83.

ON THE USE OF CROSS-GRATINGS TO OBTAIN CERTAIN DEVELOPMENTS CONNECTED WITH THE THEORY OF ELLIPTIC FUNCTIONS.

[*Johns Hopkins University Circulars,* II. (1883), pp. 43, 44.]

It will be convenient to regard the components of any partition as arranged in a natural, say a descending order of magnitude: a partition graph means a series of points, say the knots in a web or the intersections of a cross-grating, lying in lines parallel to two fixed lines: the number of points, or lines parallel to one of the boundaries chosen at will, will represent the successive components of the partition and the number of the lines themselves will be the number of parts in the partition.

The lines in question may for greater distinctness be termed *magnitude* lines and the crossing ones, *part* lines. The graph may be termed regular when the magnitude lines never increase as they recede from the rectilinear boundary to which they are parallel. This, we see intuitively, cannot happen without the same condition being true of lines parallel to the *part* boundary: so that we may say that a regular partition graph is one in which the lines and columns of points neither of them ever increase in length as they recede from their respective boundaries. If such a graph corresponds to a partition *without* repetitions, the lines drawn in the magnitude direction must continually contract (that is, contain fewer and fewer points) as they recede from their maximum boundary.

The correlation referred to in the preceding paragraph is tantamount to saying that if there be two systems of partitions in one of which a given number is set out in every possible way as a sum of i parts none exceeding j in magnitude, and another in which the same number is set out in every possible way as a sum of j parts none exceeding i in number, such partitions arranged in natural order will have a one-to-one correspondence, being representable by the same regular graphs with the names of the magnitude and part boundaries interchanged, so that there will be the same number of partitions in the two systems.

A partition is self-conjugate when its representative graph, after an interchange of the names of the part- and magnitude-lines, gives the same reading.

Such a graph, therefore, must be symmetrical.

Suppose the partible number to be n.

Then its graph may be resolved into i angles fitting into one another, consisting of continually decreasing odd numbers; and the number of such graphs will be the number of ways of composing n with unrepeated odd numbers: but it may also be analyzed into a square containing i^2 points and two similar and equal appendages each containing $\frac{n-i^2}{2}$ points; and consequently their number will be the number of ways in which $\frac{n-i^2}{2}$ may be made up with the numbers 1, 2, ... i, or what is the same thing $n-i^2$ with the numbers 2, 4, ... $2i$; it is consequently the coefficient of n in the development of

$$\frac{x^{i^2}}{(1-x^2)(1-x^4)\dots(1-x^{2i})};$$

but it is also by virtue of the preceding remark the coefficient of $x^n a^i$ in the continued product

$$(1+ax)(1+ax^3)(1+ax^5)\dots.$$

Hence this continued product

$$=1+\frac{x}{1-x^2}a+\frac{x^4}{(1-x^2)(1-x^4)}a^2+\frac{x^9}{(1-x^2)(1-x^4)(1-x^6)}a^3$$
$$+\dots+\frac{x^{i^2}}{(1-x^2)(1-x^4)\dots(1-x^{2i})}a^i+\dots.$$

The expansion of the reciprocal of

$$(1-ax)(1-ax^3)(1-ax^5)\dots$$

may be obtained in a similar manner; the coefficient of $x^n a^j$ in this product is the number of ways in which n can be composed with j free odd numbers. If we construct a graph with j angles or elbows fitting into one another, such that the number of nodes in each such angle from the outermost inward corresponds with any such partition in descending order, the graph so constructed will be still symmetrical but no longer regular; a line of nodes instead of being necessarily equal or less in number than an antecedent one may jut one degree beyond it, but if the j points in the diagonal be removed (as in the example following, the points

$$\begin{array}{cccccc}
1 & \cdot & \cdot & \cdot & \cdot & \\
\cdot & 2 & \cdot & \cdot & \cdot & \\
\cdot & \cdot & 3 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 4 & \cdot & \\
\cdot & \cdot & \cdot & \cdot & & \\
 & & \cdot & & &
\end{array}$$

whose places are supplied by the numbers 1, 2, 3, 4) then the figure that is left is decomposable into two regular graphs with one boundary line horizontal, or vertical, and the other oblique. Hence the fraction above given expanded in powers of a becomes

$$1 + \frac{x}{1-x^2}a + \ldots + \frac{x^i}{(1-x^2)(1-x^4)\ldots(1-x^{2i})}a^i + \ldots$$

the only difference from the preceding case being that i points now instead of i^2 are taken away from the graph.

I might give numerous other exemplifications of the power and grasp of this method, but the following two may suffice for the present. I propose first to prove the equation between the reciprocal of

$$(1-ax)(1-ax^2)(1-ax^3)\ldots$$

and the infinite series

$$1 + \frac{x}{1-x}\cdot\frac{a}{1-ax} + \frac{x^4}{(1-x)(1-x^2)}\cdot\frac{a^2}{(1-ax)(1-ax^2)} + \ldots$$
$$+ \frac{x^{i^2}}{(1-x)(1-x^2)\ldots(1-x^i)}\cdot\frac{a^i}{(1-ax)(1-ax^2)\ldots(1-ax^i)} + \ldots.$$

The coefficient of $x^n a^j$ in the continued product is the number of regular graphs that can be formed with n nodes, containing j lines of them with no limitations to the number of the columns. We may suppose, therefore, the number of columns to be made successively 1, 2, 3, ... Consider the case where there are i columns forming a square; the graph being regular the lines and columns will intersect in i^2 nodes, and there will be left $n - i^2$ nodes to be made up of $j - i$ quantities none greater than i (namely, the horizontal filaments of nodes in the columns underlying the square), and of other quantities not greater than i but otherwise unlimited (namely, the vertical filaments of nodes in the hollowed out indefinite parallelogram abutting alongside of the square): that number we well know is the coefficient of $x^n a^{j-i}$ in

$$\frac{1}{(1-ax)(1-ax^2)\ldots(1-ax^i)}\cdot\frac{1}{(1-x)(1-x^2)\ldots(1-x^i)}x^{i^2}.$$

Hence for every value of j the coefficient of $x^n a^j$ in the infinite product is the coefficient of $x^n a^j$ in the infinite series, and consequently the two forms when developed must be identical.

Not to weary my readers I hurry on to the development in an infinite series of the product of the two infinite products

$$(1+ax)(1+ax^3)(1+ax^5)\ldots \text{ and } (1+a^{-1}x)(1+a^{-1}x^3)(1+a^{-1}x^5)\ldots.$$

Here it will be expedient to explain what I mean by a parallel bipartition of n; it is simply a couple of sets of numbers written on opposite sides of a line of demarcation, so that the number of the numbers on the left always exceeds that on the right by a given difference δ, which may be any number from zero upwards, and such that the sum of all the elements collectively is equal to n.

When this difference is zero, such a bipartition may be called equi-parallel, in other cases parallel with a difference δ.

It is then clear that the coefficient of $x^n a^j$ or $x^n a^{-j}$ in the above product is nothing else but the number of parallel bipartitions of n to the difference j limited to contain only odd numbers which must not appear in the same arrangement more than once on the same side of the line of demarcation.

In particular the coefficient of x^n in the term not containing a will be the number of equi-parallel bipartitions of n restricted to odd numbers not repeated on the same side of the separating line.

Form a graph as follows: Supposing one of the bipartitions to consist of θ parts on each side, say $a, b, c, \ldots l$; $\alpha, \beta, \gamma, \ldots \lambda$; the parts being on each side taken in descending order, construct angles or elbows in which the horizontal sides contain

$$\frac{a+1}{2}, \frac{b+1}{2}, \ldots \frac{l+1}{2},$$

and the vertical sides

$$\frac{\alpha+1}{2}, \frac{\beta+1}{2}, \ldots \frac{\lambda+1}{2}$$

points, then these will contain

$$\frac{a+\alpha}{2}, \frac{b+\beta}{2}, \ldots \frac{l+\lambda}{2}$$

points respectively; on fitting them into one another we shall obtain a regular graph with θ lines or columns made up of $\frac{n}{2}$ points, and conversely every regular graph of $\frac{n}{2}$ points may be resolved into angles with sides p, p'; q, q'; $r, r' \ldots$ corresponding to an equi-parallel unrepeated odd-number bipartition

$$2p-1,\ 2q-1,\ 2r-1,\ \ldots;\ 2p'-1,\ 2q'-1,\ 2r'-1,\ \ldots.$$

Hence the coefficient of x^n in the term not containing a in the development is the number of regular graphs that can be formed with $\frac{n}{2}$ points; and therefore the term not containing a is

$$\frac{1}{(1-x^2)(1-x^4)(1-x^6)\ldots}.$$

Now consider the term containing a^j to which corresponds a parallel bipartition with j more elements to the left than to the right of the separating line: arrange the sets on each side of the line in descending order, strike off the j highest on the left-hand side and construct a graph G with the sets which remain, as in the last case; then subtract 1, 3, 5, $(2j-1)$ respectively from the j elements [struck off] to the left, and place, taken in ascending order, half the numbers of points remaining, over the top line of the graph G; there will result a regular graph G'; and by an obvious reverse process every such graph can be made to correspond with a bipartition of unrepeated odd numbers having j more numbers to the left than to the right. Hence the number of the parallel bipartitions to the difference j will be the number of indefinite partitions of

$$\tfrac{1}{2}\{n-(1+3+\ldots+2j-1)\} \text{ or } \frac{n-j^2}{2},$$

that is, the coefficient of x^n in

$$\frac{x^{j^2}}{(1-x^2)(1-x^4)(1-x^6)\ldots}.$$

Hence the given bi-product when developed must be identical with

$$\frac{1}{(1-x^2)(1-x^4)\ldots}\{1+x(a+a^{-1})+x^4(a^2+a^{-2})+x^9(a^3+a^{-3})+\ldots\}.$$

In the preceding volume of the *Circular* I showed how the self-same method of points (but very differently applied) serves to establish and leads to wide generalizations of the theorem of Jacobi, upon which depends the proof of the impossibility of the existence of 3-period functions.

In a future number of the *Circular*, or else in the *American Journal of Mathematics,* I propose to show how to obtain intuitively by a graphical construction the expression for the product of the two infinite products

$$\frac{1-a^k}{1-a}\cdot\frac{1-a^{3k}}{1-a^3}\cdot\frac{1-a^{5k}}{1-a^5}\ldots \text{ and } \frac{1-a^{-k}}{1-a^{-1}}\cdot\frac{1-a^{-3k}}{1-a^{-3}}\cdot\frac{1-a^{-5k}}{1-a^{-5}}\ldots.$$

The *true inwardness* of this powerful analytical method depends in the first place on the idea of *correspondence,* assisted in the second place (in some but not in all instances) by the idea of graphical representation of partition numbers restrained to assume a natural order of succession.

Mr Ferrers' method, which has lain so long dormant and sterile in mathematical soil, has after an interval of 30 years begun to germinate, and seems to be about to burst forth into luxuriant growth and efflorescence.

It is Mr Durfee's graphical determination of the number of self-conjugate partitions of n, inserted in a preceding *Circular*, that has let in the light and air and supplied the fertilizing influence needful to bring this about.