71.

ON A GEOMETRICAL PROOF OF A THEOREM IN NUMBERS.

[*Johns Hopkins University Circulars*, I. (1882), pp. 179, 180.]

THE theorem in question is the well-known one that if a, b are incommensurable and x, y integers $ax + by + c$ may be made positively and negatively indefinitely small. This is tantamount to showing that on the plane of a reticulation*, nodes may be found indefinitely near to and on each side of an irrational straight line, that is, a line not parallel to any line of nodes. The proof is based on the Lemma that no infinite parallelogram, each side of which is an irrational line containing a node, can be vacuous of nodes in its interior. If this were not true a succession of shifts of the figure in the direction of the line joining the two nodes would lead to the absurd conclusion that the whole reticulation consists of a single line of nodes.

(1) Suppose the irrational line L contains a node and that there is no other node at less than a finite distance from it on one side of it, say to the right. Let it be moved to the right parallel to itself until it passes through another node N', then there will be a vacuous parallelogram of the kind declared impossible by the Lemma. [To this it may be objected that when L has moved from the left to M through a distance δ, M might be supposed to be an asymptote to an infinite series of nodes to its right. But if this were the case a node P might be found at a less distance than δ from M, and a node, Q, nearer to M than P is; if this line of nodes PQ be followed up until we reach the first node T on the other side of M, the most elementary geometry seems to show that T in *any* case is nearer to M than P is and consequently there would be a node between L and M contrary to hypothesis.] Hence there must be a node indefinitely near to L on each side of it.

(2) Suppose the irrational line L not to contain a node. If the theorem to be proved is not true, L may as before be moved parallel to itself (through

* By a reticulation is to be understood a pair of systems of an infinite number of indefinite equidistant parallel lines in a plane whose intersections form the nodes.

a finite distance) until it pass through a single node and there would be a vacuous parallelogram of which one side contains nodes, which has already been shown to be impossible.

Dr Story and Dr Franklin took part in the discussion and the valuable critical observations of the latter, led to the consideration of the objection stated and disposed of in the passage within brackets above. Professor Cayley made a remark to the effect that the diamond point in a graver's tool however fine, drawn in a straight direction across the face of a double grating must either pass through none of the intersections of the two systems of parallel lines or through an infinite number of them. The principle established in the bracketed passage admits of being stated in the following terms: "It is impossible for a straight line in the plane of a reticulation to be asymptotic in regard to nodes on one side of it and not so in regard to the nodes on the other side"; this proposition and the Lemma being conceded, the existence of any indefinite *vacuous* strip bounded by irrational parallel lines is disproved by imagining it distended on both sides, still retaining its form (in case neither bounding line contains a node), or in the contrary case on one side only (that is, in the direction away from the nodal line) until the distended figure passes through two nodes. The asymptotic rule shows that this construction would be possible—the Lemma that it leads to an impossible result. From this it follows that every irrational line is asymptotic in respect to the nodes lying on *each* side of it which is the thing to be proved.

Let a line be termed mono-asymptotic when it is asymptotic in regard to any scheme of points lying on one side of it,—amphi-asymptotic when it is so for schemes of points lying on each side of it. The foregoing argument may then be summed up as follows. Any irrational right line in the plane of a reticulation, must be amphi-asymptotic as regards the nodes. For if not, a line parallel to it must (under pain of contradicting the Lemma) be conceded to exist, which shall be mono-asymptotic in respect to them, but the existence of such a line has been proved to be impossible*. Similarly, it may be shown for a solid network, that no indefinite open prism whose parallel edges are doubly irrational (that is, neither parallel to a nodal line nor to a plane of nodes) can be vacuous of nodes, and also that no plane can be mono-asymptotic—from which, by very similar reasoning to that previously used, may be deduced the law, that no prism of finite dimensions, *vacuous* of nodes, can be constructed about an irrational line as its axis and that consequently any such line may be regarded as a sort of asymptotic axis to a

* The form of proof is a somewhat unusual combination of an *Ex-absurdo* with a *Dilemma*. A *denial* of the amphi-asymptoticism of an irrational straight line either dashes itself against the impossibility of the existence of a vacuous parallelogram or against the equal impossibility of the existence of a mono-asymptotic line.

helical spiral of nodes. Hence it follows that if a, b, c (taken two and two) are incommensurable with each other, the quadratic function

$$\{b(z-\gamma)-c(y-\beta)\}^2+\{c(x-\alpha)-a(z-\gamma)\}^2+\{a(y-\beta)-b(x-\alpha)\}^2,$$

and as a particular case

$$(bz-cy)^2+(cx-az)^2+(ay-bx)^2$$

may be made indefinitely small with integer values of x, y, z.

Nor is this all, for not only can a node be found indefinitely near to the doubly irrational line $x:y:z::a:b:c$, but such node may be successfully sought for within any infinitesimal sector of space contained within two planes drawn through that line, or in other words a node can be found indefinitely near to the irrational line and to any plane drawn through it, that is, to any plane

$$bc(m-n)x+ca(n-l)y+ab(l-m)z,$$

where l, m, n are any quantities whatever*, so that x, y, z integer numbers can be found which shall simultaneously cause

$$(bz-cy)^2+(cx-az)^2+(ay-bx)^2$$

to be less than any positive quantity k^2 and

$$(b-c)x+(c-a)y+(a-b)z$$

to lie between 0 and h, or 0 and $-h$ where h is also any assigned quantity. And of course the proposition can be stated in more general terms by considering an irrational line which does not pass through a node.

The same geometrical property admits of being defined under another form, namely, through the assertion that if any two given planes be drawn through a completely irrational line in an infinite Nodal Block†, a node may be formed indefinitely near to each of them—and this statement translates itself into the arithmetical proposition following:

If no linear equation

$$\lambda(b\gamma-c\alpha)+\mu(c\alpha-a\gamma)+\nu(a\gamma-b\alpha)=0$$

exists for integer values of λ, μ, ν, the two expressions

$$ax+by+cz+d,$$

and

$$\alpha x+\beta y+\gamma z+\delta$$

may simultaneously be made less than any given quantity k, by integer values of x, y, z.

* A particular form of this is $(b-c)x+(c-a)y+(a-b)z$. In order to give the theorem its greatest generality it is only necessary to substitute for $x, y, z, x-\alpha, y-\beta, z-\gamma$ where α, β, γ, are any real quantities whatever.

† By a Nodal Block is to be understood three systems in space of an indefinite number of equidistant parallel planes whose intersections are the nodes.

Although it would be difficult to follow the theory of nodal schemes into regions transcending the sensible dimensions of space, there need be no hesitation in accepting the truth of the generalized arithmetical theorem corresponding to this bolder than Icarian flight, namely, that

Any number of linear functions of one more than that number of integer Variables such that the determinant to the Matrix formed by the coefficients of the Variables supplemented by a line of arbitrary integers is incapable of being made zero, can by a right assignment of the Variables be brought to lie each of them between any assigned (indefinitely narrow) limits.

This proposition admits of a partial removal of the condition imposed in the above statement.

For an irrational line even if singly irrational, that is, parallel to a *nodal* plane although not so to a line of nodes, will be asymptotic to a series of nodes *if it lies in a nodal plane*, the only difference in this case being that the nodal sheath will be plain instead of being helical. Hence the two functions

$$ax + by + cz + d, \quad a'x + b'y + c'z + d',$$

can be made simultaneously indefinitely small, even though integer numbers $A, B, C,$ *can* be found such that the determinant $\begin{vmatrix} a & b & c \\ a' & b' & c' \\ A & B & C \end{vmatrix}$ is zero, provided that a rational number D can also be found, which will cause *all* the complete minors of the Matrix $\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ A & B & C & D \end{vmatrix}$ to vanish.

A particular case of this arises when d and d' are each zero. Consequently the two twin functions

$$ax + by + cz \text{ and } a'x + b'y + c'z,$$

may *in all cases* be made each of them simultaneously to vanish, or else to become indefinitely small for integer values of x, y, z. Thus then, we have an immediate and intuitive proof of Jacobi's celebrated proposition for proving the impossibility of the existence of trebly periodic functions*.

Those gifted with the powers of a Stringham, a Newcomb, or a Charles S. Peirce to feel their way about in supersensible space, may, in like manner, obtain if not an *intuitive*, at least an immediate, or non-mediated proof of the theorem that: *Any number of* homogeneous *functions of one more than that number of integer variables may be made either* to vanish simultaneously or else *to become simultaneously less than any assignable quantity*.

* See M. Hermite's admirable *Note sur le calcul différentiel et le calcul intégral*, Paris, 1862, pp. 5—8, where the proposition in question is established by means of the theory of ternary and binary quadratic forms.

Whilst in the course of writing out the above matter the following note, addressed to him, from Dr F. Franklin, was received by Professor Sylvester:

"Your proof may be put into the following form:

Theorem.—In any stripe bounded by irrational parallels there must be a node.

For if not, let N and N' be *any* two nodes. Repeat the stripe a finite number of times, namely, until the aggregate of the stripes shall have included N and N'. No stripe can contain two nodes ν, ν', for if it did, by producing $\nu\nu'$ we see that each of the stripes must contain at least one node, which is contrary to the hypothesis. Hence we have an open parallelogram containing two nodes N, N', and only a finite number of others, which is absurd; for since the parallelogram intercepts a distance greater than NN', it must intercept on *every* nodal line parallel to NN' at least one node. Thus the theorem is proved.

It may be noted that while a stripe of finite width bounded by *rational* lines contains either no nodes or a *singly* infinite number of them, a stripe bounded by *irrational* lines always contains a *doubly* infinite number of nodes; which, although easily explicable, might at first sight strike one as paradoxical, inasmuch as the probable number in a *given finite portion* is the same for one sort of stripe as for the other."

One word in conclusion. The modes above given of presenting the theory with reference to planes passing through a singly or doubly irrational line ought not to be allowed to draw away attention from the image afforded by a doubly irrational line surrounded by an asymptotic spiral sheath (a point-helix winding round a fish-bellied-torpedo-like bobbin or core) tapering off to an indefinitely fine point in both directions, nor from the extension of the theory of continued fractions to which that image points.

Taking for greater simplicity the case of such a line passing through a node at the origin, the question invites solution to *devise an Algorithm* for finding the integer values of x, y, z which shall give the successive minima (corresponding to nodes of nearest approach) of the function

$$(bz - cy)^2 + (cx - az)^2 + (ay - bx)^2,$$

that being the problem next in the order of natural succession to the solved one of finding the successive absolute minima of $ay - bx$.

In the latter case, a and b are supposed to be incommensurable—in the former, no linear equation with rational coefficients is supposed to exist between a, b, c*.

* Which is tantamount to saying that the line $x : y : z :: a : b : c$ must be doubly irrational.