75.

ON THE PROPERTIES OF A SPLIT MATRIX.

[*Johns Hopkins University Circulars*, I. (1882), pp. 210, 211.]

SUPPOSE a square matrix split into two sets of lines which need not be contiguous and may be called ranges, say *ABC, DEFG*. Let the sum of the products of the corresponding elements of any two lines be called their product. It is well known (see Salmon's *Higher Alg.*, 3rd Ed., p. 82) that if the product of each line in the first range by every line in the other is zero, the opposite complete minors of the two ranges will be in a constant ratio, say in the ratio $l:\lambda$. Call the content of the matrix Δ: then it follows, if S, Σ denote the sums of the squares of the complete minors in the two ranges respectively, that

$$\frac{\lambda}{l}S = \frac{l}{\lambda}\Sigma = \Delta.$$

But by a theorem of Cauchy concerning rectangular matrices S is equal to the determinant $(A, B, C)^2$, that is, to the determinant

$$\begin{vmatrix} AA & AB & AC \\ BA & BB & BC \\ CA & CB & CC \end{vmatrix}$$

and similarly $$\Sigma = (D, E, F, G)^2$$

so that $$\lambda^2 : l^2 :: (D, E, F, G)^2 : (A, B, C)^2$$

and $$S\Sigma = \Delta^2.$$

Suppose now that the product of *every* two lines in the entire matrix is zero. Then into whatever two ranges the matrix be divided the ratio $\lambda^2 : l^2$ (since all but the diagonal terms in the matrices which express the ratio $l^2 : \lambda^2$ vanish) will be expressed by the ratio of one simple product to another: thus for example for the ranges $ABC : DEFG$

$$\lambda^2 : l^2 :: D^2 . E^2 . F^2 . G^2 : A^2 . B^2 . C^2; \text{ also } \Delta^2 = A^2 . B^2 . C^2 . D^2 . E^2 . F^2 . G^2.$$

If we now further suppose that the sum of the squares of the elements in each line is unity, that is, that

$$A^2 = B^2 = C^2 = D^2 = E^2 = F^2 = G^2 = 1,$$

it will follow that every minor whatever divided by its opposite will be equal to Δ $\left(\text{for on the hypothesis made, } \frac{\lambda}{l} = \frac{\Delta}{S} = \Delta\right)$.

Also Δ will be plus or minus unity since $\Delta^2 = 1$. Thus it is seen that we may pass by a natural transition from the theory of a split to that of an orthogonal or self-reciprocal matrix—to show which was the principal motive to the present communication. It is by aid of the theorem of the *split matrix* that I prove a remarkable theorem in Multiple Algebra, namely, that if the product of two matrices of the same order is a complete null, the sum of the nullities of the two factors must be at least equal to the order of the matrix—the nullity of a matrix of the order ω being regarded as unity, when its determinant simply is zero, as 2 when each first minor simply is zero, as 3 when each second minor is zero ... as $(\omega - 1)$ when each quadratic minor is zero and as ω (or absolute) when every element is zero. This theorem again is included in the more general and precise one following—*If any number of matrices of the same order be multiplied together, the nullity of their product is not less than the nullity of any single factor and not greater than the sum of the nullities of all the several factors.*

In Professor Cayley's memoir on Matrices (*Phil. Trans.*, 1858) the very important proposition is stated that if

$$\begin{matrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \\ a''' & b''' & c''' & d''' \end{matrix}$$

be any matrix of substitution, say m (here taken by way of illustration of the order 4) the determinant

$$\begin{vmatrix} a-m & b & c & d \\ a' & b'-m & c' & d' \\ a'' & b'' & c''-m & d'' \\ a''' & b''' & c''' & d'''-m \end{vmatrix}$$

is identically zero; or in other words, its nullity is complete. By means of the above theorem it may be shown that the nullity of any i distinct algebraical factors of such matrix is equal to i, i having any value from unity up to the number which expresses the order of the matrix, inclusive.