60.

A DEMONSTRATION OF THE IMPOSSIBILITY OF THE BINARY OCTAVIC POSSESSING ANY GROUNDFORM OF DEG-ORDER 10 . 4.

[*American Journal of Mathematics*, IV. (1881), pp. 62—84.]

Dr von Gall has rendered an inestimable service to algebraical science by working out, according to Gordan's method, the complete system of groundforms to the octavian binary quantic $[(x, y)^8]$. His results, published in the *Mathematische Annalen*, were at first widely discordant from those which have appeared in this *Journal*, but eventually have been brought by their author into perfect agreement with them, with the sole exception that his table includes a covariant of *deg-order* 10 . 4, not included in my list, which he states that he has not been able to decompose: it is the object of the present communication to bring the two tables into exact accord by demonstrating that no irreducible covariant to $(x, y)^8$ of that deg-order can exist. The total number of covariants of deg-order 10 . 4 obtained by multiplying together the irreducible covariants of an inferior deg-order (which appear equally in von Gall's table and in my own, and whose existence therefore may be taken for granted*) will be seen to be 32, which is the number of linearly independent covariants of that deg-order given by Cayley's law, [see p. 525, below]; hence, by the fundamental postulate, the 32 compounds in question must not be supposed subject to any linear relation, so that, according to that postulate, there exists no groundform of the deg-order in question; but my object is to use this instance as another exemplification of the validity of that same very reasonable postulate—as I have done on the three former occasions where the tables of Clebsch, Gordan and Gundelfinger comprised groundforms extraneous to the tables obtained by me on the assumption of its truth; the proof, however, on the present occasion, is much lengthier than any that has ever hitherto been employed, and involves arithmetical computations of considerable prolixity,

* As I have elsewhere remarked, since no groundforms can exist exterior to the tables furnished by Gordan's method, and no reducible forms can be contained in the tables furnished by the English method, it follows, even without assuming the truth of the fundamental postulate, that wherever the tables furnished by the two methods accord, they must, of logical necessity, be correct, *mere errors of calculation excepted.*

all necessity for which I had, in previous cases, been able to evade. It is, I may add, only after repeated trials and discomfitures, that I have succeeded at length in devising a special method adequate to prove the important point at issue.

The irreducible invariants and covariants of deg-order *inferior* to 10.4, (that is, whose degree in the coefficients and whose order in the variables are not each as great as 10 and 4 respectively), and which also can enter as factors of a covariant of deg-order 10.4 (this excludes the necessity of considering invariants of degrees 9 or 10) are as follows: the invariants are of degrees 2, 3, 4, 5, 6, 7, 8, one of each degree; the covariants are one of deg-orders 5.2, 2.4, 3.4 respectively, and two of deg-orders 4.4, 5.4, 6.4, 7.4, 8.4 respectively. We may denote the invariants by 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, and the covariants by 5.2, 2.4, 3.4, 4.4, 4.4*, 5.4, 5.4*, 6.4, 6.4*, 7.4, 7.4*, 8.4, 8.4*, and it is an easy arithmetical calculation to show (see† *Comptes Rendus*, July 25, 1881) that there are (as already stated) 32 different ways in which these duads, by their combination, can give rise to the duad 10.4. Out of these 32 it is important, with a view to what follows, to isolate those in which neither 2.0 nor 3.0 appears; their number will easily be seen to be 10, as shown in the scheme below—

4.0 + 6.4	4.0 + 6.4*	4.0 + 4.0 + 2.4	5.0 + 5.4	5.0 + 5.4*
6.0 + 4.4	6.0 + 4.4*	7.0 + 3.4	8.0 + 2.4	5.2 + 5.2.

What I have to prove is, that no equation $\Omega = 0$ exists, where Ω is a linear function of the 32 products in question, connected by numerical coefficients. Suppose it can be shown that Ω does not contain any of the 10 functions above indicated. Then Ω is either of the form $(2.0)U$ or $(3.0)V$, or is a linear function of $(2.0)U$ and $(3.0)V$. In the former two cases we should obtain $U = 0$ or $V = 0$ respectively; and in the third case the equation $\lambda(2.0)U + \mu(3.0)V = 0$, since 2.0 and 3.0 have no common factor, implies the existence of an integral equation $\lambda\frac{U}{(3.0)} + \mu\frac{V}{(2.0)} = 0$. Hence, in the three cases supposed, there would exist a syzygy of the deg-order 8.4, 7.4, 5.4 respectively between composite covariants of the inferior deg-orders; but if this were so, the number of irreducible covariants of one or another of these deg-orders would not be what it is at present, but, in order to satisfy Cayley's law, would have to be increased by a unit: or, in other words, results obtained by my method and coincident with those resulting, or capable of resulting, from the German method, would be erroneous, which never can be the case‡. Hence, the non-existence of $\Omega = 0$ will be demonstrated if it can be shown that, for some particular form of the general primitive $(x, y)^8$

[† p. 481, above.]

‡ Towards the end of this paper I establish the same conclusion by a more direct method, in which nothing extraneous to Dr von Gall's own table is assumed, except the one fact of the linearly independent covariants of deg-order 10.4 being 32 in number.

which causes the invariants of the second and third degrees each to vanish, the particular values then assumed by the 10 compounds which remain in Ω are not subject to any linear relation. Of course the converse would not be true; the fact of the existence of a syzygy between these 10, or even between the whole 32 compounds for a special form of the primitive, would not establish the existence of a syzygy between them in the general case.

The great practical gain of making the first two invariants vanish is that it leads to a computation in which only 10 instead of 32 linear functions have to be handled—but it is not possible *à priori* before the calculations have been gone through, to feel at all assured that the particular form assumed may not be such as to lead only to nugatory results. Such happily, however, turns out not to be the case with the form I am about to employ which leads to the expression of the 10 compounds as homogeneous linear functions of 11 arguments*, giving rise to a rectangular matrix 11 places wide and 10 places deep of which it can be shown that the complete minors (determinants of the 10th order) do not all vanish, so that the 10 functions cannot be subject to any syzygy; and consequently, if $\Omega = 0$ were a really existing syzygy, Ω must consist exclusively of 22 terms, every one of which contains one or both of the two first invariants; but this has been shown to be impossible, so that the non-evanescence of the minors referred to at once establishes the non-existence of a syzygy of deg-order 10.4, and, therefore, the non-existence of a *groundform of that deg-order*.

I take for the primitive the special form $(0, b, 2c, d, 0, 0, 0, 0, 1 \between x, y)^8$, that is to say, $8bx^7y + 56cx^6y^2 + 56dx^5y^3 + y^8$, with the relation $bd = 3c^2$, and proceed to form the required derivatives in conformity with von Gall's scheme of derivation. I use, as the best practical method of obtaining the "alliance" of the ith order between any two forms ϕ, ψ (of the orders μ, ν) denoted by $(\phi, \psi)_i$, the lineo-linear quadrinvariant (with respect to the variables of emanation) of the ith emanant of ϕ combined into a system with the ith emanant of ψ, taking care to reduce the result to the *parenthetical* form $(\ldots\ldots \between x, y)^{\mu+\nu-2i}$, containing only integer coefficients free from any common numerical factor. For the sake of brevity, too, I omit in general the symbolical factor containing (x, y): so that $(a_0, a_1, a_2, \ldots, a_i)$ will indicate the same thing as $(a_0, a_1, a_2, \ldots, a_i \between x, y)^i$. I shall adhere, in what follows, to the notation employed by Dr von Gall.

We have then, according to this notation,

$$f = (0, b, 2c, d, 0, 0, 0, 0, 1) \qquad (1)$$

$$\begin{aligned} i = (f, f)_4 &= (4bx^3y + 12cx^2y^2 + 4dxy^3)\,y^4 - 4dx^4(bx^4 + 8cx^3y + 6dx^2y^2) \\ &\quad + 3\,(2cx^4 + 4dx^3y)^2 \\ &= [12c^2 - 4bd, 16cd, 24d^2, 0, 0, 4b, 12c, 4d, 0][x, y]^8, \end{aligned}$$

* One of these arguments is itself a linear function of 3 combinations of the coefficients and variables, the total number of such combinations which appear in the 10 compounds being 13.

where the square bracket is employed to signify the same thing as would be indicated by the use of the round clamp, with the exception that the binomial coefficients are suppressed. We have, therefore, introducing the multipliers

$$\frac{14}{1}, \frac{14}{8}, \frac{14}{28}, \frac{14}{56}, \frac{14}{70}, \frac{14}{56}, \frac{14}{28}, \frac{14}{8}, \frac{14}{1},$$

$$i = (0,\ 28cd,\ 12d^2,\ 0,\ 0,\ b,\ 6c,\ 7d,\ 0) \tag{2}$$

$$\begin{aligned} k = (f, f)_6 &= (2bxy + 2cy^2)\, y^2 - 10d^2x^4 \\ &\equiv (\overline{20}d^2,\ 0,\ 0,\ b,\ 4c)^* \end{aligned} \tag{3}$$

$$\begin{aligned} \Delta = (k, k)_2 &= \overline{20}d^2x^2\,(2bxy + 4cy^2) - b^2y^4 \\ &\equiv (0,\ 90c^2d,\ 40cd^2,\ 0,\ 3b^2) \end{aligned} \tag{4}$$

$$C = (k, k)_4 = 20d^2 . 4c \equiv cd^2 \tag{5}$$

$$\begin{aligned} f_4 = (f, k)_4 &= 4c\,(4bx^3y + 12cx^2y^2 + 4dxy^3) - 4b\,(bx^4 + 8cx^3y + 6d)\,x^2y^2 \\ &\qquad + \overline{20}d^2y^4 \\ &= -4b^2x^4 - 16bcx^3y + (48c^2 - 24bd)\,x^2y^2 + 16cdxy^3 - 20d^2y^4 \\ &\equiv (b^2,\ bc,\ c^2,\ \overline{cd},\ 5d^2) \end{aligned} \tag{6}$$

$$\begin{aligned} f_{k,\,2} = (f_4, k)_2 &= (b^2x^2 + 2bcxy + c^2y^2)\,(2bxy + 4cy^2) - 2by^2\,(bcx^2 + 2c^2xy - cdy^2) \\ &\qquad + \overline{20}d^2x^2\,(c^2x^2 - 2cdxy + 5d^2y^2) \\ &= [\overline{20}c^2d^2,\ 2b^3 + 40cd^3,\ 6b^2c + \overline{100}d^4,\ 6bc^2,\ 4c^3 + 2bcd]\,[x, y]^4 \\ &\equiv (\overline{120}c^2d^2,\ 3b^3 + 60cd^3,\ 6b^2c + 100d^4,\ 9bc^2,\ 60c^3) \end{aligned} \tag{7}$$

$$\begin{aligned} f_{k,\,3} = (f_4, k)_3 &= (b^2x + bcy)\,(bx + 4cy) - 3by\,(bcx + c^2y) - \overline{20}d^2x\,(\overline{cd}x + 5d^2y) \\ &= (b^3 + \overline{20}cd^3)\,x^2 + (2b^2c + 100d^4)\,xy + bc^2y^2 \end{aligned}$$

$$\begin{aligned} (f_{k,\,3})^2 &= (b^6 + \overline{40}b^3cd^3 + 400c^2d^6)\,x^4 + (4b^5c\,[\overline{80}b^2c^2d^3 + 200b^3d^4] - 4000cd^7)\,x^3y \\ &\qquad + (6b^4c^2 + 400b^2cd^4 - 40bc^3d^3 + 10000d^8)\,x^2y^2 \\ &\qquad + (4b^3c^3 + 200bc^2d^4)\,xy^3 + b^2c^4y^4 \\ &= [b^6 + \overline{1080}c^7 + \overline{400}c^2d^6,\ 4b^5c + 4680c^6d + \overline{4000}cd^7, \\ &\qquad 6b^4c^2 + 3480c^5d^2 + 10000d^8,\ 4b^3c^3 + 600c^4d^3,\ b^2c^4]\,[x, y]^4 \\ &\equiv (3b^6 + \overline{3240}c^7 + 1200c^2d^6,\ 3b^5c + 3510c^6d + \overline{3000}cd^7, \\ &\qquad 3b^4c^2 + 1740c^5d^2 + 5000d^8,\ 3b^3c^3 + 450c^4d^3,\ 3b^2c^4) \end{aligned} \tag{8}$$

$$\begin{aligned} (f_\Delta) = (f, \Delta)_4 &= 3b^2\,(4bx^3y + 12cx^2y^2 + 4dxy^3) + 6\,(40cd^2)\,(2cx^4 + 4dx^3y) \\ &\qquad - 120bd^2\,(dx^4) \\ &= [480c^2d^2 - 120bd^3,\ 12b^3 + 960cd^3,\ 36b^2c,\ 12b^2d,\ 0]\,[x, y]^4 \\ &\equiv (40c^2d^2,\ b^3 + 80cd^3,\ 2b^2c,\ 3bc^2,\ 0) \end{aligned} \tag{9}$$

$$\begin{aligned} i_\Delta = (i, \Delta)_4 &= -120bd^2\,(6bx^2y^2 + 24cxy^3 + 7dy^4) \\ &\qquad + 240cd^2\,(12d^2x^4 + 4bxy^3 + 6cy^4) + 3b^2\,(112cdx^3y + 72d^2x^2y^2) \\ &= [2880cd^4,\ 336b^2cd,\ \overline{504}b^2d^2,\ \overline{1920}bcd^2,\ 1440c^2d^2 \\ &\qquad + \overline{840}bd^3]\,[x, y]^4 \\ &\equiv (240cd^4,\ 21bc^3,\ \overline{63}c^4,\ \overline{120}c^3d,\ \overline{90}c^2d^2) \end{aligned} \tag{10}$$

* The sign of equivalence ($\equiv$) is used in the above and in what follows in the sense of "may be superseded by."

$$i_4 = (i, k)_4 = \overline{20}d^2(4bx^3y + 36cx^2y^2 + 28dxy^3) - 4 . b\,(28cdx^4 + 48d^2x^3y + by^4) + (4c)(112cdx^3y + 72d^2x^2y^2)$$

$$= [\overline{112}bcd,\ 448c^2d + \overline{272}bd^2,\ \overline{432}cd^2,\ \overline{560}d^3,\ \overline{4}b^2]\,[x, y]^4$$

$$\equiv (336c^3,\ 92c^2d,\ 72cd^2,\ 140d^3,\ 4b^2) \qquad (11)$$

$$i_{k,\,2} = (i_4, k)_2 = \overline{20}d^2x^2(72cd^2x^2 + 280d^3xy + 4b^2y^2) - 2by^2(92c^2dx^2 + 144cd^2xy + 140d^3y^2) + (2bxy + 4cy^2)(336c^3x^2 + 184c^2dxy + 72cd^2y^2)$$

$$= [\overline{1440}cd^4,\ 672bc^3 + \overline{5600}d^5,\ 184bc^2d - 80b^2d^2 + 1344c^4,\ 736c^3d + \overline{144}bcd^2,\ 288c^2d^2 + \overline{280}bd^3]\,[x, y]^4$$

$$\equiv (\overline{360}cd^4,\ 42bc^3 + \overline{350}d^5,\ 49c^4,\ 19c^3d,\ 138c^2d^2) \qquad (12)$$

$$f_{k,\,\Delta} = (f_4, \Delta)_4 = -4\,(30bd^2)(\overline{cd}) + 6\,(40cd^2)\,c^2 + 3b^2 . b^2$$

$$= 3b^4 + 120bcd^3 + 240c^3d^2$$

$$\equiv b^4 + 200c^3d^2 \qquad (13)$$

$$i_{k,\,\Delta} = (i_4, \Delta)_4 = -4\,(30bd^2)(140d^3) + 6\,(40cd^2)(72cd^2) + (3b^2)(336c^3)$$

$$= 1008b^2c^3 + \overline{33120}c^2d^4 \equiv 7b^2c^3 - 230c^2d^4.$$

The term involving c^2d^4 being a multiple of the square of C (the invariant of the 4th degree) may be neglected, and, instead of $i_{k,\,\Delta}$, we may write the irreducible invariant of the 8th degree (say)

$$I_8 = b^2c^3. \qquad (14)$$

That of the 7th degree we have just found $= b^4 + 200c^3d^2$; and obviously the quadrinvariant of f is identically zero, or say

$$I_2 = 0. \qquad (15)$$

Also the cubinvariant $I_3 = (f, i)_8$, where

$$f = (0, b, 2c, d, 0, 0, 0, 0, 1)$$

and $\quad i = (0, 28cd, 12d^2, 0, 0, b, 6c, 7d, 0).$

Hence $\quad I_3 = -56bd + 336c^2 - 168bd = 504c^2 - 168bd = 0, \qquad (16)$

and we have found $I_4 \equiv cd^2$.

Also, $I_5 = (f, k^2)_8$ where

$$k^2 = (10d^2x^4 - 2bxy^3 - 2cy^4)^2$$

$$= 100d^4x^8 - 40bd^2x^5y^3 - 40cd^2x^4y^4 + 4b^2x^2y^6 + 8bcxy^7 + 4c^2y^8.$$

Hence $\quad I_5 = 100d^4 + 2c . 4b^2 - b . 8bc \equiv d^4$*.

* It will of course be recognized that the lineo-linear quadrinvariant to the system

$$(a_0, a_1, a_2, \ldots a_i \between x, y)^i,\ [b_0, b_1, b_2, \ldots b_i]\,[x, y]^i$$

is simply $\quad a_0b_i - a_1b_{i-1} + a_2b_{i-2} \ldots \pm a_ib_0$:

the disappearance of the argument b^2c from companionship with d^4 in I_5 is rather remarkable, and could not have been predicted. This circumstance considerably simplifies the subsequent calculations.

The only remaining invariant required for present purposes is I_6, represented by $(i_4, k)_4$ where

$$k = [\overline{10}d^2, 0, 0, 2b, 2c][x, y]^4,$$

and $$i_4 = (336c^3, 92c^2d, 72cd^2, 140d^3, 4b^2)(x, y)^4.$$

Hence $$I_6 = \overline{40}b^2d^2 - (2b)\, 92c^2d + 2c\,(336c^3)$$
$$= (-360 - 552 + 672)\, c^4 \equiv c^4.$$

On proceeding to form the 10 compound covariants of deg-order 10.4 obtained by suitable combinations of the invariants and covariants of inferior deg-order, it will be found that the following 13 arguments will make their appearance, in which, for greater brevity, x and y are each taken equal to unity, which in nowise affects (favourably or unfavourably) the course of the reasoning: these arguments are

$$b^6,\ c^7,\ c^2d^6;\ b^5c,\ c^6d,\ cd^7;\ b^4c^2,\ c^5d^2,\ d^8;\ b^3c^3,\ c^4d^3;\ b^2c^4,\ c^3d^4,$$

where the 5 groups of arguments, separated from one another by semicolons, are elements of the coefficients of x^4, x^3y, x^2y^2, xy^3, y^4, and when supplemented by such powers of k (of weight 8) as will bring their degrees up to the number 10, are of the respective weights 38, 39, 40, 41, 42, which is right, since the weight of the differentiant of deg-order 10.4 to $(x, y)^8$ is $\frac{10.8-4}{2}$, that is, 38; for greater brevity (in what precedes) k, the coefficient of y^8 in f, has been made unity, and it is worthy of notice that all the arguments that can appear consistently with the law of weight are represented by these 13, upon the understanding that any power of bd in an argument is replaceable by the like power of c^2.

But it is further noticeable that the 10 compounds in question, although apparently linear functions of 13 arguments, are virtually such of only 11; for it will be seen that $b^6 + 4b^5c + 6b^4c^2$ may be regarded as a single argument, none of the three simpler arguments which appear in it occurring except in two of the 10 compounds, and their coefficients in each of those two being in the ratio 1 : 4 : 6.

Had the contraction in the number of really independent arguments extended two steps further, so that the 10 compounds had been linear functions of only 9 quantities (as might, for anything that could be known *à priori*, have been the case), they would necessarily have been linearly connected, and no inference could have been drawn from the particular value assigned to f: moreover, had the 10 compounds been linear functions of only 10 quantities, although the particular form might have been sufficient for drawing a positive inference as to the non-existence of the general syzygy $\Omega = 0$, still there would have been no room for applying the all-important *test* of the correctness of the arithmetical computations upon which that inference would have reposed; and it would have been very unsatisfactory and unphilosophical to have made so important a conclusion rest upon the negative fact of a determinant of the 10th order *not vanishing*, when the undisproved existence of a single error committed in the many hundreds (or even—it might be said—thousands) of arithmetical steps involved in the calculations of the elements of that determinant would have been sufficient to account for its value differing from zero.

Fortunately, as will be seen, the correctness of the calculations may be *verified* (thanks to the existence of elements one more than barely sufficient —namely, 11 instead of 10) by the *positive* fact of a certain determinant of the 11th order being found equal to zero. It has often seemed to me that a special providence or pre-established harmony in the intellectual world brings it about that honest labour, persevering in the pursuit of an important truth in the face of doubts and difficulties and repeated disappointments, shall not in the end lose its due reward†.

Let us now denote the quantities $b^6+4b^5c+6b^4c^2$, c^7, c^2d^6; $4c^6d$, $4cd^7$; $6c^5d^2$, $6d^8$; $4b^3c^3$, $4c^4d^3$; b^2c^4, c^3d^4 by A, β, γ, δ, ϵ, ζ, η, θ, κ, λ, μ, respectively, and denote the covariants of the order 4 that have been calculated in what precedes according to their deg-order—namely, let us call

$(f_{k,3})^2$; $i_{k,2}$; i_Δ; $f_{k,2}$; f_Δ; Δ; i_4; f_4; k

10.4; 6.4; 6.4*; 5.4; 5.4*; 4.4; 4.4*; 3.4; 2.4 respectively,

then the values of 10.4, $I_4\times 6.4$, $I_4\times 6.4*$, $I_5\times 5.4$, $I_5\times 5.4*$, $I_6\times 4.4$, $I_6\times 4.4*$, $I_7\times 3.4$, $I_4^2\times 2.4$, $I_8\times 2.4$, will be as shown in the table annexed

A	β	γ	δ	ϵ	ζ	η	θ	κ	λ	μ	
3	$\overline{3240}$	1200	3510	$\overline{3000}$	1740	5000	3	450	3	.	.........(1)
.	.	$\overline{360}$	126	$\overline{350}$	49	.	.	19	.	$\overline{138}$	.........(2)
.	.	240	63	.	$\overline{63}$	.	.	$\overline{120}$	.	$\overline{90}$	.........(3)
.	.	$\overline{120}$	81	60	54	$\overline{100}$	.	27	.	60	.........(4)
.	.	40	27	80	18	.	.	9	.	.	.........(5)
.	.	.	90	.	40	.	.	.	3	.	.........(6)
.	336	.	92	.	72	.	.	140	4	.	.........(7)
1	1800	.	600	.	200	.	$\overline{3}$	$\overline{200}$	45	1000	.........(8)
.	.	$\overline{20}$	.	.	.	.	.	3	.	4	.........(9)
.	$\overline{180}$	.	.	.	.	.	1	.	4	.	.......(10)

Line (1) of course signifies $3A - 3240\beta + \ldots + 3\lambda$,

(2) .... $-360\gamma + 126\delta \ldots - 138\mu$,

† I began with taking as a special form $ax^8+by^8+cz^8$, with the relation $x+y+z=0$ (which, like the form f, contains two arbitrary ratios), and went through the very considerable labour of calculating all its inferior derivatives capable of entering into the composition of a covariant of deg-order 10.4, but the result turned out altogether nugatory.

and so for all the other lines, each being a linear function of the 11 quantities $A, \beta, \ldots, \lambda, \mu$.

If these 10 linear functions are linearly connected, all the *complete* minors of the rectangular matrix (11 by 10) must vanish.

It is not so difficult as it might at first sight appear, to calculate the actual value of any one of these minors, convenient combinations of the lines and columns having been previously effected; this arises from the number of zeros which appear in the matrix. Mr Morgan Jenkins, of the London Mathematical Society, and myself actually calculated two of them in the course of an hour or two; but the same object may be reached more expeditiously and quite as satisfactorily by proving that the minors do not vanish in respect to some judiciously or fortunately chosen modulus. I find that the number 11, taken as modulus, will accomplish the end in view. It will be found convenient to change the order of sequence of the lines and columns; to take the lines in the order 1, 8, 4, 10, 7, 6, 9, 5, 3, 2, and the columns in the order $A, \eta, \theta, \beta, \lambda, \gamma, \delta, \epsilon, \zeta, \kappa, \mu$. These transpositions having been effected, and the least positive residue of each element in respect to 11 being substituted in place of the element, the rectangular matrix above given will be replaced by the following:

3	6	3	5	3	1	1	3	2	10	.
1	.	8	7	1	.	6	.	2	9	10
.	10	.	.	.	1	4	5	10	5	5
.	.	1	7	4	.	.	.	.	.	.
.	.	.	6	4	.	4	.	6	8	.
.	.	.	.	3	.	2	.	7	.	.
.	.	.	.	.	2	.	.	.	3	4
.	.	.	.	.	7	5	3	7	9	.
.	.	.	.	.	9	8	.	8	1	9
.	.	.	.	.	3	5	2	5	8	5

It is easy to see that by proceeding as if to eliminate A between the two first lines, then β between the new line so formed and the third line, then γ between the new line again so formed and the fourth line, and so on, (always substituting the remainders to modulus 11 in lieu of the numbers themselves that arise in the process,) the first six lines may be replaced successively by the six following:

3	6	3	5	3	1	1	3	2	10	.
	5	10	5	.	10	6	8	4	6	8
		10	5	.	4	4	.	10	9	.
			10	7	7	7	.	1	2	.
				9	2	9	.	10	2	.
					5	2	.	.	5	.

Consequently, it only remains to ascertain whether the complete minors all disappear in the matrix of the dimensions (6×5) given below, namely:

$$\begin{matrix} 5 & 2 & . & . & 5 & . \\ 2 & . & . & . & 3 & 4 \\ 7 & 5 & 3 & 7 & 9 & . \\ 9 & 8 & . & 3 & 1 & 9 \\ 3 & 5 & 2 & 5 & 8 & 5 \end{matrix}$$

If all the complete minors of this matrix contain 11, the same must be true of the determinant formed by subtracting the first column in the above from the fifth and substituting the difference in place of the fifth column, that is,

$$\begin{vmatrix} 2 & . & . & . & . \\ . & . & . & 1 & 4 \\ 5 & 3 & 7 & 2 & . \\ 8 & . & 3 & 3 & 9 \\ 5 & 2 & 5 & 5 & 5 \end{vmatrix} \text{ and therefore } \begin{vmatrix} . & . & 1 & 4 \\ 3 & 7 & 2 & . \\ . & 3 & 3 & 9 \\ 2 & 5 & 5 & 5 \end{vmatrix}$$

should contain 11, and (as we may see by substituting the excess of 4 times the 3rd column over the fourth in place of the 3rd) the same must be true of

$$\text{the determinant } \begin{vmatrix} 3 & 7 & 8 \\ . & 3 & 3 \\ 2 & 5 & 4 \end{vmatrix} \text{ of which the value is } 3(12-15)+2(21-24),$$

that is, -15, and as this does not contain 11, it follows that the complete minors of the matrix which expresses the 10 compounds as linear functions of the 11 arguments $A, \beta, \gamma \ldots \lambda, \mu$ are not all zero, and they are consequently not linearly connected*. But, obviously, the calculations on which this proof depends imperatively call for a verification, as nothing would be more easy than to bring out some or all of the minors different from zero by a single error of calculation or slip of the pen. To this end I calculate the

* In the *Comptes Rendus* for 22nd August of this year, I have given a brief *résumé* of the contents of this paper. At page 367 of that fascicule, (third line from foot†), in the last line but one of the matrix, I have written 9 8 . 8 1 9 in error for 9 8 . 3 1 9 (having mistaken a 3, *covered with a blot*, for 8); consequently, the calculations which follow page 368 of the *C. R.* are erroneous. Fortunately, I did not repeat the mistake in calculating the value of the determinant subsequently given of the 11th order, in proving that it contains the divisor 11. Moreover, this determinant, or rather its remainder to modulus 11, has been calculated by an entirely different process by Mr Morgan Jenkins (whose work is before my eyes), and with the same result of its being divisible by 11. This instance shows how unsafe it would have been to have trusted to the fact of the minors not vanishing, unsupported by the positive evidence which the determinant of the 11th order affords of the preceding calculations, as regards the values of the groundforms, being unaffected with *one single error* in spite of the vast number of processes of addition, subtraction, multiplication, division, transposition, transcription and change of sign employed in working them out.

[† p. 486, above.]

value of von Gall's undecomposed covariant for the assumed special form f, and shall show that the 10 compounds and this 11th function do become linearly connected, that is, subject to a syzygy, on the assumption that the arithmetical values of the coefficients have been correctly calculated.

The function in question, Dr von Gall's i_4'', is obtained as follows:

$i'' = (i, \Delta)_2$ of deg-order $6 . 8$ is equal to

$$(168cdx^5y + 180d^2x^4y^2 + 6bxy^5 + 6cy^6)(40cd^2x^2 + 3b^2y^2)$$
$$-(180c^2dx^2 + 160cd^2xy)(28cdx^6 + 72d^2x^5y + 15bx^2y^4 + 36cxy^5 + 7dy^6)$$
$$+(12d^2x^6 + 20bx^3y^3 + 90cx^2y^4 + 42dxy^5)(180c^2dxy + 40cd^2y^2)$$
$$= [\overline{5040}c^3d^2,\ \overline{8560}c^2d^3,\ \overline{3840}cd^4,\ 504b^2cd,\ 7560c^4,\ 5640c^3d,\ 4380c^2d^2,$$
$$18b^3 + 560cd^3,\ 18b^2c] . [x, y]^8$$

which, multiplied by 28, will be seen to be equivalent to

$$(\overline{141120}c^3d^2,\ \overline{29960}c^2d^3,\ \overline{3840}cd^4,\ 756bc^3,\ 3024c^4,\ 2820c^3d,\ 4380c^2d^2,$$
$$63b^3 + 1960cd^3,\ 504b^2c).$$

Finally,

$$i_4'' = (i'', \Delta)_4 = 3b^2(\overline{141120}c^3d^2x^4 + \overline{119840}c^2d^3x^3y + \overline{23040}cd^4x^2y^2$$
$$+ 3024bc^3xy^3 + 3024c^4y^4)$$
$$+ 6 . 40cd^2(\overline{3840}cd^4x^4 + 3024bc^3x^3y + 18144c^4x^2y^2 + 11280c^3dxy^3 + 4380c^2d^2y^4)$$
$$- 4 . 90c^2d[756bc^3x^4 + 12096c^4x^3y + 16920c^3dx^2y^2 + 17520c^2d^2xy^3$$
$$+ (63b^3 + 1960cd^3)]y^4$$

which, dividing out by 144,

$$\equiv (\overline{32130}c^7 + \overline{6400}c^2d^6)x^4 + \overline{37590}c^6dx^3y + \overline{16380}c^5d^2x^2y^2$$
$$+ (63b^3c^3 + \overline{25000}c^4d^3)xy^3 + \left(\frac{\overline{819}}{2}b^2c^4 + 2400c^3d^4\right)y^4$$
$$\equiv (\overline{128520}c^7 + \overline{25600}c^2d^6,\ \overline{37590}c^6d,\ \overline{10920}c^5d^2,\ 63b^3c^3 + \overline{25000}c^4d^3,$$
$$\overline{1638}b^2c^4 + 9600c^3d^4).$$

Here it will be noticed that the arguments collected in what I have designated by A, namely, b^6, b^5c, b^4c^2, do not appear at all in i_4''. Had they made their appearance with other than coefficients bearing to each other the ratios of $1:4:6$, i_4'' could not have been a linear function of the 10 compounds which are linear functions of A and of 10 other arguments. This is in itself, to some extent, a verification of a portion at least of the preceding calculations: i_4'', as it turns out, is a linear function of only 8 out of the 11 arguments which appear in the other 10 compound covariants, namely, of β, γ, δ, ζ, θ, κ, λ, μ, neither A, ϵ nor η appearing in i_4''.

If the figuring throughout is correct, the determinant represented by the matrix constituted of the coefficients of the 11 compounds, ought to vanish identically; but it will be sufficient for all reasonable purposes (that is, to satisfy any reasonable doubts on the subject) if I show that this is the case for the value of that determinant in respect to three consecutive prime numbers 11, 13, 17 taken almost at hazard.

It must be understood that the vanishing of the determinant in question adds *no additional strength whatever* to the proof—which, by Cayley's law, is perfect without it—provided that the figures in the coefficients of the 10 compounds (excluding i_4'') have been correctly calculated. It is to authenticate these figures, and not to verify the legitimacy of the argument, that the 11th compound is calculated, and the determinant formed by all the eleven shown to contain any number taken at will. It must be remembered that the calculations have been most carefully conducted and verified at each step: consequently, if any person, after the evidence that will be given, entertains any doubt of the correctness of the result, the duty is incumbent on him to put his finger upon some one of the coefficients of the 10 first compounds and prove it to be incorrectly stated.

First, for the modulus 11. In respect to this modulus, the coefficients in i_4'' of

$$A,\ \eta,\ \theta,\ \beta,\ \lambda,\ \gamma,\ \delta,\ \epsilon,\ \zeta,\ \kappa,\ \mu$$

are congruous to 0, 0, 8, 4, 1, 8, 8, 0, 3, 3, 8.

Hence, (making use of the transformations already calculated of the upper half of the rectangular matrix), it has to be shown that 11 is a divisor of the determinant of the 9th order

$$\begin{array}{ccccccccc}
8 & 4 & 1 & 8 & 8 & . & 3 & 3 & 8 \\
10 & 5 & . & 4 & 4 & . & 10 & 9 & . \\
 & 10 & 7 & 7 & 7 & . & 1 & 2 & . \\
 & & 9 & 2 & 9 & . & 10 & 2 & . \\
 & & & 5 & 2 & . & . & 5 & . \\
 & & & 2 & . & . & . & 3 & 4 \\
 & & & 7 & 5 & 3 & 7 & 9 & . \\
 & & & 9 & 8 & . & 3 & 1 & 9 \\
 & & & 3 & 5 & 2 & 5 & 8 & 5
\end{array}$$

The first and second lines of this matrix combined give rise to the line 1 7 7 . 6 9 8, and this, combined with the 4th, to the line 5 1 . . 9 5 under which last, writing the 5 remaining lines

$$\begin{array}{cccccc}
5 & 1 & . & . & 9 & 5 \\
5 & 2 & . & . & 5 & . \\
2 & . & . & . & 3 & 4 \\
7 & 5 & 3 & 7 & 9 & . \\
9 & 8 & . & 3 & 1 & 9 \\
3 & 5 & 2 & 5 & 8 & 5
\end{array}$$

it has to be shown that the determinant to the above matrix of the 6th order contains 11.

Let the fourth line be replaced by 3 times itself + the last line, which, to the modulus 11, reduces the third column to the form of five zeros followed by 2. This shows that we may use, instead of the above, the determinant

5	1	.	9	5
5	2	.	5	.
2	.	.	3	4
2	9	4	2	5
9	8	3	1	9;

and again, replacing the fourth line of the new matrix by its double + the last line, we fall upon the matrix

5	1	9	5
5	2	5	.
2	.	3	4
2	4	5	8,

for which we may substitute

5	1	4	5
5	2	.	.
2	.	1	4
2	4	3	8,

or (as may be seen by replacing the second column by 3 times itself + the first column) $\begin{vmatrix} 8 & 4 & 5 \\ 2 & 1 & 4 \\ 3 & 3 & 8 \end{vmatrix}$, in which (to modulus 11) the first line is 4 times the second. Hence, the test is satisfied as regards the modulus 11.

I will next take the modulus 13.

The residues to modulus 13 of the coefficients in i_4'' of

will be seen to be

θ	β	λ	γ	δ	ϵ	ζ	κ	μ
11	11	.	10	6	.	.	12	6

and the matrix corresponding to the one of the same dimensions (11 × 10), previously calculated for modulus 11, will, in respect to modulus 13, become

3	8	3	10	3	4	.	3	11	8	.
1	.	10	6	6	.	2	.	5	8	12
	4	.	.	.	10	3	8	2	1	.
		1	2	4	.	.	.	.	.	.
			11	4	.	1	.	7	10	.
				3	.	12	.	1	1	.
					6	.	.	.	3	4
					1	1	2	5	9	.
					6	11	.	2	10	1
					4	9	1	10	6	5.

In place of the first six of the above lines, applying the same process as before, we may substitute

3	8	3	10	3	4	.	3	11	8	.
	5	1	8	2	9	5	10	4	3	10
		9	7	5	1	8	.	7	6	12
			11	5	12	5	12	6	7	1
				2	11	8	11	11	7	2
					6	.	7	8	7	7.

Combining the i_4'' line (that is, the coefficients of $\theta\ \beta\ \lambda \ldots \mu$ in i_4'' above given) with the third of these, we obtain the line

4 3 12 8 . 12 10 .

which, again combined with the fourth of the same, gives rise to the line

7 10 9 9 9 4.

Adding on the sixth line, namely 6 . 7 8 7 7 and the four last lines of the first matrix, namely,

*6	.	.	.	3	4
*1	1	2	5	9	.
*6	11	.	2	10	1
*4	9	1	10	6	5,

the lines marked with an asterisk, the arithmetical problem to be solved reduces itself to showing that the above determinant vanishes to modulus 13.

Substituting for the 1st column twice the 1st less three times the 6th, and for the 5th column twice the 5th less the 1st, and neglecting the factor 3, we fall upon the determinant

$$\begin{vmatrix} 2 & 10 & 9 & 9 & 11 \\ 4 & \cdot & 7 & 8 & 8 \\ 2 & 1 & 2 & 5 & 4 \\ 9 & 11 & \cdot & 2 & 1 \\ 6 & 9 & 1 & 10 & 8 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 2 & 10 & 9 & 9 & 2 \\ 4 & \cdot & 7 & 8 & \cdot \\ 2 & 1 & 2 & 5 & 12 \\ 9 & 11 & \cdot & 2 & 12 \\ 6 & 9 & 1 & 10 & 11 \end{vmatrix}.$$

Then in this last, substituting for the 4th column the 4th less twice the 1st, say M, and for the 3rd column 5 times the 1st less the 3rd, say N, we descend in like manner upon the determinant

$$\begin{matrix} 5 & 2 & 2 & 1 \\ 1 & 2 & 12 & 8 \\ 10 & 9 & 12 & 6 \\ 11 & 6 & 11 & 3 \end{matrix}$$

where the 1st column is the M with the zero in it left out, and the 4th column the N with the zero in it left out.

This, by elimination (so to say) of the first variable to the left between the successive pairs of lines, gives rise to the determinant

$$\begin{array}{ccc} 8 & 6 & . \\ 2 & 9 & 4 \\ . & 4 & 3 \end{array}$$

which (to modulus 13) $\equiv 8 \cdot 1 - 8 \cdot 3 - 6 \cdot 6 \equiv 8 - 11 - 10 \equiv 0$.

It remains only to apply the 3rd proposed test, using 17 as the modulus.

The i_4'' line here becomes

$$\begin{array}{ccccccccc} 12 & 0 & 11 & 2 & 14 & . & 11 & 7 & 12 \end{array}$$

and the grand rectangular matrix becomes

$$\begin{array}{ccccccccccccc} 3 & 2 & 3 & 7 & 3 & 10 & 8 & 9 & 6 & 8 & . & & \\ 1 & . & 14 & 15 & 11 & . & 5 & . & 13 & 4 & 14 & & \\ & 2 & . & . & . & 16 & 13 & 9 & 3 & 10 & 9 & & \\ & & 1 & 7 & 4 & . & . & . & . & . & . & & \\ & & & 13 & 4 & . & 7 & . & 4 & 4 & . & & \\ & & & & 3 & . & 5 & . & 6 & . & . & & \end{array}$$

with 4 more lines, which will be presently supplied in their proper place. For those above written may be substituted

$$\begin{array}{ccccccccccc} 3 & 2 & 3 & 7 & 3 & 10 & 8 & 9 & 6 & 8 & . \\ & 15 & 5 & 4 & 13 & 7 & 7 & 8 & 16 & 4 & 8 \\ & & 7 & 9 & 8 & 5 & 11 & . & 13 & 6 & . \\ & & & 6 & 3 & 12 & 6 & . & 4 & 11 & . \\ & & & & 2 & 14 & 15 & . & 6 & . & . \\ & & & & & 9 & 16 & . & 11 & . & . \end{array}$$

Rejecting the first two lines, and writing over the remaining ones the i_4'' line, there results

$$\begin{array}{ccccccccc} 12 & 0 & 11 & 2 & 14 & . & 11 & 7 & 12 \\ 7 & 9 & 8 & 5 & 11 & . & 13 & 6 & . \\ & 6 & 3 & 12 & 6 & . & 4 & 11 & . \\ & & 2 & 14 & 15 & . & 6 & . & . \\ & & & 9 & 16 & . & 11 & . & . \end{array}$$

which may be replaced by

$$\begin{array}{ccccccccc} 12 & 0 & 11 & 2 & 14 & . & 11 & 7 & 12 \\ & 6 & 2 & 12 & . & . & 11 & 6 & 1 \\ & & 6 & . & 2 & . & 9 & 13 & 11 \\ & & & 16 & 1 & . & 1 & 8 & 12 \\ & & & 9 & 16 & . & 11 & . & . \\ & & & *14 & . & . & . & 3 & 4 \\ & & & *\,6 & 10 & 12 & 1 & 9 & . \\ & & & *\,2 & 12 & . & 5 & 16 & 12 \\ & & & *14 & 7 & 7 & 15 & 2 & 15 \end{array}$$

to which I add in the 4 pretermitted lines distinguished by asterisks, and the determinant, represented by the square matrix (6 × 6) exhibited by the 6 lines last appearing above, ought to contain the modulus 17 as a divisor. Instead of the 3rd line from the bottom we may substitute its double less the last line, and thus, neglecting the factor 7, fall upon the matrix

$$\begin{array}{ccccc} 16 & 1 & 1 & 8 & 12 \\ 9 & 16 & 11 & . & . \\ 14 & . & . & 3 & 4 \\ 15 & 13 & 4 & 16 & 2 \\ 2 & 12 & 5 & 16 & 12. \end{array}$$

Substituting for the 4th column the sum of itself and the 1st, and for the 5th column 5 times itself + the 1st, and neglecting the factor 14, we obtain the determinant

$$\begin{array}{cccc} 1 & 1 & 7 & 8 \\ 16 & 11 & 9 & 9 \\ 13 & 4 & 14 & 8 \\ 12 & 5 & 1 & 11. \end{array}$$

Subtracting the 2nd column from the 1st and the 4th from the 2nd + the 3rd, we obtain the matrix

$$\begin{array}{cccc} 0 & 0 & 1 & 7 \\ 5 & 11 & 11 & 9 \\ 9 & 10 & 4 & 14 \\ 7 & 12 & 5 & 1, \end{array}$$

and replacing the 3rd column by 7 times the 3rd less the 4th, we descend upon the determinant

$$\begin{array}{ccc} 5 & 11 & . \\ 9 & 10 & 14 \\ 7 & 12 & . \end{array}$$ where the 1st line to modulus 17 equals 8 times the 3rd.

Hence the determinant in question contains 17, as was to be shown.

It seems needless to multiply these tests—the object being, as before stated, not a confirmation of the argument, which is wholly unnecessary, but a verification of the accuracy of the arithmetic: for this reason it has seemed to me essential that the calculations, authenticating the figures previously obtained, should be set out in considerable detail.

Instead of founding anything upon the concordance (as far as it extends) between Dr von Gall's table and my own, the proof of the non-existence of the 10 . 4 irreducible covariant may be inferred exclusively from the former and completed as follows.

I have proved that the syzygetic function Ω of the deg-order 10 . 4, if it exists, must be a consequence of the existence of a like function of the deg-order 8 . 4, 7 . 4, or 5 . 4. The last hypothesis may at once be rejected as implying an equation of the form $\frac{2 \,.\, 4}{3 \,.\, 4}$ = a numerical multiple of $\frac{2 \,.\, 0}{3 \,.\, 0}$.

Next, for the deg-order 7 . 4, again using for the primitive the same special form f, which causes 2 . 0 and 3 . 0 to vanish, the only non-vanishing arguments in the supposed syzygetic function Ω' for the particular form f will be $4 \,.\, 0 \times 3 \,.\, 4$ and $5 \,.\, 0 \times 2 \,.\, 4$, that is, $cd^2(b^2, bc, c^2, -cd, 5d^2)$, and $d^4(20d^2, 0, 0, b, 4c)$, between which obviously no syzygy is possible, so that neither of them can appear in the general form of Ω'. Hence the terms in the general form of Ω' must be divisible all by 2 . 0 or all by 3 . 0, or some by 2 . 0 and some by 3 . 0, and consequently there must exist a syzygy of the deg-order 5 . 4, 4 . 4, or 2 . 4. The first of these hypotheses has already been shown to be impossible, and the remaining two need not even have been mentioned, as there is only a single compound of the deg-order 4 . 4, namely, $2 \,.\, 0 \times 2 \,.\, 4$, and none of the deg-order 2 . 4. Lastly, for the deg-order 8 . 4, still using the same special form of f, the arguments in the supposed syzygetic functions which do not vanish are $4 \,.\, 0 \times 4 \,.\, 4$, $4 \,.\, 0 \times 4 \,.\, 4*$, $5 \,.\, 0 \times 3 \,.\, 4$, and $6 \,.\, 0 \times 2 \,.\, 4$, that is,

$$cd^2(0,\ 90c^2d,\ 40cd^2,\ 0,\ 3b^2)$$
$$cd^2(336c^3,\ 92c^2d,\ 72cd^2,\ 140d^3,\ 4b^2)$$
$$d^4(b^2,\ bc,\ c^2,\ -cd,\ 5d^2)$$

and

$$c^4(-20d^2,\ 0,\ 0,\ b,\ 4c).$$

The argument d^6 in the 3rd of these quantities has no equivalent in any of the other 3. Hence the 3rd quantity does not appear in the syzygy: moreover, the 4th compound contains one argument, namely, bc^4, which does not rationally contain d^2c $\left(\text{for } \frac{bc^3}{d^2} = \frac{b^2c}{3d}\right)$. Hence this compound also disappears, and obviously no syzygy connects together the first two. Hence in the supposed general syzygy there exist no compounds containing neither 2 . 0 nor 3 . 0, and by the same reasoning as before, this supposed syzygetic function must imply the existence of one of the deg-order 6 . 4 or 5 . 4 or 4 . 4. The two last of the three suppositions have already been seen to be impossible, and the first would imply a linear relation between $2 \,.\, 0 \times 4 \,.\, 4$, $2 \,.\, 0 \times 4 \,.\, 4*$, $3 \,.\, 0 \times 3 \,.\, 4$, $4 \,.\, 0 \times 2 \,.\, 4$, the last of which we see, by taking f for the primitive, cannot appear in the general syzygy, and the remaining 3 arguments would imply that the general covariant 3 . 4 would contain the invariant 2 . 0, which is absurd. Hence it follows from Dr von Gall's own results that the existence of a groundform of deg-order 10 . 4 is impossible. The only principle extraneous to his results made use of is Cayley's all-important rule, of which an irrefragable demonstration has been given by the author of this paper, but which still, as far as he is aware, remains unutilized, and is almost passed over in silence by invariantists of the German school.

It may be as well to make this article self-contained by showing that the number of compound irreducible groundforms of deg-order $10 \,.\, 4$ is, as stated, 32, namely the same as the number of linearly-independent covariants of that deg-order requisitioned by Cayley's rule.

Using then, for brevity's sake, i to represent the invariant $i \,.\, 0$, it is easy to see that the following is an exhaustive enumeration of all the compounded irreducibles of deg-order $10 \,.\, 4$:

$(5.2)^2$; 8×2.4; 7×3.4; 6×4.4; $6 \times 4.4*$; 5×5.4; $5 \times 5.4*$; 4×6.4; $4 \times 6.4*$; $4 \times 4 \times 2.4$; 3×7.4; $3 \times 7.4*$; $3^2 \times 4.4$; $3^2 \times 4.4*$; $3 \times 4 \times 3.4$; $3 \times 5 \times 2.4$; 2×8.4; $2 \times 8.4*$; $2 \times 3 \times 5.4$; $2 \times 3 \times 5.4*$; $2 \times 4 \times 4.4$; $2 \times 4 \times 4.4*$; $2 \times 5 \times 3.4$; $2 \times 6 \times 2.4$; $2 \times 3^2 \times 2.4$; $2^2 \times 6.4$; $2^2 \times 6.4*$; $2^2 \times 3 \times 3.4$; $2^2 \times 4 \times 2.4$; $2^3 \times 4.4$; $2^3 \times 4.4*$; $2^4 \times 2.4$.

The same number 32, it is all-important to bear in mind, is also the number of linearly independent covariants of deg-order $10 \,.\, 4$ given by Cayley's law. For this number is represented by $(w:8,\ 10) - (w':8,\ 10)$ where $w = \dfrac{10 \,.\, 8 - 4}{2} = 38$, $w' = w - 1 = 37$; that is, (by Euler's Theorem) is the coefficient of t^{38} in the development of

$$\frac{(1-t^{11})(1-t^{12})(1-t^{13})(1-t^{14})(1-t^{15})(1-t^{16})(1-t^{17})(1-t^{18})}{(1-t^2)(1-t^3)(1-t^4)(1-t^5)(1-t^6)(1-t^7)(1-t^8)},$$

which may be calculated as follows: The numerator is

$$1 - t^{11} - t^{12} - t^{13} - t^{14} - t^{15} - t^{16} - t^{17} - t^{18} + t^{23} + t^{24} + 2t^{25} + 2t^{26} + 3t^{27} + 3t^{28}$$
$$+ 4t^{29} + 3t^{30} + 3t^{31} + 2t^{32} + 2t^{33} + t^{34} + t^{35} - t^{36} - t^{37} - 2t^{38} \ldots .$$

Dividing this by $1 - t^8$, the quotient by $1 - t^7$, and so on for $1 - t^6, \ldots 1 - t^2$, we have for the numerator and the successive quotients so obtained the following values respectively:

t^0	t^1	t^2	t^3	t^4	t^5	t^6	t^7	t^8	t^9	t^{10}	t^{11}	t^{12}	t^{13}	t^{14}	t^{15}	t^{16}	t^{17}	t^{18}	t^{19}	t^{20}	t^{21}	t^{22}	t^{23}	t^{24}	t^{25}	t^{26}	t^{27}	t^{28}	t^{29}	t^{30}	t^{31}	t^{32}	t^{33}	t^{34}	t^{35}	t^{36}	t^{37}	t^{38}
1	0	0	0	0	0	0	0	0	0	0	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	0	0	0	1	1	2	2	3	3	4	3	3	2	2	1	1	$\bar{1}$	$\bar{1}$	$\bar{2}$
1	0	0	0	0	0	0	0	1	0	0	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	1	1	1	2	2	3	2	3	3	3	2	3	1	2	0
1	0	0	0	0	0	0	1	1	0	0	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	0	0	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{1}$	0	0	$\bar{1}$	$\bar{1}$	0	1	2	2	3	2	2	2	4	3	4	3
1	0	0	0	0	0	1	1	1	0	0	$\bar{1}$	0	0	1	0	0	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	0	0	0	0	0	1	2	4	3	4	3
1	0	0	0	0	1	1	1	1	0	1	0	1	1	1	1	0	$\bar{1}$	$\bar{1}$	$\bar{1}$	0	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	1	0	1	1
1	0	0	0	1	1	1	1	2	1	2	1	3	2	3	2	3	1	2	1	3	0	0	$\bar{2}$	0	$\bar{3}$	$\bar{3}$	$\bar{5}$	$\bar{3}$	$\bar{6}$	$\bar{6}$	$\bar{8}$	$\bar{6}$	$\bar{8}$	$\bar{7}$	$\bar{7}$	$\bar{6}$	$\bar{7}$	$\bar{6}$
1	0	0	1	1	1	2	2	3	3	4	4	6	6	7	8	9	8	10	10	11	10	10	9	10	7	6	5	4	0	$\bar{1}$	$\bar{4}$	$\bar{6}$	$\bar{9}$	$\overline{11}$	$\overline{13}$	$\overline{15}$	$\overline{18}$	$\overline{19}$
1	0	1	1	2	2	4	4	7	7	11	11	17	17	24	25	33	33	43	43	54	53	64	62	74	69	80	74	84	74	83	70	77	61	66	48	51	30	32

Hence the required coefficient is 32.

It is obvious that the particular method adopted in treating the grand determinant made up of 11^2 places employed in the foregoing investigation furnishes or indicates a good practical process for determining 10 out of the 32 numerical coefficients which enter into the expression of Dr von Gall's covariant i_4'' as a linear function of the 32 linearly independent covariants of its own deg-order; but, as this calculation possesses no point either of intrinsic theoretical interest or practical importance, I leave it to those who may feel any curiosity on the subject, to go through the calculations necessary to attain that end.

It may be supposed that the long calculations rendered necessary by the quadrinomial form f, attributed to the primitive in the preceding investigation, might have been evaded by using a trinomial form (of which several exist) possessing the same property of causing the two first invariants to vanish, and not less general, inasmuch as containing three independent coefficients in place of four connected by a homogeneous equation; for example, we might assume for the primitive $(0, b, 0, 0, 0, f, 0, 0, i \between x, y)^8$, where the weights of b, f, i are respectively 1, 5, 8.

The quadrinvariant vanishes because no binary combination of 1, 5, 8, with or without repetitions, will make up the required weight 8, and the cubinvariant because no ternary combination of the same will make up the weight 12. It may, however, easily be shown that such form will lead only to a nugatory conclusion, as not supplying the necessary number of arguments (10 at least are wanted) to support the independence of the 10 surviving compound covariants of deg-order 10 . 4. This may be seen as follows.

The weights of the coefficients of x^4, x^3y, x^2y^2, xy^3, y^4 in a 10 . 4 covariant are respectively 38, 39, 40, 41, 42. Let us ascertain in how many ways 10 numbers, consisting exclusively of the numbers 1, 5, 8, can be put together to make up these totals. I use the notation $a^\alpha . b^\beta . c^\gamma$ to indicate a sum of α numbers a, β numbers b, and γ numbers c.

Then the sole admissible representations of 38 are $8^4 . 1^6$, $5^7 . 1^3$,
„ 39 „ $8^3 . 5^2 . 1^5$,
„ 40 „ $8^2 . 5^4 . 1^4$,
„ 41 „ $8 . 5^6 . 1^3$,
„ 42 „ $8^4 . 5 . 1^5$, $5^8 . 1^2$,

that is, there are only at utmost 7 arguments contained in the expressions for the 10 compounds.

So, in like manner, if we assumed for the primitive

$$(0, b, 0, 0, 0, 0, g, 0, i \between x, y)^8$$

to find the number of independent arguments possible in a 10 . 4 covariant, we must ascertain the sum of the numbers of similar representations to the foregoing of the same integers 38, 39, 40, 41, 42, with 10 integers confined to be 1, 6 or 8, and we shall find that the sole representations of that kind are $8^4.1^6$; $8^2.6^3.1^5$; $6^6.1^4$; $8^4.6.1^3$; $8^3.6^2.1^5$; $8.6^5.1^4$, that is, 6 representations in all. In like manner it will be found that all the other trinomial forms of the primitive so taken that the first two invariants are null, will be incapable of yielding as many as 10 arguments to any covariant of deg-order 10 . 4†, so that the 10 compounds appurtenant to such special form will be bound to be linearly related, and no inference can be drawn from any such assumption. I have reason for believing that the quadrinomial form employed in the foregoing investigation is the most convenient and economical, as leading to the simplest calculations of any that could have been employed for the same purpose.

† On an exhaustive examination, it will be found that the only trinomial forms of the primitive which will cause the first two invariants to disappear, are those in which the surviving coefficients are

$$b, f, i;\ b, g, i$$

$$a, b, c;\ a, b, d;\ a, c, d;\ b, c, d,$$

or the complementary ones

$$g, d, a;\ h, c, a$$

$$i, h, g;\ i, h, f;\ i, g, f;\ h, g, f,$$

which, of course, are substantially equivalent to the former.

Confining our attention, then, to the upper group, it will readily be seen that the four last will cause not only the quadrinvariant and the cubinvariant, but all the other invariants as well, to vanish. Since, then, it has been shown that the b, f, i; b, g, i forms are insufficient to support the independence of the 10 compound covariants with which the reasoning is concerned, it follows that *no trinomial form* will be adequate to do so.

It may be asked what would have been the effect of using the form in which b, c, d, i are the surviving coefficients, but b, c, d are supposed mutually independent, instead of being subject to the condition employed in the refutation above: on this supposition the quadrinvariant, but not the cubinvariant, will vanish; and an easy calculation will show that of the 32 representations of the covariant of deg-order 10 . 4 as a product of inferior groundforms there will be only 16 in which the quadrinvariant does not appear as a factor. And, again, it will be found that the number of ways of representing 38, 39, 40, 41, 42, as the sum of 4 numbers, each of which is either 1, 2, 3 or 8 is 29. Hence there would arise a matrix of 16 lines and 29 columns, and to disprove the existence of the 10 . 4 groundform it would be sufficient to prove that some one of the *complete* minor determinants of this matrix differs from zero. The work involved in dealing with this and the subsequent verificatory matrix of 17 lines and 29 columns would evidently be vastly greater and more liable to error than when (as in the text) we assign the relation between b, c, d so as to make the cubinvariant vanish.

In the absence of the information as to the number of linearly independent 10 . 4's given by Cayley's rule, the direct mode of refutation would have required the calculation of the 32 compound 10 . 4's and the problematical one of von Gall for the general form of the Octavic, subject only to the simplification of taking two of the coefficients zero. There would then have remained to show that the leading terms of these 33 forms were linearly connected, which would necessarily imply that the same was true of the 33 entire forms themselves; a colossal task, probably transcending the sphere of human ability to execute.

It may be well (by way of confirmation) to determine *à priori* the number of possible arguments that can belong to the 10 . 4 covariants of the quadrinomial form of $(x, y)^8$ employed in the antecedent investigation. Since c^2 may be replaced by a numerical multiple of bd, it follows that each argument may be brought to a form in which c does not enter at all, or in which it enters only in the first degree. The total possible number (which turns out to be the actual number) of arguments is, consequently, the number of ways in which 38, 39, 40, 41, 42 can be composed with 10 parts each of them 1, 3 or 8 + the number of ways in which 36, 37, 38, 39, 40 can be composed with 9 parts, each of them also 1, 3 or 8. All the possible different compositions of these kinds are exhibited in the annexed table.

$38 = 4 . 8 + 6 . 1 = 2 . 8 + 7 . 3 + 1 . 1$	$36 = 3 . 8 + 3 . 3 + 3 . 1$
$39 = 3 . 8 + 4 . 3 + 3 . 1$	$37 = 4 . 8 + 5 . 1 = 2 . 8 + 7 . 3$
$40 = 4 . 8 + 5 . 1 + 1 . 3 = 2 . 8 + 8 . 3$	$38 = 3 . 8 + 4 . 3 + 2 . 1$
$41 = 3 . 8 + 5 . 3 + 2 . 1$	$39 = 4 . 8 + 1 . 3 + 4 . 1$
$42 = 4 . 8 + 4 . 1 + 2 . 3$	$40 = 3 . 8 + 5 . 3 + 1 . 1.$

There are thus 7 + 6, that is, 13 distinct arguments, that is, the number which actually appear distributed among the 10 surviving covariants of deg-order 10 . 4 as previously shown—it being at the same time remembered that three of the 13 enter as elements of a fixed linear combination into the 10 functions, which are thus virtually functions of only 11 independent arguments.

The method employed in what precedes suggests a mode of calculating in part at least the discriminant of the eighthic in terms of the subordinate groundforms. Thus, suppose we take for our special form,

$$(0,\ b,\ c,\ d,\ 0,\ 0,\ 0,\ 0,\ i \between x,\ y)^8$$

with b, c, d independent.

Then the quadrinvariant will vanish, and there will be no very great effort of calculation required to express the 8 remaining invariants as functions of b, c, d, i.

The discriminant is of the 14th degree and 14 may be made up in 10 (and no more than 10) ways as a sum of numbers each limited to be 3, 4, 5, 6, 7, 8, 9, or 10; as exhibited in the exhaustive table

$$14 = 10 + 4 = 9 + 5 = 8 + 6 = 8 + 3 + 3 = 7 + 7 = 7 + 4 + 3 = 6 + 4 + 4$$
$$= 6 + 5 + 3 = 5 + 5 + 4 = 4 + 4 + 3 + 3.$$

Again the weight of the discriminant is 56, and the number of ways of compounding 56 with 14 numbers each limited to be 1, 2, 3 or 8 is 11, as shown in the exhaustive table

$56 = 6.8 + 8.1 = 5.8 + 7.2 + 2.1 = 5.8 + 3.3 + 1.2 + 5.1 = 5.8 + 2.3 + 3.2 + 4.1 = 5.8 + 1.3 + 5.2 + 3.1 = 5.8 + 7.2 + 2.1 = 4.8 + 7.3 + 3.1 = 4.8 + 6.3 + 2.2 + 2.1 = 4.8 + 5.3 + 4.2 + 1.1 = 4.8 + 4.3 + 6.2 = 3.8 + 10.3 + 1.2.$

Now there will be no difficulty at all in finding by substitution and multiplication the discriminant of the assumed quantic, say Q, which is in fact the same as the resultant of $\frac{dQ}{dy}$ and $bx^6y + 3cx^5y^2 + 5dx^4y^3$. Hence there will be 11 equations for determining the coefficients of the 10 invariants of the 14th degree which are products of the inferior invariants (the quadrinvariant excepted); consequently there will be sufficient or more than sufficient equations for the purpose, unless it should (unfortunately and contrary to probability) turn out to be the case that the 10 products, although linear functions of 11 arguments, are expressible as linear functions of only 9 linear functions of those arguments.