82.

ON DR F. FRANKLIN'S PROOF OF EULER'S THEOREM CONCERNING THE FORM OF THE INFINITE PRODUCT $(1-x)(1-x^2)(1-x^3)\ldots$.

[*Johns Hopkins University Circulars*, II. (1883), p. 42.]

REVOLVING in my mind Mr Franklin's remarkable proof of Euler's theorem concerning the above infinite product inserted in the *Comptes Rendus* of the Institute of France for 1880, I have found it useful to employ a certain terminology to enable myself to seize some of the points which it contains with a firmer grasp and to clothe it in what seems to me a more purely discursive, as distinguished from what, by analogy to geometrical processes, I am wont to call a diagrammatic form of reasoning; thinking that others may find advantage in what has been useful to myself, I avail myself of the pages of the *Circular* to give it publicity.

Let us agree to understand by a *distribution* of n any combination of *unrepeated* integers in descending order, whose sum is equal to n. The number of such component integers may be termed the *order* of the distribution.

If the initial components of such distributions be $m-1, m-2, \ldots, (m-i)$ [where i may be equal to but cannot exceed the order] not followed by an element $m-i-1$, I call i (the number of terms in such initial sequence) the *consecutant* and the final (that is, the least) component, the *concluant* of the distribution.

Lemma. Any distribution of a given integer, which does not form a single sequence whereof the concluant is either equal to or greater by a unit than the consecutant, may be converted by one or the other (but not by either) of two reversible processes (say of loading or unloading) into another distribution in which the order is diminished or increased by a single unit.

By loading is to be understood the process of taking away the concluant (say ω) and increasing the ω first terms of the initial sequence each by a unit; and by unloading, that of taking away a unit from each of the components in the initial sequence and adding on an element equal to the consecutant as the new concluant.

1st. Suppose that the distribution does not form a single sequence.

If the concluant is equal to or less than the consecutant it is obvious that loading will be possible but not unloading, because the latter would give rise to a new concluant equal to or greater than the original one.

On the other hand, if the concluant is greater than the consecutant, unloading will be possible but not loading, because there will be too few terms in the initial sequence to exhaust (by the addition of one unit to each) the number of units in the concluant.

2nd. Suppose that the entire distribution forms a single sequence.

If the concluant is *less* than the consecutant loading will still be possible, because the number of terms in the sequence after taking away the concluant will still be not greater than the concluant.

Again, if the concluant is more than a unit greater than the consecutant, unloading will still be possible because the new concluant will be less than the original one even after it has lost a unit by the process of unloading.

Hence the Lemma is proved.

And as a Post-lemma, it may be stated that when the distribution forms a single sequence such that the concluant is equal to or only one unit greater than the consecutant, neither loading nor unloading will be possible. The loading on the first supposition is defeated by the fact that the diminished sequence will be one too few in number to absorb the units which make up the concluant—and the unloading on the second supposition is defeated by the fact that the new concluant will be equal to (that is, will be a repetition of) the old one when by the act of unloading it is diminished by a unit.

From the lemma and post-lemma combined, it follows as an inference that all the distributions of any number n may be taken in pairs (consisting of one of an even and one of an odd order), unless it should be the case that one of such distributions is a term in the series

$$1,\ 2,\ 3.2,\ 4.3,\ 5.4.3,\ 6.5.4,\ \ldots,\ (2i-1).(2i-2)\ldots i,\ \ 2i.(2i-1)\ldots(i+1),\ \ldots$$

which represent distributions of the several integers

$$1,\ 2,\ 5,\ 7,\ 12,\ 15,\ \ldots\ \frac{3i^2-i}{2},\ \frac{3i^2+i}{2},\ \ldots$$

to which the process either of loading or unloading (contraction or expansion by a unit) is inapplicable.

Hence if we denote by n_o, n_e, the number of distributions of n, into an odd and even number of unrepeated parts, we must have $n_o - n_e = 0$, except when $n = \frac{3i^2 \mp i}{2}$, in which case $n_e - n_o = (-)^i 1$.

Consequently we have

$$(1-x)(1-x^2)(1-x^3)\ldots,$$

that is, $$1 + \ldots + (n_e - n_o)x^n + \ldots = \sum_{i=+\infty}^{i=-\infty} (-)^i x^{\frac{3i^2+i}{2}},$$

which is Euler's theorem.

To make the demonstration absolutely objection-proof it ought to be shown that if X is convertible into Y by loading or unloading, Y will be convertible into X by the reverse process—but this is almost self-obvious; for if X has become Y by loading, the new consecutant cannot be greater than the old one and will therefore not be greater than the new concluant, but equal to or less than it, and therefore the process of *unloading* is the one applicable to Y, and if X has become Y by unloading, the new consecutant cannot be less than the old one and will therefore be greater than the new concluant, and therefore the process of *loading* is the one applicable to Y; this completes the proof, and leaves I think nothing further to be desired.

In Mr Durfee's question, treated of in the last number of the *Circulars*, the object of research is the number of self-conjugate partitions (with repeated or unrepeated components) of a given integer n; in Mr Franklin's, the object sought for is the number (1 or 0) of (so to say celibate or) unconjugate distributions of an integer: the Ferrers-law of conjugation is of universal application to all partitions—the Franklin-law only to partitions with unrepeated components.

There is, however, a singular parallelism between the two theories; let us agree to call the self-conjugate in the one, and the non-conjugate partitions in the other, in each case alike *special* partitions—and denote the number of distributions of n into an odd number and into an even number of *unrestricted* parts by $(n)_o$ and $(n)_e$ respectively. Then just as the difference between n_o and n_e is the number of special partitions in the one, so it may be shown that the difference between $(n)_o$ and $(n)_e$ (which is well-known to be the same as the total number of partitions of n into unrepeated odd parts) is the number of special partitions in the cognate theory.