of the generator-factors from Γ_0 and calling Γ_0' the group so reduced $\frac{1}{\Gamma_0'} - G$ is no longer omni-positive, this will serve to define Γ_0 absolutely; Γ_0 being so determined, Γ_1 may in like manner be limited by the condition that its quotient by any one of its generators being called Γ_1, $\frac{\Gamma_1'}{\Gamma_0} - G$ shall be no longer omni-negative; then Γ_1 is accurately determined, and, proceeding in like manner with each group in succession, the whole system of groups becomes exactly defined, and thus we obtain the necessary and sufficient condition of group-representation.

Calling $\frac{1}{\Gamma_0}, \frac{\Gamma_1}{\Gamma_0}, \frac{\Gamma_1\Gamma_3}{\Gamma_0}, \frac{\Gamma_1\Gamma_3}{\Gamma_0\Gamma_2}, \ldots \nu_0, \nu_1, \nu_2, \nu_3 \ldots$

respectively, the ν series of quantities stand to G in somewhat the same relation as the complete quotients of a continued fraction to its complete value. Observe that $\nu_0 - 1$, $\nu_1 - 1$, $\nu_2 - 1$, ... each vanish when the variables in G are each zero, and become infinite when the variables in G are each unity.

When each such variable has any value intermediate between 0 and 1, I think it almost certain that no two of the ν's can become equal, so that for all values of the variables inside those limits the parabolic lines or surfaces or hyper-surfaces, &c., represented (after introducing a new variable ω) by the equations $\omega - \nu_0 = 0$, $\omega - \nu_1 = 0$, $\omega - \nu_2 = 0$, ... (which coincide for the limiting values of the original variables at the origin and at a point at infinity) will never intersect, so that within the prescribed limits $\nu_0 - \nu_2$, $\nu_2 - \nu_4$, $\nu_4 - \nu_6$, ... will be always positive and $\nu_1 - \nu_3$, $\nu_3 - \nu_5$, ... will be always negative, the limited boundaries represented by

$$\omega - G,\ \omega - \nu_0,\ \omega - \nu_2,\ \omega - \nu_4, \ldots$$

being each external to the one that precedes it on one side of $\omega - G$, and

$$\omega - G,\ \omega - \nu_1,\ \omega - \nu_3,\ \omega - \nu_5, \ldots$$

following the same law on the other side. It is possible, moreover, that a more stringent condition than the above may be verified, namely, that

$$\nu_0 - G,\ \nu_2 - \nu_0,\ \nu_4 - \nu_2 \ldots$$
$$G - \nu_1,\ \nu_1 - \nu_3,\ \nu_3 - \nu_5 \ldots$$

may each be developable into omni-negative functions, and again (to complete the analogy with the parallel theory of continued fractions or converging continued products) that

$$\nu_0 - G,\ \ G - \nu_1,\ \ \nu_2 - G,\ \ G - \nu_3,\ \ \nu_4 - G, \ldots$$

shall form a single series of continually decreasing quantities, or even in their developed state, of functions in which the corresponding coefficients to each facient form a continually decreasing (or, at least, never-increasing) series of numbers. Then in the case of a single quantic, within the limits defined by the facient a being 0 and 1 the curves $\omega - \nu_1$, $\omega - \nu_3, \ldots \omega - G, \ldots \omega - \nu_2$, $\omega - \nu_0$, will form an infinite series of loops having one common asymptote and one common point of intersection, and except at that one point keeping clear of each other.

I annex tables (pp. [506, 507, below]) of the fundamental syzygants* or (if one pleases so to say) irreducible syzygies for the quintic and sextic, rendered more complete by inserting entries corresponding to the fundamental in- and- covariants. The positive integers correspond to these latter, the negative integers (the negative sign being set over the figure) to the irreducible syzygants. Thus, for example, in the table to the sextic the positive integer 2 found in the 6th line and 6th column, indicates that there are 2 ground-covariants of deg-order 6 . 6. The negative integer $\bar{7}$ found in the 12th line and 12th column indicates that there are 7 irreducible syzygies of deg-order 12 . 12†. The negative sign is appropriate, inasmuch as every independent syzygy of any deg-order lowers by a unit the number of linearly independent in- or- covariants of that deg-order that can be produced out of the inferior groundforms, so that syzygants may be regarded as negative existences in regard to groundforms: carrying on the same idea, counter-syzygants might be numbered by integers carrying two negative signs contradicting each other, and so on indefinitely.

* N.B.—A syzygant to a Quantic is a rational integer function of its in- or- covariants which, expressed as a function of the coefficients, vanishes identically, but we may still understand its "degree in the coefficients" to mean the degree of any one of the terms of which it is the sum.

† If j or e exceed the highest degree or order respectively found in any table, or, if without that being the case there is a blank space in the jth line and eth column of the table, the meaning is that there is no irreducible groundform or syzygy of the deg-order $j\,.\,e$. In the tables exhibited it will be seen that the deg-order $j'\,.\,e'$ of each syzygant is superior to the deg-order $j\,.\,e$ of every groundform: that is, the differences $j'-j$, $e'-e$ are neither of them less and one of them is greater than zero. The same is true for all quantics which have a finite Rep. G. F., but not necessarily and probably never actually so in other cases; thus, for example, to the seventhic belongs an irreducible invariant of degree 22 and an irreducible syzygy of degree 20, so that here the $j'\,.\,e'$ (20.0) is inferior to the $j\,.\,e$ (22.0). The fact of every $j'\,.\,e'$ being superior to the $j\,.\,e$ can be expressed by saying that the invariantive syzygetic portions of a Rep. G. F. table are not intermingled but lie totally apart and may be divided from each other by a single continuous cut.

The method of partitions or generating functions, which leads to these surprising constructions, looks at invariants and their connexions solely with regard to their deg-order or type without taking any account of their content; in other words it deals only with the *idea* or *notion* of these beings and their relations, and may therefore, I think, suitably be termed the Idealistic method*. I cannot see the faintest possibility of the symbolic method serving to determine a complete system of syzygies in any but the trivial cases of quantics of the 3rd or 4th order—the only cases where the infinite procession of beings (syzygants, counter-syzygants, anti-counter-syzygants, &c.), rising out of each other, comes to a stop—there being for those cases no procession after the 1st step, as is also true of invariants (as distinguished from covariants) for quantics of the 6th order. This is how it came to pass in the infancy of the theory that the number of ground-covariants was supposed to become infinite for quantics beyond the fourth and their ground-invariants for quantics beyond the 6th order.

I think it may be interesting to some of the readers of the *Journal* to be put in possession of the complete system of irreducible syzygies to a system of two or more quantics, and I select as an easy example the case of a combined quadratic and cubic, reserving the other combinations of which the groundform tables have been published for a subsequent number of the *Journal*. The supernumerary groundforms for the quadri-cubic system (see

* My proof in the *Phil. Trans.*, founded on the canonical form of the Quintic, of its 4th, 8th, 12th and 18th-degreed invariants forming a complete system, the late Mr Boole's discovery of the cubinvariant to the Quartic, the various disproofs in the *Comptes Rendus* and in this *Journal* of the existence of supposed groundforms, are all exemplifications of the Realistic point of view. The Symbolic lies between this and the Idealistic aspect of the subject, in so far as the operations by which invariants are engendered constitute a new and so to say finer subject-matter, capable of being itself operated upon in all respects like ordinary algebraical substance. In Professor Cayley's *Tenth Memoir on Quantics* there is a sort of half return from the Idealistic to the Realistic view—a kind of substantiality being attributed to the groundforms themselves as primary elements in the study of their syzygetic interconnections. It may be well to notice, for the benefit of the readers of that memoir (*Phil. Trans.* 1878), that in the Representative Form given at p. 657 two terms are omitted by an oversight, namely, $-a^{17}x^4$ and a^3x^{12}. I need hardly add (since the publication of my tables in this *Journal*), with reference to a doubt expressed by Prof. Cayley (*loc. cit.*), that I *had* obtained the form referred to in the paragraph following the R. G. F. in question, though *not* by dividing out the common factors from the numerator and denominator of the R. G. F.; on the contrary, the N. G. F. is first obtained from the generating function in its crude form (which if left in that form would lead to a divergent series), and then the R. G. F. is obtained from this, through multiplying its numerator and denominator by the factors needed to render the denominator a product of representative groundforms.

The Symbolic and the Idealistic (which I formerly called the fatalistic or peprotic) method alike, as far as is known, owe their conception to the same (unnecessary to be named) acute and capacious intellect. Whether very much that is essential remains to be added to the great discoveries of Gordan and Jordan in the direction of the former may reasonably be doubted, but no such misgiving can be entertained with respect to the latter, which already has given rise to many more questions than it has settled (of a kind, too, of which a solution sooner or later may reasonably be anticipated).

this *Journal**, vol. II. pp. 295, 296), are of the deg-deg-orders 3 . 4 . 0, 1 . 1 . 1, 2 . 1 . 1, 1 . 3 . 1, 2 . 3 . 1, 1 . 2 . 2, 1 . 1 . 3, 0 . 3 . 3, where the first and second numbers express the degrees in the coefficients of the quadric and cubic respectively, and the last number expresses the order in the variables. Adding each of these triads to itself and every other, rejecting the combinations 2 . 2 . 2, 3 . 2 . 2, 2 . 4 . 2, which appear in the numerator of the G. F. (and arise from the additions 1 . 1 . 1 + 1 . 1 . 1, 1 . 1 . 1 + 2 . 1 . 1, 1 . 1 . 1 + 1 . 3 . 1), replacing them by the higher combinations 1 . 1 . 1 + 1 . 1 . 1 + 1 . 1 . 1, 1 . 1 . 1 + 1 . 1 . 1 + 2 . 1 . 1, 1 . 1 . 1 + 1 . 1 . 1 + 1 . 3 . 1, that is, 3 . 3 . 3, 4 . 3 . 3, 3 . 5 . 3, and adding in the 12 types furnished by the negative terms in the numerator of the G. F., the totality of the irreducible syzygies (48 in number) to the binary quadri-cubic system is arrived at and exhibited in the annexed table, in which the exponents attached to any type signify the number of irreducible syzygies of the corresponding deg-deg-order.

Table of Irreducible Syzygies to the Quadri-cubic System.

6 . 8 . 0,	4 . 5 . 1,	4 . 7 . 1,	5 . 5 . 1,	5 . 7 . 1,	2 . 6 . 2,	$(3 . 4 . 2)^2$,
$(3 . 6 . 2)^2$,	4 . 2 . 2,	$(4 . 4 . 2)^2$,	$(4 . 6 . 2)^2$,	1 . 5 . 3,	2 . 3 . 3,	$(2 . 6 . 3)^2$,
$(3 . 3 . 3)^2$,	$(3 . 5 . 3)^2$,	3 . 6 . 3,	3 . 7 . 3,	4 . 3 . 3,	4 . 5 . 3,	1 . 4 . 4,
1 . 6 . 4,	2 . 2 . 4,	$(2 . 4 . 4)^4$,	2 . 6 . 4,	3 . 2 . 4,	3 . 4 . 4,	3 . 6 . 4,
$(1 . 5 . 5)^2$,	2 . 3 . 5,	2 . 5 . 5,	3 . 5 . 5,	0 . 6 . 6,	1 . 3 . 6,	1 . 4 . 6,
2 . 2 . 6,	4 . 7 . 6,					

there being thus one irreducible invariantive syzygy and 4, 10, 12, 11, 5, 5 covariantive syzygies of orders 1, 2, 3, 4, 5, 6 respectively.

It may be worth while just to notice that the types to the complete system of irreducible syzygies to a simultaneous linear and quartic form will consist simply of the sums of the 13 supernumerary types, (*A. M. J.* vol. II. p. 295†), 6 . 3 . 0, 3 . 1 . 1, 3 . 2 . 1, 5 . 3 . 1, 2 . 1 . 2, 2 . 2 . 2, 4 . 3 . 2, 1 . 1 . 3, 1 . 2 . 3, 3 . 3 . 3, 2 . 3 . 4, 1 . 3 . 5, 0 . 3 . 6, added each to itself and every other, together with the 14 types taken from the negative terms in the numerator of the G. F., namely, 7 . 3 . 1, 6 . 3 . 2, 5 . 3 . 3, 4 . 3 . 4, 6 . 4 . 4, 6 . 5 . 4, 3 . 3 . 5, 5 . 4 . 5, 5 . 5 . 5, 2 . 3 . 6, 4 . 4 . 6, 4 . 5 . 6, 1 . 3 . 7, 7 . 6 . 7, making $\frac{13 . 14}{2} + 14$, that is, 105 in all. In this instance there is no rejection or substitution of sums called for.

A word or two seems necessary to leave unambiguous the meaning of the term syzygants of any specified grade in what precedes.

In- or- covariants may be termed syzygants of grade zero (as already stated). Syzygants of the first grade are defined to be rational integer

[* p. 394, above.] [† p. 393, above.]

Order in the Variables.

Degree in the Coefficients.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	Degree in the Coefficients.
1						1														1
2			1				1													2
3				1		1				1										3
4	1				1		1													4
5		1		1				1				$\bar{1}$								5
6			1		1		$\bar{1}$		$\bar{1}$		$\bar{1}$		$\bar{1}$		$\bar{2}$				$\bar{1}$	6
7		1				1		$\bar{1}$		$\bar{3}$		$\bar{1}$		$\bar{2}$		$\bar{2}$				7
8	1		1				$\bar{2}$		$\bar{2}$		$\bar{3}$		$\bar{3}$				$\bar{1}$			8
9				1		$\bar{1}$		$\bar{5}$		$\bar{2}$		$\bar{2}$		$\bar{3}$						9
10					$\bar{1}$		$\bar{3}$		$\bar{4}$		$\bar{5}$				$\bar{2}$					10
11		1				$\bar{4}$		$\bar{3}$		$\bar{3}$		$\bar{3}$								11
12	1				$\bar{2}$		$\bar{4}$		$\bar{5}$		$\bar{1}$		$\bar{2}$							12
13		1		$\bar{2}$		$\bar{3}$		$\bar{3}$		$\bar{4}$										13
14			$\bar{1}$		$\bar{2}$		$\bar{6}$		$\bar{1}$		$\bar{3}$									14
15				$\bar{2}$		$\bar{3}$		$\bar{4}$		$\bar{1}$										15
16					$\bar{5}$		$\bar{2}$		$\bar{2}$		$\bar{2}$									16
17				$\bar{2}$		$\bar{3}$		$\bar{2}$		$\bar{1}$										17
18	1		$\bar{2}$		$\bar{2}$		$\bar{2}$		$\bar{2}$		$\bar{1}$									18
19				$\bar{2}$		$\bar{3}$		$\bar{1}$												19
20			$\bar{2}$		$\bar{1}$		$\bar{2}$		$\bar{1}$											20
21				$\bar{3}$		$\bar{1}$				$\bar{1}$										21
22			$\bar{1}$		$\bar{2}$		$\bar{1}$													22
23		$\bar{1}$		$\bar{1}$				$\bar{1}$				$\bar{1}$								23
24			$\bar{2}$		$\bar{1}$															24
25		$\bar{1}$				$\bar{1}$														25
26			$\bar{2}$																	26
27				$\bar{1}$																27
28																				28
29		$\bar{1}$																		29
30																				30
31		$\bar{1}$																		31
32																				32
33																				33
34																				34
35																				35
36	$\bar{1}$																			36
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	

Order in the Variables.

ORDER IN THE VARIABLES.

Degree in the Coefficients	0	2	4	6	8	10	12	14	16	18	20	22	24
1				1									
2	1		1		1								
3		1		1	1		1						
4	1		1	1		1							
5		1	1		1				$\bar{1}$				
6	1			2	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{1}$	$\bar{1}$		$\bar{1}$
7		1	1	$\bar{1}$		$\bar{1}$	$\bar{4}$	$\bar{1}$	$\bar{2}$	$\bar{2}$		$\bar{1}$	
8		1			$\bar{2}$	$\bar{4}$	$\bar{2}$	$\bar{4}$	$\bar{3}$		$\bar{2}$		
9			1		$\bar{4}$	$\bar{2}$	$\bar{5}$	$\bar{4}$		$\bar{3}$			
10	1	1	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{6}$	$\bar{5}$	$\bar{2}$	$\bar{4}$				
11					$\bar{6}$	$\bar{5}$	$\bar{2}$	$\bar{4}$					
12		1	$\bar{1}$	$\bar{4}$	$\bar{3}$	$\bar{3}$	$\bar{7}$	$\bar{1}$	$\bar{1}$				
13			$\bar{1}$	$\bar{2}$	$\bar{5}$	$\bar{5}$	$\bar{1}$	$\bar{3}$					
14			$\bar{1}$	$\bar{4}$	$\bar{6}$	$\bar{2}$	$\bar{3}$						
15	1		$\bar{3}$	$\bar{3}$	$\bar{2}$	$\bar{4}$	$\bar{1}$	$\bar{2}$					
16			$\bar{1}$	$\bar{4}$	$\bar{4}$	$\bar{1}$	$\bar{2}$						
17			$\bar{3}$	$\bar{3}$	$\bar{1}$	$\bar{2}$		$\bar{1}$					
18		$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{4}$		$\bar{1}$						
19			$\bar{2}$	$\bar{3}$		$\bar{1}$							
20		$\bar{1}$	$\bar{3}$		$\bar{1}$				$\bar{1}$				
21				$\bar{3}$									
22		$\bar{1}$	$\bar{2}$										
23		$\bar{1}$											
24			$\bar{2}$										
25		$\bar{1}$											
26													
27		$\bar{1}$											
28													
29													
30	$\bar{1}$												

ORDER IN THE VARIABLES.

functions of those of grade zero which vanish when the latter are expressed in terms of the original coefficients. It is not *necessary* to define *these* syzygants as functions of *irreducible* ones of grade zero (which vanish under the condition aforesaid), because every in- or- covariant is a rational integer function of the irreducible in- or- covariants. But when we come to syzygants of the second grade (since those of the first grade are not necessarily functions of the irreducible ones of that grade, but may be so of the in- or- covariants as well), it becomes necessary to define syzygants of the second grade (*aliter* counter-syzygants) as rational integer functions of *irreducible* ones of the first grade which vanish when they are expressed in terms of the quantities (here the in- or- covariants) which immediately precede them in the scale of generation. And so, in general, following out the defining process step by step, by a syzygant of the $(i+1)$th grade for the purpose of this theory, is to be understood a rational integer function of the *irreducible* ones of the ith grade which vanishes when these latter are expressed in terms of those of the grade $i-1$. Such at least is my present impression; but, supposing that I am labouring under a misconception on this point, it will in nowise affect the validity of the theory in what regards the computation of the irreducible in- or- covariants and the syzygants of the first grade.