81.

NOTE ON THE PAPER OF MR DURFEE'S.

[*Johns Hopkins University Circulars*, II. (1883), pp. 23, 24; 42, 43.]

MR DURFEE'S very elegant and interesting theorem above given may, by help of Euler's law of reciprocity, be expressed in the following terms.

Let fx and ϕx represent respectively:

$$\frac{1}{1-x}+\frac{x^4}{(1-x)(1-x^2)(1-x^3)}+\frac{x^{12}}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)}+\dots+\frac{x^{2i^2+2i}}{(1-x)\dots(1-x^{2i+1})}+\dots$$

and
$$\frac{x^2}{(1-x)(1-x^2)}+\frac{x^8}{(1-x)(1-x^2)(1-x^3)(1-x^4)}+\dots+\frac{x^{2i^2}}{(1-x)\dots(1-x^{2i})}+\dots,$$

then the number of self-conjugate partitions of $2m+1$ and of $2m$ are the coefficients of x^m in the ascending expansions of fx, ϕx, respectively.

Thus, suppose $2m+1=13$, the coefficient of x^6 in fx developed, that is, $\frac{6}{1,}+\frac{2}{1,\,2,}$ or 3 is the number of self-conjugate partitions of 13.

These will be found to be 7 1 1 1 1 1 1, 4 4 3 2, 5 3 3 1 1. To find the conjugate to any partition $a, b, c \dots, l$, the most expeditious method is to find n_i, the number of the elements in the partition not less than i: $n_1, n_2, \dots, n_l$ (l being supposed to be the largest value of any element) will then be its opposite.

Thus, for example, for the partition 5 3 3 1 1, $n_1=5$, $n_2=3$, $n_3=3$, $n_4=1$, $n_5=1$, and $n_1\ n_2\ n_3\ n_4\ n_5$ reproduces 5 3 3 1 1.

If $2m = 12$ we have to find the value of $\frac{4}{1, 2,}$ which is again 3, and the 3 self-conjugate or self-opposite partitions of 12 will be seen to be 4 4 2 2; 5 3 2 1 1; 6 2 1 1 1 1.

In M. Faà de Bruno's tables of symmetric functions, which are only complete for the case of equations of not higher than the 11th degree, the number of self-conjugate partitions which appear among the headings and sidings of the tables is either 1 or 2, and it was therefore reasonable to try the effect of making arrangement of the partitions such as to bring the self-conjugate or pair of self-conjugate partitions into the centre of the line or column; but as soon as that degree is passed such a kind of principle (the rule founded upon which M. de Bruno does not state) becomes *prima facie* inapplicable at all events without undergoing modifications of which at present we know nothing.

Thus M. de Bruno's tables end just where his proposed principle of arrangement becomes inapplicable, stopping short at the case of the 12th degree, which has since been tabulated by Mr Durfee in the *American Journal of Mathematics.*

The term "opposite" or "conjugate" is used by Mr Durfee in the sense in which I am in the habit of employing it to signify the relation between what M. Faà de Bruno calls *combinaisons associées.* I think it right to recall attention to the fact that the credit of calling into being this kind of conjugate relation, is due to Dr Ferrers (the present Master of Gonville and Caius College, Cambridge), who some 30 years ago or more was the first to apply it to obtain an intuitive proof of Euler's great law of reciprocity, the very same as that which I have here employed to transform Mr Durfee's theorem. Euler demonstrated his law by help of his favourite instrument of generating functions.

By instituting in the case of combinations of *unrepeated* elements quite another and more exquisite kind of conjugate relation applicable to all such with the exception of those which belong to the infinite succession 1, 2, 2 3, 3 4, 3 4 5, 4 5 6, 4 5 6 7, 5 6 7 8, Mr Franklin, of this University, succeeded in finding an instantaneous demonstration of another well-known but very much more recondite theorem in partitions, also due to Euler, expressible by the statement that the indefinite product

$$(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots$$

has for its development

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} \ldots,$$

where the indices are the complete series of direct and retrograde pentagonal numbers.

By a singular oversight in my note in the last *Circular*, I omitted to state that Mr Durfee's rule is tantamount to affirming that the number of self-conjugate partitions or (which is the same thing) of symmetrical partition graphs for n, is the coefficient of x^n in the series

$$1 + \ldots + \frac{x^{i^2}}{(1-x^2)(1-x^4)\ldots(1-x^{2i})} + \ldots$$

and since this series is identical with the infinite product

$$(1+x)(1+x^3)(1+x^5)\ldots$$

the number of self-conjugate partitions is the number of ways of distributing n into unrepeated odd-integers, a result which can be obtained directly by regarding any symmetrical partition graph as made up of a set of successively diminishing equilateral elbows or say carpenters' rules, each of which necessarily contains an odd number of points: the number of such elbows for any given graph will be the same as the number of points in the side of Mr Durfee's square nucleus, and consequently we have an intuitive proof of the theorem that the infinite product

$$(1+ax)(1+ax^3)(1+ax^5)\ldots$$

is equal to the infinite series

$$1 + \ldots + \frac{x^{i^2}}{(1-x^2)(1-x^4)\ldots(1-x^{2i})}a^i + \ldots$$

because the coefficient of $a^i x^n$ is the same in both expressions. By a similar method I obtain an intuitive and almost instantaneous solution of the problem to expand in infinite series the infinite products which express a Theta Function and its *reciprocal*, and many other questions of a similar nature.

It was the anticipation of the parallelism between the expressions for the number of special partitions in the unrepeated-numbers and the repeated-numbers theories which led me to find *à priori* the partition-into-odd-integers expression for the number of self-conjugate partitions, and thus started me on the track of the graphical method of transforming infinite products into infinite series: the light of analogy may sometimes "lead astray" but it is more often "light from heaven."