# 79.

## ON CERTAIN SUCCESSIONS OF INTEGERS THAT CANNOT BE INDEFINITELY CONTINUED.

[*Johns Hopkins University Circulars*, II. (1883), pp. 2, 3.]

A SUCCESSION of decreasing integers we know cannot be indefinitely continued, but there are also successions of increasing integers subject to certain stated conditions, but otherwise arbitrary, which are similarly incapable of indefinite extension.

The following is a simple instance of the kind. Suppose integers to be written down one after the other, no one of which is a multiple of any other, nor the sum of a multiple of any other and of a multiple of a specified one. *Such a succession cannot be indefinitely continued.*

Let $a$ be the specified integer.

(1) Suppose that all the other integers of the succession are prime to $a$.

Then if $b$ be any other of the integers, the equation $ax+by=c$ is soluble in integers if $c$ is greater than $ab$, as follows at once from the consideration that the numbers $c-b, c-2b, c-3b \ldots \overline{c}-ab$ must be all distinct residues to the modulus $a$, inasmuch as the difference of any two of them being of the form $(i-j)b$ where $i-j$ is less than and $b$ prime to $a$, cannot be divisible by $a$.

But if the succession could be indefinitely produced, it must contain a number greater than $ab$. Hence the theorem is proved for the case where $a$ is prime to every other integer in the succession.

(2) Suppose the theorem to be true for the case where the quotient of $a$ divided by $i$ prime numbers (not necessarily all distinct) is prime to all the other terms of the series: it must be true when the number of such prime numbers is $i+1$. For let $p$ be one of them and $a=pa'$, consider all the terms of the succession divisible by $p$ apart from the rest.

Let $pa'$, $pb'$, $pc'$ ... be those terms. By the law of the succession the equation $pa'x + pb'y = pc'$ cannot be satisfied for any values of $b'$, $c'$, and consequently $a'x + b'y = c'$ cannot be satisfied.

Hence by hypothesis since $a'$ divided by $i$ factors is prime to $b'$, $c'$ ... the succession of terms divisible by $p$ must be finite in number, and this will be true for every factor $p$. Hence the succession $b$, $c$, ... will contain only a finite number of terms having any factor in common with $a$. Moreover the succession containing $a$ and terms prime to $a$ exclusively, must also be finite by the preceding case. Consequently the whole succession will be finite, and the theorem if assumed to be true for $i = 0$, or any positive integer, is true for $i + 1$.

But by the preceding case the proposition is true when $i = 0$. Hence it is true universally.

In the long footnote to the Article on Subinvariants in Vol. v., pp. 92, 93 of the *Am. Journal of Math.*, will be found given the mode of extending this theorem to the case of successions of complex integers or multiplets, when a proper restriction is laid upon the ratios to one another of the simple numbers which constitute the multiplets, and a possible connexion pointed out between the finiteness of such successions and that of the system of ground-forms to a binary quantic [p. 580, above].