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Z POSIEDZEŃ WYDZIAŁU III T. N. W.

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z dnia 25 lutego 1949 r.

Wacław Sierpiński

Sur un problème de M. Zarankiewicz

Communication présentée dans la séance du 25.II 1949.

Dans une Note parue en 1928 [1]) M. Casimir Zarankiewicz a posé le problème suivant:

Existe-t-il pour tout nombre ordinal α, $1 < \alpha < \Omega$, *un ensemble linéaire ambigu de classe* α *qui ne soit pas dans aucun intervalle de classe* $< \alpha$? [2]).

Le but de cette Note est de démontrer que *la réponse à ce problème est positive*.

Je définirai d'abord, par l'induction, deux suites infinies d'ensembles parfaits disjoints $P_1, P_2, \ldots$ et $Q_1, Q_2, \ldots$ comme il suit. Soit P l'ensemble parfait non dense bien connu de G. Cantor, et soit

(1) $$\delta_1, \delta_2, \ldots$$

une suite infinie formée de tous les intervalles (fermés) aux extrémités rationnelles.

Divisons l'intervalle δ_1 en trois parties égales et soit P_1, respectivement Q_1, l'ensemble semblable (au sens géomètrique) à l'ensemble P et situé dans la première, resp. dans la troisiéne de ces parties.

Soit maintenant n un nombre naturel, et supposons que nous avons déjà défini les ensembles parfaits non denses $P_1, P_2, \ldots, P_n$ et $Q_1, Q_2, \ldots, Q_n$.

L'ensemble $S_n = P_1 + P_2 + \ldots + P_n + Q_1 + Q_2 + \ldots + Q_n$ est évidemment parfait et non dense et il existe des intervalles

[1]) *Wiadomości Matematyczne* **30**, p. 117.

[2]) M. Zarankiewicz a démontré l. c. que la réponse à ce problème est négative pour $\alpha = 1$ et il a exprimé l'hypothèse qu'il en est de même pour $\alpha > 1$.

Pour la définition des ensembles ambigus voir p. e. C. Kuratowski, *Topologie* I, Warszawa-Lwów 1933, p. 162.

de la suite (1) contenus dans δ_n et disjoints avec l'ensemble S_n: soit δ_{k_n} le premier d'entre eux. Divisons δ_{k_n} en trois parties égales et soit P_{n+1}, resp. Q_{n+1} l'ensemble semblable à P et situé dans la première, resp. dans la troisième de ces parties.

Les nombres naturels $k_1, k_2, \ldots$ et les ensembles parfaits non denses $P_1, Q_1, P_2, Q_2, \ldots$ sont ainsi définis par l'induction et ces ensembles sont évidemment disjoints.

Soit maintenant α un nombre ordinal donné, $1 < \alpha < \Omega$, et distinguons deux cas.

1) α est un nombre de première espèce, $\alpha = \beta + 1$. On a donc $1 \leqslant \beta < \Omega$ et il existe, comme on sait, pour tout n naturel un sous-ensemble E_n de l'ensemble parfait P_n qui est de classe additive β sans être de classe multiplicative β, et un sous-ensemble H_n de l'ensemble parfait Q_n qui est de classe multiplicative β sans être de classe additive β [1]).

Posons

(2) $$A = E_1 + H_1 + E_2 + H_2 + \ldots .$$

Les ensembles E_n et H_n $(n = 1, 2, \ldots)$ étant de classe β, l'ensemble A est de classe (additive) $\leqslant \beta + 1 = \alpha$.

Or, comme $E_n \subset P_n$, $H_n \subset Q_n$ et comme les ensembles $P_1, Q_1, P_2, Q_2, \ldots$ sont disjoints, on a

$$CA = \sum_{n=1}^{\infty} (P_n - E_n) + \sum_{n=1}^{\infty} (Q_n - H_n) + C(P_1 + Q_1 + P_2 + Q_2 + \ldots).$$

Vu que $P_n - E_n$ et $Q_n - H_n$ sont des ensembles de classe β et que $C(P_1 + Q_1 + P_2 + Q_2 + \ldots)$, en tant que G_δ, est un ensemble de classe 1, on conclut que CA est de classe additive $\leqslant \beta + 1 = \alpha$.

L'ensemble A est donc ambigu de classe $\leqslant \alpha$.

Or, soit δ un intervalle donné. Vu la définition de la suite (1), il existe un intervalle δ_n de cette suite, tel que $\delta_n \subset \delta$. Vu que $P_n \subset \delta_{k_n} \subset \delta_n \subset \delta$ et que les termes de la série (2) sont disjoints, on a $\delta A P_n = E_n$. Donc, si l'ensemble A était de classe multiplicative $\leqslant \beta$ dans l'intervalle δ, il en serait de même de l'ensemble E_n, ce qui n'est pas le cas. L'ensemble A n'est pas donc dans δ de classe multiplacative $< \alpha$. Pareillement, vu que

[1]) Quant à la démonstration d'existence de tels ensembles voir p. e. C. Kuratowski, l. c., p. 175.

$\delta A Q_n = H_n$, on démontre que A n'est pas dans δ de classe additive $< \alpha$.

2) α est un nombre de seconde espèce. Vu que $\alpha < \Omega$, il existe donc une suite infinie croissante de nombres ordinaux, $\alpha_1, \alpha_2, \ldots$, telle que $\alpha = \lim\limits_{n<\omega} \alpha_n$.

Soit, pour tout n naturel, E_n un sous-ensemble de P_n qui est de classe α_n sans être de classe $< \alpha_n$, et posons

$$A = E_1 + E_2 + \ldots . \tag{3}$$

En modifiant d'une façon évidente le raisonnement utilisé dans le cas 1), on démontre sans peine que l'ensemble (3) est ambigu de classe $\leqslant \alpha$.

Or, soit δ un intervalle donné et supposons que A est dans δ de classe $\beta < \alpha$. Il résulte sans peine de la définition de la suite (1) qu'il existe une infinité, de nombres naturels n, tels que $\delta_n \subset \delta$: or, comme $\beta < \alpha$ et $\alpha = \lim\limits_{n<\omega} \alpha_n$, il en résulte tout de suite qu'il existe un nombre naturel n, tel que $\delta_n \subset \delta$ et $\alpha_n > \beta$.

L'ensemble A étant, par hypothèse, dans δ de classe β, l'ensemble $\delta A P_n = E_n$ est de classe $\leqslant \beta < \alpha_n$, contrairement à la définition de E_n. L'ensemble A n'est pas donc, dans aucun intervalle, de classe $< \alpha$.

Nous avons ainsi démontré que l'ensemble A est toujours ambigu de classe α et qu'il n'est dans aucun intervalle de classe $< \alpha$. Notre assertion est ainsi démontrée.

Wacław Sierpiński

O pewnym zagadnieniu p. Zarankiewicza

Komunikat przedstawiony dnia 25 lutego 1949 r.

Streszczenie

Autor podaje dowód na to, że twierdzącą jest odpowiedź na następujące pytanie p. K. Zarankiewicza:

Czy dla każdej liczby porządkowej α, gdzie $1 < \alpha < \Omega$, istnieje zbiór liniowy będący jednocześnie klasy α addytywnej i multyplikatywnej, który w żadnym przedziale nie jest klasy $< \alpha$?

Wacław Sierpiński

Sur un exemple de M. Kunugi de la théorie des espaces abstraits

Communication présentée dans la séance du 25.II 1949.

M. Kinjiro Kunugi a donné dans son Mémoire „*La théorie des ensembles analytiques et les espaces abstraits*" paru en 1935 [1]) un exemple d'un espace (V), X, d'un espace métrique compact Y et d'un ensemble M fermé dans le produit combinatoire $X \times Y$ des espaces X et Y, dont la projection sur l'espace X n'est pas analytique dans X.

L'exemple de M. Kunugi n'est pas effectif et sa démonstration fait appel au théorème de M. Zermelo sur le bon ordre et au théorème de J. König sur la puissance du continu ($2^{\aleph_0} \neq \aleph_\omega$). Le but de cette Note est de donner un exemple effectif de même nature avec une démonstration tout à fait élémentaire (sans utiliser les nombres transfinis).

Nous pouvons, comme on sait, établir effectivement une correspondance biunivoque entre tous les ensembles $\{x, y\}$ formés de deux nombres réels différents, où $0 \leqslant x \leqslant 1$ et $0 \leqslant y \leqslant 1$, et entre tous les nombres réels t, tels que $1 < t < 2$: soit $t = f(x, y) = f(y, x)$ une telle correspondance. Il existe donc pour tout nombre réel t donné, $1 < t < 2$, un ensemble (unique) formé de deux nombres réels différents $x = \xi(t)$ et $y = \eta(t)$, $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, tel que $f(x, y) = f(y, x) = t$.

Soit X l'ensemble de tous les nombres réels x, où $0 \leqslant x < 2$ et définissons les voisinages dans l'espace X comme il suit.

A tout nombre x_0 de X, tel que $0 \leqslant x_0 \leqslant 1$, faisons correspondre un seul voisinage $U(x_0) = \{x_0\}$ (où $\{x_0\}$ désigne l'ensemble formé d'un seul point x_0).

A tout nombre x_0 de X, tel que $1 < x_0 < 2$, faisons correspondre deux (et seulement deux) voisinages: $U_1(x_0) = \{x_0, \xi(x_0)\}$ et $U_2(x_0) = \{x_0, \eta(x_0)\}$.

[1]) Journ of the Fac. of Science Hokkaido Imp. Univ. Ser. I, Vol. IV, No 1 (Sapporo 1935), p. 9—13.

Au nombre 2 de X faisons correspondre comme voisinage chaque ensemble $T+\{x\}$, où T est l'ensemble de tous les nombres réels t, tels que $1 < t \leqslant 2$ et où x est un nombre réel (variable), tel que $0 \leqslant x \leqslant 1$.

Avec une telle définition de voisinages l'ensemble X devient un espace (V) de M. Fréchet.

Comme l'espace Y prenons l'intervalle $0 \leqslant y \leqslant 1$ avec la définition ordinaire de voisinages.

Le produit combinatoire $Z = X \times Y$ (c. à. d. l'ensemble de tous les systèmes (x, y), où $x \epsilon X$ et $y \epsilon Y$) devient un espace (V) si l'on convient de regarder comme voisinage du point (x_0, y_0) de Z tout produit combinatoire $U(x_0) \times U(y_0)$, où $U(x_0)$ est un voisinage quelconque de x_0 dans X et $V(y_0)$ est un voisinage quelconque de y_0 dans Y.

Soit maintenant M l'ensemble de tous les points (x, y) de Z, tels que $0 \leqslant x \leqslant 1$ et $y = x$.

Je dis que l'ensemble M est fermé dans Z, ou, ce qui revient au même, que l'ensemble $N = Z - M$ est ouvert dans Z. Pour le démontrer il suffira de prouver qu'il existe pour tout point de N un voisinage de ce point contenu dans N.

Soit donc (x_0, y_0) un point donné de N. Distinguons trois cas.

1°. $0 \leqslant x_0 \leqslant 1$. Comme $(x_0, y_0) \epsilon N$, on a $(x_0, y_0) \operatorname{non} \epsilon M$, donc $y_0 \neq x_0$ et, vu la définition des voisinages dans Y, il existe un voisinage $V(y_0)$ de y_0 (dans Y) tel que $x_0 \operatorname{non} \epsilon V(y_0)$. Or, le point x_0 a dans X un seul voisinage $U(x_0) = \{x_0\}$. Posons $W = U(x_0) \times V(y_0)$: ce sera un voisinage du point (x_0, y_0) dans Z. Je dis que $WM = 0$. Admettons, en effet, que ce n'est pas le cas et qu'il existe un point (x, y) de Z, tel que $(x, y) \epsilon WM$. On aurait donc $(x, y) \epsilon M$, donc $y = x$ et $(x, x) \epsilon W$, ce qui donne $x \epsilon U(x_0) = \{x_0\}$, donc $x = x_0$, et $x \epsilon V(y_0)$, donc $x_0 \epsilon V(y_0)$, contrairement à la définition du voisinage $V(y_0)$. On a donc $WM = 0$, d'où $W \subset Z - M = N$.

2°. $1 < x_0 \leqslant 2$. Comme $y_0 \epsilon Y$, donc $y_0 \leqslant 1$, on a $x_0 \neq y_0$. Or, comme $\xi(x_0) \neq \eta(x_0)$ (d'après la définition des fonctions ξ et η), on a soit $\xi(x_0) \neq y_0$, soit $\eta(x_0) \neq y_0$ (soit toutes les deux inégalités à la fois). Supposons p. e. que $\xi(x_0) \neq y_0$.

Soit $U_1(x_0) = (x_0, \xi(x_0))$. Or, comme $x_0 \neq y_0$ et $\xi(x_0) \neq y_0$, il existe un voisinage $V(y_0)$ de y_0 (dans Y), tel que $x_0 \operatorname{non} \epsilon V(y_0)$ et $\xi(x_0) \operatorname{non} \epsilon V(y_0)$.

Posons $W = U_1(x_0) \times V(y_0)$: ce sera un voisinage de (x_0, y_0) dans Z. Je dis que $WM = 0$. Admettons en effet que $(x, y) \epsilon WM$. On a donc $y = x$, donc $x \epsilon U_1(x_0)$ et $x \epsilon V(y_0)$. Vu que x_0 non $\epsilon V(y_0)$, la deuxième formule donne $x \neq x_0$ et parsuite (vu que $U_1(x_0) = (x_0, \xi(x_0))$), la première donne $x = \xi(x_0)$, donc $\xi(x_0) \epsilon V(y_0)$, contrairement à la définition de $V(y_0)$. Pareillement, si $\eta(x_0) \neq y_0$, on aboutit à une contradiction. On a donc toujours $WM = 0$, d'où $W \subset N$.

3°. $x_0 = 2$. Soit x_1 un nombre réel tel que $0 \leqslant x_1 \leqslant 1$ et $x_1 \neq y_0$. Posons $U(x_0) = T + \{x_1\}$: ce sera un voisinage de x_0 dans X. Comme $x_1 \neq y_0$, il existe un voisinage $V(y_0)$ de y_0 (dans Y), tel que x_1 non $\epsilon V(y_0)$.

Posons $W = U(x_0) \times V(y_0)$: ce sera un voisinage de (x_0, y_0) dans Z. Je dis que $WM = 0$. Admettons, en effet, que $(x, y) \epsilon WM$. On a donc $x \epsilon U(x_0)$, $y \epsilon V(y_0)$, $0 \leqslant x \leqslant 1$, $y = x$. Comme $x \epsilon U(x_0) = T + \{x_1\}$ et $0 \leqslant x \leqslant 1$, on trouve, vu la définition de T, $x = x_1$. Or, $x = y \epsilon V(y_0)$: on aurait donc $x_1 \epsilon V(y_0)$, contrairement à la définition de $V(y_0)$. On a donc encore $WM = 0$ et $W \subset N$.

Nous avons ainsi démontré que l'ensemble M est fermé dans Z. Or, je dis que la projection P de M sur X n'est pas un ensemble analytique dans X. Pour le démontrer, je prouverai d'abord que tout sous-ensemble de X fermé dans X et contenant plus qu'un point, contient le point 2.

Soit, en effet, F un sous-ensemble de X fermé dans X et contenant au moins deux points différents x_1 et x_2. Si $x_1 \epsilon T$, x_1 appartient à tout voisinage de 2 dans X, d'où, F étant fermé, on trouve $2 \epsilon F$. Pareillement, si $x_2 \epsilon T$, on trouve $2 \epsilon F$. Il nous reste donc le cas, où x_1 non ϵT et x_2 non ϵT, ce qui donne $x_1 \epsilon X - T$ et $x_2 \epsilon X - T$, donc (vu la définition de X et de T) $0 \leqslant x_1 \leqslant 1$ et $0 \leqslant x_2 \leqslant 1$. Comme $x_1 \neq x_2$, $t = f(x_1, x_2)$ est un nombre bien déterminé, tel que $1 < t < 2$, et nous pouvons poser $x_1 = \xi(t)$ et $x_2 = \eta(t)$. Les ensembles (t, x_1) et (t, x_2) sont donc (les seuls) deux voisinages de t dans X. Comme $x_1 \epsilon F$, $x_2 \epsilon F$ et F est fermé dans X, il en résulte que $t \epsilon F$ (puisque tout voisinage de t contient au moins un point de F).

Or, comme $1 < t < 2$, on a $t \epsilon T$ et t appartient à tout voisinage de 2 dans X, d'où, vu que $t \epsilon F$ et que F est fermé dans X, on trouve $2 \epsilon F$, c. q. f. d.

Admettons maintenant que P est un ensemble analytique dans X, soit $P = \sum_{n_1, n_2, \ldots} F_{n_1} F_{n_1, n_2} F_{n_1, n_2, n_3} \ldots$, où $\{F_{n_1, n_2, \ldots, n_k}\}$ est un système déterminant formé d'ensembles fermés dans X. Comme P, en tant que projection de M sur X, est l'ensemble de tous les points x de X, tels que $0 < x < 1$, P est indénombrable. Il existe donc, comme on sait, une suite infinie de nombres naturels $n_1, n_2, \ldots$, telle que chacun des ensembles $F_{n_1, n_2, \ldots, n_k}$, où $k = 1, 2, \ldots$, est indénombrable. Comme fermés dans X, ces ensembles contiennent donc, comme nous avons démontré plus haut, le point 2. On a donc $2 \in F_{n_1, n_2, \ldots, n_k}$ pour $k = 1, 2, \ldots$, d'où $2 \in P$, ce qui est impossible.

L'ensemble P n'est donc pas analytique dans X, c. q. f. d.

Pareillement on démontre que l'ensemble P n'est pas un résultat d'une opération (quelconque) de F. Hausdorff effectuée sur les ensembles fermés de l'espace X, c. à. d. qu'il n'existe pas une suite infinie $F_1, F_2, \ldots$ d'ensembles fermés dans X et un ensemble N de suites infinies de nombres naturels $n_1, n_2, \ldots$, tels qu'on ait $P = \sum_N F_{n_1} F_{n_2} F_{n_3} \ldots$, la sommation s'étendant à toutes les suites $n_1, n_2, \ldots$ qui forment l'ensemble N. En effet, dans ce cas il existerait, comme on voit sans peine, une suite $n_1, n_2, \ldots$ de N pour laquelle tous les ensembles F_{n_k} $(k = 1, 2, \ldots)$ serait indénombrables [1]), donc contiendraient le point 2 et on aurait $2 \in P$, ce qui est impossible.

Or, comme on voit sans peine, l'ensemble P est ouvert dans X, donc P est un complémentaire analytique dans X. Or, on peut modifier légèrement la définition des voisinages dans X sans altérer la propriété de M d'être fermé dans Z, de sorte que P ne soit ni analytique ni complémentaire analytique dans l'espace X' ainsi modifié. A ce but il suffit, comme on le vérifie sans peine, de modifier seulement la définition de voisinage du point 0 dans X, en considérant comme un seul voisinage de 0 dans X' l'ensemble formé du nombre 0 et de tous les nombres x de X', tels que $1 < x < 2$ [2]).

[1]) Puisque dans le cas contraire il existerait pour toute suite $\nu = (n_1, n_2, \ldots)$ de N un nombre naturel $k(\nu)$, tel que $\overline{\overline{F_{n_{k(\nu)}}}} \leqslant \aleph_0$, et, vu que $P \subset \sum_\nu F_{n_{k(\nu)}}$, on trouverait $\overline{\overline{P}} \leqslant \aleph_0$, tandis que $\overline{\overline{P}} > \aleph_0$.

[2]) Après cette modification l'ensemble P reste évidemment non analytique dans X'.

Je dis que l'ensemble P n'est pas un complémentaire analytique dans X', et même que l'ensemble $X'-P$ n'est pas un résultat d'une opération de Hausdorff effectuée sur les ensembles fermés dans X'. Cela résulte tout de suite du fait que tout ensemble fermé dans X' et contenant au moins un point de $X'-P$ contient le point 0 de X' (puisque tout voisinage de 0 dans X' contient l'ensemble $X'-P$) et que $0 \text{ non} \in X'-P$.

Il est encore à remarquer qu'il est facile de donner un exemple d'un espace métrique compact X, d'un espace métrique séparable Y et d'un ensemble M fermé dans $X \times Y$ dont la projection sur l'espace X n'est pas analytique dans X. Tels sont, comme on voit sans peine, l'espace X formé de tous les nombres réels de l'intervalle $0 \leqslant x \leqslant 1$ avec la définition ordinaire de voisinages, l'espace Y formé de points d'un ensemble linéaire non analytique situé dans l'intervalle (0,1) avec la définition ordinaire de voisinages et l'ensemble M formé de tous les points (x, y) de l'espace $X \times Y$, tels que $y \in Y$ et $x = y$.

Or, si X est un espace métrique quelconque et Y en espace métrique compact, la projection sur X d'un ensemble analytique dans $X \times Y$ est toujours un ensemble analytique dans X [1]) et la projection sur X d'un ensemble fermé dans $X \times Y$ est fermée dans X [2]).

Wacław Sierpiński

O pewnym przykładzie p. Kunugi z teorii przestrzeni abstrakcyjnych

Komunikat przedstawiony dn. 25 lutego 1949 r.

Streszczenie

Autor podaje przykład efektywny, wraz z dowodem elementarnym, przestrzeni (V), X, przestrzeni metrycznej zwartej Y oraz zbioru M zamkniętego w $X \times Y$, którego rzut na przestrzeń X nie jest analityczny w X.

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[1]) Voir le mémoire cité de M. Kunugi, p. 8 (Théorème 5).

[2]) l. c., p. 4 (Théorème 2).

Posiedzenie

z dnia 29 kwietnia 1949 r.

Tadeusz Banachiewicz

Sur l'interpolation dans le cas des intervalles inégaux

Note présentée à la séance du 29 avril 1949.

La formule d'interpolation de Lagrange donne le polynôme entier en x, qui, pour les valeurs données de x, par exemple x_1, x_2, x_3, x_4, prend des valeurs données à l'avance $f(x_1)$, $f(x_2)$, $f(x_3)$, $f(x_4)$. Nous allons donner ici une forme nouvelle de la résolution du même problème en supposant le polynôme interpolateur $F(x)$, d'après Newton, en la forme

$$F(x) = a + b(x - x_1) + c(x - x_1)(x - x_2) + \\ + d(x - x_1)(x - x_2)(x - x_3). \quad (1)$$

La solution qui suit est immédiate et permet donc de se passer de la théorie des *fonctions interpolatrices.*

En posant dans (1) successivement $x = x_1$, $x = x_2$, $x = x_3$, $x = x_4$, et en résolvant les équations linéaires, on obtient:

$$a = f(x_1)$$

$$\begin{Bmatrix} b \\ c \\ d \end{Bmatrix} = \begin{Bmatrix} f(x_2) - a \\ f(x_3) - a \\ f(x_4) - a \end{Bmatrix} : \Delta \quad (2)$$

où l'on a posé

$$\Delta = \begin{Bmatrix} x_2 - x_1 & 0 & 0 \\ x_3 - x_1 & (x_3 - x_1)(x_3 - x_2) & 0 \\ x_4 - x_1 & (x_4 - x_1)(x_4 - x_2) & (x_4 - x_1)(x_4 - x_2)(x_4 - x_3) \end{Bmatrix} \quad (3)$$

L'élément général, dans la i-ème colonne et j-ème ligne de ce cracovien est

$$\Delta_{i,j} = (x_{j+1} - x_1)(x_{j+1} - x_2) \cdots (x_{j+1} - x_i) \tag{4}$$

pour un nombre arbitraire dés x_k donnés.

Cette solution paraît plus simple que celle indiquée par Whittaker, Calculus of observations, London 1926, Chapter II.

La formule (2) donne les coefficients cherchés b, c, d successivement. Mais on peut les obtenir aussi indépendamment les uns des autres, en écrivant (2) de la façon

$$\begin{Bmatrix} b \\ c \\ d \end{Bmatrix} = \begin{Bmatrix} f(x_2) - a \\ f(x_3) - a \\ f(x_4) - a \end{Bmatrix} \cdot \Delta^{-1} \tag{5}$$

Si l'on compare (5) aux formules de Gauss (Theoria interpolationis, Werke, 3, 1866 p. 274; voir aussi Tisserand, Mécanique céleste 4, 1896, p. 153, formule 3), la simplicité de cette solution saute aux yeux. Elle n'est d'ailleurs due, comme celle de la formule (2), qu'à l'emploi du calcul cracovien, rendant diverses relations numériquement explicites.

Exemple

Supposons donnés $f(1) = -18$, $f(2) = -14$, $f(4) = 54$, $f(5) = 130$.

On aura $a = -18$, et comme $-14-(-18) = 4$, $54+18 = 72$, $130+18 = 148$

$$\begin{Bmatrix} b \\ c \\ d \end{Bmatrix} = \begin{Bmatrix} 4 \\ 72 \\ 148 \end{Bmatrix} : \begin{Bmatrix} 1 & 0 & 0 \\ 3 & 6 & 0 \\ 4 & 12 & 12 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 10 \\ 1 \end{Bmatrix}$$

et l'on obtient la formule d'interpolation

$$F(x) = -18 + 4(x-1) + 10(x-1)(x-2) + (x-1)(x-2)(x-4).$$

La solution indéfinie suivant (5) serait

$$\begin{Bmatrix} b \\ c \\ d \end{Bmatrix} = \begin{Bmatrix} f(x_2) - f(x_1) \\ f(x_3) - f(x_1) \\ f(x_4) - f(x_1) \end{Bmatrix} \cdot \begin{Bmatrix} 1 & -{}^1/_2 & {}^1/_6 \\ 0 & {}^1/_6 & -{}^1/_6 \\ 0 & 0 & {}^1/_{12} \end{Bmatrix}$$

conduisant aux mêmes valeurs des coefficients.

Tadeusz Banachiewicz

O interpolacji w przypadku niejednakowych odstępów argumentu

Komunikat przedstawiony na posiedzeniu w dniu 29 kwietnia 1949 r.

Streszczenie

Autor daje wzory krakowianowe na współczynniki interpolacyjne wzoru Newtona dla przypadku nierównych odstępów argumentu, wypływające bezpośrednio z równań problemu i pozwalające się obejść bez teorii różnic dzielonych.

Tadeusz Banachiewicz

O rozwiązaniu nieoznaczonym równań normalnych metody najmniejszych kwadratów

Autor rękopisu nie nadesłał.

Roman Kozłowski

Odkrycie przedstawiciela grupy Pterobranchia, rodzaju *Rhabdopleura*, w górnej kredzie Polski

Komunikat przedstawiony dnia 29 kwietnia 1949 r.

Badając próbki skał, pochodzące z wiercenia badawczego wykonanego na zlecenie Państwowego Instytutu Geologicznego w miejscowości Góra Puławska, koło Puław i przebijającego skały piętra duńskiego i najwyższej części piętra mastrychckiego, autor stwierdził obecność w nich szczątków chitynowych rodzaju *Rhabdopleura*, dotyczczas w stanie kopalnym nie opisywanego. Wstępne badania tych szczątków wskazują, iż ma się do czynienia z formą jeżeli nie identyczną, to bardzo zbliżoną do dziś żyjącego w oceanach i kosmopolitycznego gatunku *R. normani*. Dowodzi to niezwykłego konserwatyzmu biologicznego tej formy, wyjątkowo ciekawej również tak ze względu na jej pokrewieństwo z grupą strunowców *(Chordata)* z jednej strony, jak z wygasłą grupą graptolitów *(Graptolithina)* — z drugiej.

Dziwne jest, że *Rhabdopleura*, dotychczas w stanie kopalnym nie notowana, jest jednym z najpospolitszych organizmów w górnej kredzie Góry Puławskiej, gdyż obecność jej szczątków została tam stwierdzona po przez całą miąższość 80 m przebitych skał. Należy się spodziewać, iż wykryta ona zostanie również w górnej kredzie — a zapewne i w starszych utworach — innych miejscowości Polski, jak też poza Polską.

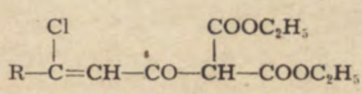
Podana tu wiadomość ma charakter komunikatu wstępnego, gdyż szczegółowe badania są dopiero w toku.

Zdzisław Macierewicz i Stefania Janiszewska-Brożek

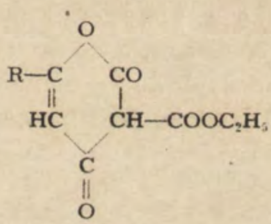
O metoksylowaniu pyrononów I. Synteza fenylopyrononu

Komunikat przedstawiony dnia 29 kwietnia 1949 r.

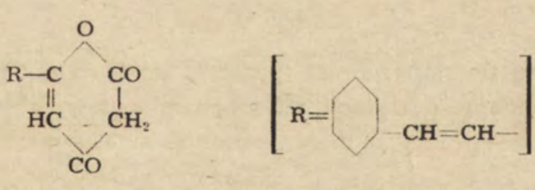
W pracy ogłoszonej przez jednego z nas w roku 1939 [1]) została przedstawiona ogólna zynteza układu pyrononu, mająca jako substrat chlorowaną acylową pochodną estru malonowego.



Ester ten pod wpływem kwasu siarkowego ulegał cyklizacji, dając karboetoksylową pochodną pyrononu:



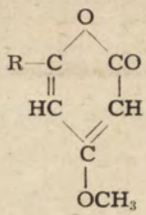
Zmydlenie i dekarboksylacja tego połączenia doprowadziły do otrzymania pyrononu, podstawionego w położeniu 6:

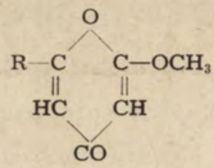


Styrylopyronon poddano metoksylowaniu.

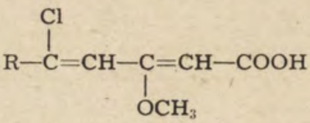
Ze względu na możliwość powstania dwóch izomerycznych metoksypochodnych, pochodzących od dwóch izomerycznych form enolowych:

[1]) Z. Macierewicz, Sprawozdania Towarzystwa Naukowego Warszawskiego. Rok XXXII, 37 (1939).

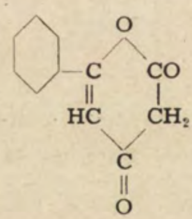
a) 

b) 

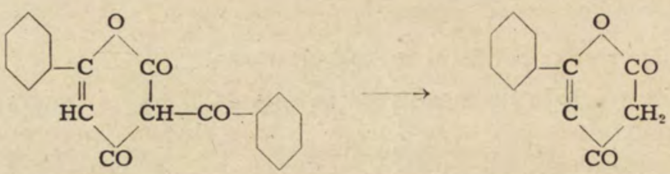
ustalono miejsce grupy eterowej na podstawie identyfikacji z produktem otrzymanym przez cyklizację kwasu:



Identyczność obu produktów wskazuje, że metoksylowanie styrylopyrononu prowadzi do γ-metoksy-α-pyronu. Chcąc sprawdzić czy rodzaj podstawnika w pierścieniu pyrononowym ma jakiś wpływ na kierunek enolizacji, a co za tym idzie, na miejsce wchodzącej grupy metoksylowej postanowiliśmy zbudować fenylopyronon i zbadać produkt jego metoksylowania:



Syntezę fenylopyrononu przeprowadził poraz pierwszy F. Arndt, poddając destylacji ester benzoilooctowy i odszczepiając z pierwotnego produktu reakcji rodnik kwasu benzoesowego [2]).



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[2]) F. Arndt, B. Eistert, H. Scholz, E. Aron: B. 69, 2373 (1936).

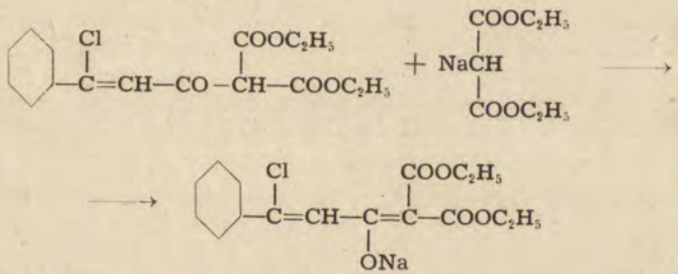
Ogólny plan naszej syntezy był identyczny z opisanym w cytowanej powyżej pracy z roku 1939. Przebieg syntezy ilustrują podane na tablicy wzory.

Chlorek kwasu β-chlorocynamonowego, kondensowany z malonianem etylowym, daje dwa produkty: I ester β-chlorocynamoilomalonowy i II ester dwu-β-chlorocynamoilomalonowy.

Budowa tego ostatniego połączenia została ustalona na podstawie rozpoznania produktów rozpadu pod wpływem działania alkoholowego roztworu ługu. Otrzymano jedną cząsteczkę kwasu β-chlorocynamonowego i ester β-chlorocynamoilomalonowy (I).

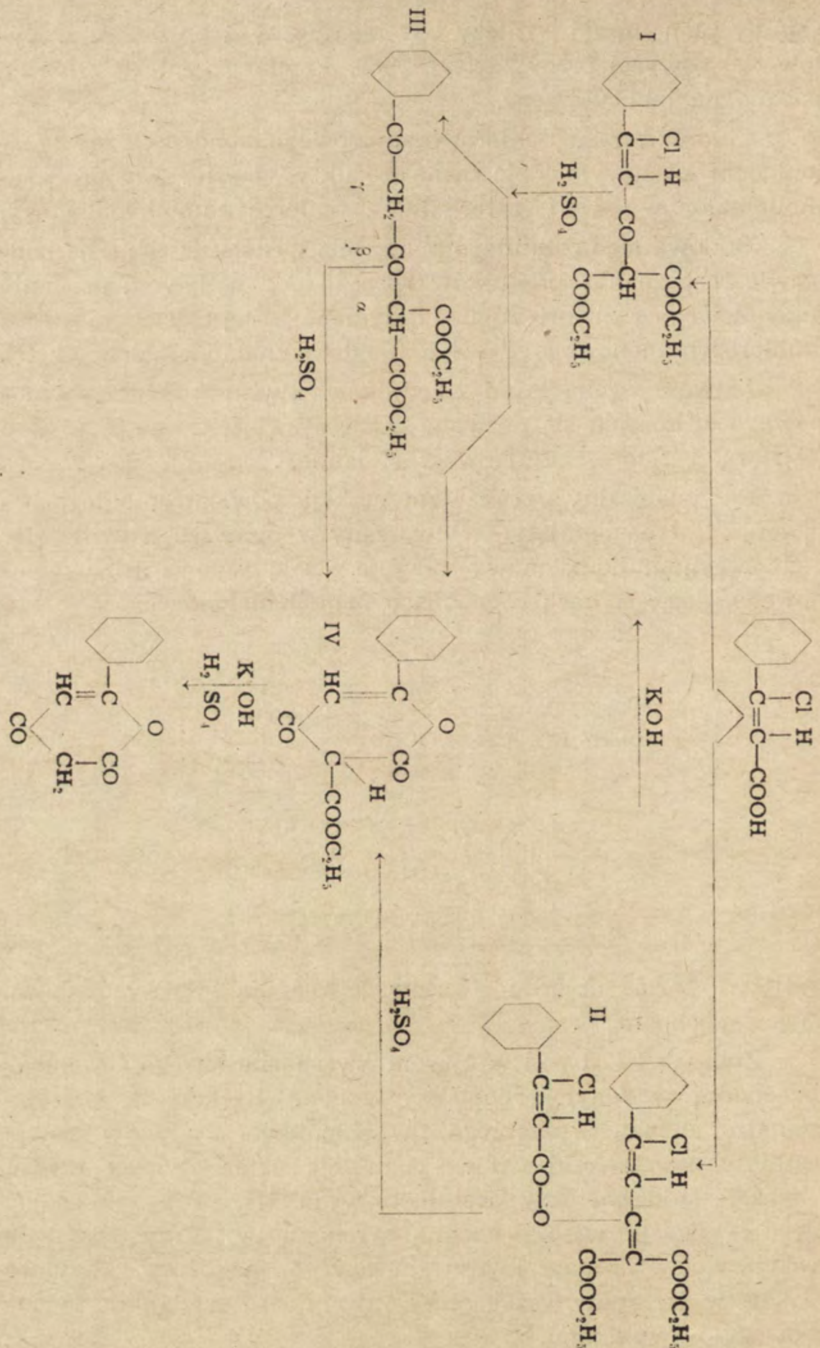
Łatwość z jaką jeden z rodników kwasu β-chlorocynamonowego odszczepia się pozwala sądzić o związaniu go przez tlen (estrowo) jak to wskazuje wzór II tablicy.

Przypuszczalny mechanizm reakcji powstawania tego połączenia jest następujący: wytwarzany w pierwszej chwili ester β-chlorocynamoilomalonowy reaguje z solą sodową estru malonowego, dając sodową pochodną cynamoilomalonową.



Następne porcje chlorku kwasowego reagują z tą solą sodową, dając o-pochodną II.

Związek I i II pod wpływem kwasu siarkowego stężonego, przechodzą w fenylokarboetoksypyronon. Reakcja ta, prześledzona na związku I, przebiega dwustopniowo. Najpierw zostaje usunięty z cząsteczki chlor i powstaje ester etylowy kwasu γ-benzoilo-α-karboetoksy acetylooctowego III. Ten zaś z kolei ulega cyklizacji, dając pochodną pyrononu IV. Przy użyciu do reakcji rozcieńczonego kwasu siarkowego, mogliśmy zatrzymać reakcję w fazie pierwszej, otrzymując dobrą wydajność łańcuchowego β-dwuketonu.



Związki tego typu — γ-acylo pochodne estru acetylooctowego — są do tej pory bardzo mało zbadane i są interesujące ze względu na obecność w cząsteczce dwóch układów enolowych:

$$— CO — CH_2 — CO — CH_2 —$$

Karboetoksypyronon IV, poddany działaniu alkoholowo-wodnego ługu lub rozcieńczonego gorącego kwasu siarkowego, wytwarza fenylopyronon.

Substancję tę zamierzamy poddać działaniu środków metylujących jak dwuazometan, siarczan dwumetylowy, jodek metylu. Miejsce zaś wchodzącej grupy metoksylowej będziemy sprawdzali przez cyklizację odpowiedniego metoksykwasu.

Całkowita treść pracy będzie opublikowana w Rocznikach Chemii.

Warszawa, Zakład Chemii Organicznej
Uniwersytetu Warszawskiego.

Posiedzenie

z dnia 27 maja 1949 r.

Wacław Sierpiński

Un théorème sur les fonctions d'ensemble

Note présentée dans la séance du 27 Mai 1949.

Soit n un nombre naturel, $E_1, E_2, \ldots, E_n$ — n ensembles donnés.

Formons tous les produits $E_{i_1} E_{i_2} \ldots E_{i_k}$ (pas nécessairement différents), où $i_1, i_2, \ldots, i_k$ est une suite croissante de nombres naturels $\leqslant n$. Il existe évidemment $2^n - 1$ tels produits. Divisons tous ces produits en deux classes, en rangeant dans la première classe tous ceux, pour lesquels k est un nombre impair, et dans la seconde tous ceux, pour lesquels k est un nombre pair. Comme on voit sans peine, la première classe contiendra

$$n + \binom{n}{3} + \binom{n}{5} + \ldots = 2^{n-1}$$

produits, et la seconde

$$\binom{n}{2} + \binom{n}{4} + \ldots = 2^{n-1} - 1$$

produits (dont la forme est distincte, mais pas nécessairement la valeur). Désignons ces produits respectivement par

(1) $$N_1, N_2, \ldots, N_{2^{n-1}} \quad \text{et} \quad P_1, P_2, \ldots, P_{2^{n-1}-1}.$$

Théorème. *Soit $E_1, E_2, \ldots, E_n$ une suite finie d'ensembles quelconques dont la somme est S. Soit $f(E)$ une fonction qui fait correspondre à tout ensemble $E \subset S$ un élément $f(E)$ d'un ensemble M dans lequel on a défini une addition de deux* *éléments de M qui est commutative et associative, et supposons que, A et B étant deux sous-ensembles disjoints de S, on a toujours $f(A+B)=f(A)+f(B)$. On a alors*

$$(2) \qquad f(E_1+E_2+\ldots+E_n)+f(P_1)+f(P_2)+\ldots+f(P_{2^{n-1}-1})=$$
$$=f(N_1)+f(N_2)+\ldots+f(N_{2^{n-1}}).$$

D é m o n s t r a t i o n. Notre théorème est évidemment vrai pour $n=1$.

Pour $n=2$ la formule (2) devient évidemment:

$$(3) \qquad f(E_1+E_2)+f(E_1E_2)=f(E_1)+f(E_2).$$

On a évidemment les décompositions du chacun des ensembles E_1 et E_2 en deux ensembles disjoints

$$E_1=(E_1-E_2)+E_1E_2 \quad \text{et} \quad E_2=(E_2-E_1)+E_1E_2$$

ainsi que la décomposition de l'ensemble E_1+E_2 en trois ensembles disjoints

$$E_1+E_2=(E_1-E_2)+(E_2-E_1)+E_1E_2.$$

Vu la propriété de la fonction f, on a donc

$$f(E_1)=f(E_1-E_2)+f(E_1E_2), \qquad f(E_2)=f(E_2-E_1)+f(E_1E_2),$$
$$f(E_1+E_2)=f(E_1-E_2)+f(E_2-E_1)+f(E_1E_2).$$

Les valeurs de la fonction f appartenant à l'ensemble M dans lequel l'addition est associative et commutative, ces formules donnent tout de suite la formule (3). Notre théorème est donc vrai pour $n=2$.

Soit maintenant n un nombre naturel $\geqslant 2$ et supposons que notre théorème est vrai pour n ensembles. Soient $E_1, E_2, \ldots, E_n, E_{n+1}$ $n+1$ ensembles donnés. On a donc la formule (2), où (1) sont des produits de la première, respectivement de la seconde classe pour la suite d'ensembles $E_1, E_2, \ldots, E_n$. On voit sans peine que pour la suite d'ensembles $E_1, E_2, \ldots, E_n, E_{n+1}$ tous les produits de la première classe seront

$$(4) \qquad N_1, N_2, \ldots, N_{2^{n-1}}, \quad P_1E_{n+1}, P_2E_{n+1}, \ldots, P_{2^{n-1}-1}E_{n+1} \quad \text{et} \quad E_{n+1},$$

et tous les produits de la seconde classe seront

(5) $P_1, P_2, ..., P_{2^{n-1}-1},\ N_1 E_{n+1},\ N_2 E_{n+1}, ..., N_{2^{n-1}} E_{n-1}$.

Notre théorème étant vrai pour deux ensembles, donc pour les ensembles $E_1+E_2+ ... +E_n$ et E_{n+1}, nous avons

$$f(E_1+E_2+ ... +E_n+E_{n+1}) + f(E_1 E_{n+1} + E_2 E_{n+1} + ... + E_n E_{n+1}) =$$
(6) $$= f(E_1+E_2+ ... +E_n) + f(E_{n+1}).$$

La formule (2) étant supposée vraie pour n ensembles quelconques, nous pouvons y remplacer les ensembles $E_1, E_2, ..., E_n$ respectivement par $E_1 E_{n+1}$, $E_2 E_{n+1}$, ..., $E_n E_{n+1}$: nous devrons alors évidemment remplacer les produits $P_1, P_2, ..., N_1, N_2, ...$ respectivement par $P_1 E_{n+1}$, $P_2 E_{n+1}$, ..., $N_1 E_{n+1}$, $N_2 E_{n+1}$, ..., ce qui donne

(7) $$f(E_1 E_{n+1} + E_2 E_{n+1} + ... + E_n E_{n+1}) +$$
$$+ f(P_1 E_{n+1}) + f(P_2 E_{n+1}) + ... + f(P_{2^{n-1}-1} E_{n+1}) =$$
$$= f(N_1 E_{n+1}) + f(N_2 E_{n+1}) + ... + f(N_{2^{n-1}} E_{n+1}).$$

D'après (6), (7) et (2) on trouve

(8) $$f(E_1+E_2+...+E_n+E_{n+1}) + f(P_1) + f(P_2) + ... + f(P_{2^{n-1}-1}) +$$
$$+ f(N_1 E_{n+1}) + f(N_2 E_{n+1}) + ... + f(N_{2^{n-1}} E_{n+1}) =$$
$$= f(N_1) + f(N_2) + ... + f(N_{2^{n-1}}) + f(P_1 E_{n+1}) +$$
$$+ f(P_2 E_{n+1}) + ... + f(P_{2^{n-1}-1} E_{n+1}) + f(E_{n+1}).$$

Vu que (4) sont tous les produits de la première classe et (5) tous les produits de la seconde classe pour la suite d'ensembles $E_1, E_2, ..., E_n, E_{n+1}$, la formule (8) prouve que notre théorème est vrai pour $n+1$ ensembles.

Notre théorème est ainsi démontré par l'induction.

En particulier, la formule (1) a lieu (pour chaque suite finie $E_1, E_2, ..., E_n$ d'ensembles), si $f(E)$ est une fonction d'ensemble, dont les valeurs sont des ensembles, pourvu qu'on ait toujours $f(A+B) = f(A) + f(B)$ pour $AB = 0$.

On obtient un autre cas particulier intéressant de notre théorème lorsque $f(E)$ désigne la puissance de l'ensemble E:

la formule (2) donne alors pour toute suite finie d'ensembles $E_1, E_2, ..., E_n$:

$$(9) \qquad \overline{\overline{E_1+E_2+...+E_n}} + \overline{\overline{P}}_1 + \overline{\overline{P}}_2 + ... + \overline{\overline{P}}_{2^{n-1}-1} =$$
$$= \overline{\overline{N}}_1 + \overline{\overline{N}}_2 + ... + \overline{\overline{N}}_{2^{n-1}}.$$

Il est à remarquer que nous avons démontré cette proposition sans faire appel à l'axiome du choix. (A l'aide de l'axiome du choix elle résulte sans peine des théorèmes connus sur les nombres cardinaux).

En particulier, pour $n=2$, la formule (9) donne

$$\overline{\overline{E_1+E_2}} + \overline{\overline{E_1 E_2}} = \overline{\overline{E}}_1 + \overline{\overline{E}}_2,$$

pour $n=3$:

$$\overline{\overline{E_1+E_2+E_3}} + \overline{\overline{E_1 E_2}} + \overline{\overline{E_1 E_3}} + \overline{\overline{E_2 E_3}} = \overline{\overline{E}}_1 + \overline{\overline{E}}_2 + \overline{\overline{E}}_3 + \overline{\overline{E_1 E_2 E_3}},$$

et, pour $n=4$:

$$\overline{\overline{E_1+E_2+E_3+E_4}} + \overline{\overline{E_1 E_2}} + \overline{\overline{E_1 E_3}} + \overline{\overline{E_1 E_4}} + \overline{\overline{E_2 E_3}} + \overline{\overline{E_2 E_4}} +$$
$$+ \overline{\overline{E_3 E_4}} + \overline{\overline{E_1 E_2 E_3 E_4}} = \overline{\overline{E}}_1 + \overline{\overline{E}}_2 + \overline{\overline{E}}_3 + \overline{\overline{E}}_4 + \overline{\overline{E_1 E_2 E_3}} +$$
$$+ \overline{\overline{E_1 E_2 E_4}} + \overline{\overline{E_1 E_3 E_4}} + \overline{\overline{E_2 E_3 E_4}}.$$

Il résulte tout de suite de la formule (9) que *si* $E_1, E_2, ..., E_n$ *est une suite finie d'ensembles finis, on a*

$$\overline{\overline{E_1+E_2+...+E_n}} = \overline{\overline{N}}_1 + \overline{\overline{N}}_2 + ... + \overline{\overline{N}}_{2^{n-1}} - \overline{\overline{P}}_1 - \overline{\overline{P}}_2 - ... - \overline{\overline{P}}_{2^{n-1}-1}$$

et, pour tout ensemble fini $E \supset E_1+E_2+...+E_n$:

$$\overline{\overline{E}} - \overline{\overline{E_1+E_2+...+E_n}} =$$
$$= \overline{\overline{E}} - \overline{\overline{N}}_1 - \overline{\overline{N}}_2 - ... - \overline{\overline{N}}_{2^{n-1}} + \overline{\overline{P}}_1 + \overline{\overline{P}}_2 + ... + \overline{\overline{P}}_{2^{n-1}-1}.$$

Comme on voit sans peine, on peut exprimer cette formule de la façon suivante:

Supposons donnés des objets quelconques en nombre fini m.

Soit m_α le nombre de ceux d'entre eux qui jouissent d'une propriété donnée α, $m_{\alpha\beta}$ le nombre de ceux qui jouissent à la fois des propriétés α et β, et ainsi de suite. Le nombre de tous ces de notres objets qui ne jouissent d'aucune des propriétés $\alpha_1, \alpha_2, \ldots, \alpha_n$ est

$$m - m_{\alpha_1} - m_{\alpha_2} - \ldots - m_{\alpha_n} + m_{\alpha_1 \alpha_2} +$$
$$m_{\alpha_1 \alpha_3} + \ldots + m_{\alpha_{n-1} \alpha_n} - m_{\alpha_1 \alpha_2 \alpha_3} - \ldots \pm m_{\alpha_1 \alpha_2 \ldots \alpha_n}$$ [1]).

Wacław Sierpiński

Pewne twierdzenie o funkcjach zbioru

Komunikat przedstawiony na posiedzeniu w dniu 27 maja 1949 r.

Streszczenie

Treścią komunikatu jest dowód następującego twierdzenia: Niech $E_1, E_2, \ldots, E_n$ oznacza ciąg skończony dowolnych zbiorów o sumie S, i niech $f(E)$ oznacza funkcję, przyporządkowywującą każdemu zbiorowi $E \subset S$ pewien element zbioru M, w którym określone jest dodawanie dwuch elementów, przemienne i łączne, i przypuśćmy, że dla każdych dwuch zbiorów rozłącznych A i B, zawartych w S, mamy $f(A+B) = f(A) + f(B)$. Zachodzi wówczas wzór

$$f\left(\sum_{i=1}^{n} E_i\right) + \sum f(E_{i_1} E_{i_2}) + \sum f(E_{i_1} E_{i_2} E_{i_3} E_{i_4}) + \ldots =$$
$$= \sum f(E_{i_1}) + \sum f(E_{i_1} E_{i_2} E_{i_3}) + \ldots,$$

gdzie $\sum f(E_{i_1} E_{i_2} \ldots E_{i_k})$ oznacza sumę, rozciągniętą na wszystkie układy k liczb naturalnych rosnących $i_1 < i_2 < \ldots < i_k$, gdzie $i_k \leq n$.

Autor podaje różne zastosowania tego twierdzenia.

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[1]) Cf. J. J. Sylvester, *C. R. Paris* **96** (1883), p. 463, U. Yule, *An introduction to the theory of statistics*, London, Griffin 1916, G. Pólya et G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, II, Berlin 1925, p. 119 (problème 21) et pp. 326—327. Ces derniers auteurs regardent cette proposition comme appartenant à la logique formelle.

Antoni Wakulicz

Sur les sommes de quatre nombres ordinaux

Note présentée par M. W. Sierpiński dans la séance du 27 Mai 1949.

M. W. Sierpiński a démontré récemment la proposition suivante:

Pour qu'il existe trois nombres ordinaux dont la somme, pour toutes les permutations de leur termes, donne k et seulement k valeurs distinctes, il faut et il suffit que k soit un nombre naturel $\leqslant 5$.

Dans le même ordre d'idées je démontrerai ici le

Théorème. *Pour qu'il existe quatre nombres ordinaux dont la somme, pour toutes les permutations de leur termes, donne k et seulement k valeurs distinctes, il faut et il suffit que k soit un nombre naturel* $\leqslant 13$.

Démonstration. α étant un nombre ordinal >0, nous désignerons par

$$\alpha = \omega^{\alpha'} a' + \omega^{\alpha''} a'' + \ldots + \omega^{\alpha^{(p)}} a^{(p)}$$

sont développement normal. p est ici un nombre naturel (dépendant de α) et $\alpha', \alpha'', \ldots, \alpha^{(p)}$ sont des nombres ordinaux (bien déterminés par α), tels que $\alpha \geqslant \alpha' > \alpha'' > \ldots > \alpha^{(p)} \geqslant 0$, et $a', a'', \ldots, a^{(p)}$ sont des nombres naturels. Nous appellerons le nombre α' *degré* du nombre α.

Comme on sait, si le nombre ordinal α est du degré supérieur à celui du nombre β, on a $\beta + \alpha = \alpha$.

Soient $\alpha_1, \alpha_2, \ldots, \alpha_s$ des nombres ordinaux donnés du même degré, et soit $l_1, l_2, \ldots, l_s$ une permutation quelconque de nombres $1, 2, \ldots, s$. On voit sans peine que

$$\alpha_{l_1} + \alpha_{l_2} + \ldots + \alpha_{l_s} = \omega^{\alpha_1'}(a_1' + a_2' + \ldots + a_s' - a_{l_s}') + \alpha_{l_s}:$$

la valeur de cette somme ne dépend donc que du nombre l_s (et pas des nombres $l_1, l_2, \ldots, l_{s-1}$). La somme $\alpha_1 + \alpha_2 + \ldots + \alpha_s$, pour toutes les permutations de leur termes, donne donc au plus s valeurs distinctes.

Soient maintenant α, β, γ et δ quatre nombres ordinaux donnés. Nous pouvons supposer que

$$\alpha' \geqslant \beta' \geqslant \gamma' \geqslant \delta',$$

et nous distinguerons 8 cas:

1. $\alpha' = \beta' = \gamma' = \delta'$. Les nombres α, β, γ et δ sont donc du même degré, et, d'après ce qui a été dit plus haut, la somme de ces quatre nombres, pour toutes les permutations de leur termes, donne au plus 4 valeurs distinctes.

2. $\alpha' > \beta' = \gamma' = \delta'$. Quatre cas peuvent se présentsr ici pour les permutations des termes α, β, γ et δ.

2_1. α est le dernier terme de la somme. Comme le degré du nombre α est supérieur à celui de chacun des nombres β, γ et δ, on a ici $\beta + \gamma + \delta + \alpha = \alpha$ et cette valeur reste la même pour une permutation quelconque de nombres β, γ et δ.

2_2. α est le troisième terme. La valeur de la somme ne dépend dans ce cas que du dernier terme qui peut être un des nombres β, γ et δ. On a donc dans le cas 2_2 (pour toutes les permutations des termes β, γ et δ) au plus 3 valeurs distinctes de la somme de nos quatre nombres ordinaux.

2_3. α est le deuxième terme. On a ici, peur toutes les permutations des nombres β, γ et δ, au plus 6 valeurs distinctes de la somme de nos quatre nombres.

2_4. α est le premier terme de la somme. Les nombres β, γ et δ étant du même degré, leur somme, lorsqu'on les prends dans un ordre quelconque, ne dépend que du dernier terme. On a donc dans le cas 2_4 au plus 3 valeurs distinctes de la somme de nos quatre nombres.

En rapprochant le cas 2_1, 2_2, 2_3 et 2_4, nous concluons que dans le cas 2 notre somme admet (pour toutes les permutations de leur termes) au plus $1 + 3 + 6 + 3 = 13$ valeurs distinctes.

3. $\alpha' = \beta' > \gamma' = \delta'$. Trois cas peuvent se présenter ici:

3_1. Aucun des termes γ et δ ne suit pas à la fois les termes α et β. La somme de nos quatre nombres est ici égale à $\alpha + \beta$ ou à $\beta + \alpha$: elle admet donc au plus 2 valeurs distinctes.

3_2. Seulement un des termes γ et δ suit à la fois les termes α et β. Il y a dans ce cas au plus 4 valeurs distinctes de la somme qui est égale à $\alpha + \beta + \gamma$, $\beta + \alpha + \gamma$, $\alpha + \beta + \delta$ ou $\beta + \alpha + \delta$.

3_3. Les termes γ et δ suivent à la fois les termes α et β: il y a dans ce cas au plus 4 valeurs distinctes de la somme qui est égale à $\alpha+\beta+\gamma+\delta$, $\alpha+\beta+\delta+\gamma$, $\beta+\alpha+\gamma+\delta$ ou $\beta+\alpha+\delta+\gamma$.

En rapprochant les cas 3_1, 3_2 et 3_2 on conclut que dans le cas 3 on a au plus $2+4+4=10$ valeurs distinctes de la somme de nos quatre nombres (pour toutes les permutations de leur termes).

4. $\alpha'=\beta'=\gamma'>\delta'$. Deux cas peuvent se présenter ici:

4_1. Le nombre δ n'est pas le dernier terme de la somme. Elle dépend alors du quatrième terme qui peut être un des nombres α, β et γ. On a donc au plus 3 valeurs distinctes de la somme.

4_2. δ est le dernier terme de la somme. Elle dépend alors de son troisième terme qui peut être un des nombres α, β et γ. On a donc au plus 3 valeurs distinctes de la somme.

Ainsi, en rapprochant les cas 4_1 et 4_2, on voit que dans le cas 4 on a au plus $3+3=6$ valeurs de la somme de nos quatre nombres (pour toutes les permutations de leur termes).

5. $\alpha'=\beta'>\gamma'>\delta'$. Trois cas peuvent se présenter ici:

5_1. Le dernier terme est α ou β. La somme de nos quatre nombres est dans ce cas égale à $\alpha+\beta$ ou à $\beta+\alpha$: elle admet donc au plus 2 valeurs distinctes.

5_2. Le troisième terme est α ou β et le quatrième est γ ou δ. La somme de nos quatre nombres est alors égale à $\alpha+\beta+\gamma$, $\alpha+\beta+\delta$, $\beta+\alpha+\gamma$ ou $\beta+\alpha+\delta$: alle admet donc au plus 4 valeurs distinctes.

5_3. Les termes α et β occupent les deux premières places. Comme $\gamma'>\delta'$, on a $\delta+\gamma=\gamma$ et la somme est égale à $\alpha+\beta+\gamma$, $\beta+\alpha+\gamma$, $\alpha+\beta+\gamma+\delta$ ou $\beta+\alpha+\gamma+\delta$, les deux premières valeurs étant comprises dans le cas 5_2.

En rapprochant les cas 5_1, 5_2 et 5_3 on trouve que dans le cas 5 on a au plus $2+4+2=8$ valeurs de la somme de nos quatre nombres (pour toutes les permutations de leur termes).

6. $\alpha'>\beta'=\gamma'>\delta'$. Quatre cas se présentent ici:

6_1. α est le dernier terme. La somme est alors égale à α.

6_2. α est le troisième terme. La somme de nos quatre nombres est alors égale à $\alpha+\beta$, $\alpha+\gamma$ ou $\alpha+\delta$: elle admet donc au plus 3 valeurs distinctes.

6_3. α est le deuxième terme. Vu que $\alpha' > \beta'$ et $\gamma' > \delta'$, on a $\beta+\alpha=\alpha$ et $\delta+\gamma=\gamma$, et la somme de nos quatre nombres est dans ce cas égale à $\alpha+\beta+\gamma$, $\alpha+\beta+\delta$, $\alpha+\gamma+\beta$, $\alpha+\gamma+\delta$, $\alpha+\beta$ ou $\alpha+\gamma$, où les deux dernières valeurs sont déjà comprises dans le cas 6_2.

6_4. α est le premier terme. La somme est alors égale à $\alpha+\beta+\gamma+\delta$, $\alpha+\gamma+\beta+\delta$, $\alpha+\beta+\gamma$ ou $\alpha+\gamma+\beta$, les deux dernières valeurs étant déjà comprises dans le cas 6_3.

En rapprochant les cas 6_1, 6_2, 6_3 et 6_4, on trouve que dans le cas 6 la somme de nos quatre nombres admet au plus $1+3+4+2=10$ valeurs distinctes (pour toutes les permutations de leur termes).

7. $\alpha' > \beta' > \gamma' = \delta'$. Quatre cas se présentent ici:

7_1. α est le dernier terme: la somme est alors égale à α.

7_2. α est le troisième terme: la somme est égale à $\alpha+\beta$, $\alpha+\gamma$ ou $\alpha+\delta$.

7_3. α est le deuxième terme. La somme est égale à $\alpha+\beta+\gamma$, $\alpha+\beta+\delta$, $\alpha+\gamma+\delta$, $\alpha+\delta+\gamma$ ou $\alpha+\beta$, cette dernière valeur étant comprise dans le cas 7_2.

7_4. α est le premier terme. La somme est égale à $\alpha+\beta+\gamma+\delta$, $\alpha+\beta+\delta+\gamma$, $\alpha+\beta+\delta$, $\alpha+\beta+\gamma$ ou $\alpha+\beta$, et les trois dernières valeurs sont comprises dans les cas 7_3 et 7_2.

En rapprochant les cas 7_1, 7_2, 7_3, et 7_4, on voit que la somme de nos quatre nombres admet dans le cas 7 au plus $1+3+4+2=10$ valeurs distinctes (pour toutes les permutations de leur termes).

8. $\alpha' > \beta' > \gamma' > \delta'$. La somme est dans ce cas égale $\alpha+\beta+\gamma+\delta$, $\alpha+\beta+\gamma$, $\alpha+\beta+\delta$, $\alpha+\gamma+\delta$, $\alpha+\beta$, $\alpha+\gamma$, $\alpha+\delta$ ou α: elle admet donc au plus 8 valeurs distinctes (pour toutes les permulations de leur termes).

En rapprochant les cas 1—8, on voit que la somme de nombres α, β, γ et δ, pour toutes les permutations de leur termes admet dans chacun de ces 8 cas au plus 13 valeurs distinctes.

Il est ainsi démontré que la condition de notre théorème est nécessaire. Pour démontrer qu'elle est suffisante, il reste à vérifier que les sommes de quatre nombres ordinaux

$$\sigma_1 = 1+2+3+4, \quad \sigma_2 = \omega+\omega+\omega+1, \quad \sigma_3 = \omega+\omega+1+1,$$

$$\sigma_4 = \omega+1+1+1, \quad \sigma_5 = \omega+1+1+2, \quad \sigma_6 = \omega+1+1+3,$$

$$\sigma_7 = \omega+1+2+3, \quad \sigma_8 = \omega+1+2+4, \quad \sigma_9 = \omega^2+\omega.2+\omega+(\omega+1),$$

$$\sigma_{10} = \omega^2+\omega+(\omega+1)+2, \quad \sigma_{11} = \omega^2+\omega.4+\omega.2+(\omega+1),$$

$$\sigma_{12} = \omega^2+\omega.4+(\omega+1)+(\omega+2), \quad \sigma_{13} = \omega^2+\omega.4+(\omega.2+1)+(\omega+2)$$

jouissent de la propriété suivante: k étant un nombre naturel quelconque $\leqslant 13$, la somme σ_k donne, pour toutes les permutations de leur termes, k et seulement k valeurs distinctes.

Ainsi, p. e. pour la somme σ_{13} on trouve 13 valeurs distinctes:

$$\omega^2+\omega.4+(\omega.2+1)+(\omega+2) = \omega^2+\omega.7+2,$$
$$\omega^2+\omega.4+(\omega+2)+(\omega.2+1) = \omega^2+\omega.7+1,$$
$$\omega^2+(\omega.2+1)+(\omega+2)+\omega.4 = \omega^2+\omega.7,$$
$$(\omega+2)+\omega^2+\omega.4+(\omega.2+1) = \omega^2+\omega.6+1,$$
$$(\omega+2)+\omega^2+(\omega.2+1)+\omega.4 = \omega^2+\omega.6,$$
$$(\omega.2+1)+\omega^2+\omega.4+(\omega+2) = \omega^2+\omega.5+2,$$
$$(\omega.2+1)+\omega^2+(\omega+2)+\omega.4 = \omega^2+\omega.5,$$
$$(\omega.2+1)+(\omega+2)+\omega^2+\omega.4 = \omega^2+\omega.4,$$
$$\omega.4+\omega^2+(\omega.2+1)+(\omega+2) = \omega^2+\omega.3+2,$$
$$\omega.4+\omega^2+(\omega+2)+(\omega.2+1) = \omega^2+\omega.3+1,$$
$$\omega.4+(\omega+2)+\omega^2+(\omega.2+1) = \omega^2+\omega.2+1,$$
$$\omega.4+(\omega.2+1)+\omega^2+(\omega+2) = \omega^2+\omega+2,$$
$$\omega.4+(\omega.2+1)+(\omega+2)+\omega^2 = \omega^2,$$

et, il résulte de ce que nous avons démontré plus haut qu'il n'y a pas plus.

Notre théorème est ainsi démontré.

L'étude des sommes d'un nombre fini quelconque de nombres ordinaux est beaucoup plus difficile et elle sera objet d'un travail spécial.

Antoni Wakulicz

O sumach czterech liczb porządkowych

Przedstawił W. Sierpiński na posiedzeniu w dniu 27 maja 1949 r.

Streszczenie

Treścią komunikatu jest dowód następującego twierdzenia:

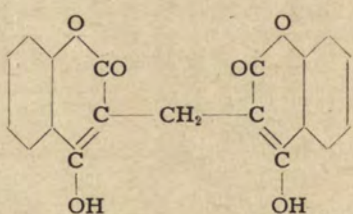
Na to żeby istniały 4 liczby porządkowe, których suma, przy wszelkich możliwych permutacjach składników przybiera k i tylko k różnych wartości, potrzeba i wystarcza aby k było liczbą naturalną $\leqslant 13$.

Irena Chmielewska

Cyklizacja zdolnych do enolizacji 1.5.dwuketonów

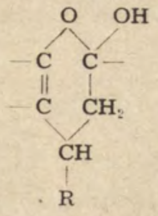
Komunikat przedstawiony dnia 27 maja 1949 r.

Do grupy zdolnych do enolizacja 1.5.dwuketonów należy 3.3'.metyleno.bis.4.hydroksy.kumaryna (I), zwana dikumarolem, czynnik wyodrębniony z zepsutej słodkiej koniczyny i wykazujący właściwości obniżania krzepnienia krwi



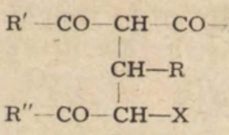
I.

W myśl mojej hipotezy czynność fizjologiczna dikumarolu i jego pochodnych nie jest związana, jak to przyjęto powszechnie, ze strukturą 4.hydroksy.kumaryny, lecz ze zdolnością do tworzenia cyklowego układu ketalowego o budowie α, β.dwuhydro. α.hydroksy.pyranu (II)



II.

W celu potwierdzenia słuszności powyższej hipotezy należało znaleźć układ, nie zawierający w swym składzie 4.hydroksy.kumaryny, a wykazujący zdolność do cyklizacji z wytworzeniem pierścienia heterocyklowego (II) i zbadać, czy posiada on biologiczną czynność dikumarolu. W tym celu otrzymałam szereg łańcuchowych, zdolnych do enolizacji 1.5.dwuketonów, o wzorze ogólnym (III)



III.

R = H, alkil, lub aryl
R′ i R″ = alkil, lub aryl
R‴ = alkil, lub grupa alkoksylowa
X = H, grupa acylowa, lub estrowa.

Badania biologiczne wykazały, że otrzymane związki nie wywierają wpływu na krzepliwość krwi. Fakt ten można wytłumaczyć tym, że nawet w łagodnych warunkach ulegają one cyklizacji z wytworzeniem pierścienia nie heterocyklowego, a cykloheksenowego.

P o s i e d z e n i e

z dnia 14 października 1949 r.

Wacław Sierpiński

Sur quelques propriétés des familles d'ensembles

Note présentée à la séance du 14 octobre 1949.

Théorème 1. *Si F est une famille quelconque d'ensembles dénombrables, telle que tout ensemble de la famille F est une somme de deux ensembles disjoints de la famille F, alors tout ensemble de la famille F est aussi une somme d'une infinité d'ensembles disjoints de la famille F.*

Démonstration. Soit F une famille d'ensembles dénombrables, telle que tout ensemble de F est une somme de deux ensembles disjoints de F, et soit E un ensemble quelconque de F.

L'ensemble E est donc dénombrable, soit $E=\{p_1,p_2,...\}$. D'après la propriété de la famille F, E est une somme de deux ensembles disjoints de F, soit $E=E_1+H_1$, et nous pouvons supposer que E_1 contient l'élément p_1. Pareillement, comme $H_1 \epsilon F$, on a $H_1=E_2+H_2$, où $E_2 \epsilon F$, $H_2 \epsilon F$ et $E_2 H_2=0$, et nous pouvons supposer que l'élément de H_1 dont l'indice est le plus petit appartient à E_2. En procédant ainsi de suite, nous obtenons deux suites infinies de sous-ensembles de E: $E_1, E_2, ...$ et $H_1, H_2, ...$, et on voit sans peine que $p_n \epsilon E_1+E_2+...+E_n$ pour $n=1,2,...$, d'où il résulte tout de suite que $E=E_1+E_2+...$ est une décomposition de l'ensemble E en ensembles disjoints de la famille F. Le théorème 1 se trouve ainsi démontré.

Remarques. Un exemple d'une famille F satisfaisant au théorème 1 est la famille de tous les ensembles infinis de nombres naturels. Cette famille est de puissance du continu.

Pour avoir une famille dénombrable F d'ensembles satisfaisant au théorème 1, il suffirait de prendre la famille F de tous les ensembles de nombres naturels dont les éléments forment une progression arithmétique infinie (c'est-à-dire la famille de tous les ensembles $\{a, a+b, a+2b, a+3b, \dots\}$, où a et b sont des nombres naturels).

Or, pour avoir une famille F d'ensembles indénombrables satisfaisant au théorème 1, il suffirait de prendre la famille F de tous les ensembles qui sont des sommes d'une infinité des intervalles $n-1 < x < n$, où $n = 1, 2, \dots$.

Théorème 2. *Il existe une famille dénombrable F d'ensembles telle que tout ensemble de la famille F est une somme de deux ensembles disjoints de la famille F, mais aucun ensemble de la famille F n'est pas une somme d'une infinité d'ensembles disjoints de la famille F.*

Démonstration. $a_1, a_2, \dots, a_n$ étant une suite finie formeé de nombres 0 ou 1, désignons par $E_{a_1 a_2 \dots a_n}$ l'ensemble de toutes les suites infinies $u_1, u_2, u_3, \dots$, formées de nombres 0 ou 1, où $u_i = a_i$ pour $i = 1, 2, \dots, n$. Soit F la famille de tous les ensembles $E_{a_1 a_2 \dots a_n}$ (où $a_1, a_2, \dots, a_n$ est une suite finie quelconque formée de nombres 0 ou 1). Je dis que la famille F satisfait aux conditions du théorème 2.

En effet, d'une part, on a évidemment pour tout ensemble $E_{a_1 a_2 \dots a_n}$ de F:

$$E_{a_1 a_1 \dots a_n} = E_{a_1 a_2 \dots a_n 0} + E_{a_1 a_2 \dots a_n 1}, \quad \text{et} \quad E_{a_1 a_2 \dots a_n 0} E_{a_1 a_2 \dots a_n 1} = 0:$$

tout ensemble de la famille F est ainsi une somme de deux ensembles disjoints de cette famille.

D'autre part, soit $E_{a_1 a_2 \dots a_n}$ un ensemble de la famille F et supposons que

$$(1) \qquad E_{a_1 a_2 \dots a_n} = H_1 + H_2 + \dots,$$

où H_k $(k = 1, 2, \dots)$ sont des ensembles disjoints de la famille F.

Il résulte tout de suite de la définition de la famille F, de (1) et de $H_k \epsilon F$ pour $k = 1, 2, \dots$, qu'on a, pour $k = 1, 2, \dots$:

$$H_k = E_{b_1 b_2 \dots b_{q_k}}, \quad \text{où} \quad q_k > n \quad \text{et} \quad b_i = a_i \quad \text{pour} \quad i = 1, 2, \dots, n,$$

soit

$$(2) \qquad H_k = E_{a_1 a_2 \dots a_n a_{n+1}^{(k)} a_{n+2}^{(k)} \dots a_{q_k}^{(k)}}.$$

Il n'existe, pour tout p naturel, qu'un nombre fini (ou nul) d'indices k tels que $q_k \leqslant p$: en effet, s'il y en avait une infinité, il existerait dans la suite $H_1, H_2, \ldots$ des termes égaux, ce qui est, comme nous savons, impossible.

Nous définirons maintenant une suite infinie $c_1, c_2, \ldots$ de nombres 0 ou 1 comme il suit.

S'il y a une infinité d'indices k tels que $q_k > n$ et $a_{n+1}^{(k)} = 0$, posons $c_1 = 0$; sinon, alors il existe une infinité d'indices k tels que $q_k > n$ et $a_{n+1}^{(k)} = 1$, et nous poserons $c_1 = 1$.

Supposons maintenant que p est un nombre naturel > 1 et que nous avons déjà défini tous les nombres $c_1, c_2, \ldots, c_{p-1}$ et qu'il existe une infinité des indices k tels que $q_k \geqslant n+p-1$ et $a_{n+i}^{(k)} = c_i$ pour $i = 1, 2, \ldots, p-1$ (ce qui est vrai pour $p = 2$).

Parmi ces indices k il n'existe, comme nous savons, qu'un nombre fini (ou nul) de tels que $q_k < n+p$, donc il en existe une infinité de tels que $q_k \geqslant n+p$. S'il en existe une infinité de tels que $a_{n+p}^{(k)} = 0$, posons $c_p = 0$, sinon, il en existe une infinité de tels que $a_{n+p}^{(k)} = 1$ et nous poserons $c_p = 1$.

Les nombres $c_1, c_2, \ldots$ sont ainsi définis par l'induction.

Soit maintenant s la suite infinie $a_1, a_2, \ldots, a_n, c_1, c_2, \ldots$: on a évidemment $s \in E_{a_1 a_2 \ldots a_n}$; d'après (1) il existe donc un indice k tel que $s \in H_k$ et, d'après (2), $s \in E_{a_1 a_2 \ldots a_n}$ et la définition des ensembles $E_{a_1 a_2 \ldots a_n}$, on trouve $a_{n+1}^{(k)} = c_1$, $a_{n+2}^{(k)} = c_2, \ldots,$ $a_{q_k}^{(k)} = c_{q_k - n}$. Or, vu la définition des nombres $c_1, c_2, \ldots,$ il existe une infinité des indices l tels que $l \geqslant q_k - n$ et $a_{n+i}^{(l)} = c_i$ pour $i = 1, 2, \ldots, q_k - n$: on aurait donc, pour ces indices l, $s \in H_l$, ce qui est impossible, les ensembles $H_1, H_2, \ldots$ étant disjoints.

L'hypothèse que l'ensemble $E_{a_1 a_2 \ldots a_n}$ est une somme d'une infinité d'ensembles disjoints de la famille F implique donc une contradiction. Le théorème 2 se trouve ainsi démontré.

Or, M. S. Mazur m'a posé le problème s'il existe un corps d'ensembles, K, tel que tout ensemble non vide de K est une somme de deux ensembles disjoints non vides de K, mais aucun ensemble de K n'est pas une somme d'une infinité d'ensembles disjoints et non vides de K.

Tel est le corps K formé de l'ensemble vide et de toutes les sommes d'un nombre fini d'ensembles de la famille F satisfaisant au théorème 2.

Pour démontrer qu'on obtient ainsi un corps d'ensembles, il suffit de remarquer que toute différence de deux ensembles de la famille F est une somme d'un nombre fini des ensembles (disjoints) de cette famille. Cela résulte du fait que la différence de deux ensembles de la famille F, $E_{a_1 a_2 \dots a_n} - E_{b_1 b_2 \dots b_m}$ est non vide seulement dans le cas où $m = n + k$, où k est un nombre naturel, et où $b_i = a_i$ pour $i = 1, 2, \dots, n$, et dans ce cas elle est égale à la somme (de k ensembles disjoints)

$$E_{a_1 a_2 \dots a_n b'_{n+1}} + E_{a_1 a_2 \dots a_n b_{n+1} b'_{n+2}} + \\ + E_{a_1 a_2 \dots a_n b_{n+1} b_{n+2} b'_{n+3}} + \dots + E_{a_1 a_2 \dots a_n b_{n+1} \dots b_{n+k-1} b'_{n+k}}$$

où $b'_j = 1 - b_j$ pour $j = n+1, n+2, \dots, n+k$.

Il en résulte sans peine que tout ensemble non vide de K est une somme d'un nombre fini d'ensembles disjoints de la famille F. Vu les propriétés de la famille F, il s'en suit que tout ensemble non vide de K est une somme de deux ensembles non vides et disjoints de F, (donc aussi de K) et qu'aucun ensemble de K n'est pas une somme d'une infinité d'ensembles non vides et disjoints de K.

Le corps K jouit donc des propriétés désirées.

Théorème 3. *Si F est une famille quelconque d'ensembles telle que tout ensemble de la famille F est une somme de deux ensembles distincts de cette famille, alors tout ensemble de la famille F est une somme d'une série infinie d'ensembles distincts de la famille F.*

Démonstration. Soit E un ensemble de la famille F: l'ensemble E est donc une somme de deux ensembles distincts de la famille F, soit $E = E_1 + H_1$. Comme $E_1 \neq H_1$, une au moins des formules $E_1 - H_1 \neq 0$ et $H_1 - E_1 \neq 0$ a lieu, et nous pouvons supposer que $E_1 - H_1 \neq 0$. Pareillement nous pouvons poser $H_1 = E_2 + H_2$, où $E_2 \in F$, $H_2 \in F$, $E_2 - H_2 \neq 0$, et ainsi de suite. On a évidemment $E = E + E_2 + E_3 + \dots$ et les ensembles $E, E_2, E_3, \dots$ appartiennent tous à la famille F et sont distincts, puisque $E - H_1 \supset E_1 - H_1 \neq 0$, $E_{n+1} \subset H_1$ pour $n = 1, 2, \dots$, donc $E - E_{n+1} \neq 0$ et $E \neq E_{n+1}$ pour $n = 1, 2, \dots$, et, d'autre part, $E_n - H_n \neq 0$, $E_{n+k} \subset H_n$, donc $E_{n+k} \neq E_n$, pour n et k naturels. L'ensemble E est donc une somme d'une série infinie d'ensembles distincts de la famille F, c. q. f. d.

Un exemple d'une famille F satisfaisant au théorème 3 est, comme on voit sans peine, la famille $F = \{E_1, E_2, ...,\}$, où E_n est l'ensemble de tous les nombres naturels $\geqslant n$. On a ici, pour n naturels: $E_n = E_n + E_{n+1}$ et $E_n \neq E_{n+1}$. Or, aucun ensemble E de cette famille F n'est pas une somme de deux ensembles de cette famille distincts de E. Cependant on a le théorème suivant:

Théorème 4. *Si F est une famille quelconque d'ensembles telle que tout ensemble E de F est une somme de deux ensembles de F distincts de E (donc aussi entre eux), alors tout ensemble E de la famille F est une somme d'une série infinie d'ensemble distincts de E et entre eux.*

Démonstration. Soit E un ensemble de la famille F: on a donc $E = E_1 + H_1$, où $E_1 \epsilon F$, $H_1 \epsilon F$, $E \neq E_1$, $E \neq H_1$, ce qui entraîne que $E_1 \neq H_1$, et nous pouvons supposer que $E_1 - H_1 \neq 0$. Pareillement on a $H_1 = E_2 + H_2$, où $E_2 \epsilon F$, $H_2 \epsilon F$, $H_1 \neq E_2$, $H_1 \neq H_2$, donc $E_2 \neq H_2$ et nous pouvons supposer que $E_2 - H_2 \neq 0$, et ainsi de suite. On aura évidemment $E = E_1 + H_1 + E_2 + E_3 + E_4 + ...$, et il suffira de démontrer que les ensembles $E, E_1, H_1, E_2, E_3, ...$ sont distincts. On a $E \neq E_1$, $E \neq H_1$, $E_{n+1} \subset H_n \subset H_1$, d'où $E - E_{n+1} \supset E - H_1 \supset E_1 - H_1 \neq 0$, ce qui donne $E \neq E_{n+1}$ pour $n = 1, 2, ...$ Ensuite on a $E_1 \neq H_1$, $E_2 \neq H_1$, $H_1 \neq H_2$, $H_1 \supset H_2$, donc $H_1 - H_2 \neq 0$ et, comme $E_{n+2} \subset H_2$, on a $H_1 - E_{n+2} \neq 0$, d'où $H_1 \neq E_{n+2}$ pour $n = 1, 2, ...$ Enfin on a $E_{n+k} \subset H_n$, $E_n - H_n \neq 0$, donc $E_n - E_{n+k} \neq 0$ et $E_n \neq E_{n+k}$ pour n et k naturels. Les ensembles $E, E_1, H_1, E_2, E_3, ...$ sont donc tous distincts et le théorème 4 se trouve démontré.

On voit sans peine que toute famille F qui satisfait au théorème 2, satisfait aussi au théorème 4.

Il est à remarquer encore qu'il existe des familles F d'ensembles, telles que tout ensemble de F est une somme de trois ensembles disjoints de F, mais aucun ensemble de F n'est pas une somme de deux ensembles disjoints de F. On peut sans peine démontrer que telle est la famille F de tous les ensembles $H_{a_1 a_2 ... a_n}$, où $a_1, a_2, ..., a_n$ est une suite finie quelconque formée de nombres 0, 1 ou 2, et où $H_{a_1 a_2 ... a_n}$ est l'ensemble de toutes les suites finies ayant au moins n termes, formées de nombres 0, 1 et 2, dont les n premiers termes sont $a_1, a_2, ..., a_n$.

Wacław Sierpiński

O pewnych własnościach rodzin zbiorów

Komunikat wygłoszony w dniu 14 października 1949 r.

Streszczenie

W komunikacie tym autor dowodzi twierdzenia, że jeżeli F jest jakąkolwiek rodziną zbiorów przeliczalnych, taką, że każdy zbiór rodziny F jest sumą dwóch zbiorów rozłącznych rodziny F, to każdy zbiór rodziny F jest również sumą nieskończenie wielu zbiorów rozłącznych rodziny F. Autor dowodzi następnie, że twierdzenie to nie da się rozciągnąć na rodziny dowolnych zbiorów, że mianowicie istnieje rodzina przeliczalna F zbiorów (nieprzeliczalnych), taka, że każdy zbiór rodziny F jest sumą dwóch zbiorów rozłącznych rodziny F, ale żaden zbiór rodziny F nie jest sumą nieskończenie wielu zbiorów rozłącznych rodziny F.

Dalej autor dowodzi twierdzenia, że jeżeli F jest dowolną rodziną zbiorów, taką, że każdy zbiór rodziny F jest sumą dwóch różnych zbiorów tej rodziny, to każdy zbiór rodziny F jest sumą szeregu nieskończonego różnych zbiorów tej rodziny, jakoteż twierdzenia, że jeżeli F jest dowolną rodziną zbiorów, taką, że każdy zbiór E tej rodziny jest sumą dwóch różnych od E (a więc i między sobą) zbiorów tej rodziny, to każdy zbiór E rodziny F jest sumą szeregu nieskończonego różnych między sobą i od E zbiorów tej rodziny.

Wacław Sierpiński

Sur quelques propositions qui entraînent l'existence des ensembles non mesurables

Note présentée à la séance du 14 octobre 1949.

1. n étant un nombre naturel donné, désignons par Z_n la proposition suivante:

Z_n. *L'axiome du choix est vrai pour les ensembles formés de n éléments.*

En d'autres termes:

Il existe une fonction τ_n qui à tout ensemble E formé de n éléments distincts fait correspondre un élément $\tau_n(E)$ de E.

Les propositions Z_n $(n = 2, 3, ...)$ ainsi que les rapports entre elles étaient édudiés par M. Mostowski et Mme Szmielew [1]).

Théorème 1. *Quel que soit le nombre naturel $n > 1$, il résulte de Z_n (sans l'aide de l'axiome du choix) l'existence d'un ensemble linéaire non mesurable au sens de Lebesgue* [2]).

Démonstration. Soit C la circonférence du cercle $x^2 + y^2 = 1$, et soit p un point quelconque de C, $\alpha(p)$ — l'angle que le vecteur $0p$ fait avec la direction positive de l'axe d'abscisses. Désignons par $f(p)$ le point q de C tel que $\alpha(q) = \alpha(p) + 1$, et soit H_p l'ensemble (dénombrable) de tous les points $f^k(p)$, où k est un entier (et où $f^0(p) = p$).

On démontre sans peine que p et q étant deux points de C, on a ou bien $H_p H_q = 0$, ou bien $H_p = H_q$. L'ensemble de tous les points de C se décompose ainsi en ensembles disjoints (superposables par rotation) que nous appellerons ensembles H.

[1]) A. Mostowski, *Axiom of choice for finite sets*, Fund. Math. 33 (1945), pp. 137—168; W. Szmielew On choices from finite sets, Fund. Math. 34 (1946), p. 75—80.

[2]) Le cas particulier de ce théorème pour $n = 2$ a été démontré par moi en 1927 dans Fund. Math. 10, p. 177.

E étant un ensemble de points de C, désignons par $E(\alpha)$ l'ensemble qu'on obtient de l'ensemble E en le tournant autour du centre 0 de l'angle α.

n étant un nombre naturel donné >1, divisons tous les ensembles H en classes, en rangeant deux ensembles H, E_1 et E_2, en une même classe dans ce et seulement dans ce cas s'il existe un entier k tel que $E_2 = E_1\left(\frac{2k\pi}{n}\right)$.

On voit sans peine que chaque classe contiendra n ensembles distincts, notamment si E est un ensemble quelconque de la classe considérée, cette classe sera formée des ensembles (tous distincts)

$$E,\ E\left(\frac{2\pi}{n}\right),\ E\left(\frac{4\pi}{n}\right), \ldots,\ E\left(\frac{(2n-2)\pi}{n}\right).$$

D'après Z_n il existe une famille F d'ensembles à laquelle appartient un et un seul ensemble de chacune de nos classes.

Soit S la somme de tous les ensembles appartenant à la famille F. Je dis que S est un ensemble non mesurable (L).

En effet, soit m un nombre naturel donné. Vu que le nombre π est irrationnel, on a, pour $k = 1, 2, \ldots, m$, les inégalités $0 < 2k\pi - \mathrm{E}\,2k\pi < 1$ et les nombres $2k\pi - \mathrm{E}\,2k\pi$ sont distincts pour k distincts. Il existe donc deux nombres k_1 et k_2 distincts de la suite $1, 2, \ldots, m$, tels que $0 < 2k_1\pi - \mathrm{E}\,2k_1\pi - (2k_2\pi - \mathrm{E}\,2k_2\pi) < \frac{1}{m}$.

En posant $k = \mathrm{E}\,2k_2\pi - \mathrm{E}\,2k_1\pi$, $l = k_1 - k_2$, nous aurons donc deux entiers k et l, tels que $0 < k + 2l\pi < \frac{1}{m}$. Vu que pour tout ensemble H on a $H(k + 2l\pi) = H(k) = H$, et vu que S est une somme de certains ensembles H, on trouve $S(k + 2l\pi) = S$. Il existe donc des nombres réels $t > 0$ aussi petits que l'on veut, tels que $S(t) = S$, ce qui donne aussi $S(it) = S$ pour i entiers.

On en conclut facilement qu'il existe une suite infinie de nombres $t_1, t_2, \ldots$ dense dans l'intervalle $(0, 2\pi)$ et telle que $S(t_i) = S$ pour $i = 1, 2, \ldots$, d'où $S = \sum_{i=1}^{\infty} S(t_i)$. Or, d'après la théorie de la mesure, si l'ensemble S n'est pas de mesure nulle, la somme $\sum_{i=1}^{\infty} S(t_i)$ (où la suite $t_1, t_2, \ldots$ est dense dans l'intervalle $(0, 2\pi))$ est un ensemble dont la mesure extérieure est dans chaque arc de la circonférence C égale à la longueur de cet arc.

D'autre part, vu que le nombre $\frac{2\pi}{n}$ est irrationnel, on voit sans peine que, E étant un ensemble H quelconque, on a $E \, . \, E\left(\frac{2\pi}{n}\right) = 0$ et il en résulte tout de suite que $S \, . \, S\left(\frac{2\pi}{n}\right) = 0$.

D'autre part, les ensembles S et $S\left(\frac{2\pi}{n}\right)$ étant superposables, ils ont dans chaque arc de C la mesure extérieure égale à la longueur de cet arc: comme ils sont disjoints, ils sont donc non mesurables (L). L'existence d'un ensemble non mesurable (L) situé sur une circonférence entraîne tout de suite l'existence d'un ensemble linéaire non mesurable (L).

Le théorème 1 se trouve ainsi démontré.

2. Il nous semble comme peu vraisemblable qu'il pourrait exister plus que $2^{\aleph_0}$ ensembles linéaires distincts superposables par translation avec l'ensemble R de tous les nombres rationnels. Or, cette hypothèse peu vraisemblable résulte de l'hypothèse qu'il n'existe aucun ensemble linéaire non mesurable (L). En effet, je démontrerai la proposition suivante:

Théorème 2. *En admettant qu'il n'existe pas plus que $2^{\aleph_0}$ ensembles linéaires distincts superposables par translation avec l'ensemble de tous les nombres rationnels, on peut, sans utiliser l'axiome du choix, démontrer l'existence des ensembles linéaires non mesurables au sens de Lebesgue.*

Démonstration. E étant un ensemble linéaire et t un nombre réel, désignons maintenant par $E(t)$ la translation de l'ensemble E le long de la droite de longueur t.

En 1917, dans le Giornale di Mat. j'ai construit, sans faire appel à l'axiome du choix, un ensemble linéaire parfait P tel que toutes les distances entre deux points de P sont irrationnelles.

Les ensembles $R(t)$, où $t \in P$, sont tous distincts pour t distincts (puisque $R(t_1) = R(t_2)$ donne $R(t_1 - t_2) = R$ ce qui ne peut avoir lieu que si le nombre $t_1 - t_2$ est rationnel).

L'ensemble P, en tant que parfait, étant de puissance $2^{\aleph_0}$, il en résulte que la famille F de tous les ensembles linéaires distincts, superposables par translation avec R, est de puissance $\geqslant 2^{\aleph_0}$. En admettant que F ne peut pas être de puissance $> 2^{\aleph_0}$, il en résulte que $\overline{\overline{F}} = 2^{\aleph_0}$.

Divisons maintenant tous les ensembles de la famille F autres que R en classes, en rangeant dans une même classe deux ensembles dans ce et seulement dans ce cas s'ils sont symétrique l'un de l'autre par rapport au point 0 comme centre de symétrie. On voit sans peine que chaque classe contiendra précisément deux ensembles distincts.

Or, comme $\overline{\overline{F}} = 2^{\aleph_0}$, il existe une correspondance biunivoque f entre les ensembles de la famille F et les nombres réels.

Dans chaque de nos classes nous pourrons donc prendre celui de deux ensembles constituant cette classe pour lequel la fonction f a une valeur plus petite. Soit S la somme de tous les ensembles ainsi obtenus. Comme $R(r) = R$ pour r rationnel, on a $R(t+r) = R(t)$ pour tout t réel et r rationnel, donc $E(r) = E$ pour tout ensemble E de la famille F, d'où $S(r) = S$ pour r rationnels. Or, on voit sans peine que l'ensemble S est disjoint avec son symétrique par rapport au point 0 comme centre. En modifiant légèrement le raisonnement utilisé dans la démonstration du théorème 1, on en déduit que l'ensemble S est non mesurable (L).

Le théorème 2 se trouve ainsi démontré.

Remarques. A l'aide de la famille F il est aisé de définir effectivement un ensemble M dont nous savons démontrer sans utiliser l'axiome du choix qu'il est de puissance du continu, mais nous ne savons pas établir une correspondance biunivoque entre les éléments de M et les nombres réels. (On pourrait dire que l'ensemble M est non effectivement de puissance du continu).

En effet, soit M l'ensemble F dans le cas où F est de puissance $2^{\aleph_0}$ et soit M l'ensemble de tous les nombres réels dans le cas contraire. On voit sans peine que l'ensemble M jouit des propriétés désirées.

D'autre part il est aisé de démontrer sans faire appel à l'axiome du choix que la famille F_1 de tous les ensembles linéaires superposables par translation avec l'ensemble E de tous les entiers est de puissance $2^{\aleph_0}$. En effet, pour chaque ensemble H de F_1 il existe (au moins) un nombre réel x, tel que $H = E(x)$, et, vu qu'on a pour k entiers, $E(k) = E$, d'où $E(x) = E(x - \mathrm{E}x)$ pour x réels, on trouve $H = E(x - \mathrm{E}x)$, donc $H = E(t)$, où $t = x - \mathrm{E}x$ est un nombre réel tel que $0 \leqslant t < 1$. La famille F_1 coïncide donc avec celle de tous les ensembles $E(t)$, où t est un nombre de l'intervalle $0 \leqslant t < 1$. D'autre part, on démontre sans peine que pour $0 \leqslant t_1 < t_2 < 1$ on a $E(t_1)E(t_2) = 0$, donc $E(t_1) \neq E(t_2)$. Il existe donc une correspondance biunivoque entre les ensembles $E(t)$ de la famille F_1 et les nombres réels t de l'intervalle $0 \leqslant t < 1$, d'où il résulte que $\overline{\overline{F}}_1 = 2^{\aleph_0}$, c. q. f. d.

Wacław Sierpiński

O paru twierdzeniach, z których wynika istnienie zbiorów niemierzalnych

Komunikat, wygłoszony na posiedzeniu w dniu 14 października 1949 r.

Streszczenie

Autor dowodzi następujących dwóch twierdzeń.

1. Niech n oznacza dowolną daną liczbę naturalną > 1.

Z założenia, że pewnik wyboru jest prawdziwy dla zbiorów o n elementach, wynika istnienie zbioru liniowego niemierzalnego w znaczeniu Lebesgue'a.

2. Z założenia, że nie istnieje więcej niż $2^{\aleph_0}$ różnych zbiorów liniowych, przystających przez przesunięcie do zbioru wszystkich liczb wymiernych, wynika (bez pomocy pewnika wyboru) istnienie zbiorów liniowych niemierzalnych w znaczeniu Lebesgue'a.

Felicjan Kępiński

Barycentric Reductions of the Sun Co-ordinates for the years 1947—1949

Note présentée à la séance du 14 octobre 1949.

The following pages contain the continuation of barycentric corrections of the Sun co-ordinates X, Y, Z, given already for the preceding years 1939 — 1946 in the Publications of the Institute Nos. 22 and 25.

The advantages of reporting to the Moon-Earth barycentrum the positions of Comets and Minor Planets which come in the neighbourhood of the Earth have therein been sufficiently explained and illustrated by an example.

The present memoir includes the barycentric reductions dX, dY, dZ for the years 1947—1949 and forms the closing of the whole series 1939—1949 having been initiated by me.

In computing the present tables I was assisted by Mr W. Opalski D. Ph. and I feel I have to express him my best thanks.

Warsaw, 15-th February 1949

Institute of Practical Astronomy of Warsaw Technical University

Felicjan Kępiński

Redukcje barycentryczne spółrzędnych Słońca na lata 1947—1949

Komunikat przedstawiony w dniu 14 października 1949 r.

Streszczenie

Przedłożona praca jest dalszym ciągiem dopełnienia przeze mnie zobowiązania, przejętego wobec Międzynarodowej Unii Astronomicznej na Kongresie w Sztokholmie w 1938 r.

W Komisji 20-ej tejże Unii podniosłem wówczas korzyści opierania się na spółrzędnych barycentrycznych, zamiast zazwyczaj stosowanych geocentrycznych, przy obliczaniu dokładnych efemeryd planetoid i komet.

Zadaniem pracy niniejszej było podanie na każdy dzień lat 1947—1949 poprawek do geocentrycznych spółrzędnych Słońca, podobnie jak poprzednio ogłosiłem tablice takich poprawek na lata 1939 do 1946.

Reductions dX, dY, dZ in units 0.0000 001, to subtract from X, Y, Z

1947		dX	dY	dZ
Jan.	0	+311	+23	—19
	1	+295	+85	+12
	2	+264	+144	+42
	3	+217	+193	+70
	4	+157	+232	+94
	5	+87	+255	+111
	6	+11	+263	+122
	7	—65	+254	+125
	8	—137	+228	+120
	9	—201	+187	+107
	10	—252	+135	+87
	11	—287	+75	+62
	12	—306	+11	+34
	13	—309	—54	+4
	14	—295	—116	—26
	15	—267	—172	—55
	16	—226	—220	—82
	17	—174	—257	—104
	18	—114	—283	—122
	19	—48	—297	—135
	20	+19	—297	—141
	21	+86	—284	—141
	22	+149	—258	—135
	23	+205	—221	—123
	24	+252	—173	—105
	25	+287	—117	—82
	26	+308	—56	—55
	27	+314	+7	—26
	28	+304	+71	+5
	29	+278	+130	+35
	30	+236	+182	+64
Jan.	31	+181	+224	+89
Feb.	1	+115	+253	+108
	2	+43	+266	+121
	3	—33	+263	+127
	4	—106	+243	+124
	5	—173	+208	+114
	6	—229	+160	+96
	7	—271	+103	+73
	8	—297	+39	+45
	9	—306	—27	+15
	10	—298	—92	—16
	11	—275	—151	—46
	12	—238	—203	—74
	13	—189	—245	—98
	14	—131	—275	—118
	15	—68	—293	—132
	16	—1	—297	—140
	17	+67	—288	—142
	18	+131	—266	—138
	19	+189	—232	—127
	20	+239	—187	—110
	21	+278	—133	—88
	22	+303	—72	—61
	23	+313	—9	—32
	24	+307	+55	—1
	25	+285	+117	+30
	26	+247	+171	+59
	27	+196	+216	+85
Feb.	28	+134	+249	+106

1947		dX	dY	dZ
Mar.	0	+134	+249	+106
	1	+64	+267	+121
	2	—9	+269	+129
	3	—83	+255	+129
	4	—151	+226	+121
	5	—210	+183	+106
	6	—257	+128	+84
	7	—289	+67	+57
	8	—304	+1	+27
	9	—302	—65	—4
	10	—284	—127	—35
	11	—251	—182	—65
	12	—206	—229	—91
	13	—151	—264	—112
	14	—89	—286	—129
	15	—22	—296	—139
	16	+45	—293	—143
	17	+110	—274	—141
	18	+171	—244	—132
	19	+223	—202	—117
	20	+265	—151	—96
	21	+295	—92	—70
	22	+309	—29	—41
	23	+308	+36	—10
	24	+290	+99	+22
	25	+256	+156	+52
	26	+209	+205	+80
	27	+149	+241	+103
	28	+80	+264	+120
	29	+7	+271	+130
	30	—66	+262	+132
Mar.	31	—135	+237	+126
Apr.	1	—197	+198	+113
	2	—247	+148	+93
	3	—283	+89	+68
	4	—303	+25	+39
	5	—307	—41	+8
	6	—294	—104	—24
	7	—265	—162	—55
	8	—224	—212	—82
	9	—171	—251	—106
	10	—111	—279	—125
	11	—45	—293	—137
	12	+22	—294	—144
	13	+89	—281	—144
	14	+151	—256	—137
	15	+206	—218	—124
	16	+252	—171	—105
	17	+286	—115	—81
	18	+305	—53	—53
	19	+309	+12	—22
	20	+297	+75	+10
	21	+268	+135	+41
	22	+223	+187	+71
	23	+166	+228	+96
	24	+99	+256	+115
	25	+26	+268	+127
	26	—48	+263	+132
	27	—120	+243	+128
	28	—184	+208	+117
	29	—238	+161	+99
Apr.	30	—278	+104	+76

1947		dX	dY	dZ
May	0	—278	+104	+76
	1	—302	+42	+48
	2	—310	—22	+17
	3	—302	—86	—14
	4	—278	—145	—45
	5	—240	—198	—74
	6	—190	—240	—99
	7	—132	—271	—120
	8	—67	—290	—134
	9	+0	—295	—143
	10	+68	—287	—145
	11	+132	—266	—141
	12	+190	—232	—130
	13	+239	—189	—113
	14	+277	—136	—91
	15	+302	—77	—65
	16	+311	—14	—35
	17	+305	+50	—3
	18	+282	+111	+28
	19	+243	+166	+59
	20	+189	+212	+86
	21	+124	+244	+107
	22	+52	+262	+122
	23	—24	+262	+130
	24	—98	+247	+129
	25	—166	+216	+120
	26	—224	+172	+104
	27	—268	+117	+81
	28	—297	+56	+55
	29	—310	—8	+25
	30	—306	—72	—7
May	31	—286	—132	—38
June	1	—252	—186	—67
	2	—206	—231	—93
	3	—150	—265	—115
	4	—86	—286	—131
	5	—19	—295	—142
	6	+48	—291	—146
	7	+114	—274	—143
	8	+174	—244	—134
	9	+227	—204	—119
	10	+269	—154	—99
	11	+298	—97	—74
	12	+313	—36	—46
	13	+313	+27	—15
	14	+296	+89	+16
	15	+263	+146	+47
	16	+215	+195	+75
	17	+154	+232	+99
	18	+84	+255	+116
	19	+8	+262	+126
	20	—68	+252	+128
	21	—140	+226	+122
	22	—202	+185	+108
	23	—252	+133	+87
	24	—287	+73	+61
	25	—305	+9	+32
	26	—306	—56	+0
	27	—291	—118	—31
	28	—261	—174	—61
	29	—219	—222	—88
June	30	—164	—258	—110

Reductions dX, dY, dZ in units 0.0000 001, to subtract from X, Y, Z

1947		dX	dY	dZ	1947		dX	dY	dZ	1947		dX	dY	dZ
July	0	—164	—258	—110	Sept.	0	+278	—142	— 92	Nov.	0	+189	+218	+ 94
	1	—103	—283	—128		1	+305	— 84	— 66		1	+125	+249	+115
	2	— 37	—295	—140		2	+318	— 22	— 37		2	+ 54	+266	+129
	3	+ 31	—294	—145		3	+315	+ 41	— 6		3	— 20	+267	+135
	4	+ 98	—280	—144		4	+297	+101	+ 26		4	— 93	+251	+134
	5	+160	—253	—137		5	+264	+157	+ 56		5	—160	+221	+124
	6	+215	—216	—124		6	+218	+204	+ 83		6	—217	+178	+107
	7	+260	—169	—105		7	+159	+240	+105		7	—262	+124	+ 84
	8	+294	—114	— 81		8	+ 92	+262	+122		8	—292	+ 63	+ 56
	9	+313	— 54	— 54		9	+ 19	+269	+131		9	—304	— 2	+ 24
	10	+318	+ 9	— 24		10	— 55	+261	+133		10	—300	— 67	— 8
	11	+307	+ 70	+ 7		11	—125	+236	+127		11	—280	—128	— 41
	12	+280	+129	+ 38		12	—188	+197	+113		12	—244	—183	— 71
	13	+237	+180	+ 67		13	—240	+146	+ 92		13	—196	—228	— 98
	14	+182	+221	+ 91		14	—276	+ 86	+ 65		14	—138	—262	—120
	15	+116	+249	+111		15	—296	+ 20	+ 34		15	— 73	—283	—136
	16	+ 42	+262	+124		16	—298	— 47	+ 1		16	— 5	—290	—145
	17	— 34	+258	+128		17	—283	—111	— 32		17	+ 64	—284	—148
	18	—108	+238	+125		18	—253	—169	— 63		18	+129	—265	—144
	19	—175	+202	+113		19	—209	—218	— 91		19	+188	—234	—133
	20	—231	+153	+ 95		20	—154	—255	—114		20	+239	—193	—116
	21	—272	+ 94	+ 70		21	— 92	—280	—131		21	+278	—142	— 94
	22	—297	+ 31	+ 41		22	— 25	—292	—142		22	+305	— 85	— 68
	23	—304	— 35	+ 9		23	+ 42	—290	—147		23	+317	— 24	— 38
	24	—294	— 99	— 23		24	+108	—274	—145		24	+314	+ 38	— 7
	25	—268	—158	— 54		25	+169	—246	—136		25	+295	+ 99	+ 25
	26	—229	—208	— 82		26	+223	—207	—121		26	+260	+154	+ 56
	27	—178	—248	—106		27	+266	—158	—100		27	+211	+201	+ 83
	28	—118	—277	—125		28	+297	—102	— 75		28	+150	+237	+106
	29	— 53	—293	—138		29	+314	— 41	— 46		29	+ 80	+258	+123
	30	+ 14	—295	—145	Sept.	30	+316	+ 22	— 14	Nov.	30	+ 5	+264	+132
July	31	+ 81	—284	—145										
Aug.	1	+145	—261	—139	Oct.	1	+302	+ 84	+ 18	Dec.	1	— 70	+254	+133
	2	+202	—226	—127		2	+273	+142	+ 49		2	—141	+228	+126
	3	+250	—181	—109		3	+229	+192	+ 77		3	—202	+188	+111
	4	+287	—128	— 87		4	+173	+232	+101		4	—252	+137	+ 89
	5	+310	— 69	— 60		5	+108	+258	+120		5	—286	+ 77	+ 62
	6	+319	— 7	— 30		6	+ 36	+270	+132		6	—304	+ 13	+ 32
	7	+313	+ 55	+ 1		7	— 38	+266	+136		7	—304	— 51	— 1
	8	+291	+115	+ 32		8	—109	+247	+132		8	—288	—114	— 33
	9	+254	+168	+ 61		9	—173	+212	+120		9	—257	—170	— 64
	10	+203	+212	+ 87		10	—228	+165	+101		10	—212	—217	— 91
	11	+141	+245	+108		11	—269	+109	+ 76		11	—157	—254	—114
	12	+ 71	+263	+122		12	—294	+ 45	+ 46		12	— 94	—279	—132
	13	— 4	+265	+130		13	—302	— 21	+ 14		13	— 26	—290	—143
	14	— 78	+251	+129		14	—292	— 86	— 19		14	+ 42	—288	—148
	15	—148	+221	+120		15	—267	—147	— 51		15	+109	—273	—145
	16	—208	+176	+103		16	—227	—199	— 81		16	+171	—246	—137
	17	—255	+121	+ 80		17	—175	—241	—106		17	+225	—208	—122
	18	—286	+ 58	+ 52		18	—115	—271	—126		18	+268	—160	—102
	19	—300	— 9	+ 21		19	— 49	—287	—140		19	+300	—106	— 77
	20	—296	— 75	— 12		20	+ 19	—290	—147		20	+317	— 46	— 49
	21	—275	—136	— 44		21	+ 86	—280	—147		21	+320	+ 15	— 18
	22	—240	—191	— 74		22	+150	—256	—140		22	+307	+ 76	+ 13
	23	—192	—235	— 99		23	+206	—221	—127		23	+279	+133	+ 44
	24	—135	—268	—120		24	+253	—176	—109		24	+235	+183	+ 73
	25	— 71	—288	—135		25	+288	—122	— 85		25	+179	+223	+ 97
	26	— 4	—294	—144		26	+309	— 63	— 57		26	+112	+250	+116
	27	+ 63	—288	—146		27	+316	+ 0	— 26		27	+ 38	+262	+128
	28	+128	—268	—142		28	+307	+ 63	+ 6		28	— 38	+258	+132
	29	+187	—236	—131		29	+282	+122	+ 38		29	—112	+237	+127
	30	+238	—193	—114		30	+242	+175	+ 68		30	—178	+201	+115
Aug.	31	+278	—142	— 92	Oct.	31	+189	+218	+ 94	Dec.	31	—233	+152	+ 95

Reductions dX, dY, dZ in units 0.0000 001, to subtract from X, Y, Z

1948		dX	dY	dZ	1948		dX	dY	dZ	1948		dX	dY	dZ
Jan.	0	—233	+152	+95	Mar.	0	—244	—172	—70	May	0	+144	—255	—142
	1	—273	+94	+69		1	—198	—220	—97		1	+202	—223	—129
	2	—297	+30	+39		2	—142	—256	—120		2	+251	—180	—111
	3	—303	—35	+6		3	—78	—279	—136		3	+288	—129	—87
	4	—290	—98	—26		4	—11	—289	—146		4	+313	—71	—59
	5	—265	—157	—58		5	+57	—285	—149		5	+322	—11	—29
	6	—224	—207	—86		6	+123	—268	—145		6	+317	+50	+3
	7	—172	—247	—110		7	+183	—239	—135		7	+296	+109	+35
	8	—111	—275	—129		8	+235	—199	—118		8	+261	+163	+65
	9	—45	—290	—141		9	+276	—150	—96		9	+212	+208	+92
	10	+24	—291	—147		10	+306	—94	—70		10	+152	+242	+114
	11	+91	—279	—147		11	+322	—34	—40		11	+84	+263	+129
	12	+154	—255	—139		12	+323	+27	—9		12	+11	+270	+138
	13	+211	—220	—126		13	+309	+87	+23		13	—62	+261	+139
	14	+258	—175	—107		14	+280	+143	+53		14	—132	+236	+131
	15	+294	—122	—84		15	+238	+192	+85		15	—193	+198	+116
	16	+316	—64	—56		16	+183	+231	+105		16	—243	+148	+94
	17	+324	—3	—26		17	+119	+258	+123		17	—279	+90	+66
	18	+317	+58	+5		18	+49	+270	+135		18	—298	+26	+35
	19	+295	+116	+36		19	—24	+268	+139		19	—300	—40	+1
	20	+257	+168	+65		20	—95	+250	+134		20	—284	—104	—33
	21	+206	+211	+90		21	—161	+217	+122		21	—252	—161	—65
	22	+144	+243	+111		22	—217	+170	+103		22	—206	—211	—94
	23	+73	+261	+125		23	—260	+113	+76		23	—149	—248	—117
	24	—2	+263	+132		24	—286	+49	+45		24	—84	—272	—134
	25	—77	+248	+130		25	—294	—18	+11		25	—15	—283	—145
	26	—147	+218	+121		26	—285	—85	—23		26	+55	—280	—148
	27	—207	+173	+103		27	—258	—146	—57		27	+122	—264	—144
	28	—254	+118	+78		28	—216	—199	—87		28	+183	—235	—134
	29	—285	+54	+49		29	—162	—240	—112		29	+236	—195	—117
	30	—298	—12	+16		30	—100	—269	—131		30	+277	—147	—95
Jan.	31	—292	—78	—17	Mar.	31	—33	—284	—143	May	31	+307	—92	—69
Feb.	1	—270	—139	—50	Apr.	1	+36	—285	—149	June	1	+322	—32	—39
	2	—233	—193	—80		2	+103	—272	—147		2	+322	+29	—8
	3	—184	—236	—105		3	+165	—247	—139		3	+306	+88	+24
	4	—125	—268	—125		4	+220	—211	—124		4	+276	+144	+54
	5	—61	—286	—140		5	+265	—164	—103		5	+231	+192	+81
	6	+7	—292	—147		6	+298	—111	—78		6	+174	+230	+106
	7	+75	—283	—148		7	+318	—52	—49		7	+108	+255	+124
	8	+139	—263	—142		8	+323	+10	—18		8	+36	+266	+134
	9	+197	—230	—130		9	+313	+70	+14		9	—39	+262	+137
	10	+248	—188	—112		10	+287	+128	+45		10	—111	+242	+132
	11	+268	—136	—90		11	+248	+179	+74		11	—176	+208	+119
	12	+312	—79	—63		12	+196	+221	+100		12	—231	+160	+98
	13	+324	—19	—33		13	+134	+252	+120		13	—271	+103	+72
	14	+321	+42	—2		14	+65	+269	+134		14	—295	+40	+41
	15	+303	+101	+29		15	—7	+271	+140		15	—302	—25	+8
	16	+271	+155	+59		16	—79	+257	+138		16	—291	—89	—25
	17	+225	+202	+86		17	—146	+229	+128		17	—264	—148	—58
	18	+167	+237	+108		18	—206	+188	+111		18	—222	—199	—87
	19	+101	+259	+124		19	—251	+134	+87		19	—168	—240	—111
	20	+28	+268	+133		20	—283	+73	+58		20	—105	—268	—130
	21	—46	+260	+135		21	—297	+7	+25		21	—37	—283	—143
	22	—117	+236	+128		22	—293	—59	—9		22	+32	—284	—148
	23	—180	+199	+113		23	—272	—122	—43		23	+101	—271	—146
	24	—234	+146	+91		24	—235	—178	—75		24	+164	—246	—137
	25	—271	+85	+62		25	—185	—224	—102		25	+220	—209	—123
	26	—291	+18	+30		26	—125	—258	—124		26	+266	—164	—102
	27	—293	—49	—4		27	—59	—278	—139		27	+300	—110	—77
Feb.	28	—276	—114	—38		28	+11	—284	—147		28	+320	—52	—49
						29	+79	—276	—148		29	+325	+8	—18
					Apr.	30	+144	—255	—142	June	30	+315	+68	+14

Reductions dX, dY, dZ in units 0.0000 001, to subtract from X, Y, Z

1948		dX	dY	dZ	1948		dX	dY	dZ	1948		dX	dY	dZ
July	0	+315	+68	+14	Sept.	0	—94	+248	+135	Nov.	0	—277	—98	—35
	1	+291	+124	+44		1	—161	+214	+121		1	—244	—157	—68
	2	+251	+175	+73		2	—217	+167	+100		2	—196	—207	—97
	3	+199	+217	+98		3	—259	+109	+72		3	—136	—244	—121
	4	+136	+247	+118		4	—284	+44	+40		4	—70	—268	—137
	5	+65	+263	+131		5	—291	—24	+5		5	+1	—277	—146
	6	—9	+264	+136		6	—280	—91	—31		6	+71	—272	—148
	7	—83	+249	+134		7	—251	—151	—64		7	+138	—254	—142
	8	—152	+219	+123		8	—207	—203	—94		8	+198	—223	—129
	9	—211	+175	+104		9	—152	—243	—118		9	+249	—181	—110
	10	—257	+120	+79		10	—88	—270	—136		10	+288	—131	—86
	11	—287	+58	+48		11	—20	—282	—146		11	+314	—76	—58
	12	—299	—8	+15		12	+49	—281	—150		12	+326	—16	—28
	13	—293	—74	—19		13	+116	—266	—146		13	+323	+44	+4
	14	—270	—134	—52		14	+178	—239	—136		14	+305	+102	+36
	15	—233	—189	—82		15	+231	—201	—119		15	+272	+156	+66
	16	—182	—231	—107		16	+275	—154	—97		16	+227	+202	+93
	17	—122	—263	—127		17	+306	—100	—70		17	+170	+238	+115
	18	—55	—281	—141		18	+324	—41	—41		18	+105	+262	+132
	19	+14	—286	—148		19	+327	+19	—9		19	+34	+273	+141
	20	+83	—277	—148		20	+316	+79	+23		20	—39	+268	+143
	21	+147	—255	—141		21	+290	+135	+54		21	—109	+249	+137
	22	+206	—221	—127		22	+251	+184	+82		22	—173	+216	+124
	23	+255	—178	—108		23	+200	+225	+106		23	—227	+170	+103
	24	+292	—127	—84		24	+139	+254	+125		24	—267	+115	+76
	25	+317	—70	—56		25	+71	+270	+138		25	—292	+53	+45
	26	+327	—10	—26		26	—1	+272	+143		26	—299	—13	+11
	27	+322	+50	+5		27	—72	+259	+140		27	—289	—78	—24
	28	+303	+108	+36		28	—140	+231	+130		28	—261	—138	—57
	29	+269	+160	+66		29	—199	+189	+111		29	—218	—190	—87
	30	+221	+205	+92	Sept.	30	—245	+136	+86	Nov.	30	—168	—232	—112
July	31	+163	+239	+113										
Aug.	1	+96	+260	+128	Oct.	1	—277	+74	+55	Dec.	1	—98	—261	—131
	2	+23	+267	+136		2	—291	+7	+21		2	—28	—275	—143
	3	—52	+258	+136		3	—286	—61	—15		3	+44	—275	—147
	4	—123	+233	+128		4	—263	—124	—50		4	+113	—261	—144
	5	—186	+194	+112		5	—224	—180	—82		5	+176	—234	—133
	6	—238	+142	+88		6	—172	—226	—108		6	+232	—197	—117
	7	—274	+81	+59		7	—110	—258	—129		7	+275	—149	—94
	8	—292	+15	+26		8	—42	—276	—143		8	+306	—95	—68
	9	—292	—52	—9		9	+28	—280	—149		9	+324	—37	—38
	10	—275	—116	—43		10	+96	—270	—148		10	+326	+23	—7
	11	—241	—173	—75		11	+160	—246	—139		11	+313	+82	+25
	12	—194	—220	—102		12	+217	—212	—124		12	+286	+137	+56
	13	—136	—255	—124		13	+263	—167	—104		13	+245	+186	+84
	14	—71	—277	—139		14	+298	—115	—78		14	+191	+225	+108
	15	—2	—286	—148		15	+320	—57	—49		15	+128	+254	+126
	16	+66	—280	—150		16	+327	+3	—18		16	+58	+268	+138
	17	+132	—262	—144		17	+319	+63	+14		17	—15	+269	+142
	18	+192	—231	—131		18	+297	+120	+46		18	—88	+254	+138
	19	+243	—190	—113		19	+261	+172	+75		19	—155	+224	+126
	20	+284	—141	—90		20	+212	+215	+101		20	—213	+181	+107
	21	+312	—85	—63		21	+153	+248	+122		21	—258	+128	+81
	22	+326	—26	—33		22	+87	+268	+136		22	—287	+67	+51
	23	+326	+34	—2		23	+16	+275	+144		23	—300	+2	+18
	24	+311	+93	+30		24	—56	+266	+143		24	—295	—63	—17
	25	+281	+147	+60		25	—124	+243	+135		25	—272	—124	—50
	26	+239	+194	+87		26	—185	+206	+120		26	—234	—178	—81
	27	+184	+232	+110		27	—236	+157	+97		27	—182	—223	—107
	28	+120	+258	+127		28	—272	+98	+68		28	—121	—255	—127
	29	+50	+269	+137		29	—292	+33	+35		29	—53	—274	—141
	30	—23	+266	+140		30	—293	—33	—0		30	+18	—278	—147
Aug	31	—94	+248	+135	Oct.	31	—277	—98	—35	Dec.	31	+89	—268	—146

Reductions dX, dY, dZ in units 0.0000 001, to subtract from X, Y, Z

1949		dX	dY	dZ
Jan.	0	+ 89	—268	—146
	1	+155	—246	—137
	2	+213	—211	—122
	3	+261	—167	—101
	4	+297	—115	— 76
	5	+320	— 58	— 47
	6	+328	+ 2	— 16
	7	+321	+ 61	+ 15
	8	+299	+118	+ 46
	9	+263	+169	+ 75
	10	+215	+212	+100
	11	+155	+244	+120
	12	+ 88	+264	+134
	13	+ 15	+269	+141
	14	— 58	+260	+139
	15	—128	+235	+130
	16	—191	+196	+113
	17	—241	+145	+ 88
	18	—276	+ 85	+ 59
	19	—295	+ 20	+ 25
	20	—295	— 46	— 10
	21	—277	—110	— 44
	22	—244	—166	— 76
	23	—196	—214	—103
	24	—137	—250	—125
	25	— 71	—272	—140
	26	— 1	—280	—148
	27	+ 69	—274	—148
	28	+136	—255	—141
	29	+197	—224	—128
	30	+248	—182	—108
Jan.	31	+288	—132	— 84
Feb.	1	+315	— 76	— 55
	2	+328	— 17	— 25
	3	+326	+ 43	+ 7
	4	+310	+100	+ 38
	5	+278	+155	+ 68
	6	+235	+199	+ 94
	7	+180	+235	+116
	8	+117	+260	+132
	9	+ 46	+271	+141
	10	— 26	+267	+143
	11	— 97	+248	+136
	12	—163	+214	+121
	13	—219	+167	+ 99
	14	—260	+110	+ 70
	15	—285	+ 45	+ 37
	16	—292	— 22	+ 2
	17	—280	— 88	— 34
	18	—251	—149	— 67
	19	—207	—200	— 97
	20	—151	—240	—120
	21	— 86	—266	—138
	22	— 17	—279	—147
	23	+ 53	—277	—150
	24	+121	—261	—144
	25	+183	—233	—132
	26	+236	—195	—114
	27	+279	—147	— 91
Feb.	28	+310	— 92	— 63

1949		dX	dY	dZ
Mar.	0	+310	— 92	— 63
	1	+326	— 34	— 33
	2	+329	+ 26	— 1
	3	+315	+ 85	+ 31
	4	+290	+140	+ 61
	5	+251	+188	+ 89
	6	+200	+227	+112
	7	+139	+256	+130
	8	+ 72	+272	+142
	9	+ 1	+274	+146
	10	— 70	+261	+142
	11	—137	+233	+130
	12	—196	+191	+111
	13	—243	+138	+ 85
	14	—275	+ 76	+ 53
	15	—289	+ 9	+ 18
	16	—284	— 59	— 18
	17	—260	—123	— 54
	18	—221	—179	— 86
	19	—167	—224	—112
	20	—104	—256	—132
	21	— 36	—273	—145
	22	+ 35	—276	—149
	23	+104	—265	—146
	24	+168	—240	—136
	25	+224	—205	—120
	26	+270	—159	— 97
	27	+303	—106	— 71
	28	+324	— 49	— 41
	29	+329	+ 11	— 9
	30	+321	+ 70	+ 23
Mar.	31	+298	+126	+ 54
Apr.	1	+262	+177	+ 83
	2	+214	+219	+108
	3	+156	+251	+128
	4	+ 91	+271	+141
	5	+ 21	+278	+148
	6	— 49	+270	+147
	7	—117	+247	+138
	8	—179	+211	+122
	9	—229	+163	+ 98
	10	—267	+105	+ 68
	11	—287	+ 40	+ 34
	12	—289	— 28	— 2
	13	—273	— 94	— 38
	14	—239	—153	— 72
	15	—189	—203	—101
	16	—129	—241	—124
	17	— 60	—264	—139
	18	+ 11	—273	—147
	19	+ 82	—266	—147
	20	+149	—246	—139
	21	+208	—214	—124
	22	+257	—171	—103
	23	+295	—120	— 78
	24	+319	— 64	— 49
	25	+329	— 5	— 17
	26	+324	+ 55	+ 15
	27	+305	+112	+ 46
	28	+272	+164	+ 76
	29	+226	+209	+102
Apr.	30	+170	+244	+123

1949		dX	dY	dZ
May	0	+170	+244	+123
	1	+106	+267	+138
	2	+ 37	+278	+147
	3	— 33	+274	+149
	4	—102	+256	+142
	5	—165	+225	+128
	6	—219	+181	+107
	7	—261	+126	+ 79
	8	—286	+ 64	+ 47
	9	—295	— 2	+ 12
	10	—285	— 68	— 23
	11	—257	—129	— 58
	12	—213	—183	— 89
	13	—156	—226	—114
	14	— 90	—255	—132
	15	— 18	—269	—143
	16	+ 55	—268	—146
	17	+124	—252	—140
	18	+187	—224	—128
	19	+241	—184	—109
	20	+283	—136	— 85
	21	+312	— 81	— 57
	22	+327	— 22	— 26
	23	+326	+ 37	+ 6
	24	+311	+ 95	+ 37
	25	+282	+149	+ 67
	26	+240	+196	+ 95
	27	+186	+234	+117
	28	+124	+261	+134
	29	+ 55	+275	+145
	30	— 16	+275	+148
May	31	— 86	+261	+143
June	1	—152	+233	+131
	2	—209	+193	+112
	3	—254	+141	+ 86
	4	—284	+ 82	+ 56
	5	—298	+ 17	+ 22
	6	—294	— 48	— 13
	7	—272	—111	— 48
	8	—234	—167	— 79
	9	—181	—213	—106
	10	—118	—246	—127
	11	— 48	—265	—140
	12	+ 25	—270	—145
	13	+ 97	—259	—142
	14	+165	—235	—132
	15	+221	—199	—115
	16	+268	—153	— 92
	17	+302	—100	— 65
	18	+322	— 42	— 35
	19	+327	+ 18	— 4
	20	+317	+ 77	+ 28
	21	+293	+132	+ 59
	22	+255	+181	+ 87
	23	+205	+222	+111
	24	+144	+253	+129
	25	+ 77	+271	+141
	26	+ 6	+275	+147
	27	— 65	+265	+144
	28	—133	+241	+134
	29	—194	+203	+116
June	30	—243	+154	+ 91

Reductions dX, dY, dZ in units 0.0000 001, to subtract from X, Y, Z

1949	dX	dY	dZ
July 0	—243	+154	+91
1	—278	+96	+62
2	—297	+32	+29
3	—298	—33	—6
4	—281	—96	—41
5	—248	—154	—73
6	—200	—203	—101
7	—140	—240	—123
8	—72	—263	—138
9	—0	—272	—145
10	+72	—266	—145
11	+140	—246	—136
12	+202	—213	—121
13	+253	—170	—100
14	+292	—119	—74
15	+317	—62	—44
16	+327	—3	—13
17	+322	+57	+19
18	+303	+114	+50
19	+270	+165	+79
20	+224	+209	+104
21	+167	+243	+124
22	+103	+266	+139
23	+33	+275	+146
24	—39	+269	+145
25	—109	+249	+137
26	—173	+215	+121
27	—226	+169	+98
28	—267	+113	+69
29	—291	+50	+36
30	—297	—16	+1
July 31	—285	—81	—34
Aug. 1	—256	—141	—67
2	—212	—193	—96
3	—156	—233	—120
4	—90	—260	—136
5	—20	—273	—146
6	+52	—271	—147
7	+121	—255	—141
8	+184	—226	—127
9	+238	—186	—107
10	+281	—137	—82
11	+311	—81	—54
12	+326	—22	—22
13	+326	+38	+10
14	+311	+96	+42
15	+283	+149	+72
16	+242	+196	+98
17	+189	+234	+120
18	+128	+260	+136
19	+60	+274	+146
20	—11	+274	+148
21	—82	+259	+142
22	—148	+230	+129
23	—205	+188	+108
24	—250	+134	+80
25	—280	+73	+48
26	—293	+6	+12
27	—287	—60	—24
28	—263	—123	—58
29	—222	—178	—89
30	—169	—222	—115
Aug. 31	—105	—254	—134

1949	dX	dY	dZ
Sept. 0	—105	—254	—134
1	—36	—270	—145
2	+36	—273	—149
3	+105	—261	—144
4	+170	—235	—132
5	+226	—198	—114
6	+272	—152	—90
7	+305	—98	—62
8	+324	—40	—31
9	+328	+20	+1
10	+318	+79	+34
11	+293	+135	+64
12	+256	+184	+92
13	+207	+225	+116
14	+149	+255	+134
15	+84	+273	+146
16	+14	+278	+151
17	—56	+269	+148
18	—123	+246	+137
19	—183	+209	+119
20	—233	+160	+94
21	—269	+101	+63
22	—288	+35	+28
23	—288	—32	—9
24	—270	—97	—45
25	—235	—157	—78
26	—185	—206	—106
27	—123	—242	—128
28	—54	—264	—142
29	+18	—271	—148
Sept. 30	+89	—263	—146
Oct. 1	+155	—241	—136
2	+214	—208	—119
3	+262	—164	—97
4	+298	—112	—70
5	+321	—55	—39
6	+329	+5	—7
7	+322	+64	+25
8	+301	+121	+57
9	+267	+172	+86
10	+221	+216	+111
11	+165	+249	+131
12	+102	+272	+145
13	+34	+281	+152
14	—35	+277	+151
15	—103	+259	+144
16	—165	+227	+128
17	—218	+183	+106
18	—259	+128	+77
19	—284	+66	+44
20	—292	—1	+8
21	—281	—67	—28
22	—251	—130	—63
23	—206	—184	—94
24	—147	—226	—118
25	—79	—253	—135
26	—7	—266	—144
27	+66	—263	—145
28	+135	—246	—138
29	+197	—216	—123
30	+249	—175	—102
Oct. 31	+289	—125	—77

1949	dX	dY	dZ
Nov. 0	+289	—125	—77
1	+316	—69	—47
2	+328	—10	—15
3	+325	+49	+17
4	+308	+107	+49
5	+277	+160	+78
6	+234	+205	+104
7	+180	+242	+126
8	+118	+268	+142
9	+51	+281	+151
10	—19	+281	+153
11	—87	+267	+147
12	—151	+238	+134
13	—207	+200	+114
14	—251	+149	+88
15	—282	+90	+57
16	—296	+26	+23
17	—292	—40	—13
18	—269	—103	—48
19	—230	—161	—80
20	—176	—207	—107
21	—110	—241	—127
22	—38	—260	—140
23	+36	—263	—143
24	+108	—251	—139
25	+175	—225	—127
26	+231	—187	—108
27	+276	—140	—83
28	+308	—86	—55
29	+325	—27	—24
Nov. 30	+327	+32	+9
Dec. 1	+314	+90	+40
2	+286	+145	+71
3	+246	+193	+98
4	+195	+232	+120
5	+135	+261	+137
6	+68	+278	+148
7	—1	+281	+152
8	—71	+271	+148
9	—137	+247	+137
10	—196	+211	+119
11	—244	+164	+94
12	—279	+107	+65
13	—298	+45	+32
14	—300	—20	—3
15	—285	—83	—38
16	—251	—142	—70
17	—202	—192	—99
18	—141	—230	—121
19	—71	—255	—136
20	+3	—264	—142
21	+77	—257	—140
22	+147	—236	—131
23	+208	—202	—114
24	+259	—158	—91
25	+296	—105	—63
26	+318	—47	—33
27	+326	+12	—1
28	+318	+72	+32
29	+296	+128	+62
30	+260	+178	+90
Dec. 31	+212	+220	+114

Felicjan Kępiński

Kometa periodyczna 1906 IV (Kopff) w czasie jej ukazania się w roku 1939 i 1945

Komunikat wygłoszony w dniu 14 października 1949 r.

Praca niniejsza jest w pewnym sensie zamknięciem poprzedzających 4 części pracy o ruchu tej komety, pracy, którą prowadziłem przez przeszło 20 lat, a której obfite archiwum przepadło w ogniu spalonej w 1944 r. Warszawy. Tylko przypadkowi złożenia w Obserwatorium Uniwersytetu Warszawskiego krótkiego odpisu precyzyjnej efemerydy na zjawienie się komety w 1945 r. i wyniesieniu jej stamtąd przez Dr J. Gadomskiego zawdzięczam możność złożenia dzisiejszego komunikatu.

Przedkładana praca przedstawia porównanie dość licznych obserwacyj komety z 1939, a zwłaszcza 1945 r., z obliczonymi przeze mnie dokładnymi efemerydami. Przedstawiały one rezultat powiązania wszystkich znanych obserwacyj komety podczas 4-krotnego powrotu jej do Słońca. W jakim stopniu powiązanie to zostało już osiągnięte, wynika m. in. z opublikowanych w Comet Notes recenzyj znanego obserwatora i łowcy komet, prof. G. Van Biesbroeck'a z Yerkes Observatory.

Osobliwością ruchu komety 1906 IV w okresie, poprzedzającym ostatnie jej zjawienie się, było wielkie zbliżenie się do Jowisza (do 0.57 jedn. astr.) i wywołane przezeń znaczne zakłócenia ruchu komety oraz okaźna zmiana elementów jej orbity. To też osiągniętą w r. 1945 różnicę w rektascenzji $+ 2^{s}.5$ i w deklinacji $+ 5''$ do $6''$ między obserwacjami a moim rachunkiem znowuż uważać można za zupełnie zadowalającą.

Przedmiotem dalszych rachunków moich jest obecnie doprowadzenie komety do perihelium w r. 1951 z uwzględnieniem zakłóceń 6 wielkich planet. Ale oprócz tego uporać się będę musiał jeszcze z usunięciem nieciągłości obliczeń, wywołanych utratą swego archiwum rachunkowego, oraz z nawiązaniem do wyników przedwojennych, które nie doczekały się opublikowania, z wyjątkiem wspomnianej na wstępie szczątkowej efemerydy.

Wiesław Opalski

Finding the azimuth of a terrestrial object by the transits of stars over the object's vertical circle

Mémoire presenté par F. Kępiński à la séance du 14 octobre 1949.

1. The design and the method of work.

The scope of this work was to put to the test the usefulness of a transit instrument for direct determinations of azimuths considerably differing from 0^0 or 180^0. For an observation of this kind a transit instrument is to be set in the object's vertical circle; then the instrumental azimuth can be determined by a star transit observation and the object's azimuth — by micrometric measurement. Such a method was applied by C. Stechert[1]) and, independently, discussed by Prof. F. Kępiński[2]) who has been suggesting the use of an impersonal micrometer for it. Stechert's observations and reductions were performed without taking care of higher accuracy. So he has been observing star transits over a single wire. Stechert's above mentioned paper was the starting point for my work; I resolved, however, to use the most accurate means I disposed of, including a transit micrometer and, accordingly, to adapt the reduction methods to the higher accuracy of results aimed at.

The task of this paper is to test the practical value of Stechert's method which I have modified with regard to the observing as well as to the computing side and, therefore, adapted for determinations with high accuracy.

2. The reduction formulae.

The reductions of my observations have been carried on with the aid of Stechert's formulae or of a set of formulae which were developed by the writer. These present some advantages because of better symmetry and clarity and of the somewhat quicker computation they allow. On the other hand they do not afford a control of computation as good as those

[1]) C. Stechert: Azimutbestimmung aus Durchgangsbeobachtungen. Hamburg, 1913.

[2]) F. Kępiński: Kilka metod astronomicznego wyznaczania azymutu. (Publikacja Z. A. P. Nr 5). Warszawa, 1931.

of Stechert do. For this reason I chose Stechert formulae for my reductions. Both sets of formulae were only applied to the observations of a small number of evenings and yielded identical results.

Both sets of formulae for the pair method are developed as a consequence of an observation of the transits of two stars over the same vertical circle.

Let us assume the following designations:

a the azimuth of the vertical circle of the central wire

α_1, δ_1, t_1 the equatorial co-ordinates and the hour angle of the first star

α_2, δ_2, t_2 the equatorial co-ordinates and the hour angle of the second star

φ the latitude of the point of observation

T_1, T_2 clock indications corresponding to the transits of the stars

u clock correction.

Then:

$$\operatorname{tg} a = -\frac{\cos\delta_1 \sin t_1}{\sin\delta_1 \cos\varphi - \cos\delta_1 \sin\varphi \cos t_1}$$
$$\operatorname{tg} a = -\frac{\cos\delta_2 \sin t_2}{\sin\delta_2 \cos\varphi - \cos\delta_2 \sin\varphi \cos t_2}. \qquad (1)$$

Hence the Stechert formulae result in the following way. After elimination of a and some simple transformations we introduce the new quantities l, L by the definition:

$$l \cdot \sin L = \sin(\delta_2 + \delta_1) \cdot \sin \tfrac{1}{2}(t_2 + t_1)$$
$$l \cdot \cos L = \sin(\delta_2 - \delta_1) \cdot \cos \tfrac{1}{2}(t_2 - t_1). \qquad (2)$$

Then we have (for more detailed demonstration see: Appendix, 1):

$$\operatorname{tg} L = \frac{\sin(\delta_2 + \delta_1)}{\sin(\delta_2 - \delta_1)} \cdot \operatorname{tg}\left[\tfrac{1}{2}(T_2 - T_1) - \tfrac{1}{2}(\alpha_2 - \alpha_1)\right] \qquad (3)$$

$$\sin\left[\tfrac{1}{2}(t_2 + t_1) - L\right] = \frac{\operatorname{tg}\varphi}{\operatorname{tg}\delta_1} \cdot \sin\left[\tfrac{1}{2}(T_2 - T_1) - \tfrac{1}{2}(\alpha_2 - \alpha_1) - L\right] \qquad (4')$$

$$\sin\left[\tfrac{1}{2}(t_2 + t_1) - L\right] = \frac{\operatorname{tg}\varphi}{-\operatorname{tg}\delta_2} \cdot \sin\left[\tfrac{1}{2}(T_2 - T_1) - \tfrac{1}{2}(\alpha_2 - \alpha_1) + L\right) \qquad (4'')$$

$$\operatorname{tg} a = \frac{\operatorname{tg}\left[\frac{1}{2}(t_2 + t_1) - L\right]}{\sin \varphi}. \tag{5}$$

The computation after these formulae run as follows: from (2) or (3) we find L; then from (4′) or (4″) we find $[\frac{1}{2}(t_2+t_1)-L]$, hence $\frac{1}{2}(t_2+t_1)$ and then $u=\frac{1}{2}(t_2+t_1)-\frac{1}{2}[(T_2-\alpha_2)+(T_1-\alpha_1)]$; finally (5) gives us a.

Still other Stechert designations will be used in the continuation and in computing schemes, namely:

$$\begin{aligned} t_1 - u &= T_1 - \alpha_1 = \lambda_1 \\ t_2 - u &= T_2 - \alpha_2 = \lambda_2 \end{aligned} \tag{6}$$

$$t_1 - t_2 = (T_1 - T_2) - (\alpha_1 - \alpha_2) = \lambda_1 - \lambda_2 = \vartheta. \tag{7}$$

As I have sought to replace the Stechert formulae (2) — (5) by somewhat clearer ones, I have introduced instead of l, L, other auxiliary quantities, n, N, which were defined by the equalities:

$$\begin{aligned} n \cdot \sin N &= \sin \lambda_1 \operatorname{tg} \delta_2 - \sin \lambda_2 \operatorname{tg} \delta_1 \\ n \cdot \cos N &= \cos \lambda_1 \operatorname{tg} \delta_2 - \cos \lambda_2 \operatorname{tg} \delta_1 . \end{aligned} \tag{8}$$

After elimination of a from (1) and after easy transformations (see: Appendix, 2) we get one equation with one unknown u:

$$\sin t_1 \operatorname{tg} \delta_2 - \sin t_2 \operatorname{tg} \delta_1 = \operatorname{tg} \varphi \cdot \sin (t_1 - t_2)$$

or, by virtue of (6) and (7):

$$\begin{aligned} &(\sin \lambda_1 \operatorname{tg} \delta_2 - \sin \lambda_2 \operatorname{tg} \delta_1) \cos u + \\ &+ (\cos \lambda_1 \operatorname{tg} \delta_2 - \cos \lambda_2 \operatorname{tg} \delta_1) \sin u = \sin \vartheta \cdot \operatorname{tg} \varphi, \end{aligned}$$

whence, on account of (8):

$$n \cdot \sin (N + u) = \sin \vartheta \cdot \operatorname{tg} \varphi. \tag{9}$$

By some transformations and by application of formulae of spherical trigonometry it can be shown (Appendix, 2), that:

$$\cos t_1 \operatorname{tg} \delta_2 - \cos t_2 \operatorname{tg} \delta_1 = \frac{\cos a \cdot \sin (z_1 \mp z_2)}{\cos \delta_1 \cos \delta_2}$$

$$\sin t_1 \operatorname{tg} \delta_2 - \sin t_2 \operatorname{tg} \delta_1 = \frac{\sin a \cdot \sin \varphi \cdot \sin (z_1 \mp z_2)}{\cos \delta_1 \cos \delta_2},$$

whence, after having divided the second equation by the first one:

$$\operatorname{tg} a \cdot \sin \varphi = \frac{\sin t_1 \operatorname{tg} \delta_2 - \sin t_2 \operatorname{tg} \delta_1}{\cos t_1 \operatorname{tg} \delta_2 - \cos t_2 \operatorname{tg} \delta_1}$$

and, by a short transformation and in view of (8), we get finally;

$$\operatorname{tg} a = \frac{\operatorname{tg} (N+u)}{\sin \varphi}. \qquad (10)$$

The course of a numerical calculation according to these formulae is as follows: at first λ_1 and λ_2 are to be found according to (6) and ϑ according to (7); then (8) give us the values of n and N; from (9) we find $N+u$, whence the clock correction u; finally we get a by the formula (10).

The formulae (8) — (10) are rather convenient for computation with the aid of an arithmometer, but they do not include any control analogous to that presented by the formulae (4′) and (4″) which afford a double determination of the angle $\frac{1}{2}(t_2+t_1)-L$. Therefore, I preferred to make the reductions of my observations with the aid of Stechert formulae (2)—(5). I used, however, both series of formulae for reductions of several preliminary observations and of the observation of 1939.I.31. The full conformity of the results is illustrated by an example quoted on page 77. Stechert recommends to adopt for a further calculation the mean of both values of $\frac{1}{2}(t_2-t_1)-L$, as obtained by (4′) and (4″). As for the equatorial stars, however, there is a small quantity in the denominator of the corresponding fraction, so the formula is not reliable in such cases. Therefore a discordance of both values can result even from a faultless reckoning. In such cases it is reasonable to use one of the formulae (4′), (4″) only, namely that which includes the declination of a northern star.

3. Reduction of transits over lateral positions of wires.

For extra-meridian transit observations a peculiar difficulty arises while reducing moments of transits over lateral wires to the moment of transit over the central wire. When observations are becing made in the way that every star is observed in both positions of the circle, then there is no need of any reduction to the central wire in the litteral sense, for the arithmetical mean of the moments T_E and T_W of transits through the same wire gives approximately the moment of transit through the fictitious mean wire. As, however, in cases of extra-meridian observations the parallels of declination are not symmetrical in relation to the maen wire, the mean moment $\frac{1}{2}(T_E + T_W)$ is not identical with the moment of transit through the mean wire and in some cases needs a correction. Let us examine this problem in detail in order to deduce some practical consequences and to get the reduction formulae.

G_1, G, G_2 (Fig. 1) represent three consecutive positions of a star on the western hemisphere; ZG — the istrument's vertical circle i. e. the vertical circle of the fictitious central wire;

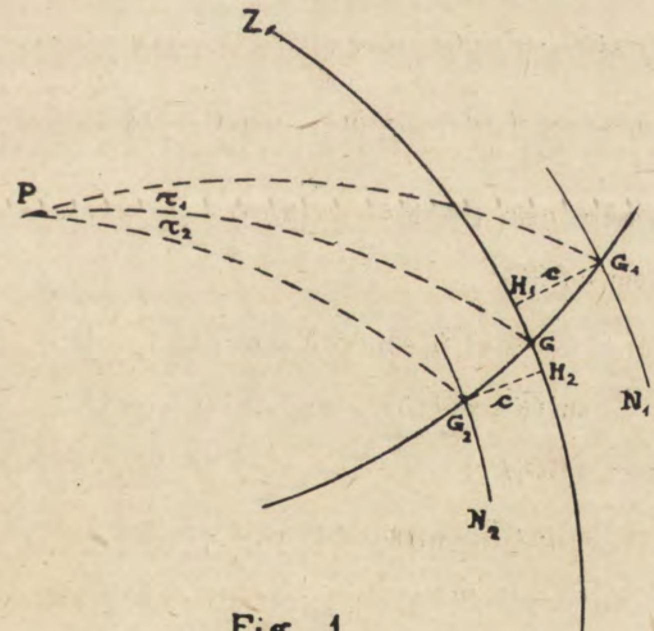


Fig. 1

$G_1 N_1$ and $G_2 N_2$ — the projections on the celestial sphere of the same lateral wire in both positions of the circle. The angular distance of the two circles $G_1 N_1$ and $G_2 N_2$ is $2c$; $\widetilde{G_1 H_1} = c$

and $\overline{G_2 H_2} = c$ are arcs of great circles perpendicular to the vertical circle ZG. The angles: $\widehat{G_1 PG} = \tau_1$ and $\widehat{GPG_2} = \tau_2$ are the differences of hour angles of the star when in G_1, G and G_2. The angle $\widehat{PGZ} = q$ is the parallactic angle of the star in its position in G. T_1 and T_2 denote the clock indications corresponding to the positions G_1 and G_2 of the star.

From $\Delta\, PGG_1$:

$$\begin{aligned} \sin \overline{GG_1} \cdot \sin \widehat{PGG_1} &= \cos \delta \cdot \sin \tau_1 \\ \sin \overline{GG_1} \cdot \cos \widehat{PGG_1} &= \sin \delta \cdot \cos \delta \cdot (1 - \cos \tau_1) \end{aligned} \tag{11'}$$

From $\Delta\, GG_1 H_1$:

$$\sin \overline{GG_1} \cdot \sin (\widehat{PGG_1} - q) = \sin c$$

or

$$\sin \overline{GG_1} \cdot (\sin \widehat{PGG_1} \cos q - \cos \widehat{PGG_1} \sin q) = \sin c. \tag{11''}$$

The elimination of $\overline{GG_1}$ and $\widehat{PGG_1}$ fron (11″) by the aid of (11′) gives us:

$$\cos \delta \cos q \sin \tau_1 - \sin \delta \operatorname{con} \delta \sin q \,(1 - \cos \tau_1) = \sin c. \tag{11}$$

From $\Delta\, PGG_2$:

$$\begin{aligned} \sin \overline{GG_2} \sin \widehat{PGG_2} &= \cos \delta \sin \tau_2 \\ \sin \overline{GG_2} \cos \widehat{PGG_2} &= \sin \delta \cos \delta \,(1 - \cos \tau_2). \end{aligned} \tag{12'}$$

From $\Delta\, GG_2 H_2$:

$$\sin \overline{GG_2} \sin (180^0 - \widehat{PGG_2} - q) = \sin c$$

or

$$\sin \overline{GG_2} (\sin \widehat{PGG_2} \cos q + \cos \widehat{PGG_2} \sin q) = \sin c. \tag{12''}$$

The elimination of $\overline{GG_2}$ and $\widehat{PGG_2}$ from (12″) by the aid of (12′) gives us:

$$\cos \delta \cos q \sin \tau_2 + \sin \delta \operatorname{con} \delta \sin q \,(1 - \cos \tau_2) = \sin c. \tag{12}$$

If we want to know the clock indication T, corresponding to the moment of transit through the fictitious central wire, i. e. through the instrumental vertical circle ZG, we have to add the correction τ_1 to T_1 or to subtract the correction τ_2 from T_2:

$$T = T_1 + \tau_1$$

$$T = T_2 - \tau_2,$$

whence

$$T = \frac{T_1 + T_2}{2} + \frac{\tau_1 - \tau_2}{2}.$$

It is obvious from (11) and (12) that exact values of τ_1 and τ_2 are not identical, or else that the correction $\frac{\tau_1 - \tau_2}{2}$ of the mean moment is, in general, different from 0. In order to examine this correction and to develop practical formulae for it let us subtract (12) from (11). Then we get:

$$\cos\delta \cos q (\sin\tau_1 - \sin\tau_2) - \sin\delta \cos\delta \sin q\, [(1 - \cos\tau_1) + + (1 - \cos\tau_2)] = 0,$$

whence:

$$\sin\tau_1 - \sin\tau_2 = \sin\delta \operatorname{tg} q\, [(1 - \cos\tau_1) + (1 - \cos\tau_2). \qquad (13)$$

It can be easily seen, that in the case $q > 90^0$ (on the western hemisphere this case occurs with stars having declinations $\delta > \varphi$ during the time interval between their upper transit and western elongation) in the formulae (11) and (12) appears the sign — at sinc; the formula (13), however, remains unaltered. From (13) we see, that for stars with declinations $\delta > 0$ the following inequalities are in force:

1^0. If $0 < q < 90^0$ then $\sin\tau_1 - \sin\tau_2 > 0$ or $\tau_2 < \tau_1$ or $\frac{\tau_1 - \tau_2}{2} > 0$

(14)

2^0. If $90^0 < q < 180^0$ then $\sin\tau_1 - \sin\tau_2 < 0$ or $\tau_1 < \tau_2$ or $\frac{\tau_1 - \tau_2}{2} < 0$.

Fig. 1 and the formulae (11) and (12) refer to a case, when G is on the western hemisphere of the celestial sphere (west of the local meridian). It can be easily seen that for a position G East of the meridian the analogous formulae can be obtained by merely changing the indices at τ. It must be kept in mind, that G_1 denotes in any case a prior position and G_2 — a later position of a star. The same remark applies to the formula (13) and to the inequality (14); the signs of corrections $\frac{\tau_1-\tau_2}{1}$ will therefore be opposite.

On the strength of the above argumentation the following practical rule can be formulated, taking into account the circumstance that stars are beeing observed on both sides of the zenith, i. e. on the werstern as well as on the eastern hemisphere:

I. If the instrument's (object's) azimuth is contained within the limits

$$0 < a < 90^0, \quad \text{then:}$$

$\frac{\tau_1-\tau_2}{2} > 0$ for stars South of the zenith and for those northern stars which were observed on the upper part of their parallels (after eastern elongation);

$\frac{\tau_1-\tau_2}{2} < 0$ for those northern stars which were observed on lower. parts of their parallels (before eastern elongation).

II. If the instrument's azimuth is contained within the limits

$$-90^0 < a < 0, \quad \text{then:}$$

$\frac{\tau_1-\tau_2}{2} < 0$ for stars South of the zenith and for those northern stars which were observed on the upper parts of their parallels (before western elongation);

$\frac{\tau_1-\tau_2}{2} > 0$ for those northern stars which were observed on lower parts of their parallels (after western elongation).

Now we are going to develop a formula for the correction $\frac{\tau_1-\tau_2}{2}$. The formula (13) can be written in the form:

$$2\sin\frac{\tau_1-\tau_2}{2}\cos\frac{\tau_1+\tau_2}{2} = \sin\delta\,\mathrm{tg}\,q\left(2-2\cos\frac{\tau_1+\tau_2}{2}\cos\frac{\tau_1-\tau_2}{2}\right). \quad (13')$$

As τ_1 and τ_2 are small quantities of the 1st order, so $\frac{\tau_1+\tau_2}{2}$ is a small quantity of the 1st order too; on the other hand, we see from (13′), that $\frac{\tau_1-\tau_2}{2}$ is a small quantity of the 2nd order. Let us adopt, for brevity, the notations:

$$\frac{\tau_1-\tau_2}{2}=x; \qquad \frac{\tau_1+\tau_2}{2}=y;$$

then we have by virtue of (13′):

$$2\sin x\cdot\cos y=2\sin\delta\,\mathrm{tg}\,q\,(1-\cos x\cdot\cos y).$$

If we replace $\sin x$, $\cos x$ and $\cos y$ by their developements and neglect all terms of order higher than the 4th, we have:

$$x\left(1-\frac{x^2}{6}\right)\left(1-\frac{y^2}{2}+\frac{y^4}{24}\right)\cong\sin\delta\,\mathrm{tg}\,q\left[1-\left(1-\frac{x^2}{2}\right)\left(1-\frac{y^2}{2}+\frac{y^4}{24}\right)\right]$$

$$x\cong\sin\delta\,\mathrm{tg}\,q\cdot\frac{1-1+\frac{y^2}{2}-\frac{y^4}{24}+\frac{x^2}{2}-\frac{x^2y^2}{4}+\frac{x^2y^4}{48}}{1-\frac{y^2}{2}+\frac{y^4}{24}-\frac{x^2}{6}+\frac{x^2y^2}{12}-\frac{x^2y^4}{144}}$$

$$x\cong\sin\delta\,\mathrm{tg}\,q\left(\frac{y^2}{2}-\frac{y^4}{24}+\frac{x^2}{2}\right)\left(1+\frac{y^2}{2}-\frac{y^4}{24}+\frac{x^2}{6}+\frac{y^4}{4}\right)\cong$$

$$\cong\sin\delta\,\mathrm{tg}\,q\left(\frac{y^2}{2}-\frac{y^4}{24}+\frac{x^2}{2}+\frac{y^4}{4}\right)$$

$$x\cong\sin\delta\,\mathrm{tg}\,q\left[\frac{y^2}{2}+\left(\frac{5}{24}y^4+\frac{1}{2}x^2\right)\right].$$

If we replace x in the second member of the last equality by $\sin\delta\,\mathrm{tg}\,q\cdot\frac{y^2}{2}$, we get:

$$x\cong\sin\delta\,\mathrm{tg}\,q\left[\frac{y^2}{2}+\left(\frac{5}{24}y^4+\frac{1}{2}\sin^2\delta\,\mathrm{tg}^2 q\cdot\frac{y^4}{4}\right)\right]$$

$$x\cong\sin\delta\,\mathrm{tg}\,q\left[\frac{y^2}{2}+\frac{1}{8}\left(\frac{5}{3}+\sin^2\delta\,\mathrm{tg}^2 q\right)\cdot y^4\right]. \tag{17}$$

$\tau_1 + \tau_2$ is the difference of star's hour angles corresponding to transits over the same wire in both positions of the instrument. $\tau_1 + \tau_2$ is therefore a quantity which can be obtained directly from observations as the difference ΔT of the clock indications at the two transit moments. If x and y are expressed in seconds of time, we have by virtue of (17):

$$\left(\frac{\tau_1 - \tau_2}{2}\right)^{s} = \frac{\sin 1^s \cdot \sin \delta \operatorname{tg} q}{8} \cdot \Delta T^2 +$$

$$+ \frac{\sin \delta \operatorname{tg} q \cdot \left(\frac{5}{3} + \sin^2 \delta \operatorname{tg}^2 q\right)}{128} \cdot \sin^3 1^s \cdot \Delta T^4. \qquad (18)$$

It is obvious that the formula (18) may be failing in cases of high values of $\operatorname{tg} q$. This danger can be prevented in anticipation when attention is paid to the rule that stars for which $q > 45^0$ should not be included in observing lists. Besides, a similar limitation is imposed also by technical regards of observation (too slow a motion of a star in azimuth, too great an inclination of star path relatively to the horizontal wires).

The formula (18) becomes simple and convenient enough when the 4[th] order term of the second member is omitted. Let this term be denoted by W_4. The two tables, Tabl. 1 and Tabl. 2 are computed for geographical latitude $\varphi \cong 52^0 13'$ (Warsaw). They afford some bearing as to values of parallactic angles q and, moreover, they let us to know within what limits of star azimuth and declination the term W_4 may be omitted without loss of accuracy.

Tabl. 1 contains values of q calculated from $\sin q = \dfrac{\cos\varphi \cdot \sin a}{\cos \delta}$.
The horizontal argument is the instrumental azimuth; the vertical argument is the star's declination. Some stars cross a given vertical circle twice above the horizon. The thick horizontal bars are marking those declination values above which the stars present twofold transits above the horizon.

Tabl. 2 is arranged according to the argument a (instrumental azimuth). It shows declination and parallactic angle values, $\overline{\delta}$, $\overline{q}$, which are adopted according to Tabl. 1 (though to a certain grade arbitrarily) as maximal values still allowed for practical use. Besides, Tabl. 2 contains approximate values of the reduction τ_0 to the central wire. τ_0 is calculated approximately according to the formula:

$$\sin\tau_0 = \sin c \cdot \sec\bar{\delta} \cdot \sec\bar{q}$$

under the assumption, that c, i. e. the distance of the extreme position of the mobile wire from its central position, has reached the greatest possible (the most disadvantageous) value of 4 revolutions of the micrometer screw:

$$c = 1^{\mathrm{m}} = 15'.$$

Tabl. 1.

a / δ	10°	20°	30°	40°	50°	60°	70°	80°	90°
−30°	7°.0	14°.0	20°.7						
−20	6.5	12.9	19.0	24°.8	29°.9				
−10	6.2	12.3	18.1	23.6	28.4	32°.6	35°.8		
0	6.1	12.1	17.8	23.2	28.0	32.1	35.1	37°.0	37°.8
+10	6.2	12.3	18.1	23.6	28.4	32.6	35.8	37.8	38.5
+20	6.5	12.9	19.0	24.8	29.9	34.4	37.8	39.9	40.7
+30	7.0	14.0	20.7	27.0	32.7	37.8	41.6	44.0	44.9
+40	8.0	15.9	23.6	30.9	37.8	43.9	48.7	51.9	53.1
+50	9.5	19.0	28.4	37.8	46.8	55.7	63.6	69.6	72.3
+60	12.3	24.8	37.8	51.9	69.6				
+70	18.3	37.8	63.6						
+75	24.3	53.0							
+80	37.8								

Tabl. 2.

a	10°	20°	30°	40°	50°	60°	70°	80°	90°
$\bar{\delta}$	+75°	+65°	+60°	+50°	+40°	+40°	+30°	+30°	+30°
$\bar{q}$	24°.3	29°.7	37°.8	37°.8	37°.8	43°.9	41°.6	44°.0	44°.9
τ_0''	3816	3454	2280	1770	1488	1632	1386	1446	1470
$10^6 W_4^s$	198	42	44	14	5	10	4	5	5

In the last line of Tabl. 2 there are confronted the values of the term W_4 from the formula (18) as calculated on the assumptions:

$$\delta = \bar{\delta}; \quad q = \bar{q}; \quad \Delta T \simeq 2\tau_0.$$

Tabl. 2 shows that, in the adopted limits for δ, the term W_4 does not reach the value $0^s.001$ even in most disadvantageous cases; moreover, it assumes for the most part far lower values than that. It results that W_4 can be omitted without any loss of accuracy in cases of star declinations not exceeding the respective values $\bar{\delta}$. Limits of allowed omission of W_4 could be even still enlarged. We can therefore safely adopt the following, rather simple and convenient working formula:

$$\left(\frac{\tau_1-\tau_2}{2}\right)^s = \frac{\sin 1^s}{8} \cdot \sin\delta \operatorname{tg} q \cdot \Delta T^2, \tag{19}$$

which affords the correction to be added with a proper sign to the mean moment $\frac{T_1+T_2}{2}$ in order to obtain the wanted moment of the transit over the mean position of the wire.

Even this correction becomes exceedingly small for low values of δ (and q). For the observation data which were used for this work the correction values exceeded $0^s.01$ for northern stars only ($\delta > \varphi$). For a half of observed stars there was therefore no need of applying even the simplified formula (19).

Formula (19) can be written in a different shape. Niethammer[3]) develops the reduction formula in the form:

$$(t_0-\bar{t})^{sec} = -\frac{2\sin^2\frac{\vartheta}{2}}{15\sin 1''}\operatorname{cotg}(\mu-\bar{t}). \tag{19'}$$

Here μ denotes the hour angle of a point wchich is the pole for the instrumental equator; $\vartheta = \frac{\Delta T}{2}$; $\bar{t} = \frac{T_1+T_2}{2}$. Both forms (19) and (19') are concordant, since it can be easily proved that

$$\operatorname{cotg}(\mu-\bar{t}) = -\sin\delta \cdot \operatorname{tg} q.$$

The form (19) is inasmuch more convenient that (19') as it does not require finding μ; the value of q must be known anyhow in order to make allowance for axis inclunation. Moreover, the deduction of (19) from (13) over (17) and (18) affords an obvious way for approximation estimates.

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[3]) Th. Niethammer: Die genauen Methoden der astronomisch-georgaphischen Ortsbestimmung, Basel, 1947, p. 40.

4. The discussion of the differential formulae.

For examining the question of the most profitable choice of stars for azimuthal observations we start from the formula:

$$\Delta a = -\frac{\sin a}{\operatorname{tg} z}\Delta\varphi + \frac{\cos\delta\cos q}{\sin z}(\Delta T + \Delta u) - \frac{\cos q\cos\delta}{\sin z}\Delta\alpha + + \frac{\sin q}{\sin z}\Delta\delta, \tag{20}$$

which results from (1) by differentiation.

For the sake of azimuth determinations by a transit of one single star when clock correction is known, stars are to be observed at considerable zenithal distances. For at small zenithal distances the errors $\Delta\varphi$, ΔT and Δu of the adopted values of φ, T and u are in the formula (20) multiplied by great coefficients. Similarly are things for $\Delta\alpha$, $\Delta\delta$ — errors of star position as taken from a catalogue.

Now we have to develop the differential formulae for the method of simultaneous determination of a and u by observation of transits of two stars over the instrumental vertical circle. For that purpose let us multiply both membres of (20) by sinz and shift the term with Δu to the first member. Then let us apply the so transformed equality (20) to both stars of a pair. Symbols concerning the two stars are discriminated by the indices 1 and 2 respectively. We obtain a system of two linear equations for the errors Δa and Δu for the pair method:

$$\begin{aligned} \sin z_1\cdot\Delta a - (\cos\delta_1\cos q_1)\,\Delta u &= -\sin a_1\cos z_1\cdot\Delta\varphi + \\ &+ \cos\delta_1\cos q_1\cdot\Delta T_1 - \cos\delta_1\cos q_1\cdot\Delta\alpha_1 + \sin q\cdot\Delta\delta_1 \\ \sin z_2\cdot\Delta a - (\cos\delta_2\cos q_2)\,\Delta u &= -\sin a_2\cos z_2\cdot\Delta\varphi + \\ &+ \cos\delta_2\cos q_2\cdot\Delta T_2 - \cos\delta_2\cos q_2\cdot\Delta\alpha_2 + \sin q\cdot\Delta\delta_2 . \end{aligned} \tag{21}$$

The two stars cross the vertical circle of the instrument on the same side of the zenith or on both sides of it. If we put $a_1 = a$, there will be $a_2 = a$ or $a_2 = a + 180^0$. We have therefore:

$$\sin a_1 = \sin a$$

$$\sin a_2 = \pm\sin a .$$

In the last equality as well as in the further developments the upper or the lower sign is in force according to whether both star transits occur on the same side or on the opposite sides of the zenith.

The determinant of the system (21) is

$$\begin{aligned} D &= -\sin z_1 \cos \delta_2 \cos q_2 + \sin z_2 \cos \delta_1 \cos q_1 = \\ &= -\sin z_1 (\sin \varphi \sin z_2 \pm \cos \varphi \cos z_2 \cos a) + \\ &+ \sin z_2 (\sin \varphi \sin z_1 + \cos \varphi \cos z_1 \cos a) = \\ &= \cos \varphi (\sin z_2 \cos z_1 \mp \cos z_2 \sin z_1) \cdot \cos a \end{aligned}$$

$$D = \cos \varphi \cdot \cos a \cdot \sin (z_2 \mp z_1). \tag{22}$$

From the system (21) we obtain:

$$D \cdot \Delta a = \begin{vmatrix} \begin{matrix}(-\sin a \cos z_1 \Delta\varphi + \cos \delta_1 \cos q_1 \Delta T_1 - \\ -\cos \delta_1 \cos q_1 \Delta\alpha_1 + \sin q_1 \Delta\delta_1),\end{matrix} & (-\cos \delta_1 \cos q_1) \\ \begin{matrix}(\mp \sin a \cos z_2 \Delta\varphi + \cos \delta_2 \cos q_2 \Delta T_2 - \\ -\cos \delta_1 \cos q_2 \Delta\alpha_2 + \sin q_2 \Delta\delta_2),\end{matrix} & (-\cos \delta_2 \cos q_2) \end{vmatrix}$$

$$D \cdot \Delta u = \begin{vmatrix} \sin z_1, & \begin{matrix}(-\sin a \cos z_1 \Delta\varphi + \cos \delta_1 \cos q_1 \Delta T_1 - \\ -\cos \delta_1 \cos q_1 \Delta\alpha_1 + \sin q_1 \Delta\delta_1)\end{matrix} \\ \sin z_2, & \begin{matrix}(\mp \sin a \cos z_2 \Delta\varphi + \cos \delta_2 \cos q_2 \Delta T_2 - \\ -\cos \delta_2 \cos q_2 \Delta\alpha_2 + \sin q_2 \Delta\delta_2)\end{matrix} \end{vmatrix}$$

and after some transformations

$$\begin{aligned} D \cdot \Delta a &= \sin a \sin \varphi \sin (z_2 \mp z_1) \Delta\varphi - \\ &- \cos \delta_1 \cos \delta_2 \cos q_1 \cos q_2 (\Delta T_1 - \Delta T_2) + \\ &+ \cos \delta_2 \cos q_2 [\cos q_1 (\cos \delta_1 \Delta\alpha_1) - \sin q_1 \Delta\delta_1] - \\ &- \cos \delta_1 \cos q_1 [\cos q_2 (\cos \delta_2 \Delta\alpha_2) - \sin q_2 \Delta\delta_2] \end{aligned}$$

$$\begin{aligned} D \cdot \Delta u &= \sin a \sin (z_2 \mp z_1) \Delta\varphi + [\sin z_1 \sin z_2 \sin \varphi (\Delta T_2 - \Delta T_1) - \\ &- \cos \varphi \cos a (\sin z_2 \cos z_1 \Delta T_1 \mp \sin z_1 \cos z_2 \Delta T_2)] + \\ &+ \sin z_2 [\cos q_1 (\cos \delta_1 \Delta\alpha_1) - \sin q_1 \Delta\delta_1] + \\ &+ \sin z_1 [\cos q_2 (\cos \delta_2 \Delta\alpha_2) - \sin q_2 \Delta\delta_2]. \end{aligned} \tag{23}$$

Let us denote for brevity the terms with errors of catalogue co-ordinates of stars by P_a and P_u respectively. If we divide the equalities (23) by D and replace D by the expression (22), we obtain:

$$\Delta a = \operatorname{tg} a \operatorname{tg} \varphi \cdot \Delta\varphi - \frac{\cos\delta_1 \cos\delta_2 \cos q_1 \cos q_2}{\cos\varphi \cos a \sin(z_2 \mp z_1)} (\Delta T_2 - \Delta T_1) + $$
$$+ \frac{P_a}{\cos\varphi \cos a \sin(z_2 \mp z_1)} \tag{24}$$
$$\Delta u = \frac{\operatorname{tg} a}{\cos\varphi} \Delta\varphi + \left[\frac{\operatorname{tg}\varphi \sin z_1 \sin z_2}{\cos a} \cdot (\Delta T_2 - \Delta T_1) - \right.$$
$$\left. - \frac{\sin z_2 \cos z_1 \Delta T_1 \mp \sin z_1 \cos z_2 \Delta T_2}{\sin(z_2 \mp z_1)} \right] + \frac{P_u}{\cos\varphi \cos a \sin(z_2 \mp z_1)}.$$

It follows from (24) that the method of simultaneous determination of azimuth and time is not advantageous in a case when the earthly object to be observed is situated near the prime vertical, since $\operatorname{tg} a$ and $\sec a$ assume high values. If the azimuth value is considerable, the latitude of the observing point ought to be known with high accuracy. The co-efficients of $\Delta\varphi$ are, for instance, for $\varphi \cong 52^0$; $a \cong 45^0$:

$$\operatorname{tg} a \cdot \operatorname{tg}\varphi = 1 \cdot 1.28 = 1.28; \quad \frac{\operatorname{tg} a}{\cos\varphi} = \frac{1}{0.616} = 1.62,$$

whereas for $\varphi \cong 52^0$; $a \cong 10^0$ they are:

$$\operatorname{tg} a \cdot \operatorname{tg}\varphi = 0.176 \cdot 128 = 0.225; \quad \frac{\operatorname{tg} a}{\cos\varphi} = \frac{0.176}{0.616} = 0.286.$$

At lower latitudes and for smaller absolute values af a the co-efficients are more convenient.

The influence on Δa of errors in clock indications can be lowered by such a choice of a star pair, that the declination of one of the stars have a rather high value and that the sum $z_1 + z_2$ be near 90^0 (as the second star ought to be observed on the opposite side of the zenith, this condition is formulated here with reference to the lower sign in (24)). It does not follow from formulae (24) that a star must not be observed at a small zenithal distance, but **both** zenithal distances must not be small. According to the secod formula (24) the influence of ΔT_1 and ΔT_2 on Δu grows smaller with lower values of z_1 or z_2. But because of axis inclination errors small zenithal distances are to be avoided. An important advantage of the method lies in the property that a systhematical error in clock indications does not influence Δa, since $(\Delta T_2 - \Delta T_1)_{syst} = 0$. As to the influence of ΔT_{syst} of Δu it can be easily seen that the first term in the square brackets in the second formula (24) vanishes; the second term, on the other hand, takes the shape:

$$-\frac{\sin z_2 \sin z_1 \mp \sin z_1 \cos z_2}{\sin (z_2 \mp z_1)} \cdot \Delta T_{syst} = -\Delta T_{syst}.$$

The whole of the systhematical errer of clock indications enters with the opposite sign in the clock correction to be determined. Besides, this is obvious beforehand.

The expressions for P_a and P_u are rather complicated. They can be replaced by somewhat simpler ones if we pass fram ordinary errors to mean errors. For this purpose we have to replace the errors $\Delta\varphi, \Delta T_1, \Delta\alpha_1, \Delta\delta_1, \ldots$ by the mean errors $\varepsilon_\varphi, \varepsilon_{T_1}, \varepsilon_{\alpha_1}, \varepsilon_{\delta_1}, \ldots$ and the algebraic sum in (24) by a sum of squares of corresponding terms. Let ε_α and ε_δ denote mean errors of catalogue position (co-ordinates) of a star; we have:

$$\cos^2\delta \cdot \varepsilon_\alpha^2 = \varepsilon_\delta^2 = \varepsilon^2. \tag{25}$$

Let Σ_a and Σ_u denote the position terms in the two expressions (corresponding to (24)) for the squares of mean errors of a and u. The expressions for Σ_a and Σ_u can be easily deduced from (23):

$$\sum\nolimits_a = \frac{1}{D^2} \{\cos^2\delta_2 \cos^2 q_2 [\cos^2 q_1 (\cos^2\delta_1 \varepsilon_{\alpha_1}^2) + \sin^2 q_1 \varepsilon_{\delta_1}^2] +$$

$$+ \cos^2\delta_1 \cos^2 q_1 [\cos^2 q_2 (\cos^2\delta_2 \varepsilon_{\alpha_2}^2) + \sin^2 q_2 \varepsilon_{\delta_2}^2]\}$$

$$\sum\nolimits_u = \frac{1}{D^2} \{\sin^2 z_2 [\cos^2 q_1 (\cos^2\delta_1 \varepsilon_{\alpha_1}^2) + \sin^2 q_1 \varepsilon_{\delta_1}^2] +$$

$$+ \sin^2 z_1 [\cos^2 q_2 (\cos^2\delta_2 \varepsilon_{\alpha_2}^2) + \sin^2 q_2 \varepsilon_{\delta_2}^2]\}$$

or, when (25) is taken into account:

$$\sum_a = \frac{\cos^2 \delta_2 \cos^2 q_2 + \cos^2 \delta_1 \cos^2 q_1}{\cos^2 \varphi \cos^2 a \cdot \sin^2 (z_2 \mp z_1)} \cdot \varepsilon^2 ;$$

$$\sum_u = \frac{\sin^2 z_2 + \sin^2 z_1}{\cos^2 \varphi \cos^2 a \cdot \sin^2 (z_2 \mp z_1)} \cdot \varepsilon . \qquad (26)$$

A pair transit observation can be used in a still different way for determining φ and a, if the clock correction is known (method „B"). In this method the accuracy of azimuth does not depend upon the azimuth value. The clock correction, however, must be determined independently. Auording to Niethammer[4]) one observation of a Zinger pair, coupled with one of a pair for determining a and φ by method „B", is equivalent as to accuracy of a with one pair observation for determining a und u (method „A") under the condition that $|a|$ does not surpass some 30° or 35° (at a station with well determined φ of moderate value).

Besides Niethammer[5]) discusses the problem of limits for azimuth in method „A", taking into account the influence of an error in φ on the error in a (see (24). He puts the condition that the m. e. in a arising from the m. e. in φ would not exceed $\pm 0''.0.5$. For $\varphi = 45^0$ he obtains the following limits for a depending on ε_φ.

ε_φ	a
$\pm 0''.05$	$\pm 45^0$
± 0.10	± 27
± 0.20	± 14

He argues, too, for the superiority of the direct methods „A" and „B" over the indirect Polaris method by confronting some simplified formulae for m. errors in both methods, indirect and direct.

5. The instrumental equipment.

The data of observation which were used as raw material for this paper were obtained with the aid of the following set of instruments:

1°. A portable transit instrument of the type of prism transit, made by the „Askania-Werke", model A. P. 50, where the object glass is mounted at one end of the axis and the eyepiece — at the other end; the istrument was standing on a solid underlying of Döllen and was provided with a striding level.

[4]) l. c. p. 125. [5]) l. c. p. 123.

2°. A transit micrometer „Askania-Werke"; the angular value of 1 revolution of its screw was 15^s; on the perimeter of the drum there were 10 contacts, one of them being triple.

3°. A needle chronograph giving half second markings in intervals of 1^{cm}.

4°. A distribution table fitted out with two relays made by „Siemens".

5°. A wireless receiver made by the „Polskie Zakłady Tele- i Radiotechniczne" for receiving the rhythmic time signals.

6°. A contact chronometer „Nardin" Nr 2103, beating half seconds.

I determined in the autumn of 1938 the value of 1^p of the striding level as a result of numerous series of measurements with the aid of a comparator made by „Maleyko". The adopted final value is

$$1^p = 1''.7805 = 0^s.1187.$$

From meridian transits of stars through the mobile wire I determined the value of 1^p of the transit micrometer drum. The final value is:

$$1^p = 2''.2893.$$

The records and reductions concerning those determinations have been lost together with a part of the library and archives of the Technical University Institute for Practical Astronomy which had been destroyed in a barbarian way by the Germans. Therefore the mean errors wchich should be ascribed to the above values cannot be quoted here. The same had been the destiny of several pieces of chronograph lent and of observation records concerning the period May—July 1939. Thus about a half of the observing data have been lost which might have been used in the present work.

I made an examination of the axis pivots with the help of the striding level. As a result of several series of measurements, the inequality of the pivots turned out to be smaller than the mean error. The axial symmetry of the pivots proved to be perfect in limits of measurement errors. Thus the assumption of the pivots' inequality beeing zero was admissible in any position of the optical axis.

In order to carry into effect my observations, I placed the transit instrument on the central pier of the observing tower of the Technical University Institute for Practical Astronomy. The pier has been protected by a dome and based on two thick walls of the tower by means of a quadruple solid brick arc. The altitude of the pier over the ground was about 30 meters. A small battery lamp located on the platform of a distant high house was serving as the terrestrial object to observe. Its distance from the observing point was about 3 kilometers. The image of the signal in the field of view of the instrument was quite alike images of stars of about 4^m. For observations of stars brighter than 3^m diaphragms were superposed on the object glass. With their help brightnesses of such stars were brought down to the interval 3^m—4^m in order to eliminate the luminosity equation.

The wireless receiver was not fit for chronographical reception of time signals. So I performed receptions by the method of the ear. The thus determined corrections of the working chronometer did not correspond to a method of observation with use of a chronograph and a transit micrometer. It therefore was to be expected in anticipation that there would be some systematical differences between corrections u_{obs} as deduced from pair observations and corrections u_{rad} as deduced from time signals. Indeed, systematical differences $u_{obs} - u_{rad}$ did appear; for individual evenings they had values comprised between $0^s.00$ and $0^s.26$. Corrections u_{rad} were used for no other purpose than determining and controlling the diurnal rate of the working chronometer. The rate was showing accelerations of the order of $0^s.1$ per day; their run was in general rather smooth with the exception of one single anomalous fluctuation about the 26.IV.1939. The chronometer itself was always kept in a room beneath the observing dome.

6. Preparation of a star list for an observation of pairs.

A preparation of an observing programme consisted in choosing stars which would cross the vertical circle of the instrument within a time interval designed for observations. The graphical method of Stechert has been applied, being very simple if a diagram of catalogue stars is at hand whereon celestial parallels and hour circles are represented as horizontal and vertical lines (Fig. 2-a). If an approximate value of the instrument's azimuth (when directed towards the object) is known, a sheet of hard paper can be cut out along the line RT (Fig. 2-b) representing a quarter of the object's vertical circle in the system of the diagram and along the stright line RU representing a quarter of the celestial equator in the same system. If we shift the edge RU of the sheet RTU along the equator of our diagram

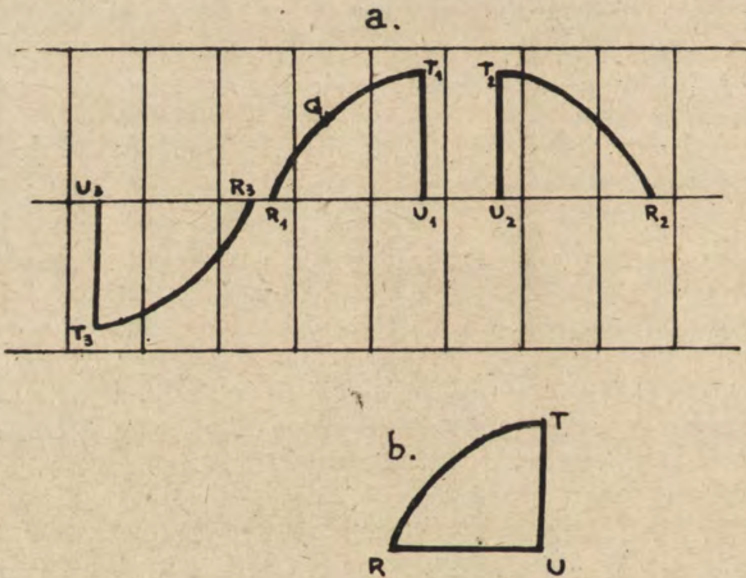

Fig. 2.

we can see in what succession several stars will be crossing the vertical circle RT. Moreover, we can graphically find moments of star transits with an accuracy of about 2^m, since the position of the equatorial point R of the vertical circle RT is marking on the equator the local sideral time of the transit of a star G plus some constant correction proper to the given vertical circle RT. The same sheet of paper can be used for various orientations, such as: $R_1T_1U_1$, $R_2T_2U_2$, $R_3T_3U_3$, which correspond to different quarters of the vertical circle.

When stars have been chosen in this way, their zenithal distances and local sideral times for transit moments are to be calculated with the help of 4-figure tables. The best way of doing it is supplied by the following convenient formulae given by Stechert in his paper quoted above:

$$\operatorname{tg} M = \sin\varphi \operatorname{tg} a; \quad \sin(M - t) = \frac{\operatorname{tg}\delta}{\operatorname{tg}\varphi}\sin M; \quad \Theta = t + \alpha$$

$$\operatorname{tg} N = \operatorname{tg}\varphi \sec a; \quad \sin(N - z) = \frac{\sec\delta}{\sin\varphi}\sin N.$$

From the formulae of the first line we calculate consecutively M, t and Θ; from those of the second line: N and z.

In order to find out stars of wanted zenithal distances, it is convenient to mark on the line RT points corresponding to several values of zenithal distance at every round 5^0. To secure the elimination of stars which at transit have too large values of the parallactic angle q, an adequate mark on the line RT is to be drawn too. If we try to match the chosen stars in pairs for a simultaneous determination of azimuth and time by the method of pairs, we must see to it that consecutive stars cross the vertical circle at two points on opposite sides of the zenith in an angular distance of about 90^0 from one another (as far as it is possible) and not too near to the zenith. As an example the following list of stars is given below forming the observing programme for the evening of the 1939.VI.7.

Observing programme for 1939.VI.7^d.

Θ		✻	z	
$13^h\,37^m$	$(4^m.2)$	ι Vir	$58^0\,25'$	S
54	(4 .7)	5 H. Cam	55 8	N
14 9	(3 .8)	109 Vir	50 30	S
20	(6 .0)	Grb 848	50 4	N
40	(2 .7)	β Lib	61 53	S
57	(4 .0)	γ Lib	67 22	S
15 15	(4 .4)	9 Cam	60 19	N
29	(5 .4)	4 Cam	70 9	N
40	(3 .3)	ε Oph	57 14	S

7. The observations.

The observations which form the observational data for this work were performed during 12 evenings. This is hardly a half of all observations performed till the end of July 1939. Unfortunately, the chronographical lents and observation records concerning observations in May, June and July 1939 have been spoiled by the occupants. The results of the evening 1939.VI.7 have exceptionally been saved as I had made a partical reduction of them soon after the observation itself, for the sake of control.

Observations have been made according to schemes conserving as far as possible symmetry within observations of a pair. The positions of the instrument's circle will be marked, for convenience, by the letters E and W in analogy with meridian observations; similarly stars and their zenithal distances will be marked ty the letters S and N in comformity with the side of the zenith on which the star was crossing the vertical circle. Here is an example of such a scheme:

Object	Circle position	Kind of observation
Signal	W	4-fold fixing the mobile wire on the signal image
	E	„ „ „ „ „ „ „ „ „
Star S	E	Observations of the level in both positions
		Transit micrometer observation of the star
	W	Transit micrometer observation of the star
		Observations of the level in both positions
Star N	W	Level observation
		Transit observation
	E	Transit observation
		Level observation
Signal	E	4-fold fixing the mobile wire on the signal image
	W	„ „ „ „ „ „ „ „ „

Each star has been observed in both positions of the circle in order to eliminate the error of collimation and of pen parallax. The distribution table with relays was provided with a switch. In the first position of the switch the chronometer was beeing switched on in the circuit of the 1 chronograph needle and the I relays; the transit micrometer — in that of the II needle and the II relays. In the second position of the switch the reversed combination of the circuits was at work. Changing the circuits has been applied during the observation of each star; after the first half of a star's observation, say in the circle position E, the two circuits have been interchanged with the help of the switch and so the second half of the obseervation, say in the circle position W, was performed with a reversed combination of the circuits. Owing to circuit changing and the thus occasioned reversion of the sign of the pen (needle) parallax, its influence will be eliminated if we only take the arithmetical mean of transit times for both circle positions E and W.

The following examples refer to the observations of two stars observed at the date 1939.IV.26: 4 Cas and χ Leo. K denotes the co-efficient of T^2 in the formula (18). The first column referring to 4 Cas contains contact numbers, the second and third columns, headed T_W and T_E, give chronograph indications for transits; the fourth — their arithmetical mean; the fifth, headed ΔT, — time intervals between two moments of transits over the same contact position of the mobile vire; the sixth — correction $K \cdot \Delta T^2$ to be applied to $\frac{T_W + T_E}{2}$; the seventh — corrected mean transit times T with their general arithmetical mean as given at the bottom; the eighth and ninth columns show deviations from that mean and squares of the same. At the bottom of the table there are: μ — the mean error of a single value of T and, finally, ε — the mean error of the arithmetical mean. The table referring to χ Leo is shorter because $K = 0$ and so there is no need of applying any correction to $\frac{T_W + T_E}{2}$.

($5^m.2$) 4 Cas			$K = 1928 \cdot 10^{-9}$			$\delta = +61^\circ 57'$		
Contact Nr	Chronograph readings		$\frac{T_W + T_E}{2}$	ΔT	$K.\Delta T^2$	T	v	vv
	T_W	T_E						
6^{rev}	9^h	10^h	10^h			10^h		
III	$59^m41^s.36$	$1^m54^s.82$	$0^m48^s.09$	133	$0^s.03$	$0^m48^s.12$	$+0^s.02$	0.0004
II	44.62	51.80	.21	127	3	.24	— 10	100
I	47.64	48.30	47.97	121	3	.00	+ 14	196
6_2	51.16	45.10	48.13	114	3	.16	— 2	4
IX	54.50	41.86	.18	107	2	.20	— 6	36
VIII	57.88	38.36	.12	100	2	.14	0	0
VII	60 1.30	34.80	.05	94	2	.07	+ 7	49
VI	4.66	31.78	.22	87	1	.23	— 9	81
V	7.78	28.40	.09	81	1	.10	+ 4	16
IV	11.08	25.14	.11	74	1	.12	+ 2	4
						$T_{mean} = 10^h0^m48^s.14$	+ 2	0.0490
	$\mu = \pm 0^s.07$					$\varepsilon = \pm 0^s.02_3$		

($4^m.7$) χ Leo		$K = 0$		$\delta = +7^\circ 40'$	
7^{rev}	10^h	10^h	10^h	v	vv
VII	$30^m50^s.54$	$32^m35^s.93$	$31^m43^s.24$	$-0^s.05$	0.0025
VI	51.99	34.37	.18	+ 1	1
V	53.60	32.94	.27	— 8	64
IV	55.07	31.24	.16	+ 3	9
III	56.68	29.70	.19	0	0
II	58.13	28.22	.18	+ 1	1
I	59.92	26.47	.20	— 1	1
7_3	31 0.74	25.70	.22	— 3	9
7_2	1.20	25.15	.18	+ 1	1
7_1	1.69	24.45	.07	+ 12	144
		$T_{mean} = 10^h31^m43^s.19$		+ 1	0.0255
	$\mu = \pm 0^s.05$		$\varepsilon = \pm 0^s.01_7$		

We quote herewith the differences between the azimuth of the light signal (a_{δ}) and that of the instrument (a) as observed in the evening 1939.IV.26. Original micrometer readings cannot be quoted here, as the observation records have been spoiled.

When transit observations had been finished I made, for the sake of control and comparison, a fourfold determination of the azimuth of the same signal by observations of the star Polaris with the help of the universal instrument Heyde Nr 4944. The observations were performed in 4 sets corresponding to 4 position of the isfrument's horizontal circle. By one set we mean here an observing series consisting of 8 pointings at the signal and of 8 transits of Polaris. The sets I and II were made in the evening 1939.VII.31 and III and IV — 1939.VIII.3[d].

$\Delta a = a_{\delta} - a$
+18.6
17.1
17.7
17.7
17.9
17.5
17.1
17.2
17.5
17.4

Before the transit instrument was taken from the pier, I determined as accurately as possible the projection of the middle of the instrument's axis on the surface of the pier in order to secure a due centering of the universal instrument. This function has caused some practical difficulties because there was no centering arrangement on the transit instrument. The error of centering can be estimated as not exceeding $\pm 0^{cm}.5$. In a most inconvenient case the corresponding error in azimuth would be

$$\frac{0.5}{300\ 000 \cdot \sin 1''} = 0''.3$$

since the signal was at the distance of about 3 km.

8. The results.

I reduced my observations by the method of pairs. Pairs were formed by matching two consecutively observed stars: a northern and a southern one, in turn. The reductions for the pair: τ Dra and 10 Mon are quoted below as an example. The first table contains auxiliary calculations dealing with:

a) Influence of inclination, b, of the horizontal axis according to the formula:

$$\tau_b = \frac{\pm b \cdot \cos z}{\cos \delta \cos q},$$ where the sign

+ is to be put for a star observed southward from the zenith or northward, if near the lower culmination;

— is to be put for a star observed northward from the zenith, if near the upper culmination.

b) Influence of the end play and contact breadth of the transit micrometer according to the formula:

$$\delta T^s = 0^s.086 : \cos \delta \cdot \cos q.$$

The adopted co-efficient $0^s.086$ is a mean of numerous determinations.

c) Allowance for rates of the working chronometer for time intervals from some initial moment T_0 till the transit moment T according to the formula:

$$\Delta u = (T - T_0)^m \cdot \Delta u_{1m}.$$

d) Corrections of equatorial coordinates for diurnal aberration:

$$(\alpha' - \alpha)^s = \frac{1}{15}\,[9.294] \cdot \sec \delta \cdot \cos (\Theta - \alpha)$$

$$(\delta' - \delta) = [9.294] \cdot \sin \delta \cdot \sin (\Theta - \alpha)$$

The correction $[(\tau_b + \delta T) + \Delta u]$ has been subsequently added to the arithmetical mean T of the chronograph transit moments $\left(\text{corrected, if need be, for } \frac{\tau_1 - \tau_2}{2}\right)$. Thus chronograph reduced indications T' have been obtained.

Auxiliary reductions for 1939.I.31.

	τ Dra	10 Mon
$\cos q$	9.966·	9.997·
$\cos \delta$	9.460	9.998·
$\sin t$	9.693·	9.179
$\cos \delta \cdot \sin t$	9.153·	9.177
$\sin z$	9.902	9.925·
$z =$	$52^0.9$	$57^0.6$
$\cos z$	9.780	9.731·
b^p	$0.193\cdot_n$	0.255
$(1^{pars})^s$	9.074·	9.074·
$b^s \cdot \cos z$	9.048_n	9.061
$b^s \cdot \cos z =$	— 0.111·	+ 0.115
$0^s.086 \pm b^s \cdot \cos z =$	— 0.025·	+ 0.201
$\sin q$	9.579	9.040·
$q =$	$22^0.3$	$6^0.3$
$\cos \delta \cdot \cos q$	9.426·	9.996
$\sec \delta \cdot \sec q$	0.573·	0.004
$\sec \delta \cdot \sec q =$	3.74·	1.01
$(\tau_b + \delta T) =$	$-0^s.09\cdot$	$+0^s.20\cdot$
$\Delta u =$	+ 0.10	+ 0.14
$T =$	$5^h19^m51^s.67$	$5^h51^m49^s.76$
$\|(\tau_b - \delta T) + \Delta u\| =$	+ 0.00·	+ 0.34·
$T' =$	5 19 51 .67·	5 51 50 10·
$\Theta - \alpha =$	$150^0.0$	$-8^0.7$
$\cos(\Theta - \alpha)$	$9.939\cdot_n$	9.995
$\sec \delta$	0.540	0.001·
$(\alpha' - \alpha)^s$	$8.597\cdot_n$	8.114·
$(\alpha' - \alpha)^s =$	$-0^s.04$	$+0^s.01\cdot$
α^{app}_{FK3}	$19^h16^m39^s.96\cdot$	$6^h24^m58^s.91\cdot$
$\alpha' =$	19 16 39 .92·	6 24 58 .93
$\sin(\Theta - \alpha)$	9.693·	9.179_n
$\sin \delta$	9.981	8.916_n
$(\delta' - \delta)''$	8.968·	7.389
$(\delta' - \delta)'' =$	$+0''.09\cdot$	$0''.00$
$\delta^{app}_{FK3} =$	$+73^0 14' 35''.21\cdot$	$-4^0 43' 36''.12\cdot$
$\delta' =$	+73 14 35 .30·	— 4 43 36 .12·

In the next following table we quote the proper reduction as performed by the two methods: on the left there is the reduction according to the Stechert formulae; on the right — according to the writer's formulae.

The reduction for the observation of 1939.I.31.

as per Stechert's formulae

	τ Dra	10 Mon
$T' =$	$5^h19^m51^s.67$	$5^h51^m50^s.10$
$\alpha' =$	19 16 39 .92	6 24 58 .93
$\lambda = T' - \alpha' =$	10 3 11 .75	—0 33 8 .82
$-\vartheta = \lambda_2 - \lambda_1 =$	$-10^h36^m20^s.57$	
$-\frac{1}{2}\vartheta =$	— 5 18 10 .29	
$\delta_2' =$	$-4^0 43' 36''.12$	
$\delta_1' =$	+ 73 14 35 .30	
$\delta_2' + \delta_1' =$	+ 68 30 59 .18	
$\delta_2' - \delta_1' =$	— 77 58 11 .43	
$\sin(\delta_2' + \delta_1')$	9.968 7269	
$\mathrm{tg}(-\frac{1}{2}\vartheta)$	0.733 8503	
$\sin(\delta_2' - \delta_1')$	$9.990\ 3557_n$	
$\mathrm{tg}\, L$	$0.712\ 2215_n$	
$L =$	$180^0 + 79^0 1' 17''.6$	
$-\frac{1}{2}\vartheta =$	— 79 32 34 .3	
$-\frac{1}{2}\vartheta - L =$	+ 21 26 8 .0	
$-\frac{1}{2}\vartheta + L =$	180^0— 0 31 16 .7	
$\sin(-\frac{1}{2}\vartheta + L)$	7.958 9638	
$-\mathrm{tg}\,\delta_2'$	8.917 4223	
Diff.	9.041 5415	
$\sin(-\frac{1}{2}\vartheta - L)$	9.562 8333	
$\mathrm{tg}\,\delta_1'$	0.521 2944	
Diff.	9.041 5388	
$\mathrm{tg}\,\varphi$	0.110 6698	
$\sin[\frac{1}{2}(t_2 + t_1) - L]$	9.152 2086	
$\frac{1}{2}(t_2 + t_1) - L =$	$180^0 - 8^0\ 9' 43''.5$	
$\frac{1}{2}(t_2 + t_1) =$	70 51 34 .1	
$\frac{1}{2}(t_2 + t_1) =$	$4^h43^m26^s.27$	

as per Stechert's formulae

$\frac{1}{2}(\lambda_2+\lambda_1)=$	$4^h45^m\ 1^s.46$
$u=$	$-1\ 35\ .18\cdot$
$\mathrm{tg}\,[\frac{1}{2}(t_2+t_1)-L]$	$9.156\ 6302_n$
$\sin\varphi$	$9.897\ 8445\cdot$
$\mathrm{tg}\,a$	$9.258\ 7856\cdot_n$
$a=$	$-10^0 17'\ 6''.3$
$\Delta a=$	$-\ \ 36\ .6$
$a_\delta=$	$-10\ 17\ 42\ .9$

as per the writer's formulae

	τ Dra, 10 Mon		τ Dra, 10 Mon
$\vartheta=$	$180^0-20^0 54'51''.3$	$n\cdot\cos N$	$0.507\ 0922\cdot_n$
$\sin\lambda_1\,\mathrm{tg}\,\delta_2'$	$8.605\ 7313\cdot_n$	$\mathrm{tg}\,N$	$9.134\ 7157$
$\sin\lambda_1$	$9.688\ 3090\cdot$	$N=$	$180^0-\ 7^0 45'55''.7$
$\mathrm{tg}\,\delta_2'$	$8.917\ 4223_n$	n	$0.511\ 0933\cdot$
$\cos\lambda_1$	$9.940\ 9711_n$	$\sin\vartheta$	$9.552\ 6325\cdot$
$\cos\lambda_1\,\mathrm{tg}\,\delta_2'$	$8.858\ 3934$	$\mathrm{tg}\,\varphi$	$0.110\ 6698$
$\sin\lambda_2\,\mathrm{tg}\,\delta_1'$	$9.680\ 0420\cdot_n$	$1:n$	$9.488\ 9066\cdot$
$\sin\lambda_2$	$9.158\ 7476_n$	$\sin(N+u)$	$9.152\ 2090$
$\mathrm{tg}\,\delta_1'$	$0.521\ 2944\cdot$	$N+u=$	$180^0-\ 8^0\ 9'43''.5\cdot$
$\cos\lambda_2$	$9.995\ 4417\cdot$	$N=$	$180\ -\ 7\ 45\ 55\ .7$
$\cos\lambda_2\,\mathrm{tg}\,\delta_1'$	$0.516\ 7362$	$u=$	$-\ 0\ 23\ 47\ .7\cdot$
$\sin\lambda_1\,\mathrm{tg}\,\delta_2'=$	$-0.0403\ 3958$	$u=$	$-\ 1^m 35^s.18\cdot$
$\sin\lambda_2\,\mathrm{tg}\,\delta_1'=$	$-0.4786\ 764$	$\mathrm{tg}\,(N+u)$	$9.156\ 6309_n$
$n\cdot\sin N=$	$+0.4383\ 368$	$\sin\varphi$	$9.897\ 8445\cdot$
$\cos\lambda_1\,\mathrm{tg}\,\delta_2'=$	$+0.0721\ 7610$	$\mathrm{tg}\,a$	$9.258\ 7863\cdot_n$
$\cos\lambda_2\,\mathrm{tg}\,\delta_1'=$	$+3.2865\ 196$	$a=$	$-10^0 17'\ 6''.3\cdot$
$n\cdot\cos N=$	$-3.2143\ 435$	$\Delta a=$	$-\ \ 36\ .6$
$n\cdot\sin N$	$9.641\ 8079\cdot$	$a_\delta=$	$-10\ 17\ 42\ .9\cdot$
$\cos N$	$9.995\ 9989_n$		

Supplementary reductions have been made for comparison for each star separately as per formula (1), modified in the well known manner, by the use of an auxiliary angle M, to the shape:

$$\mathrm{tg}\,M=\frac{\mathrm{tg}\,\delta}{\cos t};\qquad \mathrm{tg}\,a=\frac{\cos M\cdot\mathrm{tg}\,t}{\sin(\varphi-M)}.$$

An example of these computations is given below referring to the same observations of 1939.I.31. Chronometer correction has been adopted in every case, as determined by the pair method.

	τ Dra	10 Mon
$T'+u=$	$5^h18^m16^s.49$	$5^h50^m14^s.92$
$\alpha'=$	19 16 39 .92	6 24 58 .93
$t=$	10 1 36 .56	—0 34 44 .01
	150°24′ 8″.48	—8°41′10″.15
tg δ'	0.521 2944	$8.917\,4223_n$
cos t	$9.939\,2772_n$	9.994 9932
tg M	$0.582\,0172_n$	$8.922\,4290_n$
$M=$	180°— 75°19′43″.7	— 4°46′52″.5
$\varphi-M=$	— 52 26 55 .8	+ 57 0 13 .5
cos M	$9.403\,5865_n$	9.998 4861
tg t	$9.754\,3672_n$	9.183 9089
sin $(\varphi-M)$	$9.899\,1680_n$	$9.923\,6098_n$
tg a	$9.258\,7858_n$	$9.258\,7851_n$
$a=$	—10°17′ 6″.3	—10°17′ 6″.2
$\Delta a=$	— 36 .6	— 36 .6
$a_{♁}=$	—10 17 42 .9	—10 17 42 .8

Tabl. 4 shows the values of $a_{♁}$ (object's zimuth) as calculated from individual pair observations. The means of each evening's results are entered as well as mean errors: the m. e. of a single determination of $a_{♁}$ (in brackets) and the m. e. referring to an evening's arithmetical mean of $a_{♁}$ (without brackets). These mean errors characterize the internal accuracy of observations in a given evening.

Tabl. 5 contains an analogous confrontation of chronometer corrections as determined by the pair method. The tabulated values are not the observed corrections themselves, u_{obs}, but the difference values: u_{obs} minus the correction u_{rad} for the same initial moment as determined by wireless rhythmic signals *GBR* and *FYL*. High values of $u_{obs}-u_{rad}$ are calling attension. They are easily to be interpreted by the fact, above all, that wireless signals have been received by ear in default of an arrangement for a chronographic reception. So the reception method was not adequate to the chronographical method of observations. Chronometer corrections u_{rad} do not play any essential role in this work. They only have been used for determining and controlling chronometer diurnal rates.

Tabl. 4.

Date 1939	a♁ −10°17′	
I.31d	42.9″	—
II.23	41.6	
	40.7 ·	
	40.0 ·	(± 0.99″)
	41.3	
	42.7	
	41.3	± 0.44
II.25	41.0	
	44.1	
	40.9	(± 1.29)
	41.8 ·	
	42.0	
	41.9 ·	± 0.58
II.26	43.1 ·	
	43.2	(± 0.10)
	43.1 ·	
	43.1 ·	± 0.00
III.4	44.4	
	43.5	
	43.8	(± 0.45)
	43.8 ·	
	43.2	
	43.7 ·	± 0.20

Date 1939	a♁ −10°17′	
III.5d	44.9″	
	43.0	
	44.2	(± 0.96″)
	42.9 ·	
	42.6 ·	
	43.5 ·	± 0.43
IV.4	42.2 ·	
	43.0 ·	
	43.0 ·	(± 0.70)
	43.9 ·	
	43.9 ·	
	43.2 ·	± 0.32
IV.11	42.2 ·	
	44.4	
	42.2 ·	(± 0.91)
	43.4 ·	
	43.3	
	43.1 ·	± 0.41
IV.21	41.9	
	44.0	(± 1.36)
	41.4 ·	
	42.4 ·	± 0.78

Date 1939	a♁ −10°17′	
IV.26d	42.5″	
	42.9	(± 1.00″)
	44.1	
	41.7	
	42.8	± 0.50
IV.27	42.1	
	40.8	(± 1.30)
	43.4	
	42.1	± 0.75
VI.7	40.5 ·	
	41.9	(± 0.75)
	42.2 ·	
	41.5 ·	± 0.52

Tabl. 5.

Date 1939	$u_{obs}-u_{rad}$	
I.31[d]	$-0^{s}.11$	—
II.23	0.00	
	− 1	
	+ 7	$(\pm 0^{s}.056)$
	+ 3	
	− 8	
	0.00	±0.023
II.25	−0.06	
	− 18	
	− 12	(±0.050)
	− 16	
	− 17	
	−0.14	±0.022
II.26	−0.20	
	− 16	(±0.035)
	− 13	
	−0.16	±0.020
III.4	−0.30	
	− 25	
	− 28	(±0.057)
	− 19	
	− 17	
	−0.24	±0.025

Date 1939	$u_{obs}-u_{rad}$	
III.5[d]	$-0^{s}.24$	
	− 15	
	− 28	$(\pm 0^{s}.077)$
	− 8	
	− 16	
	−0.18	±0.035
IV.4	−0.24	
	− 25	
	− 26	(±0.030)
	− 32	
	− 26	
	−0.26	±0.014
IV.11	−0.06	
	− 28	
	− 2	(±0.100)
	− 15	
	− 13	
	−0.13	±0.045
IV.21	−0.10	
	− 16	(±0.076)
	− 1	
	−0.09	±0.044

Date 1939	$u_{obs}-u_{rad}$	
IV.26[d]	$-0^{s}.24$	
	− 15	$(\pm 0^{s}.054)$
	− 25	
	− 28	
	−0.23	±0.027
IV.27	−0.12	
	− 2	(±0.097)
	− 21	
	−0.12	±0.056
VI.7	−0.05	
	− 8	(±0.019)
	− 8	
	−0.07	±0.011

For reduction as per formula (1) corrections u_{obs} have been used exclusively.

Tabl. 6 shows the values of the object's azimuth as obtained by single star observations (formula (1)), their mean errors, the mean values for separate evening and their mean errors.

Tabl. 6.

Date 1939	a♁ −10°17′	
I.31d	42″.9	
	42.8 ·	
	42.9	
II.23	42.1	
	41.4	
	42.3	
	38.6	
	40.6	(± 1″.18)
	41.4 ·	
	42.7 ·	
	41.6	
	42.0 ·	
	40.8	
	41.3 ·	± 0.37
II.25	42.6	
	41.2 ·	
	42.1 ·	
	42.7	
	41.9 ·	(± 1.05)
	39.6	
	42.8	
	42.7	
	40.8	
	41.0	
	41.7 ·	± 0.33
II.26	42.5 ·	
	43.5	
	42.6 ·	(± 0.74)
	43.1 ·	
	43.4	
	44.6	
	43.3	± 0.31
III.4	44.3	
	43.4 ·	
	42.2 ·	
	42.5	
	42.8 ·	(± 0.85)
	43.5	
	44.6	
	44.7	
	44.0	
	43.8 ·	
	43.6	± 0.27

Date 1939	a♁ −10°17′	
III.5d	44″.4	
	44.1	
	43.2 ·	
	44.1	
	42.0 ·	(± 0″.94)
	43.5	
	44.0	
	45.2	
	42.3	
	43.3 ·	
	43.6	± 0.30
IV.4	43.0 ·	
	44.1	
	43.1 ·	
	43.5 ·	
	43.3	(± 0.45)
	43.1 ·	
	43.5 ·	
	44.4	
	43.9	
	43.8	
	43.6	± 0.14
IV.11	42.0	
	43.6	
	42.7 ·	
	41.8	
	42.9	(± 0.62)
	41.8	
	43.0 ·	
	42.4 ·	
	43.1	
	43.1	
	42.6 ·	± 0.20
IV.21	41.8	
	41.5	
	43.7	(± 0.84)
	42.7	
	43.0	
	41.9 ·	
	42.4 ·	± 0.34

Date 1939	a♁ −10°17′	
IV.26d	41″.6 ·	
	42.9 ·	
	43.4 ·	
	44.2	(± 0″.99)
	43.7	
	43.7	
	45.1	
	43.3	
	43.5	± 0.35
IV.27	42.3 ·	
	43.3 ·	
	41.8	(± 0.68)
	42.6	
	43.1	
	41.6 ·	
	42.5	± 0.28
VI.7	40.5 ·	
	40.8	
	41.0	(± 0.46)
	41.2 ·	
	41.5	
	40.2	
	40.9	± 0.18

The mean azimuth values for separate evinings are confronted in Tabl. 7. The left half of this table refers to the method of pairs; the right one — to the determinations by single star observations. At the bottom of both columns there are the arithmetical means. μ denotes the m. e. of a single evening value and ε — the m. e. of their arithmetical mean.

$$a_{♁} = -10^0\,17'\,42''.66 \pm 0''.23$$

is adopted as the definitive result of the azimuth determinations by the method of pairs.

Tabl. 7.

Date 1939	From star pairs: Number of pairs	$a_{♁}$	v	vv	From single stars: Number of stars	$a_{♁}$	v	vv
I.31^d	1	$10^0\,17'\,42''.9$	$-0''.2$	0.06	2	$10^0\,17'\,42''.9$	$-0''.2$	0.04
II.23	5	41.3	+1.3	1.82	10	41.3	+1.3	1.69
25	5	41.9	+0.7	0.49	10	41.7	+0.9	0.90
26	3	43.1	−0.5	0.25	6	43.3	−0.6	0.36
III.4	5	43.7	−1.1	1.21	10	43.6	−0.9	0.81
5	5	43.5	−0.9	0.81	10	43.6	−0.9	0.90
IV.4	5	43.2	−0.6	0.36	10	43.6	−0.9	0.81
11	5	43.1	−0.5	0.25	10	42.6	+0.0	0.00
21	3	42.4	+0.2	0.04	6	42.4	+0.2	0.06
26	4	42.8	−0.1	0.02	8	43.5	−0.8	0.64
27	3	42.1	+0.5	0.30	6	42.5	+0.2	0.04
VI.7	3	41.5	+1.1	1.21	6	40.9	+1.8	3.24
		10 17 42.66	−0.1	6.82		10 17 42.68	−0.3	9.49
		$\mu = \pm 0''.79$				$\mu = \pm 0''.93$		
		$\varepsilon = \pm 0.23$				$\varepsilon = \pm 0.27$		

9. The discussion of accuracy.

The m. e. of an azimuth value obtained in a single evening (4 pairs in the average) amounts to $\mu = \pm 0''.79$ (Tabl. 7), whereas the m. errors of the results obtained in individual evenings (Tabl. 5) have lower values. The square root of the arithmetical mean of squares of these values is $\pm 0''.50$. This is clear since the individual evening's m. e. characterizes the internal accuracy and does not enclose any systematical error which may be involved in observations of the given evening. On the contrary the m. e. calculated from the whole of the results brings out, at least partially, systematical errors peculiar to separate evenings, namely those parts of them which are of accidental character with reference to the whole of the evenings (for instance those arising from atmospherical conditions). Therefore the error value $\mu = \pm 0''.79$ gives a more reliable estimate of accuracy.

The mean of mean values and its error ε were calculated without taking weights into account. It is true that numbers of pairs are ranging from 1 to 5. I thought it well, however, to disregard weights, in consideration of a rahter marked influence of systematical evening errors. Introducing weights would have caused an undesired preponderance of systematical errors associated with the better filled evenings. As on the whole 47 pairs were observed during 12 evenings, we have an average of 4 pairs per evening. Thus the accuracy of these observations can be characterised in the following manner: in a value of azimuth as determined from observations of 4 pairs the mean error

$$\mu = \pm 0''.8$$

is to be expected; the m. e. of the final azimuth value (12 evenings) is

$$\varepsilon = \pm 0''.23.$$

If weights for separate evenings are introduced, a weight p being equal to the number of observed pairs, it follows from the deviations (Tabl. 7, left half):

$$\mu^{(1)} = \pm \sqrt{\frac{[pvv]}{n-1}} = \pm \sqrt{\frac{30.27}{11}} = \pm 1''.66;$$

$$\mu^{(4)} = \pm \sqrt{\frac{[pvv]}{4\,(n-1)}} = \pm 0''.83,$$

where $\mu^{(1)}$ and $\mu^{(4)}$ denote the m. e. of a result of one evening with weights 1 and 4 respectively. The *internal* m. errors of the separate evenings (Tabl. 4) are considerably *smaller*. The value of $\mu^{(4)}$ is consistent with that of μ. As for the value of $\mu^{(1)}$, it reflects also systematical errors of separate evenings with exception of such systemat. errors which may be common for all the evenings.

It is to be emphasized that Tabl. 7 seems to show some run of azimuth values with a maximum in March. This being the case, the deviations probably do not represent pure accidental errors. If the above mentioned run was caused by some real alterations, the accuracy of the method would appear higher than it does if merely the method of least squares is applied to the deviations of Tabl. 7.

We quote below some instances of accuracy as reached by observations of Polaris with the help of universal instruments.

In Polish reports for the Geodetic and Geophysical Union, „Travaux géodésiques exécutés de...à..." there are the following data regarding accuracy reached in determinations of azimuths in Poland (ε_m denotes mean error);

in „Travaux géod."	of 1930	ε_m		$\pm 0''.56$		
in „ „	of 1933	ε_m	from	$\pm 0\ .2$	to	$\pm 0''.5$
in „ „	of 1936	ε_m	from	$\pm 0\ .18$	to	$\pm 0\ .45$
in „ „	of 1939	ε_m	from	$\pm 0\ .15$	to	$\pm 0\ .30$.

The above mentioned determinations are not uniform as to weights, nor is the number of sets and evenings upon which the respective azimuth value is based stated in each case.

In the „Veröffentlichungen des Finnischen Geod. Institutes", Nr 7, there are given the results of azimuth observations in the years 1924—1926, for the South-Finnish triangulation. The azimuth values are based on 5 to 12 sets (2 transits of Polaris and 2 pointings at the object in each set) which have been performed during one evening. The one evening's average internal accuracy is expressed by the m. e.

$$\varepsilon_m = \pm 0''.62.$$

Auording to Nr 18 of the afore named finnish publications in the years 1926—1931 internal m. errors of 1 evening's results (in the average 14.4 sets, 2 star transits and 2 pointings at the object in each) have been reaching

$$\text{from } \pm 0''.34 \quad \text{to} \quad \pm 0''.62.$$

In the „Rikets Almanna Kartverk", Medellande Nr 1, 1943, B. Wideland gives the m. e. of an azimuth value as obtained from observations for the triangulation of the Swedish net Signilskar — Omberg. Observations of one azimuth have consisted of 10 to 20 sets, 2 pointings at the object and 4 at the Polaris in each. The m. e. amounts to

$$\varepsilon_m = \pm 1^{cc}.35 = \pm 0''.44 .$$

In Nr 5 (1945) of the same publications L. Asplund gives the accuracy of azimuth values as determined in the South — Swedisch adjustment region of the Baltic Geodetic Commission. Here the m. errors are ranging

$$\text{from} \quad \pm 0^{cc}.61 = \pm 0''.20$$

$$\text{to} \quad \pm 1^{cc}.4 = \pm 0''.45 .$$

In 1938 Stamatin has made azimuth determinations in Rumenia by transits through object's verticals. Clock corrections have been determined independently by wireless time signals. Some of the above mentioned observations were chosen and their reductions performed by Niethammer[6]. The m. e. of the azimuth, as determined by 4 stars, is estimated to be

$$\pm \sqrt{(\pm 0.25)^2 + (\pm 0.25)^2} = \pm 0''.35 .$$

The corresponding value for 8 stars (4 pairs) is therefore $\pm 0''.25$. This is a value which characterizes the internal accuracy of 1 evening's observations and it corresponds to the value $\pm 0''.50$ as resulting from the writer's observations (Tabl. 4). On account of lack of the original Stamatin paper[7] the accuracy analysis of his results cannot be done here along lines as adopted for the writer's observations. At any rate, the low value of the m. e. gives evidence of advantageousness of direct azimuth determinations.

10. Qualities and disadvantages of the method.

The most important quality of the method is that readings of any graduated circles are entirely eliminated. Circle readings may dangerously spoil results of observations on account of graduation errors. The common way of eliminating systematical

[6] i. c. p. 145.

[7] Azimut astronomique direct, par le Capitaine J. Stamatin. Imprimerie de l'Institut géographique militaire (Roumanie), 1941.

errors of this kind is observation in several sets, so that the unknown angle be measured by the use of different parts of the circle. In cases of azimuth determinations with the help of a transit instrument that sort of errors does not come into consideration: here the small azimuth difference between the vertical of the object and that of the optical axis is being measured with a filar micrometer. If, however, m e r i d i a n transits are being observed whereas the terrestrial object is far distant from the meridian, an auxiliary mark must be placed in the immediate vicinity of the meridian. The horizontal angle between the mark and the object must be measured independently and thus circle readings come into play again. On the contrary, transit observations in the object's vertical eliminate circle readings altogether.

In connexion with circle graduation errors it should be kept in mind that systematical errors arising from this source may sometimes exceed random errors. It should be remembered that, for instance, a m. e. of an azimuth value as determined from a Polaris observation with a universal instrument in one single set does not characterize truly the accuracy of this method. A more real appreciation can be afforded by a m. e. as calculated from deviations of several set results from their mean. A still better characteristic of accuracy can be obtained if observations are distributed on several evenings. For in such a case some possible errors which may be of a systematical character during a single evening become random errors in relation to the whole of evenings.

Therefore, when discussing the accuracy of the pair method, I adopt as the m. e. of a single evening's result (4 star pairs) the above quoted value (Tabl. 7):

$$\mu = \pm 0''.79,$$

disregaring the much smaller m. e. values of the Tabl. 4 (without brackets) as they are representing internal accuracies of individual evening results; their average is ± 0.50. So my appreciation of accuracy of the method is rather cautious.

In order to have some control and comparison I have also determined the azimuth of the same direction by observations of the Pole Star with the help of the 2″ universal instrument of Heyde Nr 9844. These observations were made in 4 sets. By one set is meant an observing series consisting of 8 transits and of 8 pointings at the object. The results were as follows:

Date	Set	a_{δ}	Intern. m. e.	v	vv
1939.VII.31	I	$-10^{\circ}17'45''.5$	$\pm 0''.9_5$	$-1''.7$	2.89
31	II	44 .3	± 1.1	-0.5	0.25
VIII 3	III	43 .8	± 1.1	-0.0	0.00
3	IV	41 .4	± 1.0	$+2.3$	5.52
	Mean	-10 17 43 .8		$+0.1$	8.66

$\mu = \pm 1.7$ (the m. e. of a single set).

$\varepsilon = \pm 0.85$ (the m. e. of the mean of 4 sets).

Further, if allowance is made for position (co-ordinates) error of the star, the value of ε will exceed $\pm 1''$. In connexion with that, let us take into account the fact that in the pair method position errors cancel one another as random errors; thus the above value $\mu = \pm 0''.79$ (4 star pairs) can hardly be influenced by position errors. We shall come back to this point once more.

A further quality of the pair method is that precise clock corrections are not needed. In fact, the clock correction can be determined together with the azimuth from the observation itself. The faculty of determining clock corrections with a m. e. of the order of $\pm 0^{s}.04$ from 4 pairs is an important advantage too. When an azimuth determination only is intended, a systematical error in chronometer records does not matter. In such cases a transit micrometer is not indispensable, though allways desirable: it reduces random errors and makes the observing process easy.

Still another quality of the method is the above mentioned circumstance that position errors of observed stars cancel one another in their effects on the determined azimuth and clock correction. In the Polaris method the position error of this star may influence the result as a systematical error, no matter how numerous the sets are, unless the sets have been distributed in time so as to secure a symmetry of hour angles, a rather tedious condition which is hardly beeing taken into account in field work. In the expression for the square of the m. e. of the azimuth from Polaris (cfr. (20)) there is a term of the form: $\operatorname{cosec}^2 z \cdot \varepsilon^2_{pos}$. For the latitude of Warsaw it is

$$(\operatorname{cosec} z \cdot \varepsilon_{pos})^2 = (1.6 \cdot 0.3)^2 = 0.23 \tag{27}$$

if we put $\varepsilon_{pos} = \pm 0''.3$.

In order to see clearly the effect of position errors in the method of pairs, we have to use the expression for Σ_a from formula (26). As an example, the values of the co-efficient of ε^2 for all four pairs observed on 1939.IV.26 were calculated; the following values resulted: 4.31, 5.14, 4.98, 3.92. Their mean is 4.59. The effect of position errors on the square of the mean error of the azimuth from 1 pair amounts therefore to

$$4.59 \cdot \varepsilon^2 = 4.59 \cdot 0.3^2 = 0.41.$$

This value looks disadvantageous in comparison with value (27) for the azimuth by the Polaris method. But the position error's effect on the square of the m. e. of the azimuth as derived from 4 pairs is

$$0.41 : 4 = 0.10;$$

by observing a still greater number of pairs the effect of position errors can entirely be neglected.

With regard to the economy of time in applying the method of pairs, there are three moments to be taken into consideration: the preparation of an observation, the observation itself and its reduction.

As I had at hand a star map appended to the above alleged Stechert paper, it has taken me about ½ hour to chose 4 star pairs for observation and to make a 4-figure computation of transit times and zenithal distances. A pair, however, can be observed in several evenings and so the preparation time would be reduced.

An observation of 4 pairs lasted about 2 hours. This length of time would be considerably reduced if stars had been taken from more abundant lists than that of the „Berliner Jahrbuch".

In order to secure the uniformity of star positions, I have used but the star list of B. J. taking into account corrections to the FK_3 system. It follows from the foregoing discussion of accuracy that an observation of 4 pairs is equivalent to an observation of the Pole Star in 4 sets, if by a set an observation series is meant, consisting of 8 transits and of 8 pointings at the object. This makes 3⅓ hours if we reckon 50 minutes for each set. Although the above adopted equiponderance of 4 pairs and 4 sets may suggest some doubts, as being supported by too scarce observational data, it nevertheless seems that the pair method leads to some economy of observing time. To avoid any confusion it should be noticed that in geodetic papers the term „set" is often used for an observing series composed of 2 pointings at the object and 2 transits of Polaris. Thus an observing series such as called here 1 set is equiponderent with 4 sets such as are commonly meant by geodesists.

With regard to reductions, the pair method requires perhaps somewhat more time than the Polaris method does. Especially the auxiliary computations (corrections of chronograph transit moments, axis inclinations) are somewhat more complicated since the parallactic angle is involved in the formulae. The principal computations, on the other hand, are rather shorter, considering that 4 computing columns for 4 pairs correspond to $4 \cdot 8 = 32$ columns for 4 sets. I hardly can give any real orientation figures, because my reductions have been partially performed before the war, i. e. 9 years ago. The whole of reductions requires in sum not more or but little more time than reductions of equivalent Polaris observations do.

The qualities of the pair method can therefore be confronted shortly as follows:

1°. It eliminates errors of circle graduation,

2°. It does not require knowledge of the precise time and, if need be, can afford a clock correction.

3°. Star position errors cancel one another if many pairs are used.

4°. In the whole it is not less economic in time than the Polaris method and it shortens the time of observation in comparison with the latter.

As compared with meridian transits, the method of pairs is in general less accurate and its reduction formulae are more complicated. On the other hand, it presents a superiority of exempting us from measurements of the angle between a near mark and a distant terrestrial object. Moreover, the constancy of the instrumental azimuth is postulated for an observation of one single pair only.

Now, as we pass to disadventages of the method of pairs, we have to note the following moments.

The accuracy of results depends on the azimuth of the object: it falls as the object's azimuth varies from 0^0 or 180^0 to 90^0 or 270^0, especially in high latitudes. To apply the method for objects more than 45^0 distant from the meridian seems not to be advisable, not only with regard to accuracy but also because of the more restricted choice of stars (see Tabl. 1). The influence of an error in φ on the error in a has formerly been under discussion (cfr. p. 65).

In principle any adequately equipped universal instrument can be used for the pair method. Though, if a high accuracy is aimed at, better results would be secured by the use of a transit instrument. Its simpler and solid construction creates better conditions for stability in azimuth. A transit instrument, if destined for azimuth determinations in field work, ought to be equipped with an arrangement for easy centering. It requires a solid stand. But it should be noticed that the pair method is by far less exacting in this respect than the common meridian transit observations are; the constancy of the instrumental azimuth is postulated for the time interval of one pair observation only (2 stars and twice the object) but not during observations of several stars as in the case of meridian observations for time (and azimuth) determination.

The disadvantages of the pair method can, therefore, shortly be confronted as follows:

1^0. Its sphere of applicability is restricted in azimuth.

2^0. It requires the use of a transit instrument which is less commonly in use and requires a more solid stand than a universal instrument does.

3^0. At high latitudes it can afford accurate results only when an accurate latitude value is known.

11. Proposed improvements.

A small transit instrument of the type like „Askania-Werke A. P. 50", provided with a striding level, is excellently suitable for pair method observations. The telescope can be reversed in the Y's so easily and quickly as to allow every star's observation in both positions of the circle without hurry. The instrument, though of strong construction, has relatively small dimensions. If considered as an istrument for azimuth determinations its only fault is perhaps that it has no centering arrangement. This deficiency could be remedied with without serious construction difficulties. The optical axis of A. P. 50 is broken at the end of the horizontal axis. Flexion deformations are by far less at the end than they are in the middle of the axis. So it should be expected that the variable term of collimation, caused by flexion and proportional to $\cos z$, is likely to vanish in such a type of a broken telescope.

An important improvement of the observing process would be brought by an arrangement for driving the telescope in zenithal distances in order to keep the star's image between the horizonthal wires during the whole of the observation. Such an arrangement would be especially desirable in cases of stars crossing the horizonthal wires at considerable angles.

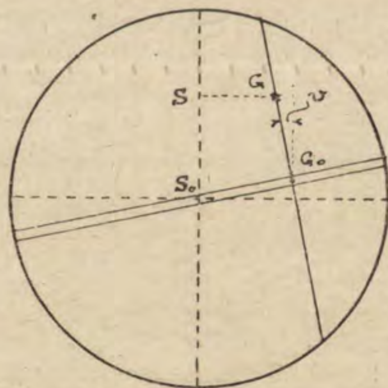

Fig. 3.

On account of a small error in the position angle of the micrometer, the mobile wire G_0G (Fig. 3) is slightly inclined to the vertical circle S_0S. Let the inclination error be υ. Thus the star transit is observed at G, at the distance $G_0G = d$ from the horizontal wires S_0G_0, instead of being observed at G_0, between the wires. The position G_0G of the mobile wire corresponds to the distance SG of the star from S_0S instead of corresponding to the d stance S_0G_0. The resulting error in azimuth is $d \cdot \sin \upsilon$. Let us put, with Niethammer[5]), $\upsilon = 1'$ and assume that a star had been observed, before axis reversing, between the horizontal wires. Then, if we want the above error not to exceed $0^s.001$, we shall take care to observe the star, after axis reversing at distances d not exceeding $3^s.4 = 0'.85$.

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[5]) l. c. p. 29.

An electrical arrangement for guiding the telescope vertically would allow to keep always the star image between the horizontal wires and so the position angle error would be eliminated.

As to programs of observations, it is more appropriate to observe during many evenings, 4 pairs at each, rather than to increase the number of pairs in the same evening at the cost of the number of evenings. Besides, the same remark is in force for every other method, since it aims at reducing the effects of such errors which are of systematical character with reference to a single evening, but are of a random character relatively to the whole of the evenings.

For completing lists of stars to be observed, some introductory work is to be done in order to establish the limits of declination for stars which are suitable for observation from a given point and for a given approximate value of the azimuth (with regard to the values of q and z). This task can be settled with the help of the simple formulae:

$$\sin q = \frac{\cos \varphi \cdot \sin a}{\cos \delta}; \quad \sin \delta = \sin \varphi \cos z \mp \cos \varphi \sin z \cos a,$$

where δ is to be cosidered as the unknown. After the auxiliary quantities n and N are introduced by the system of equations:

$$\sin \varphi = n \cdot \sin N$$

$$\cos \varphi \cos a = n \cdot \cos N$$

the formula with z assumes the very simple form:

$$\sin \delta = n \cdot \sin (N \mp z) .$$

It would perhaps be profitable if some simple nomographical way were advised for establishing in every definite case limits of declination for pair method observations.

I wish to thank very warmly Professor F. Kępiński, the Director of the Polytechnic Institute of Practical Astronomy, for his having counselled me to attack the subject of this paper as well as for his numerous advices during the performance of the work.

Warsaw, April 1948.

APPENDIX

1. Demonstration of the Stechert formulae.

It follows from (1):

$$\frac{\sin t_1}{\operatorname{tg}\delta_1\cos\varphi-\sin\varphi\cos t_1}=\frac{\sin t_2}{\operatorname{tg}\delta_2\cos\varphi-\sin\varphi\cos t_2}$$

$$\sin t_1(\operatorname{tg}\delta_2\cos\varphi-\sin\varphi\cos t_2)=\sin t_2(\operatorname{tg}\delta_1\cos\varphi-\sin\varphi\cos t_1)$$

$$(\sin t_1\operatorname{tg}\delta_2-\sin t_2\operatorname{tg}\delta_1)\cos\varphi=\sin(t_1-t_2)\sin\varphi. \qquad \text{(a)}$$

Let us denote:

$$\tau=\frac{t_2+t_1}{2};\qquad \Theta=\frac{t_2-t_1}{2};$$

then:

$$t_1=\tau-\Theta;\qquad t_2=\tau+\Theta.$$

After substituting for t_1 and t_2 in the equation (a) these expressions and multiplying it by

$$\frac{\cos\delta_1\cos\delta_2}{\cos\varphi}$$

it takes the shape:

$$(\sin\tau\cos\Theta-\sin\Theta\cos\tau)\sin\delta_2\cos\delta_1-(\sin\tau\cos\Theta+$$
$$+\sin\Theta\cos\tau)\sin\delta_1\cos\delta_2=-2\sin\Theta\cos\Theta\operatorname{tg}\varphi\cos\delta_1\cos\delta_2$$

$$\sin\tau\cos\Theta\sin(\delta_2-\delta_1)-\cos\tau\sin\Theta\sin(\delta_2+\delta_1)=$$
$$=-\operatorname{tg}\varphi\sin\Theta\cos\Theta\cdot 2\cos\delta_1\cos\delta_2.$$

We introduce now, in conformity with (2), the quantities l and L:

$$\sin(\delta_2+\delta_1)\sin\Theta=l\cdot\sin L$$

$$\sin(\delta_2-\delta_1)\cos\Theta=l\cdot\cos L.$$

Then we obtain:

$$l\sin(\tau-L)=\begin{cases}\dfrac{\operatorname{tg}\varphi}{\operatorname{tg}\delta_1}\cdot\sin\Theta\cos\Theta\cdot(-2)\sin\delta_1\cos\delta_2\\[2mm] \dfrac{\operatorname{tg}\varphi}{-\operatorname{tg}\delta_2}\cdot\sin\Theta\cos\Theta\cdot 2\sin\delta_2\cos\delta_1\end{cases}$$

$$l\sin(\tau-L)=\begin{cases}\dfrac{\operatorname{tg}\varphi}{\operatorname{tg}\delta_1}\cdot\sin\Theta\cos\Theta\cdot[\sin(\delta_2-\delta_1)-\sin(\delta_2+\delta_1)]\\[2mm] \dfrac{\operatorname{tg}\varphi}{-\operatorname{tg}\delta_2}\cdot\sin\Theta\cos\Theta\cdot[\sin(\delta_2-\delta_1)+\sin(\delta_2+\delta_1)]\end{cases}$$

$$l\sin(\tau-L)=\begin{cases}\dfrac{\operatorname{tg}\varphi}{\operatorname{tg}\delta_1}\cdot(\sin\Theta\cdot l\cos L-\cos\Theta\cdot l\sin L)\\[2mm] \dfrac{\operatorname{tg}\varphi}{-\operatorname{tg}\delta_2}\cdot(\sin\Theta\cdot l\cos L+\cos\Theta\cdot l\sin L)\end{cases}$$

$$\sin(\tau-L)=\begin{cases}\dfrac{\operatorname{tg}\varphi}{\operatorname{tg}\delta_1}\cdot\sin(\Theta-L) & \text{(b}'\text{)}\\[2mm] \dfrac{\operatorname{tg}\varphi}{-\operatorname{tg}\delta_2}\cdot\sin(\Theta+L). & \text{(b}''\text{)}\end{cases}$$

These equations are identical with (4′) and (4″).

The formula (5) results from (1) in the following way,

$$\operatorname{tg}a=\frac{-\sin t_1}{\sin\varphi\left(\dfrac{\operatorname{tg}\delta_1}{\operatorname{tg}\varphi}-\cos t_1\right)}.$$

On the basis of (b′) we obtain:

$$\operatorname{tg}a=\frac{1}{\sin\varphi}\cdot\frac{\sin t_1}{\cos t_1-\dfrac{\sin(\Theta-L)}{\sin(\tau-L)}}=$$

$$=\frac{\operatorname{tg}(\tau-L)}{\sin\varphi}\cdot\frac{\cos(\tau-L)\cdot\sin t_1}{\sin(\tau-L)\cdot\cos t_1-\sin(\Theta-L)}.$$

After substituting $t_1 = \tau - \Theta$ it can be easily seen that the second factor equals 1 and therefore we have, in conformity with (5):

$$\operatorname{tg} a = \frac{\operatorname{tg}(\tau - L)}{\sin \varphi}. \tag{c}$$

2. Demonstration of the writer's formulae.

The equation (a) can be written in the form:

$$\sin t_1 \operatorname{tg} \delta_2 - \sin t_2 \operatorname{tg} \delta_1 = \sin(t_1 - t_2) \operatorname{tg} \varphi .$$

If we use the substitutions expressed by the formulae (6) and (7):

$$\begin{aligned} t_1 &= \lambda_1 + u \\ t_2 &= \lambda_2 + u \end{aligned} \qquad t_1 - t_2 = \lambda_1 - \lambda_2 = \vartheta,$$

then we have:

$$(\sin \lambda_1 \operatorname{tg} \delta_2 - \sin \lambda_2 \operatorname{tg} \delta_1) \cos u -$$
$$- (\cos \lambda_1 \operatorname{tg} \delta_2 - \cos \lambda_2 \operatorname{tg} \delta_1) \sin u = \sin \vartheta \cdot \operatorname{tg} \varphi,$$

and, after introducing n and N, as in (8), by the equalities:

$$\begin{aligned} \sin \lambda_1 \operatorname{tg} \delta_2 - \sin \lambda_2 \operatorname{tg} \delta_1 &= n \cdot \sin N \\ \cos \lambda_1 \operatorname{tg} \delta_2 - \cos \lambda_2 \operatorname{tg} \delta_1 &= n \cdot \cos N \end{aligned} \tag{d}$$

we obtain the equation for determining u, as in (9):

$$n \cdot \sin(N + u) = \sin \vartheta \cdot \operatorname{tg} \varphi. \tag{e}$$

The formula (10) can be proved in the following manner. Let us transform the following expressions which are analogous to the expressions (d):

$$A = \cos t_1 \operatorname{tg} \delta_2 - \cos t_2 \operatorname{tg} \delta_1$$

$$B = \sin t_1 \operatorname{tg} \delta_2 - \sin t_2 \operatorname{tg} \delta_1$$

$$A = \frac{1}{\cos\delta_1 \cos\delta_2} (\cos t_1 \cos\delta_1 \sin\delta_2 - \cos t_2 \cos\delta_2 \sin\delta_1)$$

$$B = \frac{1}{\cos\delta_1 \cos\delta_2} (\sin t_1 \cos\delta_1 \sin\delta_2 - \sin t_2 \cos\delta_2 \sin\delta_1).$$

Now, let us apply the formulae of spherical trigonometry in order to transform the expressions:

$$\cos\delta_1 \cos t_1, \quad \cos\delta_1 \sin t_1, \quad \sin\delta_1,$$

$$\cos\delta_2 \cos t_2, \quad \cos\delta_2 \sin t_2, \quad \sin\delta_3.$$

$$A = \frac{1}{\cos\delta_1 \cos\delta_2} \{(\cos z_1 \cos\varphi - \sin z_1 \sin\varphi \cos a)(\cos z_2 \sin\varphi \mp \mp \sin z_2 \cos\varphi \cos a) - (\cos z_2 \cos\varphi \pm \pm \sin z_2 \sin\varphi \cos a)(\cos z_1 \sin\varphi - \sin z_1 \cos\varphi \cos a)\}.$$

The upper or the lower sign is to be put according to whether both stars cross the vertical on the same side of the zenith or on the opposite sides of it. After a simple reduction we have:

$$A = \frac{\cos a}{\cos\delta_1 \cos\delta_2} (\sin z_1 \cos z_2 \mp \cos z_1 \sin z_2)$$

$$A = \frac{\cos a}{\cos\delta_1 \cos\delta_2} \sin(z_1 \mp z_2). \tag{f}$$

In a similar way we can transform the expression for B:

$$B = \frac{1}{\cos\delta_1 \cos\delta_2} \{\sin z_1 \sin a \cdot (\cos z_2 \sin\varphi \mp \sin z_2 \cos\varphi \cos a) \mp \mp \sin z_2 \sin a \cdot (\cos z_1 \sin\varphi - \sin z_1 \cos\varphi \cos a)\}$$

$$B = \frac{\sin a}{\cos\delta_1 \cos\delta_2} \cdot \sin\varphi \cdot (\sin z_1 \cos z_2 \mp \cos z_1 \sin z_2)$$

$$B = \frac{\sin a \cdot \sin\varphi}{\cos\delta_1 \cos\delta_2} \cdot \sin(z_1 \mp z_2). \tag{g}$$

Then we obtain by dividing B by A:

$$\operatorname{tg} a \sin\varphi = \frac{B}{A} = \frac{\sin t_1 \operatorname{tg}\delta_2 - \sin t_2 \operatorname{tg}\delta_1}{\cos t_1 \operatorname{tg}\delta_2 - \cos t_2 \operatorname{tg}\delta_1} =$$

$$= \frac{\sin(\lambda_1 + u)\operatorname{tg}\delta_2 - \sin(\lambda_2 + u)\operatorname{tg}\delta_1}{\cos(\lambda_1 + u)\operatorname{tg}\delta_2 - \cos(\lambda_2 + u)\operatorname{tg}\delta_1} =$$

$$= \frac{(\sin\lambda_1 \cos u + \cos\lambda_1 \sin u)\operatorname{tg}\delta_2 - (\sin\lambda_2 \cos u + \cos\lambda_2 \sin u)\operatorname{tg}\delta_1}{(\cos\lambda_1 \cos u - \sin\lambda_1 \sin u)\operatorname{tg}\delta_2 - (\cos\lambda_2 \cos u + \sin\lambda_2 \sin u)\operatorname{tg}\delta_1} =$$

$$= \frac{(\sin\lambda_1 \operatorname{tg}\delta_2 - \sin\lambda_2 \operatorname{tg}\delta_1)\cos u + (\cos\lambda_1 \operatorname{tg}\delta_2 - \cos\lambda_2 \operatorname{tg}\delta_1)\sin u}{(\cos\lambda_1 \operatorname{tg}\delta_2 - \sin\lambda_2 \operatorname{tg}\delta_1)\cos u - (\sin\lambda_1 \operatorname{tg}\delta_2 - \sin\lambda_2 \operatorname{tg}\delta_1)\sin u} =$$

$$= \frac{n \cdot \sin N \cos u + n \cdot \cos N \sin u}{n \cdot \cos N \cos u - n \cdot \sin N \sin u} = \frac{n \cdot \sin(N+u)}{n \cdot \cos(N+u)},$$

whence, finally:

$$\operatorname{tg} a = \frac{\operatorname{tg}(N+u)}{\sin\varphi}$$

in conformity with (10).

Wiesław Opalski

Wyznaczanie azymutu narzędziem przejściowym w wertykale przedmiotu ziemskiego i dyskusja obserwacyj własnych

Komunikat przedstawiony przez czł. F. Kępińskiego
dnia 14 października 1949 r.

Streszczenie

Przedmiotem tej pracy jest zagadnienie wyznaczania azymutu z dużą dokładnością metodą bezpośrednią tj. wolną od odczytywania kół poziomych. Rozpatrywana metoda polega na obserwacjach przejść pary gwiazd przez wertykał narzędzia przejściowego, ustawionego w przybliżeniu zgodnie z wertykałem przedmiotu ziemskiego, tak, że drobną różnicę w azymucie wyznacza się pomiarem mikrometrycznym. Z obserwacji pary gwiazd można wyznaczyć jednocześnie azymut i czas. Metoda ta stanowi udoskonalenie metody S t e c h e r t'a przez zwielokrotnienie obserwacyj przejść.

Część teoretyczna obejmuje wyprowadzenie własnych wzorów redukcji przejść przez boczne położenia nitki mikrometru bezosobowego, a także wyprowadzenie własnego układu zasadniczych wzorów redukcyjnych, równoległego do układu wzorów Stechert'a; zawiera też dyskusję wzorów różnicowych metody.

Część praktyczna podaje opis wyposażenia instrumentalnego, użytego do obserwacji, którego główną częścią było narzędzie przejściowe „Askania-Werke A. P. 50" z mikrometrem bezosobowym, opis sposobu przygotowania programu obserwacji i schemat jej wykonania. Następują przykłady redukcji według wzorów Stecherta i autora oraz zestawienia wyników. Błąd średni wyznaczenia azymutu z obserwacji 4 par w ciągu 1 wieczoru wypadł $\pm 0''.5$; po uwzględnieniu zaś odchyłek wyników poszczególnych wieczorów od średniej — wypada $\pm 0''.8$. Błąd średni wyznaczenia czasu z 4 par wyniósł $\pm 0^s.03$. Po dyskusji dokładności i porównaniu jej z dokładnościami cytowanymi z szeregu publikacji astronomiczno-geodezyjnych następuje dyskusja zalet i wad metody. Z zalet podkreślić należy: wyeliminowanie błędów podziału kół i jednoczesne wyznaczanie 2 wielkości: azymutu i czasu. Głównym niedostatkiem metody jest jej ograniczone pole zastosowania w azymucie: naogół nie jest już korzystna dla azymutów przekraczających 45^0.

Drobniejszymi czcionkami odbito ustępy włączone do tekstu po złożeniu tej pracy, jako doktorskiej, na Wydziale Geodezyjnym Politechniki Warszawskiej.

Halina Milicer-Grużewska

On the law of probability and the characteristic function of the standardized sum of equivalent variables

Mémoire présenté par W. Sierpiński à la séance du 14 octobre 1949.

Introduction. I supposed, in my work entitled „On the co-efficient of correlation of equivalent variables" [1]), that the law of approximation [2]) of the Gaussian Laplace integral by the distribution function of the standardized sum of the equivalent variables is the same as for the independent ones.

Variables are equivalent if the probability that n among them fulfil definite condition depends upon the number n and upon this condition only. In detail $E(X_{i_1}^{\nu_1} \dots X_{i_n}^{\nu_n}) = M_{\nu_1 \dots \nu_n}$ for any indexes $i_1, \dots i_n$.

This article is an attempt to prove this supposition, then to prove the theorem of approximation for equivalent variables. I suppose that they are correlated.

My previous results [3]) suggest that we cannot expect such a general theorem as in the case of independence. In fact, the necessary and sufficient conditions of the central limit theorem for the equivalent variables are quite special. We cannot expect less suppositions for a stronger result, such as in the theorem of approximation in comparison with the central limit theorem.

The greater part of this work is devoted to investigations of the distribution and characteristic functions of the standardized sum of equivalent variables.

We know that the characteristic $\varphi(t)$ and distribution $F(x)$ functions are connected with the following relations

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[1]) Comtes rendus de la Société de Sciences et des Lettres de Varsovie, Classe III, Année XXXIX, 1946.

[2]) H. Cramer — Random Variables and Probability Distributions, Cambridge University Press, Ch. VII, 1937.

[3]) C. R. d. l. Soc. Polonais de Math., 1946, p. 244—246, and Atti d. Acc. Naz. Lincei — Sulla legge limite d. variabili causali equivalenti, Roma, 1948, s. VIII, vol. II, fasc. 2.

$$F(x+h)-F(x)=\frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^{+T}\frac{1-e^{-ith}}{it}e^{-itx}\varphi(t)\,dt$$

or, if $\varphi(t)$ is integrable, and $F'(x)=f(x)$ (i. e. $F(x)$ has an integrable density) then:

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-itx}\varphi(t)\,dt, \tag{1}$$

or, if we only know that:

$$\int_{|t|>A}\left|\frac{\varphi(t)}{t}\right|dt,\ A>0 \tag{2}$$

exists, then:

$$F(x)-\Phi(x)=-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\varphi(t)-e^{-t^2/2}}{t}e^{-itx}\,dt \tag{3}$$ [4])

where $\Phi(x)$ is the Gaussian-Laplace integral.

Of course the existence of integral (2) neads less suppositions then the theorem of the approximation. Still less suppositions needed to prove the uniform existence of integral:

$$\int_{|t|>c}\frac{\varphi(t)\,e^{-it\xi}}{t}dt,\quad c>0.$$

In the first paragraphs we have to deal with the distribution function and the characteristic function of the standardized sums of the equivalent variables, in the last ones — with the theorem of the approximation.

In § 8 I discuss the problem of the impossibility of the approximation and I even prove that *the central limit theorem is wrong in case the equivalent, strongly correlated variables are bounded.*

I shall use the formula (3) in my proofs, and consequently the existence of the integral (2) will be investigated.

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[4]) H. Cramer, loc. cit. Ch. IV Remark p. 34.

§ 1. Notations and abbreviations.

Abbreviations

I shall write for	probability	p.
,,	law of the probability	p. p.
,,	distribution of the probability	d. p.
,,	probability density	p. d.
,,	frequency function	fr. f.
,,	characteristic function	ch. f.
,,	equivalent variables	e. v.
,,	equivalent correlated variables (i. e. the total coeff. of corr. $\neq 0$)	e. c. v.
,,	random variable	r. v.
,,	standardized sum	st. s.
,,	mathematical expectation	m. exp.
,,	coordinate	co.
,,	random coordinate	r. co.

Notations

$X_1, X_2, \ldots, X_n, \ldots$ mean e. c. v.

$E(X)$ means m. exp. of the r. v. X

(1,1) $$X_1 + X_2 + \cdots + X_n = S_n$$

(2,1) $$E\,[S_n - E\,(S_n)]^2 = B_n^2$$

(3,1) $X_n' = \dfrac{S_n - E\,(S_n)}{B_n}$ is the st. s. of the e. c. r. X_i, $i = 1, 2, \ldots n$

$\varkappa_n = (x_1, \ldots x_n)$ is a point in the R_n space of co. $x_1, \ldots x_n$

$\mathrm{X}_n = (X_1, \ldots X_n)$ is a random point in R_n space of r. co. $X_1, \ldots X_n$

$F(x_1, \ldots x_n) = F(\varkappa_n)$ the d. f. of the r. v. $(X_1, \ldots X_n) = \mathrm{X}_n$, then

(3',1) $$F(\varkappa_n) = P(\mathrm{X}_n \leqslant \varkappa_n) = P(X_1 \leqslant x_1, \ldots X_n \leqslant x_n)$$

$\Delta_n F$ is the p. of the inequalities:

$$x_i < X_i \leqslant x_i + \Delta x_i, \qquad i = 1, 2, \ldots n$$ i. e.

$$(4,1) \qquad \varkappa_n < X_n \leqslant \varkappa_n + \Delta \varkappa_n$$ [5]

$\Delta_n F = \Delta_n F(x_1, \ldots x_n)$ is the difference of the function $F(\varkappa_n)$ on the half-opened interval defined by (4,1).

We write

$$d_n F(\varkappa_n) = d_n F = d_n F(x_1, \ldots x_n).$$

If, for instance, $h(\varkappa_n)$ is integrable over R_n with respect to the p. p. P then:

$$(5,1) \qquad E[h(X_n)] = \int_{R_n} h(\varkappa_n)\, d_n F = \underbrace{\int \cdots \int}_{n-times} h(x_1, \ldots x_n)\, d P =$$
$$= \int \cdots \int h(x_1, \ldots x_n)\, d_n F = \int \cdots \int h(x_1, \ldots x_n)\, d_n F(x_1, \ldots x_n).$$

If the partial derivative

$$f(x_1, \ldots x_n) = \frac{\partial^n F(x_1, \ldots x_n)}{\partial x_1 \ldots \partial x_n} = f(\varkappa_n) = \frac{\partial^n F(\varkappa_n)}{\partial x_1 \ldots \partial x_n}$$

exists and is integrable, then $f(x_1, \ldots x_n) = f(\varkappa_n)$ represents the density of mass at the point $\varkappa_n$. This function is called the probability density (p. d.) or the frequency function (fr. f.).

We write:

$$(5',1) \qquad E[h(X_n)] = \int \cdots \int h(x_1, \ldots x_n) f(x_1, \ldots x_n)\, dx_1 \ldots dx_n =$$
$$= \int_{R_n} h(\varkappa_n) f(\varkappa_n)\, d\varkappa_n .$$

If $f(\varkappa_n)$ and $h(\varkappa_n)$ are continuous functions, then $E[h(X_n)]$ is a n-dimensional Rieman integral [6]).

$$(6,1) \qquad E(e^{itX}) = \varphi(t)$$

is the ch. f. of the r. v. X.

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[5]) v. H. Cramer loc. cit. Ch. II.

[6]) Or, if the functions $h(\varkappa_n)$ and $f(\varkappa_n)$ have at most finite sets of discontinuous „points" without any common „points".

We have then for $X = X_n'$ where X_n' means the st. s.:

(7,1) $$E(e^{itX_n'}) = \varphi_n(t) = \int \cdots \int e^{it\frac{x_1 + \ldots x_n}{B_n}} dF(x_1, \ldots x_n)$$

but we write, when the density function exists:

(8,1) $$E(e^{itX_n'}) = \varphi_n(t) = \int \cdots \int e^{it\frac{x_1 + \ldots + x_n}{B_n}} f(x_1, \ldots x_n)\, dx_1 \ldots dx_n .$$

Wu put, as usually:

(9,1) $$E(X_{i_1}^{l_1} \ldots X_{i_k}^{l_k}) = m_{l_1, \ldots l_k}; \quad E\,|X_{i_1}^{l_1} \ldots X_{i_k}^{l_k}| = M_{l_1 \ldots l_k}, \qquad k = 1, \ldots$$

(9′,1) $$m_{\underset{n-times}{1 \ldots 1}} = M_1^{(n)}; \quad m_2 - m_1^2 = \sigma^2; \quad m_{11} - m_1^2 = \sigma^2 R_{11};$$

(9″,1) $$m_{\underset{(n-1)\ times}{21 \ldots 1}} = M_{21}^{(n)}.$$

Note that the moments (9,1), (9′,1), (9″,1) are independent upon the order of the indexes; this results from the definition of the e. v.

We indicate also by

$$f(x \mid \varkappa_{n-1}) = f(\varkappa_n) : f(\varkappa_{n-1})$$

the conditional frequency of the e. v. X_n, relative to the hypothesis that $X_{n-1} = \varkappa_{n-1}$. We shall write its derivate as follows:

$$\left(\frac{\partial f(x \mid \varkappa_{n-1})}{\partial x}\right)_{x=a} = f'(x \mid \varkappa_{n-1})_{x=a}$$

and the set defined by the relation

$$|x_1 + \cdots + x_{n-1}| < \sqrt{a}\,.\,B_{n-1}, \qquad a > 0$$

with K_a, i. e.:

(11,1) $$K_a = \underset{x_1, \ldots x_{n-1}}{E} \left[\, |x_1 + \cdots + x_{n-1}| < \sqrt{a}\, B_{n-1} \right]$$

and as $\bar{K}_a$ its complement, i. t.:

(11′,1) $$\bar{K}_a = \underset{x_1, \ldots x_{n-1}}{E} \left[\, |x_1 + \cdots + x_{n-1}| > \sqrt{a}\, B_{n-1} \right].$$

§ 2. The distribution function of the standardized sum of equivalent variables and its properties.

The distribution function of the st. s., i. e. the p. that $X_n' \leqslant a$, i. e.:

$$F_n(a) = P(X_n' \leqslant a) \tag{1,2}$$

is a Stieltjes-Lebesgue n-dimensional integral over this part of the space R_n, for which:

$$X_n' \leqslant a$$

and in respect to the p. p. (3′,1). This means that:

$$F_n(a) = \underset{X_n' \leqslant a}{\int \cdots \int} dF(x_1, \dots x_n) = \int_{X_n' \leqslant a} d_n F \tag{1',2}$$

and, in case the density function $f(x_1, \dots x_n) = f(\varkappa_n)$ exists, this may be writen:

$$F_n(a) = \underset{X_n' \leqslant a}{\int \cdots \int} f(x_1, \dots x_n)\, dx_1 \dots dx_n = \int_{X_n'} f(\varkappa_n)\, d\varkappa_n . \tag{1'',2}$$

The inequality:

$$X_n' \leqslant a \quad \text{or} \quad X_1 + \cdots + X_n \leqslant a \,.\, B_n$$

gives

$$X_n \leqslant a\, B_n - (X_1 + \dots + X_{n-1})$$

or symbolically:

$$X_n \leqslant a\, B_n - \varkappa_{n-1}$$

then:

$$F_n(a) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \int_{-\infty}^{aB_n - \varkappa_{n-1}} f(x_1, \dots x_n)\, dx_n\, dx_{n-1} \dots dx_1 = \tag{1''',2}$$

$$= \int_{R_{n-1}} \int_{-\infty}^{aB_n - \varkappa_{n-1}} f(x_1, \dots x_n)\, dx_n\, d\varkappa_{n-1} .$$

If $f(x_1, \dots x_n)$ were a continuous function, then this integral would be a Rieman integral.

We shall prove now:

Lemma I. *If the conditional density* $f(x \mid \varkappa_{n-1})$ *is continuous and bounded for every* $\varkappa_{n-1} \subset R_{n-1}$ *then* $f_n(a)$, *the density of the d. f.* $F_n(a)$.

1^0 *exists and it is:*

$$f_n(a) = B_n \cdot \int_{R_{n-1}} f(x_1, \dots x_{n-1}, \tag{2,2}$$

$$aB - (x_1 + x_2 + \dots + x_{n-1}))\, dx_{n-1} \dots dx_1$$

which we write symbolically:

$$f_n(a) = B_n \cdot \int_{R_{n-1}} f(\varkappa_{n-1}, aB - \varkappa_{n-1})\, d\varkappa_{n-1}. \tag{2',2}$$

2^0 *this density* $f_n(a)$ *is continuous in* R_1.

Proof:

$$\frac{F_n(a+h) - F_n(a)}{h} = \frac{1}{h} \cdot \int_{R_{n-1}} f(\varkappa_{n-1}) \int_{aB_n - \varkappa_{n-1}}^{(a+h)B_n - \varkappa_{n-1}} f(x \mid \varkappa_{n-1})\, dx\, d\varkappa_{n-1} =$$

$$= B_n \int_{R_{n-1}} f(\varkappa_{n-1}) \frac{1}{h \cdot B_n} \int_{aB_n - \varkappa_{n-1}}^{(a+h)B_n - \varkappa_{n-1}} f(x \mid \varkappa_{n-1})\, dx\, d\varkappa_{n-1} =$$

$$= B_n \int_{R_{n-1}} f(\varkappa_{n-1}) \cdot f[(a + \Theta h) B_n - \varkappa_{n-1} \mid \varkappa_{n-1}]\, d\varkappa_{n-1}, \qquad 0 < \Theta < 1$$

$$\frac{F_n(a+h) - F_n(a)}{h} - B_n \int_{R_{n-1}} f(\varkappa_{n-1}, aB_n - \varkappa_{n-1})\, d\varkappa_{n-1} =$$

$$= -B_n \int_{R_{n-1}} f(\varkappa_{n-1}) \{ f(aB_n - \varkappa_{n-1} \mid \varkappa_{n-1}) -$$

$$- f[(a + \Theta h) B_n - \varkappa_{n-1} \mid \varkappa_{n-1}] \}\, d\varkappa_{n-1}.$$

It results from the suppositions of the continuity and boundedness of the function $f(x \mid \varkappa_{n-1})$ that: 1^0 the expressions in the brackets of the integral approaches to zero when $h \to 0$, 2⁰ the modulus of the function under the sign of the integral is not greater than $c \cdot f(\varkappa_{n-1})$ where $0 < c <$ const and $f(\varkappa_{n-1})$ is integrable in R_{n-1}:

$$f(\varkappa_{n-1}) > 0, \int\limits_{R_{n-1}} f(\varkappa_{n-1})\, d\varkappa_{n-1} = 1$$

then [7]):

$$\lim_{h\to 0} \frac{F_n(a+h) - F_n(a)}{h} = f_n(a) = B_n \cdot \int\limits_{R_{n-1}} f(\varkappa_{n-1},\ a B_n - \varkappa_{n-1})\, d\varkappa_{n-1}$$

thus the Lemma I, 1⁰ is proved.

When the variables $X_1, \ldots X_n$ are independent, then

$$f_n(a) = B_n \cdot \int\limits_{R_{n-1}} f(\varkappa_{n-1}) f(a B_n - \varkappa_{n-1})\, d\varkappa_{n-1} = B_n \cdot E\,[f(a B_n - \varkappa_{n-1})].$$

For instance, if $n = 2$, then it is:

$$f_2(a) = B_2 \int\limits_{R_1} f(x) f(a B_2 - x)\, dx = B_2 \cdot E\,[f(a B_2 - x)].$$

We may still write $f_n(a)$ as follows:

$$(2'',2) \qquad f_n(a) = B_n \int\limits_{R_{n-1}} f(\varkappa_{n-1}) f(a B_n - \varkappa_{n-1} \mid \varkappa_{n-1})\, d\varkappa_{n-1}$$

or

$$f_n(a+h) = B_n \int\limits_{R_{n-1}} f(\varkappa_{n-1}) f\,[(a+h) B_n - \varkappa_{n-1} \mid \varkappa_{n-1}]\, d\varkappa_{n-1},$$

and

$$(3,2) \qquad f_n(a+h) - f_n(a) = B_n \int\limits_{R_{n-1}} f(\varkappa_{n-1}) \cdot$$

$$\cdot \{f\,[(a+h) B_n - \varkappa_{n-1} \mid \varkappa_{n-1}] - f(a B_n - \varkappa_{n-1} \mid \varkappa_{n-1})\}\, d\varkappa_{n-1}.$$

Similary we deduce:

$$f_n(a+h) \to f_n(a), \quad h \to 0$$

and obtain 2⁰ part of Lemma I.

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[7]) See f. ex. H. Cramer, Mathematical Methods of Statistics Princeton 1946, Ch. 7, p. 67.

Lemma II. *If* $f(x \mid \varkappa_{n-1})$ *is continuous and bounded in* R_1 *for every* $\varkappa_{n-1} \subset R_{n-1}$ *and* $f(\varkappa_n) = f(\varkappa_{n-1}, x_n) \to 0$, $|x_n| \to \infty$ *uniformly in respect to* $\varkappa_{n-1}$, *then the density* $f_n(a)$, *of the st. s., is uniformly continuous in* R_1.

Proof. We replace the integral of the formula (3,2) with two other as follows:

$$f_n(a+h) - f_n(a) = B_n \int_{\overline{K}_\alpha} f(\varkappa_{n-1}) \cdot r(\varkappa_{n-1})\, d\varkappa_{n-1} +$$

$$+ B_n \int_{K_\alpha} f(\varkappa_{n-1}) \cdot r(\varkappa_{n-1})\, d\varkappa_{n-1} = I_1 + I_2$$

where K_α and $\overline{K}_\alpha$ define the formulae (11,1), (11',1), for $a = \alpha$ and $r(\varkappa_{n-1})$ replaces the expression of the gteat bracket from the equality (3,2).

It results from the supposition of the boundedness of $f(x \mid \varkappa_{n-1})$ that:

$$|I_1| \leqslant 2\,C \,.\, B_n \int_{\overline{K}_\alpha} f(\varkappa_{n-1})\, d\varkappa_{n-1} \qquad (C = \text{const}). \tag{4,2}$$

But Tchebycheff's inequality gives here:

$$\int_{\overline{K}_\alpha} f(\varkappa_{n-1})\, d\varkappa_{n-1} \leqslant \frac{1}{\alpha} \qquad \text{i. e.}$$

$$|I_1| \leqslant \frac{2\,C \,.\, B_n}{\alpha}. \tag{4',2}$$

Now we write the integral I_2 in the form (2,2):

$$I_2 = B_n \int_{K_\alpha} f\,[\varkappa_{n-1},\ (a+h)\,B_n - \varkappa_{n-1}]\, d\varkappa_{n-1} - \tag{4'',2}$$

$$- B_n \int_{K_\alpha} f\,[\varkappa_{n-1},\ a\,B_n - \varkappa_{n-1}]\, d\varkappa_{n-1}.$$

Since we have in K_α:

$$|a\,B_n - \varkappa_{n-1}| > (|a| - \sqrt{\alpha})\,B_n\,,$$

because:

$$B_n > B_{n-1}$$

and because the integral

$$\int_{K_\alpha} f(\varkappa_{n-1}, a B_n - \varkappa_{n-1})\, d\varkappa_{n-1} \leqslant C \int_{R_{n-1}} f(\varkappa_{n-1})\, d\varkappa_{n-1} = C,$$

and

$$\lim_{x_n \to \infty} f(\varkappa_{n-1}, x_n) = 0 \text{ }^{8)}$$

then

$$\lim_{|a| \to \infty} B_n \int_{K_\alpha} f(\varkappa_{n-1}, a B_n - \varkappa_{n-1})\, d\varkappa_{n-1} = 0, \quad 0 < \alpha \leqslant a.$$

We get a similar result for the first integral of the expression (4″,2) if h bounded, for example

$$|h| \leqslant 1;$$

$$\lim_{a \to \infty} B_n \int_{K_\alpha} f[\varkappa_{n-1}, (a+h) B_n - \varkappa_{n-1}]\, d\varkappa_{n-1} = 0.$$

We see then, that to each $\varepsilon > 0$ corresponds such a positive number $A(\varepsilon)$, that:

$$|a| \geqslant A(\varepsilon) \rightarrow |f_n(a+h) - f_n(a)| \leqslant \left(\varepsilon + \frac{c}{\alpha}\right) B_n.$$

But for $|a| < A(\alpha)$ it will be, uniformly in respect to a:

$$\lim_{h \to 0} I_2 = 0 \qquad \text{i. e.}$$

$$|h| \leqslant H(\varepsilon) \rightarrow |f_n(a+h) - f_n(a)| \leqslant \left(\varepsilon + \frac{c}{\alpha}\right) B_n, \qquad \text{i. e.}$$

$$\lim_{h \to 0} |f_n(a+h) - f_n(a)| \leqslant \frac{c B_n}{\alpha}$$

uniformly in respect to a. But α is as great as we wish, then

$$\lim_{h \to 0} [f_n(a+h) - f_n(a)] = 0$$

uniformly in respect to a.

[8]) v. 7).

Remark I. *We may replace in Lemma II the supposition that* $\lim_{|x_n|\to\infty} f(\varkappa_{n-1}, x_n) = 0$), *uniformly in respect to* $\varkappa_{n-1} \subset R_{n-1}$, *by the supposition that*

$$\lim_{|a|\to\infty} \int_{R_\alpha} f(\varkappa_{n-1}, a B_n - \varkappa_{n-1})\, d\varkappa_{n-1} = 0$$

or by a weaker one:

$$(5,2) \qquad \lim_{|a|\to\infty} \int_{K_\alpha} f(\varkappa_{n-1}, a B_n - \varkappa_{n-1})\, d\varkappa_{n-1} = 0$$

where $0 < \alpha \leqslant |a|$.

Remark II. *In case* (5,2) *has place uniformly in respect to n, and if in* K_α *the function of a,* $f(\varkappa_{n-1}, a - \varkappa_{n-1})$, accomplisches the Lipschitz condition i. e.:

$$|f(\varkappa_{n-1}, a+h-\varkappa_{n-1}) - f(\varkappa_{n-1}, a-\varkappa_{n-1})| \leqslant c\, h^\beta f(\varkappa_{n-1}), \quad \beta > 0,$$

then

$$f_n(a) : B_n^{1+\beta} = \frac{1}{B_n^\beta} \int_{R_{n-1}} f(\varkappa_{n-1}, a B_n - \varkappa_{n-1})\, d\varkappa_{n-1}$$

is an uniformly continuous function of $a \subset R_1$ *in respect to n,* i. e. there exists such a constant $H(\varepsilon)$, that

$$(5',2) \qquad |h| < H(\varepsilon) \to |f_n(a+h) - f_n(a)| : B_n^{1+\beta} \leqslant \varepsilon, \quad n = 1, 2, \ldots.$$

It is enough to suppose that for $0 < \alpha \leqslant |a|$

$$\lim_{h\to 0} \int_{K_\alpha} \{f(\varkappa_{n-1}, a B_n - \varkappa_{n-1}) - f[\varkappa_{n-1}, (a+h) B_n - \varkappa_{n-1}]\}\, d\varkappa_{n-1} = 0$$

uniformly in respect to n and *then* $f_n(a) : B_n$, *is an uniformly continuous function of* $a \subset R_1$ *in respect to n,* i. e. that there exists such a constant $H(\varepsilon)$ that:

$$(5'',2) \qquad |h| \leqslant H(\varepsilon) \to |f_n(a+h) - f_n(a)| : B_n^{1+\beta} \leqslant \varepsilon, \quad n = 1, 2, \ldots.$$

Lemma III. *If the density function* $f(x \mid \varkappa_{n-1})$ *and its derivatives are uniformly bounded:*

$$(6,2) \qquad f(x \mid \varkappa_{n-1}) \leqslant C; \quad \left| \frac{\partial f(x \mid \varkappa_{n-1})}{\partial x} \right| \leqslant C; \qquad x \subset R_1; \quad \varkappa_{n-1} \subset R_{n-1}$$

then the density function of the standardized sum X_n' is derivable and

$$(6',2) \qquad |f_n'(a)| \leqslant C \,.\, B_n^2, \quad n=1,2,\ldots; \quad a \subset R_1.$$

Proof. We find from formula (2″,2) and the supposition (6,2)

$$(6'',2) \qquad f_n'(a) = B_n^2 \int\limits_{R_{n-1}} [f(\varkappa_{n-1}) \cdot f'(x \mid \varkappa_{n-1})]_{x=aB_n-\varkappa_{n-1}} \, d\varkappa_{n-1},$$

$$|f_n'(a)| \leqslant CB_n^2.$$

Remark III. *According to formula* (6,2) *we may represent the derivative* $f_n'(a)$ *as follows:*

$$(6''',2) \qquad f'(a) = B_n^2 \Bigg(C \,.\, \Theta' \int\limits_{\overline{K}_\alpha} f(\varkappa_{n-1})\, d\varkappa_{n-1} + \\ + \Theta \max_{\varkappa_{n-1} \subset K_\alpha} f'(x \mid \varkappa_{n-1})_{x=aB_n-\varkappa_{n-1}} \Bigg),$$

$$|\Theta|, \ |\Theta'| \leqslant 1.$$

Remark IV. *If the density $f(\varkappa_{n-1})$ fulfils the condition:*

$$(7,2) \qquad f(\varkappa_{n-1}) \cdot \left\{ \prod_{i=1}^{n-1} (1+|x_i|) \right\}^{5+2\alpha} \leqslant C_1$$

$$\alpha > 0, \quad C_1 = \text{const}, \quad \varkappa_{n-1} \subset R_{n-1}$$

then

$$I_a = \int\limits_{\overline{K}_\alpha} f(\varkappa_{n-1})\, d\varkappa_{n-1} = 0(B_n^{-2}\, a^{-1-\alpha}).$$

Proof:

$$0 < I_\alpha \leqslant C_1 : \min_{\varkappa_{n-1} \subset \overline{K}_a} \left[\prod_{i=1}^{n-1} (1+|x_i|) \right]^{2(1+\alpha)} \cdot \int\limits_{\overline{K}_a} \prod_{i=1}^{n-1} (1+|x_i|)^{-3}\, d\varkappa_{n-1},$$

but

$$\prod_{i=1}^{n-1} (1+|x_i|) > \sum_{i=1}^{n-1} |x_i| \geqslant \left| \sum_{i=1}^{n-1} x_i \right| \geqslant \sqrt{a} \,.\, B_n, \quad \varkappa_{n-1} \subset \overline{K}_a \qquad \text{i. e.}$$

$$0 < I_a < (a B_{n-1}^2)^{-(1+\alpha)}$$

because

$$\int\limits_{\overline{K}_a} \prod_{i=1}^{n-1} (1+|x_i|)^{-3} \, d\varkappa_{n-1} \leqslant \prod_{i=1}^{n-1} \int\limits_{-\infty}^{\infty} (1+|x_i|)^{-3} \, dx_i = 1,$$

and

$$B_n \to \infty, \quad n \to \infty,$$

then

$$|I_a \cdot a^{1+\alpha} B_n^2| \leqslant B_n^{-2\alpha} \to 0, \quad n \to \infty, \qquad \text{i. e.}$$

$$I_n = 0\,(B_n^{-2} a^{-1-\alpha}).$$

Remark V. *If the densities* $f(\varkappa_{n-1})$ *and* $f(x|\varkappa_{n-1})$ *fulfil the conditions:*

1. $f(\varkappa_{n-1}) \cdot \prod_{i=1}^{n-1} (1+|x_i|)^{5+2\alpha} \leqslant C_1, \quad \alpha > 0; \quad \varkappa_{n-1} \subset R_1$

2. $|f'(x|\varkappa_{n-1})_{x=aB_n-\varkappa_{n-1}}| \leqslant C, \qquad \varkappa_{n-1} \subset R_1$

3. $f'(x|\varkappa_{n-1})_{x=aB_n-\varkappa_{n-1}} = 0\,(B_n^{-2} a^{-1-\alpha}), \qquad \varkappa_{n-1} \subset K_a$

then

(8,2) $$f_n'(a) = 0\,(a^{-1-\alpha}), \quad a \subset R_1$$

uniformly in respect to n.

Proof. Formula (8,2) is a simple result of Remarks III, IV and supposition 3. In fact:

$$|f_n'(a)| \leqslant C B_n^2 \int\limits_{\overline{K}_a} f(\varkappa_{n-1}) \, d\varkappa_{n-1} + B_n^2 \cdot 0\,(B_n^{-2} \cdot a^{-1-\alpha})$$

then:

$$|f_n'(a)| \leqslant (B_n^{-2\alpha} + 1) \cdot 0\,(a^{-1-\alpha}),$$

and

$$|f_n'(a)| = 0\,(a^{-1-\alpha}), \quad a \subset R_1$$

uniformly in respect to n.

§ 3. The properties of the characteristic function of the standardized sum of equivalent variables.

Lemma IV. *If*

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx$$

where

1°. $f(x)$ *is bounded and integrable in* R_1, *and*

2°:

$$\frac{1}{h}\int_0^h [f(\xi + x) + f(\xi - x) - 2f(\xi)]\, dx \to 0, \quad h \to 0$$

uniformly in respect to ξ, *then the integral*

$$\int_{|t|>C} \frac{\varphi(t)}{t} \cdot e^{-it\xi}\, dt, \qquad C = \text{const} > 0$$

exists uniformly in respect to ξ; *especially, if we have* 2° *only for* $\xi = 0$, *then the integral*

$$\int_{|t|\geqslant C} \frac{\varphi(t)}{t}\, dt$$

exists.

Proof. Il results from the suppositions 1° and 2° that

$$\int_A^B \frac{\varphi(t)}{t}\, dt \to 0, \quad A \to \infty, \quad B \geqslant A,$$

and

$$\int_A^B \frac{\varphi(t)}{t}\, e^{-it\xi}\, dt \to 0, \quad A \to \infty, \quad B \geqslant A$$

uniformly in respect to ξ.

Really:

$$I_A(\xi) = \int_{-\infty}^{\infty}\int_A^B f(x)\,\frac{e^{it(x-\xi)}}{t}\,dt\,dx = \int_{-\infty}^{\infty}\int_A^B f(x+\xi)\,\frac{e^{itx}}{\xi}\,dt\,dx =$$

$$= \int_{|x|\geqslant 1:lg A} f(x+\xi)\int_{Ax}^{Bx}\frac{e^{iy}}{y}\,dy\,dx + \int_{|x|\leqslant 1:lg A} f(x+\xi)\int_A^B \frac{e^{itx}}{t}\,dt\,dx = I_1 + I_2$$

$$|I_1| \leqslant \int_{|x|\geqslant 1:lg A} f(x+\xi)\,dx \cdot \underset{|Ax|\geqslant A:lg A}{\mathrm{Max}} \left|\int_{Ax}^{Bx}\frac{e^{iy}}{y}\,dy\right| \to 0, \quad A\to\infty$$

uniformly in respect to ξ

$$I_2 = i\int_{|x|<1:lg A} f(x+\xi)\cdot\int_{Ax}^{Bx}\frac{\sin y}{y}\cdot dy\,dx +$$

$$+ \int_{|x|<1:lg A} f(x+\xi)\int_A^B\frac{\cos tx}{t}\,dt\,dx = I_3 + I_4$$

$$|I_3| \leqslant \pi\cdot\int_{|x|<1:lg A} f(x+\xi)\,dx \leqslant \pi\cdot C : lg A \to 0, \quad A\to\infty$$

unifoomly to respect to ξ.

$$I_4 = \int_0^{1:lg A} \underbrace{[f(x+\xi)+f(-x+\xi)-2f(\xi)]}_{r(x)}\cdot\int_A^B\frac{\cos tx}{t}\,dt\,dx +$$

$$+ 2f(\xi)\int_A^B\frac{dt}{t}\cdot\int_0^{1:lg A}\cos tx\,dx =$$

$$= \int_0^{1:lg A} r(x)\int_A^B\frac{\cos tx}{t}\,dt + 2f(\xi)\cdot\int_A^B\frac{1}{t^2}\sin(t:lg A)\,dt =$$

$$= I_5 + 2\,\Theta\,f(\xi)\cdot A^{-1}, \qquad |\Theta|\leqslant 1$$

$$I_5 = \int_0^{1:B} r(x) \int_A^B \frac{\cos tx}{t}\, dt + \int_{1:B}^{1:lg A} r(x) \int_A^B \frac{\cos tx}{t}\, dt = I_6 + I_7$$

$$|I_6| \leqslant \operatorname*{Max}_{0<x<1:B} |r(x)| \cdot \frac{lg B}{B} \to 0, \quad B \to \infty$$

uniformly in respect to ξ

$$|I_7| = \left| \int_{1:B}^{1:lg A} r(x) \cdot \int_{Ax}^{Bx} \frac{\cos y}{y}\, dy\, dx \right| \leqslant$$

$$\leqslant \int_{1:B}^{1:lg A} |r(x)| \cdot \left[lg A + lg x + \int_1^{Bx} \frac{\cos y}{y}\, dy \right] dx =$$

$$= ly A \cdot \int_{1:B}^{1:lg A} |r(x)|\, dx + \int_{1:B}^{1:lg A} |r(x)|\, lg x\, dx +$$

$$+ \int_{1:B}^{1:lg A} |r(x)| \int_1^{Bx} \frac{\cos y}{y}\, dy\, dx = I_8 + I_9 + I_{10}$$

$I_8 \to 0$, $A \to \infty$, uniformly in respect to ξ, according to 2^0 or $I_8 \to 0$, for $\xi = 0$ if we suppose only 2^0; for

$$|I_9| \leqslant \operatorname*{Max}_{0<x\leqslant 1:lg A} |r(x)| \cdot x \left| (lg x - 1) \right|_{1:B}^{1:lg A} \to 0, \quad A \to \infty$$

uniformly in respect to ξ

$$|I_{10}| \leqslant \int_{|x|\leqslant 1:lg A} |r(x)|\, dx \cdot \operatorname*{Max}_{1:B<x<1:lg A} \left| \int_1^{Bx} \frac{\cos y}{y}\, dy \right| \leqslant$$

$$\leqslant \pi \int_{|x|\leqslant 1:lg A} |r(x)|\, dx \leqslant \pi : lg A \to 0, \quad A \to \infty$$

uniformly in respect to ξ.

Lemma V. *If the density of the d. f. $F(x)$ of the st. v. X exists and if its derivative $f'(x)$ is integrable, i. e.:*

(1,3)
$$\int_{R_1} |f'(x)|\, dx$$

exists, and $\varphi(x)$ is ch. f. of this r. v. X, then the integral:

(2,3)
$$\int_{|t|>A} \left| \frac{\varphi(t)}{t} \right| dt, \quad A > 0.$$

exists.

Proof. If X is a st. v. then:

$$E(X) = 0; \quad D(X) = E\left[(X - E(X))^2\right] = 1,$$

and

$$P\{|x| \geqslant t\} \leqslant \frac{1}{t^2}.$$

We now may write for $B > A > 0$:

$$\int_A^B \left| \frac{\varphi(t)}{t} \right| dt \leqslant \int_A^B \frac{dt}{t} \cdot \left| \int_{R_1} e^{itx}\, dF(x) \right| =$$

$$= \int_A^B \frac{dt}{t} \cdot \left| \int_{|x|<t} e^{itx}\, dF(x) + \int_{|x|>t} e^{itx}\, dF(x) \right| \leqslant$$

$$\leqslant \int_A^B \frac{dt}{t} \left\{ \left| \int_{|x|<t} e^{itx}\, dF(x) \right| + \int_{|x|>t} dF(x) \right\} \leqslant$$

$$\leqslant \int_A^B \frac{dt}{t} \left\{ \left| \int_{|x|<t} e^{itx}\, dF(x) \right| + \frac{1}{t^2} \right\},$$

but

$$\int_{|x|<t} e^{itx}\, dF(x) = \frac{e^{itx} f(x)}{it} \Big|_{-t}^{+t} - \frac{1}{it} \int_{-t}^{+t} e^{itx} f'(x)\, dx = 0\,(t^{-1}),$$

i. e.:

$$(3,3) \qquad \int_A^B \left| \frac{\varphi(t)}{t} \right| dt < C \cdot \int_A^B \frac{dt}{t^2} \leqslant \frac{C}{A} \to 0, \quad A \to \infty, \quad B > A.$$

Similary we have:

$$\int_{-B}^{-A} \left| \frac{\varphi(t)}{t} \right| dt \to 0, \quad A \to \infty, \quad B > A > 0.$$

We receive as a corollary from Lemma III and V the:

Teorem I. *If the d. f.* $F(\varkappa_n)$, $\varkappa_n \subset R_n$ *fulfils the suppositions of the Remark V and* $\varphi_n(t)$ *is the ch. f. of the st. s.* X_n', *then the integral:*

$$(4,3) \qquad \int_{|t| \geqslant A} \left| \frac{\varphi_n(t)}{t} \right| dt, \quad A > 0$$

exists uniformly in respect to n.

Proof. In fact, we have proved in § 2 that the integrability of the derivative of the density function $f_n(a)$ results from the supposition 1, 2, 3 of. Remark V because we have:

$$f_n'(a) = 0(|a|^{-1-\alpha}), \qquad \alpha > 0.$$

The existence of the integral (4,3) results now from Lemma V.

Theorem II. *If the d. f.* $F(\varkappa_n)$ *of the e. c. v.* $X_1, \dots X_n$ *fulfils the suppositions of Remark V, then the d. f.* $F_n(a)$ *of the st. s.* X_n' *of these variables may be expressed by its ch. f.* $\varphi_n(t)$ *and the Gaussian-Laplace integral* $\Phi(a)$ *as follows:*

$$F_n(a) = \Phi(a) - \frac{1}{2\pi i} \cdot \int_{-\infty}^{+\infty} \frac{\varphi_n(t) - e^{-\frac{t^2}{2}}}{t} \cdot e^{-ita} dt$$

where the integral exists uniformly in respect to n.

Proof. Indeed:

$$\int_{-\infty}^{\infty}\left|\frac{\varphi_n(t)-e^{-\frac{t^2}{2}}}{t}\right|dt \leqslant \int_{|t|<1}\left|\frac{\varphi_n(t)-1}{t}\right|dt+\int_{|t|<1}\left|\frac{1-e^{-\frac{t^2}{2}}}{t}\right|dt+$$

$$+\int_{|t|>1}\left|\frac{\varphi_n(t)}{t}\right|dt+\int_{|t|>1}\frac{e^{-\frac{t^2}{2}}}{|t|}dt.$$

Evidently the second and the fourth integrals exists. The third one exists also according to Theorem I. The proof of the existence of the first integral is also elementary, because:

$$|\varphi_n(t)-1|=\left|\int_{-\infty}^{+\infty}(e^{itx}-1)\,dF_n(x)\right|=\left|\int_{-\infty}^{+\infty}it\,x\,e^{-i\Theta tx}\,dF_n(x)\right|\leqslant$$

$$\leqslant|t|\cdot\int_{-\infty}^{+\infty}|x|\,dF_n(x),\quad \text{i. e.:}$$

$$\left|\frac{\varphi_n(t)-1}{t}\right|\leqslant\left\{\int_{-\infty}^{+\infty}x^2\,dF_n(x)\right\}^{\frac{1}{2}}=1,$$

since the r. v. X_n' is standardized, therefore

$$\int_{|t|<1}\left|\frac{\varphi_n(t)-1}{t}\right|dt\leqslant 1$$

wich proves Theorem II.

§ 4. The properties of the characteristic function $\varphi_n(t)$ of the standardized sum X_n', with a distribution function converging to the law of Gauss-Laplace.

Property I. *If $F_n(x)$ are the d. f. of the r. v. X_n, and $F_n(x)\to\Phi(x)$, $n\to\infty$, the corresponding ch. f. $\varphi_n(t)\to 0$, $|t|\to\infty$, uniformly in respect to n, beginning from an index N, which is independent from t.*

Proof. We know that the convergence of $F_n(x)$ to $\Phi(x)$, $n \to \infty$ is uniform in respect to x, if only $\Phi(x)$ is a d. f. [9]). In that case $\varphi_n(t) \to \varphi(t)$, $n \to \infty$, where $\varphi(t)$ means the ch. f., corresponding to $\Phi(t)$. In our case $\varphi(t) = e^{-\frac{t^2}{2}}$ and consequently $\varphi(t) \to 0$, $|t| \to \infty$ but:

$$\varphi_n(t) - e^{-t^2/2} = \int_{-\infty}^{+\infty} e^{itx}\, d[F_n(x) - \Phi(x)] =$$

$$= \int_{|x| \leqslant A} e^{itx}\, d[F_n(x) - \Phi(x)] + \int_{|x| > A} e^{itx}\, d[F_n(x) - \Phi(x)] = I_1 + I_2$$

$$|I_2| \leqslant (1 - F_n(A)) + (1 - \Phi(A)) + F_n(-A) + \Phi(-A) \leqslant$$

$$\leqslant 2(1 - \Phi(A)) + 2\Phi(-A) + |F_n(A) - \Phi(A)| +$$

$$+ |F_n(-A) - \Phi(-A)| \leqslant \frac{\varepsilon}{4}$$

for $n \geqslant N_1$ $(\varepsilon > 0)$, where A and N_1 are sufficiently great constants. Also when $n \geqslant N_2(A)$, and N_2 is a sufficiently great constants:

$$|I_1| \leqslant \frac{\varepsilon}{4}, \quad \text{for} \quad n \geqslant N_2$$

i. e. that

$$|\varphi_n(t) - e^{-t^2/2}| \leqslant \frac{\varepsilon}{2}, \quad \text{for} \quad n \geqslant \text{Max}\,[N_1, N_2(A)] = N.$$

Also for T sufficiently large it will be:

$$|t| \geqslant T \to e^{-t^2/2} \leqslant \frac{\varepsilon}{2}$$

then it will be:

$$|t| \geqslant T; \quad n \geqslant N \to$$

(1,4) $$|\varphi_n(t)| \leqslant \varepsilon.$$

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[9]) v. M. Frèchet: Généralités s. l. Probabilité, Variables aleatoires. Paris 1937, p. 276.

Property II. *There exists such a constant t_0, that:*

$$\varphi_n(t) = e^{-t^2/2} + 0\left(\frac{1}{n}\right), \quad |t| \leqslant t_0 \tag{2,4}$$

when:

$$F_n(x) \to \Phi(x), \quad n \to \infty,$$

and:

$$|m_{l_1 \dots l_n}| \leqslant r!\, S^r, \quad r = l_1 + \dots + l_n, \quad S = const. \text{ [10]}$$

This property is a simple consequence of Theorem I from my paper quoted above sub 3), p. 28—29.

We obtain now from formula (3,3), Property II and Theorems I and II the:

Property III. *If the d. f. $F(x_n)$ of e. c. v. $X_1, \dots X_n$ fulfils the suppositions of Theorem II and Property II, then:*

$$F_n(a) = \Phi(a) + 0\left(\frac{1}{n}\right) - \frac{1}{2\pi i} \int\limits_{t_0 \leqslant t \leqslant n} \frac{\varphi_n(t) - e^{-t^2/2}}{t} \cdot e^{-it \cdot a}\, dt =$$

$$= \Phi(a) + 0\left(\frac{1}{n}\right) - \frac{1}{2\pi i} \int\limits_{t_0}^{n} \frac{\varphi_n(t) - \varphi_n(-t)}{t} \cos ta\, dt +$$

$$+ \frac{1}{2\pi} \int\limits_{t_0}^{n} \frac{\varphi_n(t) - \varphi_n(-t) - 2e^{-t^2/2}}{t} \sin ta\, dt.$$

§ 5. The properties of the ch. f. of the st. s. of e. c. v., which fulfil still more suppositions.

Theorem III. *If the moments of the r. v. X are as follows:*

$$E(X^l) \cdot \begin{cases} = 0, & l = 2k+1, \quad k = 1, 2, \dots; \quad 0 < a < 2, \\ \leqslant C a^k \dfrac{(2k)!}{2^k k!}, & l = 2k, \quad 0 < C = \text{const.} \end{cases}$$

then the function:

$$\varphi(z) = E(e^{izx})$$

is an integer function of the complex variable z; its order is not greater than 2.

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[10]) $t_0 = \sqrt{m_{1,1}} : S.$

I shall name $\varphi(z)$ — the characteristic function of the r. v. X.

Proof. If $F(x)$ is d. f. of the r. v. X, then

$$\varphi(z) = \int_{R_1} e^{izx}\, dF(z)\,.$$

We shall prove some properties of this function $\varphi(z)$.

1. $\varphi(z)$ exists for every complex value of z.

We know that:

$$|izx| \leqslant |iz|^2 + \left|\frac{x}{2}\right|^2$$

and

$$(1,5) \qquad |\varphi(z)| \leqslant \int_{R_1} e^{|izx|}\, dF(x) \leqslant e^{|z|^2} \cdot \int_{R_1} e^{(x/2)^2}\, dF(x)$$

if only the last integral exists. But we shall prove that it is so.

The series:

$$e^{(x/2)^2} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{x}{2}\right)^{2k}$$

is uniformly convergent in any finite interval, f. ex. if

$$-A \leqslant x \leqslant B,$$

where $A > 0$, $B > 0$, then

$$\int_{-A}^{B} e^{(x/2)^2}\, dF(x) = \sum_{k=0}^{\infty} \frac{1}{2^{2k} k!} \int_{-A}^{B} x^{2k}\, dF(x) \leqslant$$

$$\leqslant \sum_{k=0}^{\infty} \frac{1}{2^{2k} k!} \int_{-\infty}^{\infty} x^{2k}\, dF(x) \leqslant C \cdot \sum_{k=0}^{\infty} \frac{a^k}{2^{2k} k!} \cdot \frac{(2k)!}{2^k\, k!} = C \cdot \sum_{k=0}^{\infty} \frac{(2k)!\, a^k}{2^{3k} (k!)^2},$$

but:

$$(2k)! \sim (2k)^{2k} \sqrt{4\pi k} \cdot e^{-2k}, \qquad 2^{3k} (k!)^2 \sim 2^{3k}\, k^{2k+1} \cdot 2\pi \cdot e^{-2k}, \qquad \text{i. e.}$$

$$\frac{(2k)!\, a^k}{2^{3k} (k!)^2} \sim \frac{a^k}{2^k \sqrt{\pi k}}\,.$$

For every A, $B > 0$:

$$\int_{-A}^{B} e^{(x/2)^2}\, dF(x) \leqslant 1 + \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} \left(\frac{a}{2}\right)^k \cdot \frac{\varepsilon_k}{\sqrt{k}},$$

$$0 < \varepsilon_k < 1, \quad k \to \infty,$$ i. e.

(2,5) $$K = \int_{-\infty}^{\infty} e^{(x/2)^2}\, dF(x)$$

exists, since $0 < a < 2$.

I'. If $|a| < 1$, then the integral

$$\int_{-\infty}^{\infty} e^{x^2/2}\, dF(x)$$

exists.

The proof is the same as in I.

II. The order of $\varphi(z)$ is not greater than 2.

This proposition results directly from (1,5) and (2,5).

III. The derivatives:

$$\varphi_{(z)}^{(l)} = i^l \cdot \int e^{izx}\, x^l\, dF(x), \qquad l = 0, 1, 2, \ldots$$

exists.

Proof.

$$|\varphi_{(z)}^{(l)}| \leqslant \left(\int x^{2l}\, dF(x)\right)^{\frac{1}{2}} \cdot \left(\int |e^{2izx}|\, dF(x)\right)^{\frac{1}{2}},$$

$$|2izx| \leqslant |2z|^2 + \left(\frac{x}{2}\right)^2,$$

$$|e^{2izx}| \leqslant e^{|2z|^2} \cdot e^{(x/2)^2}$$

$$|\varphi_{(z)}^{(l)}|^2 \leqslant K \cdot C \frac{a^l (2l)!}{2^l l!} e^{|2z|^2}.$$

In particular it is:

(3,5) $$\varphi_{(0)}^{(l)} = i^l \cdot \int_{R_1} x^l\, dF(x) = i^l \cdot E(x^l) = \begin{cases} 0, & l = 2k+1 \\ (-1)^k E(x^{2k}), & l = 2k. \end{cases}$$

IV. $\varphi(z)$ is an integer function.

Proof:

$$e^{izx} = \sum_{l=0}^{n-1} \frac{(izx)^l}{l!} + \frac{(izx)^n}{n!} e^{i\Theta zx}, \qquad |\Theta| \leqslant 1$$

$$\left| \int x^n e^{i\Theta zx} dF(x) \right| \leqslant \left| \varphi^{(n)}_{(\Theta z)} \right| \leqslant \left\{ K \cdot C \frac{a^n (2n)!}{2^n n!} \cdot e^{(2|z|)^2} \right\}^{1/2}$$

$$\varphi(z) = \sum_{l=0}^{n-1} \frac{(iz)^l}{l!} E(x^l) + \frac{z^n}{n!} \varphi^{(n)}_{(\Theta z)},$$

but

$$\left| \frac{z^n}{n!} \varphi^{(n)}_{(\Theta z)} \right| \leqslant (K \cdot C)^{1/2} \cdot \frac{|z|^n}{n!} e^{2|z|^2} \left(\frac{a}{2} \right)^{n/2} \left(\frac{(2n)!}{n!} \right)^{1/2},$$

but

$$\frac{1}{n!} \left(\frac{(2n)!}{n!} \right)^{1/2} \sim \frac{2^{n-1/4} e^{n/2}}{n^{n/2} (\pi n)^{1/4}}, \qquad \text{i. e.}$$

$$e^{2|z|^2} \frac{|z|^n}{n!} \left(\frac{a}{2} \right)^{n/2} \left(\frac{(2n)!}{n!} \right)^{1/2} \sim e^{2|z|^2} \cdot \left(\frac{2ae|z|^2}{n} \right)^{n/2} \frac{2^{-1/4}}{\sqrt{\pi n}} \to 0, \; n \to \infty,$$

if only $z = \text{const}$.

If

(4,5) $$|z| \leqslant \frac{\sqrt[4]{n}}{2e}$$

then the remainder is as follows:

$$\frac{z^n}{n!} \varphi^{(n)}_{(\Theta z)} = 0 \left(\left(\frac{a}{2} \right)^{n/2} \frac{1}{\sqrt{n}} \right)$$

IV' $$\varphi(z) = \sum_{l=0}^{\infty} \frac{(-1)^l z^{2l}}{(2l)!} \cdot E(X)^{2l}$$

this series is convergent for every z, and

$$\varphi(z) = \sum_{l=0}^{\left[\frac{n-1}{2}\right]} (-1)^l \frac{z^{2l}}{(2l)!} E(X)^{2l} + R_n(z)$$

where

$$R_n(z) \to 0, \quad n \to \infty, \quad |z| \leqslant \frac{\sqrt{n}}{2e},$$

uniformly in respect to z.

Theorem IV. *If the e. c. v. $X_1, \dots X_n$ have following moments:*

$$m_{l_1, \dots l_k} \begin{cases} = 0, & l_1 + \cdots + l_k = 2p + 1 \\ \leqslant C \cdot a^p \dfrac{(2p)!}{2^p \, . \, p!}, & l_1 + \cdots + l_k = 2p, \quad \dfrac{a}{m_{11}} < 2 \end{cases}$$

then the ch. f. $\varphi_n(z)$ of the st. s. of these variables is an integer function of the complex variable z. Its order does not exceed 2.

Proof. We suppose for the sake of the simplicity of transformations:

$$E(X_l) = 0, \quad l = 0, 1, 2, \dots . \tag{6,5}$$

We find

$$B_n^2 = n^2 \cdot m_{11} \left\{ 1 + \frac{1}{n} (R_{11}^{-1} - 1) \right\} = n^2 R_{11} \sigma^2 \cdot b_n \tag{6',5}$$

where

$$b_n = 1 + \frac{1}{n} (R_{11}^{-1} - 1). \tag{6'',5}$$

We know that, X_n' is a st. s. of X_l, $l = 1, 2, \dots n$ then [11])

$$E(X_n')^r = \frac{1}{B_n^r} \cdot \sum_{v_1 + \dots + v_n = r} \frac{r!}{v_1! \dots v_n!} E(X_1^{v_1}, \dots X_n^{v_n}), \tag{7,5}$$

$$\sum_{v_1 + \dots + v_n = r} \frac{r!}{v_1! \dots v_n!} = n^r,$$

$$E(X_n')^r \begin{cases} = 0, & r = 2k + 1 \\ \leqslant C \left(\dfrac{a}{m_{11} \cdot b_n} \right)^k \cdot \dfrac{(2k)!}{2^k \, k!}, & r = 2k, \quad n = 1, 2, \dots . \end{cases}$$

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[11]) v. H. Milicer-Grużewska, loc. cit. p. 28 formula 6.

For
$$b=\frac{a}{m_{11}}<2$$

the moments of the st. s. X_n', $n=1,2,\ldots$ fulfil the suppositions of the Theorem III uniformly in respect to n.

Corollary. *If the r. v. $\{X_l\}$ fulfil the suppositions of Theorem IV then $\varphi_n(z)$ the ch. f. of the st. s. X_n' is representable with a series, uniformly convergent in respect to n for every complex variable z, viz.*

(7′,5)
$$\varphi_n(z)=\sum_{r=0}^{\infty}\frac{(-1)^r\cdot z^{2r}}{(2r)!}E(X_n')^{2r}$$

and

(7″,5)
$$\varphi_n(z)=\sum_{r=0}^{\left[\frac{m-1}{2}\right]}\frac{(-1)^r\cdot z^{2r}}{(2r)!}E(X_n')^{2r}+R_m^{(n)}(z)$$

where

(7‴,5)
$$R_m^{(n)}(z)=0\left(\left(\frac{b}{2}\right)^{m/2}\frac{1}{\sqrt{m}}\right)$$

for

(7⁗,5)
$$|z|\leqslant\frac{\sqrt{m}}{2e}$$

uniformly in respect to n.

$\varphi(z)$ is an integer function of the complex variable z of an order not greater them 2.

Theorem V. *If the e. c. v. $X_l\,(l=1,2,\ldots n)$ fulfil the suppositions of Theorem IV and*

(8,5)
$$M_1^{(2r)}:m_{11}^r=\frac{(2r)!}{2^r\cdot r!}$$

then the ch. f. $\varphi_n(z)$ of the st. s. of these variables approaches the ch. f. of the normal law of p., when $n\to\infty$, i. e.

(8′,5)
$$\varphi_n(z)\to e^{-z^2/2},\quad n\to\infty$$

for every complex z; beyond that the $n-th$ remainder of the series that represents $\varphi_n(z)$ aproaches to zero, uniformly in respect to z in the interval:

(8″,5) $$|z| \leqslant \frac{\sqrt{n}}{2e}$$

with n tending to infinity.

Proof. We know [12]) that under suppositions of Theorems IV and V:

(9,5) $$E(X_n')^{2r} \to \frac{(2r)!}{2^r r!}, \quad n \to \infty, \quad 0 \leqslant 2r \leqslant n$$

uniformly in respect to r.

If we suppose in formula (7″,5) $n \geqslant m$, then we find:

(9′,5) $$\lim_{n \to \infty} \varphi_n(z) = \sum_{r=0}^{\left[\frac{m-1}{2}\right]} \frac{(-1)^r z^{2r}}{2^r r!} + \lim_{n \to \infty} R_m^{(n)}(z),$$

but

$$\lim_{n \to \infty} |R_m^{(n)}(z)| \leqslant 0 \left(\left(\frac{b}{2}\right)^{m/2} \frac{1}{\sqrt{m}} \right) < \frac{\varepsilon}{2}, \quad \varepsilon > 0$$

for $m \geqslant M \geqslant 4e^2 |z|^2$, and M sufficiently great.

Similary it is for sufficiently large M, and $m > M$

$$\left| \sum_{r=0}^{\left[\frac{m-1}{2}\right]} \frac{(-1)^r \cdot z^{2r}}{2^r \cdot r!} - e^{-z^2/2} \right| < \frac{\varepsilon}{2},$$

then

$$\left| \lim_{n \to \infty} \varphi_n(z) - e^{-z^2/2} \right| < \frac{\varepsilon}{2}, \qquad \text{i. e.}$$

(10,5) $$\lim_{n \to \infty} \varphi_n(z) = e^{-z^2/2}.$$

For $m = n$ we receive the second part of Theorem V.

[12]) H. Milicer-Grużewska, loc. cit. Theorem I.

§ 6. The properties of the ch. f. $\varphi_n(z)$ of the st. s. X_n' of the e. c. v. $X_1, \dots X_n$, that fulfil the suppositions of § 5 and other special ones.

Remark (1,6). *The zeros of the c. f. $\varphi_n(z)$ approaches to $\pm\infty$ when n increases indefinitely, because $e^{-z^2/2} \neq 0$ in every finite region* [13]).

Theorem VI. *If the integer functions $\varphi_n(z) \to e^{-z^2/2}$, $n \to \infty$, $\varphi_n(0) = 1$ are of orders $\leqslant 2$, the orders of their canonical products do not exceed* 1, *and if $\varphi_n(t) \to 0$, $|t| \to \infty$, $t \subset R_1$, uniformly in respect to n, and $0 < \alpha < 1$, then*

$$\varphi_n(t) = 0\,(e^{-\alpha t^2/2})$$

uniformly in respect to n, i. e. to every constant α, $0 < \alpha < 1$ correspond such numbers N and T that:

$$\left.\begin{matrix} n \geqslant N(\alpha) \\ t \geqslant T > 1 \end{matrix}\right\} \longrightarrow |\varphi_n(t)| \leqslant e^{-\alpha t^2/2}. \tag{1,6}$$

Proof. If $z_l^{(n)}$ are zeros of the functions $\varphi_n(z)$ and $r_l^{(n)}$ are the moduli of $z_l^{(n)}$; $|z_l^{(n)}| = r_l^{(n)}$, then

$$\varphi_n(z) = \prod_l \left(1 - \frac{z}{z_l^{(n)}}\right) e^{\frac{z}{z_l^{(n)}}} \cdot e^{\alpha_n z + \beta_n z^2} = P_n(z)\, e^{Q_n(z)}, \tag{2,6}$$

$P_n(z)$ is of order not greater than 1, and

$$Q_n(z) = \alpha_n z + \beta_n z^2. \tag{2',6}$$

This means that it will be for every index l:

$$l \leqslant k\,(r_l^{(n)})^{3/2}, \quad k = \text{const}$$

$$(r_l^{(n)})^{-2} \leqslant k^{4/3}\, l^{-4/3}$$ and

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[13]) v. Hurwitz: Über d. Nullstellen d. Bessel'schen Funktionen. Math. Annalen s. ed. 33 (1889) 246—66).

$$(2'',6) \qquad \sum_{l=1}^{\infty} (r_l^{(n)})^{-2} \leqslant k^{4/3} \sum_{l=1}^{\infty} l^{-4/3} \text{ [14]}.$$

Then the series:

$$\sum_{l=1}^{\infty} (r_l^{(n)})^{-2}$$

converge uniformly in respect to n. It results from Remark (1,6) that:

$$\lim_{n\to\infty} \sum_{l=1}^{\infty} (r_l^{(n)})^{-2} = \sum_{l=1}^{\infty} \lim_{n\to\infty} (r_l^{(n)})^{-2} = 0$$

i. e. it is such N that:

$$n \geqslant N(\varepsilon) \to \sum_{l=0}^{\infty} (r_l^{(n)})^{-2} \leqslant \frac{\varepsilon}{6}.$$

But we know that:

$$lg\,|P_n(z)| = 0\left(|z|^2 \sum_{l=1}^{\infty} (r_l^{(n)})^{-2}\right), \qquad \text{i. e.}$$

$$n \geqslant N \to lg\,|P_n(z)| = 0(|z|^2 \cdot \varepsilon/6)$$

$$P_n(z) = e^{0(|z|^2 \cdot \varepsilon/6)}.$$

It results from the last dependance that:

$$|\varphi_n(z)| = |e^{Q_n(z)}| \cdot e^{0(|z|^2 \varepsilon/6)}$$

but

$$\varphi_n(z) \to e^{-z^2/2}, \qquad n \to \infty \qquad \text{i. e.}$$

$$\varphi_n(z) \cdot e^{+z^2/2} \to 1, \quad n \to \infty \qquad \text{i. e.}$$

$$e^{Q_n(z) + z^2/2} \to 1, \qquad n \to \infty$$

because:

$$e^{0(|z|^2 \cdot \varepsilon/6)} \sim 1 \quad \text{for} \quad z = \text{const.}$$

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[14]) v. for instance Titchmarsh: The Theory of Functions, Oxford U. Press s. ed. 1939, p. 249—250.

It will be:

$$n \geqslant N \to |\alpha_n| \leqslant \frac{\varepsilon}{6}, \qquad \left|\beta_n + \frac{1}{2}\right| \leqslant \frac{\varepsilon}{6}$$

provided N is sufficiently great i. e.

$$\alpha_n = \frac{\Theta_n' \cdot \varepsilon}{6}, \quad \beta_n = -\frac{1}{2} + \Theta_n'' \cdot \frac{\varepsilon}{6}, \quad |\Theta_n'| \leqslant 1, \quad |\Theta_n''| \leqslant 1 .$$

Now, if

$$|z|^{-1} < 1, \quad \text{then} \quad \left|\frac{\alpha_n}{z}\right| \leqslant \frac{\varepsilon}{6},$$

and

$$lg\,|\varphi_n(z) \cdot e^{z^2/2}| = R \cdot \left\{ \left[\frac{\alpha_n}{z} + \left(\beta_n + \frac{1}{2}\right) + \Theta_n \frac{\varepsilon}{6} \right] z^2 \right\}, \quad |\Theta_n| < 1$$

$$|z| \geqslant 1; \quad n \geqslant N \to \frac{|\Theta_n|}{|z|} + |\Theta_n''| + |\Theta_n| < 3$$

$$lg\,|\varphi_n(z)| < \frac{-1+\varepsilon}{2} \cdot R(z^2),$$

and

$$|\varphi_n(z)| < e^{\frac{-1+\varepsilon}{2} \cdot R(z^2)} .$$

Name $1 - \varepsilon = \alpha$, $0 < \alpha < 1$, and then

$$\left.\begin{array}{l} n \geqslant N(\alpha) \\ \\ |z| \geqslant 1 \end{array}\right\} \to \begin{array}{l} \varphi_n(z) \cdot e^{\frac{\alpha R(z^2)}{2}} = 0(1) \\ \varphi_n(z) = 0\left(e^{\frac{-\alpha . R(z^2)}{2}}\right) \end{array}$$

Remark (2,6). *If the canonical product* $P_n(z)$ *is of order* 2, *but the series*

$$\sum_{l=1}^{\infty} (r_l^{(n)})^{-2}$$

are uniformly convergent in respect to n, then the Theorem VI is still true [15]). This is obvious.

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[15]) v. Titchmarsh, loc. cit. p. 251—252.

Theorem VI is still true when the limit of the following expressions

$$u_n(t) = \sum_{l=1}^{M(t)} (r_l^{(n)})^{-2+1:lg|t|}, \quad M(t) = [|t|^{4 lg_2 t}] + 1, \quad t \subset R_1$$

exists uniformly in respect to n, i. e. if we have:

$$\lim_{|t| \to \infty} u_n(t) = \gamma_n$$

uniformly in respect to n.

(The proof is omitted here, because it is too complicated).

Remark (3,6). *It results from Theorem VI that the integral:*

$$\int_{|t|>c} \left| \frac{\varphi_n(t)}{t} \right| dt, \quad c = \text{const} > 0$$

exists uniformly in respect to n.

§ 7. Other properties of the ch. f. $\varphi_n(t)$ of the st. s. X_n' of the e. c. v. $X_1, \ldots X_n$.

Suppose that

$$\varphi_n(t) = e^{-t^2/_2} \cdot \left(1 + \frac{t^2 \cdot H_2(t)}{n}\right) + \frac{\Psi_n(t)}{n^2},$$

where $H_2(t)$ is a polynomial of the second order in respect to t. We know that to each number $0 < \alpha < 1$ corresponds such a number $N(\alpha)$ for which:

$$n \geqslant N(\alpha) \rightarrow \varphi_n(t) = 0\,(e^{-\alpha t^2/_2}), \qquad \text{i. e.}$$

$$|\varphi_n(t) \cdot e^{\alpha t^2/_2}| \leqslant \text{const},$$

if only:

$$t \geqslant T(\alpha) \quad \text{and} \quad n \geqslant N(\alpha) \qquad \text{i. e.}$$

$$\left| e^{-t^2/_2(1-\alpha)} \cdot \left(1 + \frac{t^2 H_2(t)}{n}\right) + e^{\alpha t^2/_2} \cdot \frac{\Psi_n(t)}{n^2} \right| \leqslant \text{const},$$

but
$$\lim_{|t|\to\infty} e^{-t^2/2(1-\alpha)} \cdot \left(1 + \frac{t^2 \cdot H_2(t)}{n}\right) = 0, \qquad \text{i. e.}$$

$$\left.\begin{array}{l} n \geqslant N_1(\alpha) \\ |t| \geqslant T_1(\alpha) \end{array}\right\} \longrightarrow e^{\alpha t^2/2} \cdot \frac{\Psi_n(t)}{n^2} \leqslant \text{const} \qquad \text{i. e.}$$

$$\Psi_n(t) = 0\,(n^2 \cdot e^{-\alpha t^2/2}). \tag{2,7}$$

Corollary 1.7. *If* $\varphi_n(t)$ *satisfies the suppositions of Theorem VI, then the function* $\Psi_n(t)$ *defined by the equation* (1,7) *fulfils the conditions* (2,7).

Corollary (2,7). *If* $\varphi_n(t)$ *fulfils the suppositions of Theorem VI, the function* $\Psi_n(t)$ *defined by the equation* (1,7) *satisfies the following conditions:*

$$1) \quad |\Psi_n(t)| \leqslant c\,|t|, \quad |t| \leqslant \text{const},$$

for integers n sufficiently large

$$2) \quad |\Psi_n(t)| \leqslant t^8 \cdot e^{t^2/2}, \quad t \leqslant \sqrt{2\,lg\,n},$$

then:

$$F_n(\xi) = \Phi(\xi) - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\varphi_n(t) - e^{-t^2/2}}{t} \cdot e^{-it\xi}\,dt = \tag{3,7}$$

$$= \Phi(\xi) + 0\,(n^{-\alpha}\,lg^{-1} n), \quad 0 < \alpha < 1,$$

where:

$$\Phi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi} e^{-t^2/2}\,dt.$$

Proof. Let us write:

$$\varphi_n(t) - e^{-t^2/2} = \overline{\varphi}_n(t)$$

$$\int_{-\infty}^{\infty} \overline{\varphi}_n(t) \cdot t^{-1} e^{-it\xi}\,dt = \int_{|t| \leqslant \sqrt{2\,lg\,n}} \overline{\varphi}_n(t) \cdot t^{-1} \cdot e^{-it\xi}\,dt + \tag{4,7}$$

$$+ \int_{|t| > \sqrt{2\,lg\,n}} \overline{\varphi}_n(t) \cdot t^{-1} \cdot e^{-it\xi}\,dt = J_1 + J_2\,.$$

Since exist N and T such that:

$$n \geqslant N, \quad t \geqslant T \rightarrow |\bar{\varphi}_n(t)| e^{\alpha t^2/_2} \leqslant C, \quad 0 < \alpha < 1,$$

then for $lg\, n \geqslant T$ i. e. for $n \geqslant \mathrm{Max}((N, e^T)$ it is:

$$|J_2| \leqslant C \cdot \int\limits_{|t| \geqslant \sqrt{2\, lg\, n}} e^{-\alpha t^2/_2} \cdot |t|^{-1}\, dt = 2\, C \int\limits_{u \geqslant \sqrt{\alpha\, lg\, n}} e^{-u^2} u^{-1}\, du =$$

$$= C \int\limits_{v > \alpha\, lg\, n} e^{-v}\, v^{-1}\, dv \leqslant \frac{C e^{-\alpha\, lg\, n}}{\alpha\, lg\, n},$$

$$|J_2| \leqslant \frac{C}{\alpha\, lg\, n} \left(\frac{1}{n}\right)^{\alpha} \leqslant 0\, (n^{-\alpha}\, lg^{-1}\, n).$$

But it results from (1,7) that:

$$J_1 = \frac{1}{n} \int\limits_{|t| < \sqrt{2\, lg\, n}} t \cdot H_2(t) \cdot e^{-t^2 - it\xi}\, dt + \frac{1}{n^2} \int\limits_{|t| < 1} \Psi_n(t) \cdot t^{-1} \cdot e^{-it\xi}\, dt +$$

$$+ \frac{1}{n^2} \int\limits_{1 < |t| < \sqrt{2\, lg\, n}} \Psi_n(t) \cdot t^{-1} \cdot e^{-it\xi}\, dt = J_3 + J_4 + J_5\,.$$

It is obvious that:

$$J_3 = 0\left(\frac{1}{n}\right) = 0\,(n^{-\alpha}\, lg^{-1} n).$$

We receive from the supposition 2) that exists N_1, such that:

$$n \geqslant N_1 \rightarrow J_4 = \frac{C}{n^2},$$

and:

$$|J_5| \leqslant \frac{2}{n^2} \int\limits_{1 < t < \sqrt{2\, lg\, n}} t^7 \cdot e^{t^2/_2}\, dt = \frac{16}{n^2} \int\limits_{\frac{1}{2} < v < lg\, n} e^v \cdot v^3\, dv =$$

$$= 0\left(\frac{lg^4\, n}{n}\right) = 0\,(n^{-\alpha}\, lg^{-1}\, n), \quad 0 < \alpha < 1.$$

Il results from this that

$$F_n(\xi) = \Phi(\xi) + 0\,(n^{-\alpha}\, lg^{-1} n), \quad 0 < \alpha < 1$$

for $n \geqslant \mathrm{Max}\,(N, e^T, N_1)$, where N and T depend only upon α.

Theorem VII. *If e. c. v. $X_1, \ldots, X_n$ fulfil the suppositions of Theorem V for $0 < b \leqslant 1$, and*

$$M_{21}^{(2r)} : m_{11}^r = R_{11}^{-1} \cdot \frac{(2r)!}{2^r \cdot r!}$$

then the ch. f. $\varphi_n(t)$ of the st. s. X_n' of these variables is of the form (1,7) *and satisfies the suppositions of the Corollary* (2,7) *if only the order of the canonical product of the ch. f. $\varphi_n(t)$ deos not exceed the number* 1 [16]).

Proof. We have proved that it is for:

$$|t| \leqslant \frac{\sqrt{m}}{2e}$$

$$\varphi_n(t) = E\left(e^{itX_n'}\right) = \int_{-\infty}^{\infty} e^{itx}\, dF_n(x) =$$

$$= \sum_{1=0}^{\left[\frac{m-1}{2}\right]} (-1)^l \cdot \frac{t^{2l}}{(2l)!} \cdot E\,(X_n')^{2l} + R_m^{(n)}(t)\,,$$

$$R_m^{(n)}(t) = 0\left(\left(\frac{b}{2}\right)^{m/2} \frac{1}{\sqrt{m}}\right), \quad |t| \leqslant \frac{\sqrt{m}}{2e}, \quad 0 < b < 2, \qquad \text{i. e.}$$

$$\varphi_n(t) \doteq \sum_{l=0}^{\infty} (-1)^l \cdot \frac{t^{2l}}{(2l)!} \cdot E\,(X_n')^{2l}, \quad \text{for every } t.$$

---

[16]) or accomplishes the suppositions of the generalization of the Theorem VI.

We also know that:

$$E(X_n')^{2r} = B_n^{-2r} \cdot \sum_{v_1+\dots+v_n=2r} \frac{(2r)!}{v_1!\,v_2!\dots v_n!} \cdot E(X_1^{v_1}, \dots X_n^{v_n}),$$

$$\sum_{v_1+\dots+v_n=2r} \frac{(2r)!}{v_1!\dots v_n!} = n^{2r}.$$

Suppose that $2r \leqslant n$. Exclude the components that correspond to $v_1 = \cdots = v_{2r} = 1$. We have supposed that:

$$M_{\underbrace{1\dots1}_{2r\ times}} = M_1^{(2r)} = \frac{(2r)!}{2^r \cdot r!} m_{11}^r; \quad 2r \leqslant n.$$

The number of these components will be:

$$n(n-1)\dots(n-2r+1) \sim n^{2r}.$$

Similarly the number of the components with one exponential index equal to 2 and the other ones equal to 1, is as follows:

$$(2r-1)\frac{(2r)!}{r!}\binom{n}{2r-1} = r(2r-1)\,n(n-1)\dots(n-2r+2) \sim$$

$$\sim r \cdot (2r-1)\, n^{2r-1}, \quad 1 < 2r \leqslant n.$$

All these components are equal to

$$M_{2\underbrace{11\dots1}_{(2r-2)\ times}} = M_{21}^{(2r)}.$$

The number of the remaining components is:

$$n^{2r} - n(n-1)\cdots(n-2r+1) - r(2r-1)\,n(n-1)\cdots(n-2r+2) =$$

$$= n^{2r}\left[1 - \left(1-\frac{1}{n}\right)\cdots\left(1-\frac{2r-1}{n}\right) - \frac{r(2r-1)}{n}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{2r-2}{n}\right)\right] =$$

$$= n^{2r}\left\{1 - \left(1-\frac{1}{n}\right)\cdots\left(1-\frac{2r-2}{n}\right)\left[1-\frac{2r-1}{n}+\frac{r(2r-1)}{n}\right]\right\} =$$

$$= n^{2r}\left\{1-\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{2r-2}{n}\right)\left[1-\frac{(1-r)(2r-1)}{n}\right]\right\}=$$

$$= n^{2r}\left\{1-\left[1-\frac{(r-1)(2r-1)}{n}+0\left(\frac{r^3}{n^2}\right)\right]\cdot\left[1-\frac{(1-r)(2r-1)}{n}\right]\right\}=$$

$$= n^{2r}\left[1-1+\frac{(r-1)(2r-1)}{n}+\frac{(1-r)(2r-1)}{n}+0\left(\frac{r^4}{n^2}\right)\right]=n^{2r}\cdot 0\left(\frac{r^4}{n^2}\right)$$

$$1 < 2r \leqslant n.$$

Since:

$$B_n^2 = n^2 m_{11}\, b_n = n^2 R_{11}\,\sigma^2\, b_n, \quad b_n \sim 1, \quad \text{then for} \quad r > 1:$$

$$E(X_n')^{2r} = \frac{(2r)!}{2^r\cdot r!\, b_n^r}\cdot\left(1-\frac{(2r-1)r}{n}\right)+\frac{(2r-1)\cdot r\cdot M_{21}^{(2r)}}{n\cdot m_{11}^r\cdot b_n^r}+$$

$$+\frac{C\cdot r^4\cdot(2r)!\, b^r}{n^2\cdot 2^r\cdot r!\, b_n^r}\Theta_r, \quad |\Theta_r| < 1.$$

But:

$$b_n = 1+\frac{1}{n}(R_{11}^{-1}-1);$$

$$b_n^{-r} = 1-\frac{r}{n}(R_{11}^{-1}-1)+\frac{(R_{11}^{-1}-1)^2\, r(r+1)}{n^2\cdot\left[1+\frac{\Theta}{n}(R_{11}^{-1}-1)\right]^{r+1}}, \quad 0 < \Theta < 1,$$

$$b_n^{-r} = 1-\frac{r}{n}(R_{11}^{-1}-1)+0\left(r(r+1)\,n^{-2}\right)$$

$$E(X_n')^{2r} = \frac{(2r)!}{2^r\cdot r!}\left\{1-\frac{r(2r-1)}{n}\cdot\left[1-\frac{M_{21}^{(2r)}\cdot 2^r\cdot r!}{m_{11}^r\,(2r)!}\right]+0\left(\frac{r^4}{n^2}\right)\right\}\cdot$$

$$\cdot\left[1-\frac{r}{n}(R_{11}^{-1}-1)+0\left(\frac{r^2}{n^2}\right)\right]=$$

$$=\frac{(2r)!}{2^r\cdot r!}\cdot\left\{1-\frac{r(2r-1)}{n}(1-a_r)+0\left(\frac{r^4}{n^2}\right)\right\}\cdot\left\{1-\frac{r}{n}(R_{11}^{-1}-1)+0\left(\frac{r^2}{n^2}\right)\right\}$$

where:

$$a_r = \frac{M_{21}^{(2r)} \cdot 2^r \cdot r!}{m_{11}^r \cdot (2r)!}, \quad r > 1$$

then

$$E(X_n')^{2r} = \frac{(2r)!}{2^r \cdot r!} \cdot \left\{1 - \frac{r}{n}(R_{11}^{-1} - 1) - \frac{r(2r-1)}{n}(1 - a_r) + 0\left(\frac{r^4}{n^2}\right)\right\},$$

$$r > 1$$

$$E(X_n')^2 = 1, \qquad \text{ex. def.}$$

. But it is:

$$\varphi_n(t) = 1 + \sum_{r=1}^{\left[\frac{n-1}{2}\right]} E(X_n')^{2r} \frac{(it)^{2r}}{(2r)!} + R_n(t) = 1 - \frac{t^2}{2} + \sum_{r=2}^{\left[\frac{n-1}{2}\right]} \frac{\left(-\frac{t^2}{2}\right)^r}{r!} \cdot$$

$$\cdot \left\{1 - \frac{r}{n}\left[(2r-1)(1-a_r) + (R_{11}^{-1} - 1)\right] + 0\left(\frac{r^4}{n^2}\right)\right\} + R_n(t),$$

$$R_n(t) = 0\left(\left(\frac{b}{2}\right)^{n/2} \frac{1}{\sqrt{n}}\right), \quad |t| \leqslant \frac{\sqrt{n}}{2e}, \quad b < 2$$

according to the Corollary of Theorem IV (p. 124).

Similary:

$$\varphi_n(t) = e^{-t^2/4} + \frac{R_{11}^{-1} - 1}{2n} \cdot t^2 \sum_{r=1}^{\left[\frac{n-1}{2}\right]-1} \frac{(-1)^r}{r!}\left(\frac{t^2}{2}\right)^r +$$

$$+ \frac{t^2}{2n} \sum_{r=1}^{\left[\frac{n-1}{2}\right]-1} \frac{(-1)^r}{r!}(2r+1)(1-a_{r+1})\left(\frac{t^2}{2}\right)^r + \overline{R}_n(t) + 0\left(\frac{1}{n^2}\right)$$

where:

$$0\left(\frac{1}{n^2}\right) = \sum_{r=2}^{\left[\frac{n-1}{2}\right]} \left(-\frac{t^2}{2}\right)^r \cdot \frac{1}{r!} 0\left(\frac{r^4}{n^2}\right),$$

$$\overline{R}_n(t) = R_n(t) + \sum_{r=\left[\frac{n-1}{2}\right]+1}^{\infty} \left(-\frac{t^2}{2}\right)^r \cdot \frac{1}{r!} = 0\left(\frac{1}{n^2}\right), \quad |t| \leqslant \frac{\sqrt{n}}{2e}.$$

Similary:

$$\varphi_n(t) = e^{-t^2/2} + \frac{R_{11}^{-1} - 1}{2n} \cdot t^2 (e^{-t^2/2} - 1) +$$

$$+ \frac{t^2}{2n} \cdot \frac{d}{dt} \left[\sum_{r=1}^{\left[\frac{n-3}{2}\right]} \frac{(-1)^r}{r!} (1 - a_{r+1}) \frac{t^{2r+1}}{2^r} \right] + \overline{\overline{R}}_n(t) + 0\left(\frac{1}{n^2}\right), \text{ [17]}$$

$$|t| \leqslant \frac{\sqrt{n-1}}{2e}.$$

According to the suppositions of Theorem VII we have

$$M_{21}^{(2r)} : m_{11}^r = R_{11}^{-1} \cdot \frac{(2r)!}{2^r \cdot r!},$$

then

$$a_r = R_{11}^{-1}, \qquad 1 - a_r = 1 - R_{11}^{-1}, \qquad r = 2, 3, \ldots.$$

We shall put in addition

$$a_1 = R_{11}^{-1}.$$

It is then:

$$\varphi_n(t) = e^{-t^2/2} \left(1 + \frac{R_{11}^{-1} - 1}{2n} t^2\right) +$$

$$+ \frac{t^2}{2n} \frac{d}{dt} \left[t \cdot \sum_{r=0}^{\left[\frac{n-3}{2}\right]} \left(-\frac{t^2}{2}\right)^r \cdot \frac{1}{r!} (1 - R_{11}^{-1}) \right] + \overline{\overline{R}}_n(t) + 0\left(\frac{1}{n^2}\right),$$

and

$$\varphi_n(t) = e^{-t^2/2} \left(1 + \frac{R_{11}^{-1} - 1}{2n} t^2\right) - (R_{11}^{-1} - 1) \frac{t^2}{2n} \cdot \frac{d}{dt} (t \cdot e^{-t^2/2}) +$$

$$+ t^2 \overline{\overline{\overline{R}}}_n(t) + 0\left(\frac{1}{n^2}\right), \quad |t| \leqslant \frac{\sqrt{n-3}}{2e} \text{ [18]}$$

---

[17] $\overline{\overline{R}}_n(t) = -\frac{R_{11}^{-1} - 1}{2n} \cdot t^2 \cdot \sum_{r=\left[\frac{n-1}{2}\right]}^{\infty} \frac{1}{r!} \left(-\frac{t^2}{2}\right)^r + \overline{R}_n(t).$

[18] $\overline{\overline{\overline{R}}}_n(t) = (R_{11}^{-1} - 1) \frac{1}{2n} \frac{d}{dt} \left[t \cdot \sum_{r=\left[\frac{n-1}{2}\right]}^{\infty} \left(-\frac{t^2}{2}\right)^r \cdot \frac{1}{r!} \right] + \overline{\overline{R}}_n : t^2.$

$$(3,7)\qquad \varphi_n(t) = e^{-t^2/2}\left(1 + \frac{R_{11}^{-1} - 1}{2n}\cdot t^4\right) + t^2\,\overline{\overline{\overline{R}}}_n(t) + 0\left(\frac{1}{n^2}\right),$$

$$|t| \leqslant \frac{\sqrt{n-3}}{2e}.$$

Remark (1,7). *If we put on the contrary for* $R_{11} > \frac{1}{3}$:

$$1 - a_r = (1 - R_{11}^{-1})^r, \quad r = 1, 2, \ldots \qquad \text{i. e.}$$

$$a_r = 1 - (1 - R_{11}^{-1})^r;$$

$$(M_{21}^{(2r)} : m_{11}^r)\cdot\frac{2^r\cdot r!}{(2r)!} = 1 - (1 - R_{11}^{-1})^r \qquad \text{i. e.}$$

$$M_{21}^{(2r)} : m_{11}^r = \frac{(2r)!}{2^r\cdot r!}\left[1 + (-1)^{r-1}(R_{11}^{-1} - 1)^r\right]$$ [19]),

then

$$\varphi_n(t) = e^{-t^2/2}\left(1 + \frac{R_{11}^{-1} - 1}{2n}\cdot t^2\right) +$$

$$+ \frac{t^2}{2n}\cdot\frac{d}{dt}\left\{t\cdot\sum_{r=0}^{\left[\frac{n-3}{2}\right]} -(R_{11}^{-1} - 1)\left[\frac{t^2(R_{11}^{-1} - 1)}{2}\right]^r\frac{1}{r!}\right\} + \overline{\overline{R}}_n(t) + 0\left(\frac{1}{n^2}\right)$$

$$\varphi_n(t) = e^{-t^2/2}\left(1 + \frac{R_{11}^{-1} - 1}{2n}\cdot t^2\right) - \frac{t^2}{2n}\cdot\frac{d}{dt}\cdot$$

$$\cdot\left\{(R_{11}^{-1} - 1)\left[e^{\frac{t^2(R_{11}^{-1} - 1)}{2}} - \sum_{\left[\frac{n-1}{2}\right]}^{\infty}\left(\frac{t^2(R_{11}^{-1} - 1)}{2}\right)^r\frac{1}{r!}\right]\right\} +$$

$$+ \overline{\overline{R}}_n(t) + 0(n^{-2}), \quad |t| \leqslant \frac{\sqrt{n-3}}{2e}, \qquad \text{and}$$

$$(3',7)\qquad \begin{aligned}\varphi_n(t) &= e^{-t^2/2}\cdot\left\{1 + \frac{R_{11}^{-1} - 1}{2n}\cdot t^2\left[1 - (1 + t^2(R_{11}^{-1} - 1))\cdot e^{\frac{t^2 R_{11}^{-1}}{2}}\right]\right\} + \\ &+ t^2\,\overline{\overline{\overline{R}}}_n(t) + 0(n^{-2}), \quad |t| \leqslant \frac{\sqrt{n-5}}{2e}, \quad R_{11} > \frac{1}{3}.\end{aligned}$$

---

[19]) If $M_{21}^{(2r)} > 0$, then this supposition is possible only for $R_{11} > \frac{1}{2}$.

In we put $R_{11} > \frac{1}{3}$:

$$1 - a_r = (-1)^{r-1}(1 - R_{11}^{-1})^r = -(R_{11}^{-1} - 1)^r, \qquad \text{i. e.}$$

$$a_r = 1 + (R_{11}^{-1} - 1)^r \qquad \text{i. e.}$$

$$M_{21}^{(2r)} : m_{11}^r = \frac{(2r)!}{2^r \cdot r!} \cdot [1 + (R_{11}^{-1} - 1)^r],$$

then

$$\varphi_n(t) = e^{-t^2/2}\left(1 + \frac{R_{11}^{-1} - 1}{2n} t^2\right) - \frac{R_{11}^{-1} - 1}{2n} \cdot t^2 \cdot$$

$$\cdot \frac{d}{dt}\left\{t \cdot \sum_{r=0}^{\left[\frac{n-3}{2}\right]} \left[-\frac{t^2(R_{11}^{-1} - 1)}{2}\right]^r \frac{1}{r!}\right\} + \overline{R}_n(t) + 0(n^{-2}),$$

$$|t| \leqslant \frac{\sqrt{n-3}}{2e}, \qquad \text{i. e.}$$

$$\varphi_n(t) = e^{-t^2/2} \cdot \left\{1 + \frac{R_{11}^{-1} - 1}{2n} \cdot t^2\left[1 - (1 - t^2(R_{11}^{-1} - 1)) \cdot e^{\frac{-t^2(R_{11}^{-1} - 2)}{2}}\right]\right\} +$$

(3″,7)

$$+ t^2 \overline{\overline{\overline{R}}}_n(t) + 0(n^{-2}), \quad |t| \leqslant \frac{\sqrt{n-5}}{2e}.$$

In thit case, similary as in the second one:

$$e^{-\frac{t^2(R_{11}^{-1} - 2)}{2}} \to \infty$$

if

$$R_{11}^{-1} < 2 \quad \text{i. e.} \quad R_{11} > \frac{1}{2}.$$

It is easy to verify, that it will be for $b < 2$, $R_{11} > \frac{1}{3}$, $|t| \leqslant \frac{\sqrt{n-5}}{2e}$

$$|t^2 \cdot \overline{\overline{\overline{R}}}_n(t)| \leqslant C : n^2$$

where C is a positive constant, and for $n \geqslant N_0(b, R_{11})$ where $N_0(b, R_{11})$ is an index that depends only upon b and R_{11}, i. e. upon the statistics of the d. f. $F(\varkappa_n)$.

The terms $0(n^{-2})$ of formulae (3,7), (3',7) and (3'',7) may be evaluated as follows:

$$0(n^{-2}) = n^{-2} \cdot \sum_{r=2}^{\infty} \frac{(-1)^r \Theta_r r^4}{r!} \left(\frac{b \cdot t^2}{2}\right)^r, \qquad 0 < \Theta_r < 1;$$

$$|0(n^{-2})| \leqslant n^{-2} \left[\Theta_2 \cdot 2^3 \left(\frac{bt^2}{2}\right)^2 + \Theta_3 \cdot \frac{3^3}{2} \left(\frac{bt^2}{2}\right)^3 + \right.$$

$$\left. + \sum_{r=4}^{\infty} \frac{\Theta_r (bt^2 : 2)^r}{(r-4)! \left(1 - \frac{3}{r}\right)\left(1 - \frac{2}{r}\right)\left(1 - \frac{1}{r}\right)} \right],$$

but:

$$\left(1 - \frac{3}{r}\right)\left(1 - \frac{2}{r}\right)\left(1 - \frac{1}{r}\right) > \left(1 - \frac{3}{r}\right)^3 > \left(\frac{1}{4}\right)^3, \qquad r \geqslant 4,$$

then

$$|0(n^{-2})| \leqslant n^{-2} \cdot 4^3 \left[\left(\frac{bt^2}{2}\right)^2 + \left(\frac{bt^2}{2}\right)^3 + \left(\frac{bt^2}{2}\right)^4 \cdot e^{bt^2/_2} \right].$$

We then see that the term $0(n^{-2})$ is of the form:

$$0\left(n^{-2} \cdot t^8 \cdot e^{\frac{bt^2}{2}}\right) \leqslant 0\left(n^{-2} \cdot t^8 \cdot e^{t^2/_2}\right) \quad \text{if} \quad b \leqslant 1.$$

Then in the case of the suppositions of Theorem VII the ch. f. $\varphi_n(t)$ satisfies the suppositions of Corollary (2,7) and this proves Theorem VII.

Corollary (3,7). *If the moments $M_{21}^{(2r)}$ do not fulfil the suppositions of Theorem VII, but the followings one:*

$$M_{21}^{(2r)} : m_{11}^r = \frac{(2r)!}{2^r \cdot r!} [1 + (R_{11}^{-1} - 1)^r],$$

then for $R_{11} > \frac{1}{3}$ Theorem VII is true only when $R_{11} < \frac{1}{2}$.

Remark (2,7). *The evaluation of the rest* $R_n(t)$ *may be improved for* $b < 1$.

We then see that, if the e. c. v. $X_1, \ldots, X_n$ *fulfil the suppositions of Theorem VII, then the d. f. of the st. s.* X_n' *of this variables is as follows:*

$$F_n(\xi) = \Phi(\xi) + 0\,(n^{-\alpha} \cdot lg^{-1} n), \qquad 0 < \alpha < 1$$

$\Phi(\xi)$ *is here the Gaussian-Laplace integral.*

This results from: Corollary (2,7), Remark (3,6), Property II § 4 and Cramer's Theorem, quoted in the Introduction, according to which, if the integral

$$\int\limits_{|t|>0} \left| \frac{\varphi_n(t)}{t} \right| dt$$

exists, then the d. f. $F_n(\xi)$ connected with the ch. f. $\varphi_n(t)$ may be represented as follows:

$$F_n(\xi) = -\frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{\varphi_n(t) - e^{-t^2/_2}}{t} \cdot e^{-it\xi}\, dt + \Phi(\xi) \text{ [20]}.$$

§ 8. Negative Results.

A. If the d. f. $F_n(x)$ of the st. s. X_n' were of the form:

$$(1,8) \qquad F_n(x) = \Phi(x) \cdot \left[1 + \frac{1}{n} \cdot H_2(x)\right] + 0\left(\frac{1}{n^2}\right)$$

where $H_2(x)$ is the fourth Hermite's polynomial, and

$$0\left(\frac{1}{n^2}\right) = \frac{1}{n^2} \Psi_n(x); \qquad \int\limits_{-\infty}^{\infty} d\,\Psi_n(x) \leqslant \text{const},$$

[20]) Cramer, loc. cit. sub. 4.

then it necessarily would be

$$(2,8) \qquad \varphi_n(t) = e^{-t^2/_2}\left[1 + \frac{1}{n}\overline{H}_2(t^2)\right] + 0\left(\frac{1}{n^2}\right),$$

where $\overline{H}_2(t)$ is a polynomial of the 2-d order in respect to t.

In fact we obtain this by integrating by parts the second component of the ch. f. $\varphi_n(t)$

$$\varphi_n(t) = \int e^{-itx}\,dF_n(x) = e^{-t^2/_2} + \frac{1}{n}\int e^{itx}\,d\,[\Phi(x)\cdot H_2(x)] +$$

$$+ \frac{1}{n^2}\int e^{itx}\,d\,\Psi_n(x).$$

We have from (2,8):

$$(\varphi_n(t) - e^{-t^2/_2})\,n = e^{-t^2/_2}\cdot\overline{H}_2(t^2) + 0\left(\frac{1}{n}\right), \qquad \text{or}$$

$$(3,8) \qquad [e^{t^2/_2}\cdot\varphi_n(t) - 1]\,n = \overline{H}_2(t^2) + 0\left(\frac{1}{n}\right)e^{t^2/_2}, \qquad \text{i. e.}$$

$$(3',8) \qquad \lim_{n\to\infty}[e^{t^2/_2}\cdot\varphi_n(t) - 1]\,n = \overline{H}_2(t^2) = 0\,(t^4).$$

Formula (3,7) gives the expression of the same form:

$$(4,8) \qquad \lim\,[e^{t^2/_2}\cdot\varphi_n(t) - 1]\,n = \frac{R_{11}^{-1} - 1}{2}\cdot t^4 = 0\,(t^4).$$

Contrary to that, formula (3″,7) *gives the expression:*

$$(4',8) \qquad \lim_{n\to\infty}[e^{t^2/_2}\cdot\varphi_n(t) - 1]\,n =$$

$$= \frac{R_{11}^{-1} - 1}{2}\cdot t^2\cdot[1 - (1 - t^2(R_{11}^{-1} - 1))\cdot e^{-t^2/_2(R_{11}^{-1} - 2)}] \neq 0\,(\overline{H}_2(t^2)),$$

for $R_{11} > \frac{1}{2}$, *then gives an expression of a different form.*

Consequently we see that the d. f. $F_n(x)$ *of the st. s.* X_n' *takes a suitable form only when the moments* $M_1^{(r)}$ *and* $M_{21}^{(r)}$, $r=1,2,\dots$ *are conveniently chosen; beyond that all moments of the d. f.* $F(\varkappa_n)$ *ought to fulfil certain conditions of a very strong boundedness.*

B. *The limit theorem is surely false if the e. c. v.* $X_1, \dots X_n$ *are bounded,* because the st. s. X_n' is then also bounded.

It would be for instance:

$$a \leqslant X_n' \leqslant b, \quad a, b \text{ — const.}$$

and then, if $f_n(x)$ means the density function of the st. s. X_n' we may write:

$$\varphi_n(t) = \int_a^b e^{itx} \cdot f_n(x)\, dx .$$

If we had

$$f_n(x) \longrightarrow \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}, \quad n \to \infty$$

then it must also be:

$$\varphi_n(t) \to e^{-t^2/2} .$$

Contrary to that we have:

$$\varphi_n(t) \longrightarrow \frac{1}{\sqrt{2\pi}} \int_a^b e^{itx} \cdot e^{-x^2/2}\, dx \neq e^{-t^2/2} .$$

Corollary (1,8). *The d. f. of the st. s. of the e. c. v. and bounded variables do not tend to the normal distribution.*

Corollary (2,8). *If the e. c. v. are bounded, their moments:*

$$m_{\underbrace{1 \dots 1}_{-\ times}} = M_1^{(r)}$$

de not fulfil the conditions:

$$M_1^{(r)} = \begin{cases} 0 & , \quad r = 2p+1 \\ \dfrac{(2p)!}{2^p \cdot p!} \cdot m_{11}^p , & r = 2p \end{cases}$$

$$p = 0, 1, 2, \dots .$$

Halina Milicer-Grużewska

O prawie prawdopodobieństwa i funkcji charakterystycznej standaryzowanej sumy zmiennych ekwiwalentnych parami skorelowanych

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na posiedzeniu w dniu 14 października 1949 r.

Streszczenie

W pracy tej autorka zajmuje się repartycją prawdopodobieństwa i funkcją charakterystyczną standaryzowanej sumy zmiennych ekwiwalentnych parami skorelowanych, z punktu widzenia twierdzenia o aproksymacji prawa granicznego prawdopodobieństwa (§§ 2—6).

Przy dość specjalnych założeniach twierdzenie to okazuje się prawdziwe (§ 7) — przy innych jednak założeniach nie jest tak (§§ 7, 8). Gdyby zmienne ekwiwalentne parami skorelowane były ograniczone, to nie zachodziłoby nawet prawo graniczne prawdopodobieństwa (§ 8).

Tadeusz Penkala

Sur la transformation de la projection gnomonique en projection stéréographique à l'aide d'une equerre

Note présentée par T. Wojno à la séance du 14 octobre 1949.

Une simple méthode dans laquelle on projette sur le même plan et du même centre les points situés sur deux sphères superposées à la distance de leur rayon.

Tadeusz Penkala

Przejście z rzutu gnomonicznego do rzutu stereograficznego z pomocą ekierki

Komunikat przedstawiony przez czł. T. Wojno
na posiedzeniu w dniu 14 października 1949 r.

Prosta metoda, polegająca na rzutowaniu z tego samego środka zbieżności na tę samą płaszczyznę rzutu punktów powierzchni dwóch kul, przesuniętych o długość ich promienia.

Maria Turnau-Morawska
with assistance of Maria Jahn

Optical character of quartz grains in a silificated trunk of the environs of Chrzanów

Note présentée par T. Wojno à la séance du 14 octobre 1949.

The authors have performed a microscopic study of the silificated trunk Araucariaxylon schroellianus Goepp, appearing in situ in the Permian sediments of the environs of Chrzanów. The microscopic examinations of thin sections have shown, that the silica impregnating the fossil plant in question is exclusively quartz. 40% of the quartz grains are marked by strain shadows, which optical anomaly is mightily interpreted till now as a result of mountain-building pressures.

The authors conclude that during crystallization of silica in organic cells some phenomena of deformation in the crystalline lattice may occur, which appears in the strain shadows of quartz grains. The resistance of the membrane produce stresses and strains, the result of which is a preferred optic orientation of quartz grains, similar to that appearing in crystalline schists.

Maria Turnau-Morawska
ze współpracą Marii Jahn

Orientacja optyczna ziarn kwarcu w drzewie skamieniałym okolic Chrzanowa

Komunikat przedstawiony przez czł. T. Wojno
na posiedzeniu w dniu 14 października 1949 r.

Streszczenie

Przeprowadzono analizę mikroskopową płytek cienkich wykonanych z pnia drzewa skamieniałej araukarii (Araucarioxylon schroellianus Goepp), występującej „in situ" wśród osadów permskich okolic Chrzanowa i znanej z literatury paleobotanicznej. Badania w świetle spolaryzowanym wykazują, że substancją mineralizującą jest wyłącznie kwarc, którego naogół dobrze wykształcone ziarna wypełniają komórki drewna. Około $40^0/_0$ ziarn wykazuje faliste znikanie światła, którą to anomalię optyczną interpretuje się dotąd przeważnie jako wynik ciśnień górotwórczych. Ponieważ osady permu chrzanowskiego nie są zmetamorfizowane, przeto jedynym ciśnieniem wchodzącym tu w grę byłby opór błon komórkowych, ujawniający się w czasie krystalizacji kwarcu w tkankach roślinnych. Pomiary kątów znikania światła ziarn kwarcu (200 pomiarów) względem osi cewek wykazały, że orientacja optyczna ziarn kwarcu nie jest bezładna, ale istnieje pewne uporządkowane rozmieszczenie tych kierunków, przy czym uzyskuje się podobny diagram jak w przypadku łupków krystalicznych, czyli skał zmienionych pod działaniem ciśnienia kierunkowego.

Stanisław Józef Thugutt

Scolecite, Episcolecite, Thomsonite
Internal Structure and Origin

Note présentée par T. Wojno à la séance du 14 octobre 1949.

The fact, recently confirmed, of polymerism occurring between the natrolite: $11\,Na_2Al_2Si_3O_{10}\,.\,22\,H_2O$, and and the epinatrolite: $8\,Na_2Al_2Si_3O_{10}\,.\,16\,H_2O$ appeared also among the calcic substitution products of these minerals, namely between the scolecite: $11\,CaAl_2Si_3O_{10}\,.\,33\,H_2O$ and the episcolecite: $8\,CaAl_2Si_3O_{10}\,.\,24\,H_2O$, the symmetry of the paternal types being preserved. Hemiedria and the pyroelectric properties of the natrolite found their expression in the scolecite as well.

Besides epinatrolite, the thomsonite: $8\,CaAl_2Si_3O_{10}.4Na_2Al_2O_4.\,.\,24\,H_2O$ may also serve as the source of episcolecite delivering easily its aluminate link to water.

M. H. Hey's hypothesis about the dimorphism of thomsonite cannot be applied here, because the supposed dimorphism concerns the specimens showing different degrees of hydration.

Stanisław Józef Thugutt

Skolecyt, episkolecyt, tomsonit
Ustrój i pochodzenie

Komunikat przedstawiony przez czł. T. Wojno
na posiedzeniu w dniu 14 października 1949 r.

Streszczenie

Stwierdzony świeżo fakt polimerii, zachodzącej pomiędzy natrolitem: $11 Na_2Al_2Si_3O_{10} . 22 H_2O$ i epinatrolitem: $8 Na_2Al_2Si_3O_{10} . 16 H_2O$ powtórzył się również pośród wapniowych produktów podstawień obu powyższych minerałów, mianowicie pomiędzy skolecytem $11 CaAl_2Si_3O_{10} . 33 H_2O$ i episkolecytem $8 CaAl_2Si_3O_{10} . 24 H_2O$ nawet z zachowaniem symetrii minerałów macierzystych. Hemiedria i własności piroelektryczne natrolitu znalazły swój wyraz również w skolecycie.

Poza epinatrolitem źródłem episkolecytu może być także tomsonit: $8 CaAl_2Si_3O_{10} . 4 Na_2Al_2O_4 . 24 H_2O$, łatwo uwalniający swe ogniwo glinianowe.

Hipoteza M. H. Hey'a o dwupostaciowości tomsonitu niema tu zastosowania, albowiem dotyczy okazów, wykazujących różny stopień uwodnienia.

Stanisław Józef Thugutt

The views of Gustav Bischof on the origin of zeolites
A historical note

Note présentée par T. Wojno à la séance du 14 octobre 1949.

According to Bischof the zeolites represent a combination of sodium-, potassium-, calcium- and aluminum-silicates, drawing their essential compounds from sodiocalcic and from alkalic feldspars. Only the natrolite makes exception, being a product of transformation of an anhydrous mineral (in this case nephelite) in a hydrated one. This reaction becomes realized not by means of addition of silica to nephelite, but by detaching the alumina and the alkalies from it. Hence the paragenetical connections between the natrolite and the diaspore or the hydrargillite.

Concerning the pseudomorphosies Bischof stated a general rule for transformations occuring without change of the volume. In such a case the molecule wights of the paternal mineral and that of its product ought remain in the ratio of their specific gravities.

Bischof established also the great importance of colloid solutions in the nature, facilitating the migration of zeolites fròm primary to the secondary layer, without disturbing their internal structure.

Stanisław Józef Thugutt

Poglądy Gustawa Bischofa na genezę zeolitów

Komunikat przedstawiony przez czł. T. Wojno
na posiedzeniu w dniu 14 października 1949 r.

Streszczenie

Według Bischofa zeolity stanowią kombinację krzemianów sodowych, potasowych, wapniowych i glinowych, powstałych z rozkładu skaleni alkalicznych i sodowowapniowych. Wyjątek stanowi natrolit jako jedyny przykład przeobrażenia się bezwodnego minerału, w danym przypadku nefelinu, w minerał uwodniony. Reakcja dochodzi do skutku nie przez przyłączenie się krzemionki, lecz drogą ubytku glinki i ługowców. Stąd paragepeza natrolitu z diasporem i hydrargilitem.

W stosunku do postaci zapożyczonych Bischof podaje ogólną zasadę interpretacji przeobrażeń zachodzących bez zmiany objętości. W tym przypadku ciężary cząsteczkowe minerału macierzystego i wytworzonego zeń produktu tak się winny mieć do siebie jak ich ciężary właściwe.

Bischof wykazał również wielką rolę roztworów koloidalnych w przyrodzie, umożliwiających wędrówkę zeolitów ze złoża pierwotnego na złoże wtórne, bez naruszenia ich ustroju wewnętrznego.

Tadeusz Wieser

The ophiolite from Osielec

Note présentée par T. Wojno à la séance du 14 octobre 1949.

A ophiolite rock, found by Prof. M. Książkiewicz in the stream at Osielec near Maków (Western Flysh Carpathians) was by the author microscopicaly examined and determined as an albite amphibolite, or as a hornblende prasinite in the terminology of Niggli. Its mineral composition: albite, epidote (beside rare clinozoisite and zoisite-β), chlorite and hornblende as main constituents, while sphene, calcite and quartz are additional — and ilmenite, leucoxene, hematite and zircon — accessory compounds.

The preserved relicte structure of a deap — seated rock indicates, that the primary rock must have been a gabbro-dioritic type. The comparison of the described ophiolite with other similar rocks permits to find analogous types among Alpine ophiolites and that of Dobšiná (internal zone of Western Carpathians).

On the basis of the convergence in stratigraphic development in the Alpine-Carpathian arc, the ophiolite from Osielec may be regarded as a prim-orogenic eruption, corresponding cither to Lower Cretacerus Alpine ophiolites, er to the antecarboniferous ophiolites of the internal Carpathians.

Tadeusz Wieser

Ofiolit z Osielca

Komunikat przedstawiony przez czł. T. Wojno
na posiedzeniu w dniu 14 października 1949 r.

Streszczenie

Autor zbadał próbki skały egzotycznej, znalezionej przez prof. M. Książkiewicza w postaci bloku o przeszło dwumetrowej średnicy w potoku Osielczyk na północ od miejscowości Osielec, koło Makowa. Blok ten tkwi wśród zlepieńców i piaskowców środkowo-eoceńskich serii magórskiej i stanowi odmianę ofiolitu, złożonego z albitu, amfibolu, epidotu i chlorytu jako składników głównych, z tytanitu, kalcytu, klinozoizytu, zoizytu i kwarcu jako składników pobocznych, oraz z akcesorycznego ilmenitu, leukoksenu, hematytu i cyrkonu.

Obecność klasycznie wykształconej budowy prazynitowej skłania do określenia omawianej skały jako prazynit w terminologii Niggli'ego.

Struktura reliktowa tej skały głębinowej wskazuje, że skałą pierwotną powyższego ofiolitu był gabro-dioryt.

Podobnie jak kredowe ofiolity alpejskie i przedkarbońskie ofiolity karpackie, skałę z Osielca, na zasadzie konwergencji w rozwoju stratygraficznym, uważać można za erupcję primorogeniczną, która uległa późniejszej dyzlokacyjnej epimetamorfozie w czasie jednego z ruchów orogenetycznych, obejmujących trzon krystaliczny Prakarpat.

Mieczysław Dominikiewicz

The behavior of kaolin towards dyes

Note présentée par T. Wojno à la séance du 14 octobre 1949.

The author stated that the kaolin thanks to its own high molecular weight shows a considerable affinity towards basic dyes, having a smaller molecular weight. Its affinity to the basic dyes is doubtless and results from the chemical reaction.

The kaolin however should not be coloured by acid dyes, according to his acid nature.

Mieczysław Dominikiewicz

Stosunek kaolinu do barwników

Komunikat przedstawiony przez czł. T. Wojno
na posiedzeniu w dniu 14 października 1949 r.

Streszczenie

Autor, badając stosunek kaolinu, mającego duży ciężar cząsteczkowy, do barwników organicznych o małym ciężarze cząsteczkowym, stwierdził, że jego powinowactwo jest niewątpliwe mianowicie do barwników zasadowych i że wynika z reagowania chemicznego.

Względem barwników o charakterze kwasowym kaolin zachował się obojętnie, zresztą zgodnie z jego naturą kwasową.

Jerzy St. Paduszyński

The Dustfall in Southern Poland, April 11—22, 1948

Note présentée par T. Wojno à la séance du 14 octobre 1949.

On April 11th and 12th, 1948, a fall of aeolian dust took place in the district of Kraków and the Podhale (Poland). The intensity of the phenomenon was comparatively small, so that the samples taken in the Tatra Mts. and Kraków were extremly small and allowed only microscopical research. As a result of the investigations the qualitative and quantitative mineral composition of the dust was ascertained (quartz + felspars 61%; clayey aggregates 25%; heavy minerals 4%; limonite (?) 7%; dark minerals 3%) as well as the distribution of its grain size (maximum value $= 54\,\mu$; modal value $= 6\,\mu$; arithmetic mean $= 11\,\mu$).

The results of this investigation and their comparison with research on analogous dust materials fallen in Poland on Aprill 26th and 27th, 1928, show far-reached analogies of their nature and origin. The similarity of the qualitative and quantitative mineral composition as well as the analogies of the shapes of the cumulative curves for the distribution of component grain size allow to ascertain — with great probability — that the parent material of both dusts was similar in both cases (a brown soil) and that the material was efflated from the same or nearly situated areas (Azow Sea region). These conclusions were corroborated by an analysis of the weather during the dust fall of 1948 which (as well as direct observations) shows the existance of winds blowing from eastern directions.

Only the existance of more accurate and numerous data from other areas would make possible more detailed and sure statements. Unfortunately such data are not available till now.

Jerzy St. Paduszyński

Opad pyłu eolicznego w Polsce południowej w dniach 11—12 kwietnia 1948 roku

Komunikat przedstawiony przez czł. T. Wojno na posiedzeniu w dniu 14 października 1949 r.

Streszczenie

W okresie 11-12 kwietnia 1948 r. miał miejsce opad pyłu eolicznego w rejonie Podhala i w okręgu krakowskim (Polska). Nasilenie zjawiska było jednak stosunkowo słabe, tak, że próbki zebrane w Tatrach i w Krakowie były znikomo małe i zezwoliły jedynie na wykonanie badań mikroskopowych. W wyniku tych badań poznano skład mineralny jakościowy i ilościowy (kwarzec+skalenie 61%; agregaty ilaste 25%; minerały ciężkie 4%; limonit (?) 7%; minerały ciemne 3%), rozsiew wielkości ziarn składowych (wartość maksymalna = 54 μ; wartość modalna = 6 μ; średnia arytmetyczna = 11 μ).

Wyniki opracowania i ich porównanie z wynikami badań analogicznych materiałów pyłowych opadłych w Polsce w d. 26-27 kwietnia 1928 r. wykazują daleko posunięte analogie ich natury i genezy. Podobieństwo jakościowego i ilościowego składu mineralnego oraz analogie kształtu krzywych sumacyjnych rozsiewu wielkości ziarn składowych pozwalają — z dużym prawdopodobieństwem — stwierdzić, iż macierzysty materiał pyłów w obu wypadkach był taki sam (gleba brunatna) i że materiał został wywiany z tych samych lub bliskich sobie obszarów (rejon Morza Azowskiego). Wnioski powyższe potwierdzone zostały przez analizę stanu pogody panującej w okresie opadu pyłu w 1948 r., która to analiza, jak i bezpośrednie obserwacje, stwierdzają obecność wiatrów wiejących z kierunków wschodnich.

Bardziej pewne stwierdzenia możliwe są jedynie w wypadku istnienia dokładnych danych z innych terenów. Danych takich niestety dotychczas brak.

Posiedzenie

z dnia 18 listopada 1949 r.

Michał Kamieński

Researches on the origin of the Comet Wolf I
Part V—A

The motion of the Comet in the period

1839 Apr. 20.5 — 1825 Sept. 30.5

Mémoire présenté à la séance du 18 novembre 1949.

1. Towards the end of 1939 the Comet Wolf I approached nearly the borded of the activity sphere of Jupiter. In November 1939 it attained its minimum distance from this planet, namely:

$$1839 \text{ Nov. } 13.1 \qquad \triangle = 0.5382.$$

The distance $\triangle$, then slowly increasing, attained its maximum $\triangle = 8.89$ in the beginning of 1832, as can be seen from the Tables below. Consequently, the perturbations of its motion, caused by the influence of Jupiter during the period 1839 Apr. 20.5 — 1825 Sept. 30.5, were not very great.

Since the paths of Mercury, Venus, the Earth and Mars lie deep inside the comet's orbit, the influence of these planets on the motion of the comet is comparatively small and has no greater importance for the solution of the problem of the comet's origin.

The influence of Uranus and Neptune may in this case be neglected as well. Having this in view, the author made up his mind to compute the perturbations caused by Jupiter and Saturn only, what is quite sufficient for our purpose.

During the computations, the method of variation of arbitrary constants was thoroughly applied. The systems of elements were changed every $12\frac{1}{2}$, 25 or 50 days, in conformity with the distances of the comet from Jupiter.

2. The system of elements P_{-11} deduced in part IV of the author's „Researches on the origin of the Comet Wolf I" was taken as a basis for all further investigations (1839—1800):

1839 April 20.5 Greenwich Mean Time

$$P_{-11} \dots \begin{array}{ll} M = 187^\circ 58' 0''.6 & \Omega = 211^\circ 29' 20''.1 \\ \varphi = 24^\circ 10' 50''.0 & \pi = 12^\circ 12' 32''.4 \\ n = 429''.7627 & i = 27^\circ 36' 59''.4 \end{array} \Bigg\} \; 1950.0$$

After the integration of the differentials of perturbations given in the adjacent Tables, the author obtained the following results:

1839 Apr. 20.5 — 1834 March 27.5			
	Jupiter	Saturn	Total
δM	— 3987″.7	— 76″.5	— 67′ 44″.2
$\delta \varphi$	— 0.2	+ 17.6	+ 0 17.4
$\delta \Omega$	+ 4001.1	— 2.3	+ 66 38.8
$\delta \pi$	+ 2680.4	+ 36.0	+ 45 16.4
δi	— 2376.2	+ 2.0	— 39 34.2
δn	— 0″.0385	+ 0″.0041	— 0″.0344

1834 March 27.5 — 1825 Sept. 30.5			
	Jupiter	Saturn	Total
δM	+ 4085″.6	+ 65″.2	+ 69′ 10″.0
$\delta \varphi$	— 339.3	— 67.8	— 6 47.1
$\delta \Omega$	+ 99.4	+ 65.7	+ 2 45.1
$\delta \pi$	+ 158.4	+ 67.3	+ 3 45.7
δi	— 162.9	— 23.4	— 3 6.3
δn	— 2″.2462	— 0″.1045	— 2″.3507

When adding the totals given above to the system P_{-11}, the author derived the following perturbed systems of elements:

System	P_{-12}	P_{-13}
Epoch and oscul. Gr. M. T.	1834 March 27.5	1825 Sept. 30.5
M	329° 59′ 15″.4	317° 5′ 48″.5
φ	24 11 7.4	24 4 20.3
Ω	212 35 58.9	212 38 44.0
π	12 57 48.8	13 1 34.5
i	26 57 25.2	26 54 18.9
n	429″.7283	427″.3776
Aequin	1950.0	1950.0

This last system P_{-13} was taken as a basis for the computation of the perturbations in the motion of the comet for the next period 1825—1817.

Cracow, Aug. 26, 1949.

JUPITER

1834 March. 27.5 — 1825 Sept. 30.5

	$d\delta\Omega$	'f	$d\delta i$	'f	$d\delta\varphi$	'f	Δ
1825 Aug. 11.5	— 6″8	+ 102″6	— 1″4	— 162″4	— 0″9	— 339″2	7·9506
Sept. 30.5	— 6·4	+ 96·2	— 0·9	— 163·3	— 0·1	— 339·3	7·9078
Nov. 19.5	— 6·0	+ 90·2	— 0·4	— 163·7	+ 0·6	— 338·7	7·8793
1826 Jan. 8.5	— 5·5	+ 84·7	0·0	— 163.7	+ 1·0	— 337·7	7·8628
Feb. 27.5	— 4·8	+ 79·9	+ 0·4	— 163·3	+ 1·1	— 336·6	7·8547
Apr. 18.5	— 4·0	+ 75·9	+ 0·8	— 162·5	+ 0·8	— 335·8	7·8495
June 7.5	— 3·1	+ 72·8	+ 1·0	— 161·5	— 0·1	— 335·9	7·8401
July 27.5	— 2·2	+ 70·6	+ 1·2	— 160·3	— 1·3	— 337·2	7·8195
Sept. 15.5	— 1·2	+ 69·4	+ 1·3	— 159·0	— 2·7	— 339·9	7·7807
Nov. 4.5	— 0·4	+ 69·0	+ 1·3	— 157·7	— 4·1	— 344·0	7·7208
Dec. 24.5	+ 0·3	+ 69·3	+ 1·2	— 156·5	— 5·3	— 349·3	7·6404
1827 Feb. 12.5	+ 0·8	+ 70·1	+ 1·0	— 155·5	— 5·9	— 355·2	7·5458
Apr. 3.5	+ 1·2	+ 71·3	+ 0·8	— 154·7	— 6·1	— 361·3	7·4435
May 23.5	+ 1·3	+ 72·6	+ 0·6	— 154·1	— 5·9	— 367·2	7·3400
July 12.5	+ 1·2	+ 73·8	+ 0·4	— 153·7	— 5·5	— 372·7	7·2422
Aug. 31.5	+ 1·1	+ 74·9	+ 0·2	— 153·5	— 4·8	— 377·5	7·1548
Oct. 20.5	+ 0·8	+ 75·7	+ 0·1	— 153·4	— 4·0	— 381·5	7·0797
Dec. 9.5	+ 0·5	+ 76·2	0·0	— 153·4	— 3·3	— 384·8	7·0193
1828 Jan. 28.5	+ 0·1	+ 76·3	0·0	— 153·4	— 2·4	— 387·2	6·9754
March 18.5	— 0·4	+ 75·9	0·0	— 153·4	— 1·5	— 388·7	6·9477
May 7.5	— 0·8	+ 75·1	0·0	— 153·4	— 0·6	— 389·3	6·9362
June 26.5	— 1·3	+ 73·8	+ 0·1	— 153·3	+ 0·4	— 388·9	6·9410
Aug. 15.5	— 1·9	+ 71·9	+ 0·3	— 153·0	+ 1·5	— 387·4	6·9605
Oct. 4,5	— 2·4	+ 69·5	+ 0·4	— 152·6	+ 2·6	— 384·8	6·9944
Nov. 23.5	— 2·9	+ 66·6	+ 0.6	— 152·0	+ 3·8	— 381·0	7·0413
1829 Jan. 12.5	— 3·5	+ 63·1	+ 0·9	— 151·1	+ 5·0	— 376·0	7·1003
March 3.5	— 4·1	+ 59·0	+ 1·2	— 149·9	+ 6·2	— 369·8	7·1697
Apr. 22.5	— 4·6	+ 54·4	+ 1·5	— 148·4	+ 7·5	— 362·3	7·2490
June 11.5	— 5·1	+ 49·3	+ 1·9	— 146·5	+ 8·7	— 353·6	7·3362
July 31.5	— 5·5	+ 43·8	+ 2.3	— 144·2	+ 9·9	— 343·7	7·4305
Sept. 19.5	— 5·9	+ 37·9	+ 2·8	— 141·4	+ 11·1	— 332·6	7·5305
Nov. 8.5	— 6·2	+ 31·7	+ 3·3	— 138·1	+ 12·3	— 320·3	7·6344
Dec. 28.5	— 6.5	+ 25·2	+ 3·7	— 134·4	+ 13·3	— 307·0	7.7412
1830 Feb. 16.5	— 6·6	+ 18·6	+ 4·2	— 130·2	+ 14·3	— 292·7	7·8502

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1834 March. 27.5 — 1825 Sept. 30.5

	$d\delta\pi$	'f	$\lambda.d\delta n$	'f	''f	P	'f
1825 Aug. 11.5	— 26·4	+ 171·0	— 1·571	—111·550	+ 4371·202	+19·5	— 183·0
Sept. 30.5	— 25·0	+ 146·0	— 1·506	—113·056	+ 4259·652	+17·8	— 165·2
Nov. 29.5	— 23·1	+ 122·9	— 1·402	—114·458	+ 4146·596	+15·7	— 149·5
1826 Jan. 8.5	— 20·9	+ 102·0	— 1.247	—115·705	+ 4032·138	+13·2	— 136·3
Feb. 27.5	— 18·5	+ 83·5	— 1·024	—116·729	+ 3916·433	+10·5	— 125·8
Apr. 18.5	— 16·0	+ 67·5	— 1·723	—117·452	+ 3799·704	+ 7·9	— 117·9
June 7.5	— 13·8	+ 53·7	— 0·337	—117·789	+ 3682·252	+ 5·7	— 112·2
July 27.5	— 12·3	+ 41·4	+ 0·123	—117·666	+ 3564·463	+ 4·3	— 107·9
Sept. 15.5	— 12·0	+ 29·4	+ 0·621	—117·045	+ 3446·797	+ 4·4	— 103·5
Nov. 4.5	— 12·9	+ 16·5	+ 1·107	—115·938	+ 3329·752	+ 5·9	— 97·6
Dec. 24.5	— 14·8	+ 1·7	+ 1.529	—114·409	+ 3213·814	+ 8·6	— 89·0
1827 Feb. 12.5	— 17·4	— 15·7	+ 1·855	—112·554	+ 3099·405	+12·1	— 76·9
Apr. 3.5	— 20·1	— 35·8	+ 2·083	—110·471	+ 2986·851	+15·9	— 61·0
May 23.5	— 22·5	— 58·3	+ 2·222	—108·249	+ 2876·380	+19·5	— 41·5
July 12.5	— 24·5	— 82·8	+ 2·296	—105·953	+ 2768·131	+22·9	— 18·6
Aug. 31.5	— 26·2	— 109·0	+ 2·324	—103·629	+ 2662·178	+25·9	+ 7·3
Oct. 20.5	— 27·6	— 136·6	+ 2·322	—101·307	+ 2558·549	+28·6	+ 35·9
Dec. 9.5	— 28·8	— 165·4	+ 2·302	— 99·005	+ 2457·242	+31·1	+ 67·0
1828 Jan. 28.5	— 29·7	— 195·1	+ 2·273	— 96·732	+ 2358·237	+33·3	+ 100·3
March 18.5	— 30·6	— 225·7	+ 2·238	— 94·494	+ 2261·505	+35·3	+ 135·6
May 7.5	— 31·3	— 257·0	+ 2·201	— 92·293	+ 2167·011	+37·0	+ 172·6
June 26.5	— 32.0	— 289·0	+ 2·165	— 90·128	+ 2074·718	+38·6	+ 211·2
Aug. 15.5	— 32·4	— 321·4	+ 2·128	— 88·000	+ 1984·590	+39·9	+ 251·1
Oct. 4.5	— 32·7	— 354·1	+ 2·093	— 85·907	+ 1896·590	+40·8	+ 291·9
Nov. 23.5	— 32·9	— 387·0	+ 2 058	— 83.849	+ 1810·683	+41·5	+ 333·4
1829 Jan. 12.5	— 32·7	— 419·7	+ 2·025	— 81·824	+ 1726·834	+41·7	+ 375·1
March 3.5	— 32·3	— 452·0	+ 1·992	— 79·832	+ 1645·010	+41·6	+ 416·7
Apr. 22.5	— 31·7	— 483·7	+ 1·961	— 77·871	+ 1565·178	+41·0	+ 457·7
June 11.5	— 30·7	— 514·4	+ 1·930	— 75·941	+ 1487·307	+40·0	+ 497·7
July 31.5	— 29·4	— 543·8	+ 1·901	— 74·040	+ 1411·366	+38·5	+ 536·2
Sept. 19.5	— 27·7	— 571·5	+ 1·874	— 72·166	+ 1337·326	+36·6	+ 572·8
Nov. 8.5	— 25·8	— 597.3	+ 1·848	— 70·318	+ 1265·160	+34·3	+ 607·1
Dec. 28.5	— 23·6	— 620·9	+ 1·825	— 68·493	+ 1194·842	+31·6	+ 638·7
1830 Feb. 16.5	— 21·1	— 642·0	+ 1·805	— 66·688	+ 1126·349	+28·5	+ 667·2

	$d\delta\Omega$	'f	$d\delta i$	'f	$d\delta\varphi$	'f	Δ
1830 Feb. 16.5	— 6″6	+ 18″6	+ 4″2	— 130″2	+ 14″3	— 292″7	7·8502
Apr. 7.5	— 6·6	+ 12·0	+ 4·7	— 125·5	+ 15·1	— 277·6	7·9588
May 27.5	— 6·5	+ 5·5	+ 5·2	— 120·3	+ 15·9	— 261.7	8·0668
July 26.5	— 6·3	— 0·8	+ 5·7	— 114·6	+ 16·5	— 245·2	8·1730
Sept. 4.5	— 6·0	— 6·8	+ 6·1	— 108·5	+ 17·0	— 228·2	8·2758
Oct. 24.5	— 5·5	— 12·3	+ 6·5	— 102·0	+ 17·4	— 210·8	8·3740
Dec. 13.5	— 5·0	— 17·3	+ 6·8	— 95·2	+ 17·6	— 193·2	8·4668
1831 Feb. 1.5	— 4·4	— 21·7	+ 7·1	— 88·1	+ 17·7	— 175·5	8·5534
March 23.5	— 3·7	— 25·4	+ 7·3	— 80·8	+ 17·6	— 157·9	8·6320
May 12.5	— 2·9	— 28·3	+ 7·4	— 73·4	+ 17·4	— 140·5	8·7016
July 1.5	— 2·1	— 30·4	+ 7·4	— 66·0	+ 17·0	— 123·5	8·7620
Aug. 20.5	— 1·3	— 31·7	+ 7·3	— 58·7	+ 16·6	— 106·9	8·8118
Oct. 9.5	— 0·4	— 32·1	+ 7·2	— 51·5	+ 15·9	— 91·0	8·8500
Nov. 28.5	+ 0·4	— 31·7	+ 6·9	— 44·6	+ 15·2	— 75·8	8·8752
1832 Jan. 17.5	+ 1·1	— 30·6	+ 6·6	— 38·0	+ 14·3	— 61·5	8·8875
March 7.5	+ 1·8	— 28·8	+ 6·2	— 31·8	+ 13·1	— 48·4	8·8858
Apr. 26.5	+ 2·4	— 26·4	+ 5·8	— 26·0	+ 12·3	— 36·1	8·8692
June 15.5	+ 2·9	— 23·5	+ 5·2	— 20·8	+ 11·1	— 25·0	8·8365
Aug. 4.5	+ 3·3	— 20·2	+ 4·7	— 16·1	+ 9·9	— 15·1	8·7872
Sept. 23.5	+ 3·5	— 16·7	+ 4·1	— 12·0	+ 8·6	— 6·5	8·7204
Nov. 12.5	+ 3·6	— 13·1	+ 3·4	— 8·6	+ 7·2	+ 0·7	8·6358
1833 Jan. 1.5	+ 3·5	— 9·6	+ 2·8	— 5·8	+ 5·8	+ 6·5	8·5321
Feb. 20.5	+ 3·3	— 6·3	+ 2·2	— 3·6	+ 4·3	+ 10·8	8·4080
Apr. 11.5	+ 2·8	— 3·5	+ 1·6	— 2·0	+ 2·9	+ 13·7	8·2633
May 31.5	+ 2·3	— 1·2	+ 1·1	— 0·9	+ 1·4	+ 15·1	8·0972
July 20.5	+ 1·7	+ 0·5	+ 0·7	— 0·2	0·0	+ 15·1	7·9084
Sept. 8.5	+ 1·0	+ 1·5	+ 0·3	+ 0·1	— 1·4	+ 13·7	7·6955
Oct. 28.5	+ 0·3	+ 1·8	+ 0·1	+ 0·2	— 2·7	+ 11·0	7·4575
Dec. 17.5	— 0·3	+ 1·5	— 0·1	+ 0·1	— 3·8	+ 7·2	7·1928
1834 Feb. 5.5	— 0·9	+ 0·6	— 0·1	0·0	— 4·7	+ 2·5	6·9005
March 27.5	— 1·3	— 0·7	0·0	0·0	— 5·1	— 2·6	6·5790
May 16.5	— 1·4		+ 0·1		— 4·9		6·2253

	$d\delta\pi$	$'f$	$\lambda d\delta n$	$'f$	$''f$	P	$'f$
1830 Feb. 16.5	— 21″1	— 642″0	+ 1″805	— 66″688	+ 1126″349	+28″5	+ 667″2
Apr. 7.5	— 18·4	— 660·4	+ 1·788	— 64·900	+ 1059·661	+25·1	+ 692·3
May 27.5	— 15·4	— 675·8	+ 1·774	— 63·126	+ 994·761	+21·4	+ 713·7
July 16.5	— 12·2	— 688·0	+ 1·765	— 61·361	+ 931·635	+17·4	+ 731·1
Sept. 4.5	— 8·9	— 696·9	+ 1·761	— 59·600	+ 870·274	+13·2	+ 744·3
Oct. 24.5	— 5·4	— 702·3	+ 1·762	— 57·838	+ 810·674	+ 8·9	+ 753·2
Dec. 13.5	— 1·8	— 704·1	+ 1·769	— 56·069	+ 752·836	+ 4·4	+ 757·6
1831 Feb. 1.5	+ 1·9	— 702·2	+ 1·781	— 54·288	+ 696·767	— 0·2	+ 757·4
March 23.5	+ 5·5	— 696·7	+ 1·800	— 52·488	+ 642·479	— 4·8	+ 752·6
May 12.5	+ 9·2	— 687·5	+ 1·826	— 50·662	+ 589·921	— 9·3	+ 743·3
July 1.5	+ 12·8	— 674·7	+ 1·858	— 48·804	+ 539·329	—13·9	+ 729·4
Aug. 20.5	+ 16·6	— 658·1	+ 1·897	— 46·907	+ 490·525	—18·3	+ 711·1
Oct. 9.5	+ 19·9	— 638·2	+ 1·943	— 44·964	+ 443·618	—22·5	+ 688·6
Nov. 28.5	+ 23·2	— 615·0	+ 1·996	— 42·968	+ 398·654	—26·5	+ 662·1
1832 Jan. 17.5	+ 26·3	— 588·7	+ 2·056	— 40·912	+ 355·686	—30·3	+ 631·8
March 7.5	+ 29·2	— 559·5	+ 2·122	— 38·790	+ 314·774	—33·7	+ 598·1
Apr. 26.5	+ 31·9	— 527·6	+ 2·194	— 36·596	+ 275·984	—36·9	+ 561·2
June 15.5	+ 34·4	— 493·2	+ 2·272	— 34·324	+ 239·388	—39·6	+ 521·6
Aug. 4.5	+ 36·5	— 456·7	+ 2·354	— 31·970	+ 205·064	—42·0	+ 479·6
Sept. 23.5	+ 38·4	— 418·3	+ 2·440	— 29·530	+ 173·094	—43·8	+ 435·8
Nov. 12.5	+ 40·0	— 378·3	+ 2·529	— 27·001	+ 143·564	—45·2	+ 390·6
1833 Jan. 1.5	+ 41·1	— 337·2	+ 2·619	— 24·382	+ 116·563	—46·1	+ 344·5
Feb. 20.5	+ 42·0	— 295·2	+ 2·707	— 21·675	+ 92·181	—46·5	+ 298·0
Apr. 11.5	+ 42·4	— 252·8	+ 2·792	— 18·883	+ 70·506	—46·3	+ 251·7
May 31.5	+ 42·4	— 210·4	+ 2·869	— 16·014	+ 51·623	—45·5	+ 206·2
July 20.5	+ 42·0	— 168·4	+ 2·932	— 13·082	+ 35·609	—44·1	+ 162·1
Sept. 8.5	+ 41·1	— 127.3	+ 2·975	— 10·107	+ 22·527	—42·1	+ 120·0
Oct. 28.5	+ 39·6	— 87·7	+ 2·985	— 7·122	+ 12·420	—39·3	+ 80·7
Dec. 17.5	+ 37·4	— 50·3	+ 2·946	— 4·176	+ 5·298	—35·7	+ 45·0
1834 Feb. 5.5	+ 34·5	— 15·8	+ 2·835	— 1·341	+ 1·122	—31·4	+ 13·6
March 27.5	+ 30·9	+ 15·1	+ 2·633	+ 1·292	— 0·219	—26·2	— 12·6
May 16.5	+ 26·5		+ 2·236		+ 1·073	—20·3	

JUPITER
1839 April 20.5 — 1834 March. 27.5

	$d\delta\Omega$	'f	$d\delta i$	'f	$d\delta\varphi$	'f	Δ
1834 Feb. 5.5	— 0″9	+4001″7	— 0″1	—2376″2	— 4″6	+ 2″3	6·8962
March 27·5	— 1·3	+4000·4	0·0	—2376·2	— 5·1	— 2·8	6·5790
May 16.5	— 1·4	+3999·0	+ 0·1	—2376·1	— 4·9	— 7·7	6·2253
July 5.5	— 1·2	+3997·8	+ 0·2	—2375·9	— 3·5	— 11·2	5·8430
Aug. 24.5	— 0·7	+3997·1	+ 0·2	—2375·7	— 0·9	— 12·1	5·4360
Oct. 13.5	0·0	+3997·1	0·0	—2375·7	+ 3·3	— 8·8	5·0079
Dec. 2.5	+ 0·9	+3998·0	— 0·7	—2376·4	+ 9·3	+ 0·5	4·5704
1835 Jan. 21.5	+ 1·2	+3999·2	— 2·0	—2378·4	+ 16·3	+ 16·8	4·1416
March 2.5	— 0·1	+3999·1	— 4·1	—2382·5	+ 23·3	+ 40·1	3·7424
May 1.5	— 4·1	+3995·0	— 7·0	—2389·5	+ 28·5	+ 68·6	3·3915
June 20.5	— 12·3	+3982·7	— 10·3	—2399·8	+ 30·5	+ 99·1	3·1034
Aug. 9.5	— 24·9	+3957·8	— 13·2	—2413·0	+ 29·0	+ 128·1	2·8840
Sept. 28.5	— 40·7	+3917·1	— 14·8	—2427·8	+ 24·7	+ 152·8	2·7309
Nov. 17.5	— 57·3	+3859·8	— 14·5	—2442·3	+ 19·2	+ 172·0	2·6339
1836 Jan. 6.5	— 72·3	+3787·5	— 12·4	—2454·7	+ 13·9	+ 185·9	2·5823
Feb. 25.5	— 84·4	+3703·1	— 8·9	—2463·6	+ 9·8	+ 195·7	2·5615
Apr. 15.5	— 93·3	+3609.8	— 4·1	—2468·2	+ 7·1	+ 202·8	2·5595
June 4·5	— 99·4	+3510·4	— 0·1	—2468·3	+ 5·6	+ 208·4	2·5664
July 24.5	— 103·7	+3406·7	+ 4·6	—2463·7	+ 5·0	+ 213·4	2·5749
Sept. 12.5	— 106·9	+3299·8	+ 9·2	—2454·5	+ 5·1	+ 218·5	2·5796
Nov. 1.5	— 109·6	+3190·2	+ 13·8	—2440·7	+ 5·5	+ 224·0	2·5773
Dec. 21.5	— 112·2	+3078·0	+ 18·5	—2422·2	+ 6·3	+ 230·3	2·5651
1837 Feb. 9.5	— 115·0	+2963·0	+ 23·4	—2398·8	+ 7·3	+ 237·6	2·5421
March 31.5	— 118·3	+2844·7	+ 28·7	—2370·1	+ 8·4	+ 246·0	2.5073
May 20.5	— 122·3	+2722·4	+ 34·5	—2335·6	+ 9·5	+ 255·5	2·4608
July 9.5	— 127·0	+2595·4	+ 41·1	—2294·5	+ 10·8	+ 266·3	2·4024
Aug. 28.5	— 132·7	+2462·7	+ 48·6	—2245·9	+ 12·0	+ 278·3	2·3322
Oct. 17.5	— 139·5	+2323·2	+ 57·5	—2188·4	+ 13·0	+ 291·3	2·2509
Dec. 6.5	— 147·5	+2175·7	+ 68·0	—2120·4	+ 13·9	+ 305·2	2·1595
1838 Jan. 25.5	— 156·9	+2018·8	+ 80·7	—2039·7	+ 14·4	+ 319·6	2·0584
March 16.5	— 168·0	+1850·8	+ 96·2	—1943·5	+ 14·2	+ 333·8	1·9482

JUPITER

1839 April 20.5 — 1834 March. 27.5

	dδπ	'f	λdδn	'f	''f	P	'f
1834 Feb. 5.5	+ 34·5	+2664·6	+ 2·828	— 3·256	+ 166·667	— 31·3	—4137·7
March 27.5	+ 30·9	+2695·5	+ 2·616	— 0·640	+ 163·411	— 26·2	—4163·9
May 16.5	+ 26·5	+2722·0	+ 2·236	+ 1·596	+ 162·771	— 20·3	—4184·2
July 5.5	+ 21·7	+2743·7	+ 1·633	+ 3·229	+ 164·367	— 14·0	—4198·2
Aug. 24.5	+ 16·8	+2760·5	+ 0·739	+ 3·968	+ 167·596	— 8·1	—4206·3
Oct. 13.5	+ 13·1	+2773·6	— 0·518	+ 3·450	+ 171·564	— 3·8	—4210·1
Dec. 2.5	+ 11·8	+2785·4	— 2·130	+ 1·320	+ 175·014	— 2·7	—4212·8
1835 Jan. 21.5	+ 14·1	+2799·5	— 3·995	— 2·675	+ 176·334	— 6·7	—4219·5
March 2.5	+ 20·1	+2819·6	— 5·834	— 8·509	+ 173·659	— 16·2	—4235·7
May 1.5	+ 27·4	+2847·0	— 7·246	— 15·755	+ 165·150	— 29·4	—4265·1
June 20.5	+ 32·0	+2879·0	— 7·808	— 23·563	+ 149·395	— 42·0	—4307·1
Aug. 9.5	+ 29·9	+2908·9	— 7·281	— 30·844	+ 125·832	— 48·8	—4355·9
Sept. 28.5	+ 19·6	+2928·5	— 5·765	— 36·609	+ 94·988	— 46·4	—4402·3
Nov. 17.5	+ 3·0	+2931·5	— 3·658	— 40·267	+ 58·379	— 34·6	—4436·9
1836 Jan. 6.5	— 16·3	+2915·2	— 1·458	— 41·725	+ 18·112	— 16·6	—4453·5
Feb. 25.5	— 34·4	+2880·8	+ 0·470	— 41·255	— 23·613	+ 3·6	—4449·9
Apr. 15.5	— 49·6	+2831·2	+ 1·976	— 39·279	— 64·868	+ 23·1	—4426·8
June 4.5	— 61·4	+2769·8	+ 3·057	— 36·222	— 104·147	+ 40·4	—4386·4
July 24.5	— 70·0	+2699·8	+ 3·788	— 32·434	— 140·369	+ 55·1	—4331·3
Sept. 12.5	— 76·3	+2623·5	+ 4·258	— 28·176	— 172·803	+ 67·5	—4263·8
Nov. 1.5	— 80·8	+2542·7	+ 4·545	— 23·631	— 200·979	+ 77·9	—4185·9
Dec. 21.5	— 84·2	+2458·5	+ 4·711	— 18·920	— 224·610	+ 87·1	—4098·8
1837 Feb. 9.5	— 87·0	+2371·5	+ 4·798	— 14·122	— 243·530	+ 95·3	—4003·5
March 31.5	— 89·3	+2282·2	+ 4.833	— 9·289	— 257·652	+102·9	—3900·6
May 20.5	— 91·5	+2190·7	+ 4·836	— 4·453	— 266·941	+110·3	—3790·3
July 9·5	— 93·8	+2096·9	+ 4·821	+ 0·368	— 271·394	+118·1	—3672·2
Aug. 28.5	— 96·3	+2000·6	+ 4·787	+ 5·155	— 271·026	+126·2	—3546·0
Oct. 17.5	— 99·0	+1901·6	+ 4·736	+ 9·891	— 265·871	+135·2	—3410·8
Dec. 6.5	— 102·5	+1799·1	+ 4·662	+ 14·553	— 255·980	+145·6	—3265·2
1838 Jan. 25.5	— 106·7	+1692·4	+ 4·544	+ 19·097	— 241·427	+158·0	—3107·2
March 16.5	— 112·0	+1580·4	+ 4·345	+ 23·442	— 222·330	+173·0	—2934·2

	$d\delta\Omega$	'f	$d\delta i$	'f	$d\delta\varphi$	'f	Δ
1837 Dec. 31.5	— 76·0	+2137·1	+ 37·0	—2101·1	+ 7·1	+ 308·8	2·1103
1838 Jan. 25.5	— 78·4	+2058·7	+ 40·3	—2060·8	+ 7·2	+ 316·0	2·0584
Febr. 19.5	— 81·1	+1977·6	+ 44·0	—2016·8	+ 7·2	+ 323·2	2·0043
March 16.5	— 84·0	+1893·6	+ 48·1	—1968·7	+ 7·1	+ 330·3	1·9482
Apr. 10.5	— 87·1	+1806·5	+ 52·7	—1916·0	+ 6·8	+ 337·1	1·8901
May 5.5	— 90·5	+1716·0	+ 57·8	—1858·2	+ 6·4	+ 343·5	1·8300
May 30.5	— 94·1	+1621·9	+ 63·5	—1794·7	+ 5·8	+ 349·3	1·7684
June 24.5	— 98·1	+1523·8	+ 70·0	—1724·7	+ 4·9	+ 354·2	1·7049
July 19.5	— 102·4	+1421·4	+ 77·3	—1647·4	+ 3·6	+ 357·8	1·6401
Aug. 13.5	— 107·0	+1314·4	+ 85·6	—1561·8	+ 1·7	+ 359·5	1·5737
Sept. 7.5	— 112·0	+1202·4	+ 95·1	—1466·7	— 0·7	+ 358·8	1·5063
Oct. 2.5	— 117·4	+1085·0	+ 106·0	—1360·7	— 4·1	+ 354·7	1·4377
Oct. 27.5	— 123·2	+ 961·8	+ 118·4	—1242·3	— 8·6	+ 346·1	1.3686
1838 Nov. 21.5	— 129·4		+ 132·9		— 14·6		1·2986
		+1173·4		—1441·0		+ 358·0	
1838 Oct. 2.5	— 58·7	+1114·7	+ 53·0	—1388·0	— 2·0	+ 356·0	1·4377
Oct. 15.0	— 60·1	+1054·6	+ 56·0	—1332·0	— 3·1	+ 352·9	1·4033
Oct. 27.5	— 61·7	+ 992·9	+ 59·2	—1272·8	— 4·3	+ 348·6	1·3686
Nov. 9.0	— 63·1	+ 929·8	+ 62·7	—1210·1	— 5·7	+ 342·9	1·3337
Nov. 21.5	— 64·7	+ 865·1	+ 66·5	—1143·6	— 7·3	+ 335·6	1·2986
Dec. 4.0	— 66·4	+ 798·7	+ 70·5	—1073·1	— 9·2	+ 326·4	1·2634
Dec. 16.5	— 68·0	+ 730·7	+ 74·8	— 998·3	— 11·4	+ 315·0	1·2282
Dec. 29.0	— 69·7	+ 661·0	+ 79·5	— 918·8	— 13·9	+ 301·1	1·1930
1839 Jan. 10.5	— 71·5	+ 589·5	+ 84·5	— 834·3	— 16·8	+ 284·3	1·1578
Jan. 23.0	— 73·2	+ 516·3	+ 89·8	— 744·5	— 20·2	+ 264·1	1·1228
Feb. 4.5	— 74·9	+ 441·4	+ 95·6	— 648·9	— 24·3	+ 239·8	1·0879
Feb. 17.0	— 76·7	+ 364·7	+ 101·8	— 547·1	— 28·8	+ 211·0	1·0531
March 1.5	— 78·3	+ 286·4	+ 108·4	— 438·7	— 34·1	+ 176·9	1·0184
March 14.0	— 80·0	+ 206·4	+ 115·5	— 323·2	— 40·4	+ 136·5	0·9840
March 26.5	— 81·5	+ 124·9	+ 123·1	— 200·1	— 47·8	+ 88·7	0·9497
Apr. 8.0	— 82·9	+ 42·0	+ 131·1	— 69·0	— 56·4	+ 32·3	0·9160
Apr. 20.5	— 84·1	— 42.1	+ 139·5	+ 70·5	— 66·5	— 34·2	0·8828
May 3.0	— 85·0	— 127·1	+ 148·2	+ 218·7	— 78·1	— 112·3	0·8505
1839 May 15.5	— 85·5		+ 157·0		— 91·7		0·8186

	$d\delta\pi$	'f	$\lambda d\delta n$	'f	"f	P	'f
1837 Dec. 31.5	— 52·3	+1772·7	+ 1·153	+ 7·849	— 248·992	+ 75·8	—3226·5
1838 Jan. 25·5	— 53·3	+1719·4	+ 1·136	+ 8·985	— 241·143	+ 79·0	—3147·5
Feb. 19.5	— 54·6	+1664·8	+ 1·114	+10·099	— 232·158	+ 82·5	—3065·0
March 16.5	— 56·0	+1608·8	+ 1·086	+11·185	— 222·059	+ 86·5	—2978·5
Apr. 10.5	— 57·6	+1551·2	+ 1·052	+12·237	— 210·874	+ 91·0	—2887·5
May 5.5	— 59·5	+1491·7	+ 1·008	+13·245	— 198·637	+ 96·1	—2791·4
May 30·5	— 61·8	+1429·9	+ 0·953	+14·198	— 185·392	+102.0	—2689·4
June 24.5	— 64·4	+1365·5	+ 0·880	+15·078	— 171·194	+108·8	—2580·6
July 19·5	— 67·5	+1298·0	+ 0·786	+15·864	— 156·116	+116·7	—2463·9
Aug. 13.5	— 71·4	+1226·6	+ 0·664	+16·528	— 140·252	+126·0	—2337·9
Sept. 7·5	— 76·1	+1150·5	+ 0·505	+17·033	— 123·724	+137·2	—2200·7
Oct. 2.5	— 81·7	+1068·8	+ 0·296	+17·329	— 106·691	+150·6	—2050·1
Oct. 27.5	— 89·1	+ 979·7	+ 0.025	+17·349	— 89·362	+166·9	—1883·2
1838 Nov. 21.5	— 98·3		— 0·344		— 72·013	+186·9	
1838 Oct. 2.5	— 40·8	+1130·5	+ 0·074	+ 8·561	— 106·666	+ 75·3	—2163·9
Oct. 15.0	— 42·6	+1089·7	+ 0·042	+ 8·635	— 98·031	+ 79·2	—2088·6
Oct. 27.5	— 44·5	+1047·1	+ 0·005	+ 8·677	— 89·354	+ 83·5	—2009·4
Nov. 9.0	— 46·7	+1002·6	— 0·037	+ 8·682	— 80·672	+ 88·2	—1925·9
Nov. 21.5	— 49·1	+ 955·5	— 0 086	+ 8·645	— 72·027	+ 93·4	—1837·7
Dec. 4.0	— 51·9	+ 906·8	— 0·143	+ 8·559	— 63·468	+ 99.2	—1744 3
Dec. 16.5	— 55·0	+ 854·9	— 0·209	+ 8·416	— 55·052	+105·8	—1645·1
Dec. 29.0	— 58·7	+ 799·9	— 0·284	+ 8·207	— 46·845	+113·1	—1539·3
1339 Jan. 10.5	— 62·6	+ 741·2	— 0·371	+ 7·923	— 38·922	+121·3	—1426·2
Jan. 23.0	— 67·2	+ 678·6	— 0·474	+ 7·552	— 31·370	+130·5	—1304·9
Feb. 4.5	— 72·4	+ 611·4	— 0·595	+ 7·078	— 24·292	+140·7	—1174·4
Feb. 17.0	— 78·6	+ 539·0	— 0·733	+ 6·483	— 17·709	+152·5	—1033·7
March 1.5	— 85·8	+ 460·4	— 0·892	+ 5 750	— 11·859	+166·1	— 881·2
March 14.0	— 94·0	+ 374·6	— 1·081	+ 4·858	— 7·001	+181·2	— 715·1
March 26.0	— 103·5	+ 280·6	— 1·304	+ 3·777	— 3·224	+198·2	— 533·9
Apr. 8.0	— 114·6	+ 177·1	— 1·566	+ 2·473	— 0·751	+217·7	— 335·7
Apr. 20.5	— 127·4	+ 62·5	— 1·870	+ 0·907	+ 0·156	+239·9	— 118·0
May 3.0	— 142·5	— 64·9	— 2·227	— 0·963	— 0·807	+265·2	+ 121·9
1139 May 15 5	— 160·1	— 207·4	— 2·643	— 3·190	— 3·997	+294·2	+ 387·1

SATURN

1834 March 27.5 — 1825 Sept. 30.5

	dδΩ	'f	dδi	'f	dδφ	'f	Δ
1825 Aug. 11.5	—0″·4	+65″·9	—0″·1	—23″·3	—0″·5	—67″·5	12·20
Sept. 30.5	—0·4	+65·5	—0.1	—23·4	—0·6	—68·1	11·90
Nov. 19.5	—0·4	+65·1	0·0	—23·4	—0·6	—68·7	11·60
1826 Jan. 8.5	—0·4	+64·7	0·0	—23·4	—0·6	—69·3	11·30
Feb. 27.5	—0·3	+64·4	0·0	—23·4	—0·6	—69·9	10·90
Apr. 18.5	—0·2	+64·2	0·0	—23·4	—0·5	—70·4	10·42
June 7.5	—0·1	+64·1	0·0	—23·4	—0·3	—70·7	9·92
July 27.5	0·0	+64·1	0·0	—23·4	—0·1	—70·8	9·41
Sept. 15.5	0·0	+64·1	0·0	—23·4	+0·2	—70·6	8·90
Nov. 4.5	0·0	+64·1	—0·1	—23·5	+0·5	—70·1	8·39
Dec. 24.5	0·0	+64·1	—0·2	—23·7	+0·8	—69·3	7·90
1827 Feb. 12.5	—0·2	+63·9	—0·3	—24·0	+1·0	—68·3	7·45
Apr. 3.5	—0·5	+63·4	—0·4	—24·4	+1·2	—67·1	7·05
May 23.5	—0·9	+62·5	—0·4	—24·8	+1·2	—65·9	6·67
July 12.5	—1·4	+61·1	—0·4	—25·2	+1·1	—64·8	6·40
Aug. 31.5	—1·9	+59·2	—0·4	—25·6	+1·0	—63·8	6·23
Oct. 20.5	—2·4	+56·8	—0·3	—25·9	+1·0	—62·8	6·08
Dec. 9.5	—2·8	+54·0	—0·2	—26·1	+1·0	—61·8	5·98
1828 Jan. 28.5	—3·2	+50·8	—0·1	—26·2	+1·0	—60·8	5·93
March 18.5	—3·4	+47·4	+0·1	—26·1	+1·0	—59·8	5·90
May 7.5	—3·6	+43·8	+0·2	—25·9	+1·1	—58·7	5·90
June 26.5	—3·6	+40·2	+0·4	—25·5	+1·1	—57·6	5·94
Aug. 15.5	—3·6	+36·6	+0·5	—25·0	+1·2	—56·4	5·98
Oct. 4.5	—3·5	+33·1	+0·6	—24·4	+1·3	—55·1	6·02
Nov. 23.5	—3·4	+29·7	+0·7	—23·7	+1·4	—53·7	6·08
1829 Jan. 12.5	—3·2	+26·5	+0·8	—22·9	+1·5	—52·2	6·14
March 3.5	—3·1	+23·4	+0·9	—22·0	+1·6	—50·6	6·19
Apr. 22.5	—2·9	+20·5	+1·0	—21·0	+1·7	—48·9	6·25
June 11.5	—2·7	+17·8	+1·0	—20·0	+1·7	—47·2	6·31
July 31.5	—2·5	+15·3	+1·1	—18·9	+1·8	—45·4	6·38
Sept. 19.5	—2·3	+13·0	+1·1	—17·8	+1·8	—43·6	6·45
Nov. 8.5	—2·1	+10·9	+1·1	—16·7	+1·9	—41·7	6·51
Dec. 28.5	—1·9	+ 9·0	+1·1	—15·6	+1·9	—39·8	6·57
1830 Feb. 16.5	—1·7	+ 7·3	+1·1	—14·5	+2·0	—37·8	6·63

SATURN

1834 March 27.5 — 1825 Sept. 30.5

	$d\delta\pi$	'f	$\lambda d\delta n$	'f	"f	P	'f
1825 Aug. 11.5	+1.4″	+66.6″	+ 0.160″	— 5.310″	+164.802″	—2.1″	— 93.3″
Sept. 30.5	+1.4	+68.0	+ 0.172	— 5.138	+159.492	—2.0	— 95.3
Nov. 19.5	+1.5	+69.5	+ 0.184	— 4.954	+154.354	—1.8	— 97.1
1821 Jan. 8.5	+1.5	+71.0	+ 0.193	— 4.761	+149.409	—1.7	— 98.8
Feb. 27.5	+1.4	+72.4	+ 0.186	— 4.575	+144.639	—1.4	— 100.2
Apr. 18.5	+1.1	+73.5	+ 0.155	— 4.420	+140.064	—1.0	— 101.2
June 7.5	+0.9	+74.4	+ 0.100	— 4.320	+135.644	—0.6	— 101.8
July 27.5	+0.8	+75.2	+ 0.030	— 4.290	+131.324	—0.3	— 102.1
Sept. 15.5	+0.7	+75.9	— 0.043	— 4.333	+127.034	—0.3	— 102.4
Nov. 4.5	+0.9	+76.8	— 0.120	— 4.453	+122.701	—0.4	— 102.8
Dec. 24.5	+1.1	+77.9	— 0.205	— 4.658	+118.248	—0.8	— 103.6
1827 Feb. 12.5	+1.4	+79.3	— 0.275	— 4.933	+113.590	—1.4	— 105.0
Apr. 3.5	+1.6	+80.9	— 0.308	— 5.241	+108.657	—1.9	— 106.9
May 23.5	+1.5	+82.4	— 0.298	— 5.539	+103.416	—2.3	— 109.2
July 12.5	+1.1	+83.5	— 0.262	— 5.801	+ 97.877	—2.3	— 111.5
Aug. 31.5	+0.4	+83.9	— 0.196	— 5.997	+ 92.076	—2.0	— 113.5
Oct. 20.5	—0.5	+83.4	— 0.119	— 6.116	+ 86.079	—1.5	— 115.0
Dec. 9.5	—1.4	+82.0	— 0.038	— 6.154	+ 79.963	—0.8	— 115.8
1828 Jan. 28.5	—2.3	+79.7	+ 0.041	— 6.113	+ 73.809	0.0	— 115.8
March 18.5	—3.1	+76.6	+ 0.110	— 6.003	+ 67.696	+0.9	— 114.9
May 7.5	—3.8	+72.8	+ 0.167	— 5.836	+ 61.693	+1.7	— 113.2
June 26.5	—4.2	+68.6	+ 0.211	— 5.625	+ 55.857	+2.4	— 110.8
Aug. 15.5	—4.5	+64.1	+ 0.243	— 5.382	+ 50.232	+3.0	— 107.8
Oct. 4.5	—4.7	+59.4	+ 0.268	— 5.114	+ 44.850	+3.5	— 104.3
Nov. 23.5	—4.8	+54.6	+ 0.282	— 4.832	+ 39.736	+3.8	— 100.5
1829 Jan. 12.5	—4.8	+49.8	+ 0.290	— 4.542	+ 34.904	+4.2	— 96.3
March 3.5	—4.7	+45.1	+ 0.295	— 4.247	+ 30.362	+4.4	— 91.9
Apr. 22.5	—4.6	+40.5	+ 0.297	— 3.950	+ 26.115	+4.5	— 87.4
June 11.5	—4.5	+36.0	+ 0.294	— 3.656	+ 22.165	+4.6	— 82.8
July 31.5	—4.2	+31.8	+ 0.287	— 3.369	+ 18.509	+4.7	— 78.1
Sept. 19.5	—4.0	+27.8	+ 0.279	— 3.090	+ 15.140	+4.6	— 73.5
Nov. 8.5	—3.7	+24.1	+ 0.270	— 2.820	+ 12.050	+4.6	— 68.9
Dec. 28.5	—3.5	+20.6	+ 0.262	— 2.558	+ 9.230	+4.5	— 64.4
1830 Feb. 16.5	—3.2	+17.4	+ 0.253	— 2.305	+ 6.672	+4.4	— 60.0

	$d\delta\Omega$	'f	$d\delta i$	'f	$d\delta\varphi$	'f	Δ
1830 Feb. 16.5	—1″7	+7″3	+1″1	—14″5	+2″0	—37″8	6·63
Apr. 7.5	—1·6	+5·7	+1·1	—13·4	+2·0	—35·8	6·69
May 27.5	—1·4	+4·3	+1·1	—12·3	+2·0	—33·8	6·75
July 16.5	—1·2	+3·1	+1·1	—11·2	+2·0	—31·8	6·81
Sept. 4.5	—1·0	+2·1	+1·0	—10·2	+2·0	—29·8	6·87
Oct. 24.5	—0·9	+1·2	+1·0	— 9·2	+2·0	—27·8	6·92
Dec. 13.5	—0·7	+0·5	+1·0	— 8·2	+2·0	—25·8	6·98
1831 Feb. 1.5	—0·6	—0·1	+1·0	— 7·2	+2·0	—23·8	7·04
March 23.5	—0·5	—0·6	+0·9	— 6·3	+1·9	—21·9	7·11
May 12.5	—0·3	—0·9	+0·9	— 5·4	+1·9	—20·0	7·18
July 1.5	—0·2	—1·1	+0·8	— 4·6	+1·8	—18·2	7·28
Aug. 20.5	—0·1	—1·2	+0·7	— 3·9	+1·7	—16·5	7·35
Oct. 9.5	0·0	—1·2	+0·7	— 3·2	+1·7	—14·8	7·42
Nov. 28.5	0·0	—1·2	+0·6	— 2·6	+1·6	—13·2	7·51
1832 Jan. 17.5	+0·1	—1·1	+0·5	— 2·1	+1·5	—11·7	7·62
March 7.5	+0·1	—1·0	+0·5	— 1·6	+1·4	—10·3	7·73
Apr. 26.5	+0·2	—0·8	+0·4	— 1·2	+1·3	— 9·0	7·85
June 15.5	+0·2	—0·6	+0·3	— 0·9	+1·2	— 7·8	7·98
Aug. 4.5	+0·2	—0·4	+0·3	— 0·6	+1·1	— 6·7	8·11
Sept. 23.5	+0·2	—0·2	+0.2	— 0·4	+1·0	— 5·7	8·26
Nov. 12.5	+0·2	0·0	+0·2	— 0·2	+0·9	— 4·8	8·41
1833 Jan. 1.5	+0·2	+0·2	+0·1	— 0·1	+0·8	— 4·0	8·59
Feb. 20.5	+0·1	+0·3	+0·1	0·0	+0·7	— 3·3	8·81
Apr. 11.5	+0·1	+0·4	0·0	0·0	+0·6	— 2·7	9·02
May 31.5	0·0	+0·4	0·0	0·0	+0·5	— 2·2	9·27
July 20.5	0·0	+0·4	0·0	0·0	+0·4	— 1·8	9·48
Sept. 8.5	0·0	+0·4	0·0	0·0	+0·4	— 1·4	9·73
Oct. 28.5	—0·1	+0·3	0·0	0·0	+0·4	— 1·0	9·98
Dec. 17.5	—0·1	+0·2	0·0	0·0	+0·4	— 0·6	10·23
1834 Feb. 5.5	—0·1	+0·1	0·0	0·0	+0·4	— 0·2	10·50
March 27.5	—0·1	0·0	0·0	0·0	+0·4	+ 0·2	10·80
May 16.5	—0·1		0·0		+0·5		11·10

	$d\delta\pi$	$'f$	$\lambda d\delta n$	$'f$	$''f$	P	$'f$
1830 Feb. 16.5	—3″·2	+17″·4	+0″·253	—2″·305	+6″·672	+4″·4	—60″·0
Apr. 7.5	—2·9	+14·5	+0·243	—2·062	+4·367	+4·3	—55·7
May 27.5	—2·6	+11·9	+0·233	—1·829	+2·305	+4·1	—51·6
July 16.5	—2·4	+9·5	+0·222	—1·607	+0·476	+3·9	—47·7
Sept. 4.5	—2·1	+7·4	+0·214	—1·393	—1·131	+3·7	—44·0
Oct. 24.5	—1·8	+5·6	+0·205	—1·188	—2·524	+3·5	—40·5
Dec. 13.5	—1·5	+4·1	+0·195	—0·993	—3·712	+3·3	—37·2
1831 Feb. 1.5	—1·2	+2·9	+0·187	—0·806	—4·705	+3·1	—34·1
March 23.5	—1·0	+1·9	+0·176	—0·630	—5·511	+2.8	—31·3
May 12.5	—0·7	+1·2	+0·165	—0·465	—6·141	+2·6	—28·7
July 1.5	—0·5	+0·7	+0·155	—0·310	—6·606	+2·4	—26·3
Aug. 20.5	—0·2	+0·5	+0·144	—0·166	—6·916	+2·2	—24·1
Oct. 9.5	0·0	+0·5	+0·136	—0·030	—7·082	+1·9	—22·2
Nov. 28.5	+0·2	+0·7	+0·125	+0·095	—7·112	+1·7	—20·5
1832 Jan. 17.5	+0·3	+1·0	+0·113	+0·208	—7·017	+1·6	—18·9
March 7.5	+0·4	+1·4	+0·102	+0·310	—6·809	+1.4	—17·5
Apr. 26.5	+0·5	+1·9	+0·088	+0·398	—6·499	+1·3	—16·2
June 15.5	+0·6	+2·5	+0·075	+0·473	—6·101	+1.2	—15·0
Aug. 4.5	+0·6	+3·1	+0·061	+0·534	—5·628	+1·1	—13·9
Sept. 23.5	+0·6	+3·7	+0·047	+0·581	—5·094	+1·0	—12·9
Nov. 12.5	+0·5	+4·2	+0·933	+0·614	—4·513	+1.0	—11·9
1833 Jan. 1.5	+0·4	+4·6	+0·016	+0·630	—3·899	+1·0	—10·9
Feb. 20.5	+0·2	+4·8	—0·005	+0·625	—3·269	+1·1	—9·8
Apr. 11.5	—0·1	+4·7	—0·026	+0·599	—2·644	+1·2	—8·6
May 31.5	—0·2	+4·5	—0·044	+0·555	—2·045	+1·2	—7·4
July 20.5	—0·4	+4·1	—0·061	+0·494	—1·490	+1·2	—6·2
Sept. 8.5	—0·6	+3·5	—0·079	+0·415	—0·996	+1·3	—4·9
Oct. 28.5	—0·8	+2·7	—0·096	+0·319	—0·581	+1·4	—3·5
Dec. 17.5	—1·0	+1·7	—0·116	+0·203	—0·262	+1·4	—2·1
1834 Feb. 5.5	—1·1	+0·6	—0·132	+0·071	—0·059	+1·4	—0·7
March 27.5	—1·2	—0·6	—0·144	—0·073	+0·012	+1·4	+0·7
May 16.5	—1·3		—0·149		—0·061	+1·3	

SATURN

1839 Apr. 20.5 — 1834 March 27.5

	$d\delta\Omega$	$'f$	$d\delta i$	$'f$	$d\delta\varphi$	$'f$	Δ
1834 Feb. 5.5	−0″1	−2″2	0″0	+2″0	+0″4	+17″4	10·50
March 27.5	−0·1	−2·3	0·0	+2·0	+0·4	+17·8	10·80
May 16.5	−0·1	−2·4	0·0	+2·0	+0·5	−18·3	11·10
July 5.5	−0·1	−2·5	0·0	+2·0	+0·5	+18·8	11·30
Aug. 24.5	−0·1	−2·6	0·0	+2·0	+0·4	+19·2	11·60
Oct. 13.5	−0·1	−2·7	0·0	+2·0	+0·4	+19·6	11·80
Dec. 2.5	−0·1	−2·8	0·0	+2·0	+0·2	+19·8	11·90
1835 Jan. 21.5	0·0	−2·8	0·0	+2·0	+0·1	+19·9	12·00
March 12.5	0·0	−2·8	0·0	+2·0	−0·1	+10·8	12·00
May 1.5	0·0	−2·8	0·0	+2·0	−0·2	+19·6	12·10
June 20.5	0·0	−2·8	0·0	+2·0	−0·3	+19·3	12·00
Aug. 9.5	+0·1	−2·7	0·0	+2·0	−0·3	+19·0	11·90
Sept. 28.5	+0·1	−2·6	0·0	+2·0	−0·3	+18·7	11·70
Nov. 17.5	0·0	−2·6	0·0	+2·0	−0·3	+18·4	11·50
1836 Jan. 6.5	0·0	−2·6	0·0	+2·0	−0·3	+18·1	11·40
Feb. 25.5	0·0	−2·6	0·0	+2·0	−0·2	+17·9	11·20
Apr. 15.5	0·0	−2·6	0·0	+2·0	−0·2	+17·7	11·00
June 4.5	0·0	−2·6	0·0	+2·0	−0·2	+17·5	10·80
July 24.5	0·0	−2·6	0·0	+2·0	−0·2	+17·3	10·60
Sept. 12.5	0·0	−2·6	0·0	+2·0	−0·2	+17·1	10·40
Nov. 1.5	0·0	−2·6	0·0	+2·0	−0·3	+16·8	10·20
Dec. 21.5	0·0	−2·6	0·0	+2·0	−0·3	+16·5	10·05
1837 Feb. 9.5	0·0	−2·6	0·0	+2·0	−0·4	+16·1	9·88
March 31.5	0·0	−2·6	0·0	+2·0	−0·4	+15·7	9·72
May 20.5	0·0	−2·6	0·0	+2·0	−0·5	+15·2	9·56
July 9.5	0·0	−2·6	0·0	+2·0	−0·6	+14·6	9·42
Aug. 28.5	+0·1	−2·5	0·0	+2·0	−0·6	+14·0	9·28
Oct. 17.5	+0·1	−2·4	0·0	+2·0	−0·7	+13·3	9·12
Dec. 6.5	+0·1	−2·3	0·0	+2·0	−0·8	+12·5	8·97
1838 Jan. 25.5	+0.2	−2·1	−0·1	+1·9	−0·9	+11·6	8·84
March 16.5	+0·2	−1·9	−0·1	+1·8	−1·0	+10·6	8·71
May 5.5	+0·2	−1·7	−0·1	+1·7	−1·1	+ 9·5	8·58
June 24.5	+0·2	−1·5	−0·1	+1·6	−1·2	+ 8·3	8·46
Aug. 13.5	+0·2	−1·3	−0·2	+1·4	−1·3	+ 7·0	8·34

SATURN

1839 Apr. 20.5 — 1834 March 27.5

	dδπ	'f	λdδn	'f	''f	P	'f
1834 Feb. 5.5	—1″·1	+36″·6	— 0″·129	+ 0″·275	—15″·778	+1″·4	—61″·7
March 27.5	—1·2	+35·4	— 0·144	+ 0·131	—15·503	+1·3	—60·4
May 16.5	—1·3	+34·1	— 0·149	— 0·018	—15·372	+1·2	—59·2
July 5.5	—1·2	+32·9	— 0·145	— 0·163	—15·390	+1·0	—58·2
Aug. 24.5	—1·1	+31·8	— 0·131	— 0·294	—15·553	+0·8	—57·4
Oct. 13.5	—1·0	+30·8	— 0·105	— 0·399	—15·847	+0·6	—56·8
Dec. 2.5	—0·9	+29·9	— 0·067	— 0·466	—16·246	+0·4	—56·4
1835 Jan. 21.5	—0·8	+29·1	— 0·021	— 0·487	—16·712	+0·3	—56·1
March 12.5	—0·9	+28·2	+ 0·026	— 0·461	—17·199	+0·4	—55·7
May 1.5	—1·1	+27·1	+ 0·068	— 0·393	—17·660	+0·6	—55·1
June 20.5	—1·2	+25·9	+ 0·101	— 0·292	—18·053	+0·8	—54·3
Aug. 9.5	—1·4	+24·5	+ 0·124	— 0·168	—18·345	+1·1	—53·2
Sept. 28.5	—1·5	+23·0	+ 0·137	— 0·031	—18·513	+1·3	—51·9
Nov. 17.5	—1·6	+21·4	+ 0·144	+ 0·113	—18·544	+1·5	—50·4
1836 Jan. 6.5	—1·6	+19·8	+ 0·143	+ 0·256	—18·431	+1·7	—48·7
Feb. 25.5	—1·6	+18·2	+ 0·137	+ 0·393	—18·175	+1·8	—46·9
Apr. 15.5	—1·5	+16·7	+ 0·128	+ 0·521	—17·782	+1·8	—45·1
June 4.5	—1·4	+15·3	+ 0·118	+ 0·639	—17·261	+1·9	—43·2
July 24.5	—1·3	+14·0	+ 0·109	+ 0·748	—16·622	+1·9	—41·3
Sept. 12.5	—1·2	+12·8	+ 0·095	+ 0·843	—15·874	+1·9	—39·4
Nov. 1.5	—1·0	+11·8	+ 0·080	+ 0·923	—15·034	+1·9	—37·5
Dec. 21.5	—0·9	+10·9	+ 0·067	+ 0·990	—14·108	+1·9	—35·6
1837 Feb. 9.5	—0·8	+10·1	+ 0·052	+ 1·042	—13·118	+1·9	—33·7
March 31.5	—0·7	+ 9·4	+ 0·038	+ 1·080	—12·076	+1·9	—31·8
May 20.5	—0·6	+ 8·8	+ 0·024	+ 1·104	—10·996	+1·9	—29·9
July 9·5	—0·5	+ 8·3	+ 0·010	+ 1·114	— 9·892	+1·9	—28·0
Aug. 28.5	—0·5	+ 7·8	— 0·004	+ 1·110	— 8·778	+1·9	—26·1
Oct. 17.5	—0·4	+ 7·4	— 0·020	+ 1·090	— 7·668	+1·9	—24·2
Dec. 6.5	—0·4	+ 7·0	— 0·034	+ 1·056	— 6·578	+2·0	—22·2
1838 Jan. 25.5	—0·4	+ 6·6	— 0·049	+ 1·007	— 5·522	+2·0	—20·2
March 16.5	—0·4	+ 6·2	— 0·064	+ 0·943	— 4·515	+2·1	—18·1
May 5.5	—0·5	+ 5·7	— 0·079	+ 0·864	— 3·572	+2·1	—16·0
June 34·5	—0·5	+ 5·2	— 0·093	+ 0·771	— 2·708	+2·2	—13·8
Aug. 13.5	—0·6	+ 4·6	— 0·108	+ 0·663	— 1·937	+2·2	—11·6

	dδΩ	'f	dδi	'f	dδφ	'f	Δ
1838 Aug. 13.5	+0·2″	—1·3″	—0·2″	+1·4″	—1·3″	+7·0″	8·34
Oct. 2.5	+0·2	—1·1	—0·2	+1.2	—1·4	+5·6	8·23
Nov. 21.5	+0·3	—0·8	—0·3	+0·9	—1·5	+4·1	8·09
1839 Jan. 10.5	+0·3	—0·5	—0·3	+0·6	—1·6	+2·5	7·96
March 1.5	+0·3	—0·2	—0·4	+0·2	—1·7	+0·8	7·85
Apr. 20.5	+0·3	+0·1	—0·4	—0·2	—1·7	—0·9	7·76
June 9.5	+0·2	+0·3	—0·5	—0·7	—1·7	—2·6	7·66

	$d\delta\pi$	'f	$\lambda d\delta n$	'f	''f	P	'f
1838 Aug. 13.5	−0″6	+4″6	− 0″108	+ 0″663	− 1″937	+2″2	−11″6
Oct. 2.5	−0·8	+3·8	− 0·121	+ 0·542	− 1·274	+2·4	− 9·2
Nov. 21.5	−0·9	+2·9	− 0·137	+ 0·405	− 0·732	+2·5	− 6·7
1839 Jan. 10.5	−1·0	+1·9	− 0·151	+ 0·254	− 0·327	+2·6	− 4·1
March 1.5	−1·2	+0·7	− 0·166	+ 0·088	− 0·073	+2·7	− 1·4
Apr. 20.5	−1·4	−0·7	− 0·170	− 0·091	+ 0·015	+2·8	+ 1·4
June 9.5	−1·7	−2·4	− 0·192	− 0·283	− 0·076	+2·9	+ 4·3

Michał Kamieński

Researches on the origin of the Comet Wolf I Part V—B

Heliocentric perturbations in the motion of the Comet, due to Jupiter and Saturn during the period 1825 Sept. 30.5 — 1817 July 14.5

Mémoire présente à la séance du 18 novembre 1949.

1. In the period 1825—1817 the mutual distance between the Comet and Jupiter was very large. It was not less than 3.0 A. U., amounting to $\triangle = 10.22$ on March 10.5, 1822. Consequently, the heliocentric perturbations, caused by Jupiter in the motion of the comet, were comparatively small. They are given, together with those due to Saturn, in the adjacent Tables.

All the perturbations were computed with the help of the fundamental system P_{-13} of elements:

1825 Sept. 30.5 Gr. M. T.

$$P_{-13} \ldots \begin{array}{ll} M = 317^\circ 5' 48''.5 & \Omega = 212^\circ 38' 44''.0 \\ \varphi = 24^\circ 4' 20''.3 & \pi = 13^\circ\ 1' 34''.5 \\ n = 427''.3776 & i = 26^\circ 54' 18''.9 \end{array} \Bigg\} \ 1950.0$$

The systems of elements were changed every 50 days, in order to avoid the influence of higher order perturbations. This interval is quite sufficient for our purpose.

After the integration of the differentials in question, the author obtained the following result:

1825 Sept. 30.5 — 1817 July 14.5			
	Jupiter	Saturn	Total
δM	$-1726''.7$	$+103''.8$	$-27'\ 2''.9$
$\delta \varphi$	$+\ 253.4$	$+\ 27.2$	$+\ 4\ 40.6$
$\delta \Omega$	$+\ 254.9$	$+\ 13.8$	$+\ 4\ 28.7$
$\delta \pi$	$-\ 542.3$	$-\ 60.8$	$-10\ \ 3.1$
δi	$+\ 97.4$	$+\ 0.3$	$+\ 1\ 37.7$
δn	$+1''.7322$	$+0''.0349$	$+1''.7671$

Adding the totals given above to system P_{-13}, the author deduced the following perturbed system of elements for 1817 July 14.5:

1817 July 14.5 Gr. M. T.

$$P_{-14} \ldots \begin{array}{ll} M = 320^0 29' 52''.8 & \Omega = 212^0 43' 12''.7 \\ \varphi = 24^0\ 9'\ 0''.9 & \pi = 12^0 51' 31''.4 \\ n = 429''.1447 & i = 26^0 55' 56''.6 \end{array} \Bigg\} \; 1950.0$$

This system was taken as a basis for further researches into the motion of the comet for the period 1817—1809.

2. The small influence of the planets: Mercury—Venus—the Earth — Mars — Uranus — Neptune on the motion of the Comet Wolf I was neglected in the above investigations, because their comparatively small perturbations do not play any role for the solution of the problem of the comet's ogigin.

Cracow, Aug. 29, 1949.

JUPITER

1825 Sept. 30.5 — 1817 July 14.5

	$d\delta\Omega$	'f	$d\delta i$	'f	$d\delta\varphi$	'f	Δ
1817 May 25.5	−15″5	+ 261″2	−2″7	+ 98″1	− 3″7	+ 255″7	3·4405
July 14.5	−12·0	+ 249·2	−1·3	+ 96·8	− 4·6	+ 251·1	3·7225
Sept. 2.5	− 8·5	+ 240·7	−0·3	+ 96·5	− 4·3	+ 246·8	4·0246
Oct. 22.5	− 5·4	+ 235.3	+0·3	+ 96·8	− 3·0	+ 243·8	4·3465
Dec. 11.5	− 2·8	+ 232·5	+0·4	+ 97·2	− 1·0	+ 242·8	4·6866
1818 Jan. 30.5	− 0.8	+ 231·7	+0·2	+ 97·4	+ 1·5	+ 244·3	5·0406
March 21.5	+ 0.4	+ 232·1	−0·2	+ 97·2	+ 4·1	+ 248·4	5·4032
May 10.5	+ 0·9	+ 233·0	−0·7	+ 96·5	+ 6·5	+ 254·9	5·7661
June 29.5	+ 0.7	+ 233·7	−1·3	+ 95·2	+ 8·2	+ 263·1	6·1196
Aug. 18.5	− 0·1	+ 233·6	−1·7	+ 93·5	+ 9·3	+ 272·4	6·4558
Oct. 7.5	− 1·2	+ 232·4	−2·1	+ 91·4	+ 9·7	+ 282·1	6·7690
Nov. 26.5	− 2·7	+ 229·7	−2·3	+ 89·1	+ 9·5	+ 291·6	7·0579
1819 Jan. 15.5	− 4.3	+ 225·4	−2·2	+ 86·9	+ 9·0	+ 300·6	7·3242
March 6.5	− 5·8	+ 219·6	−2·1	+ 84·8	+ 8·3	+ 308·9	7·5708
Apr. 25.5	− 7·2	+ 212·4	−1·8	+ 83·0	+ 7·9	+ 316·8	7·8005
June 14.5	− 8·5	+ 203·9	−1·4	+ 81·6	+ 7·3	+ 324·1	8·0162
Aug. 3.5	− 9.6	+ 194·3	−1·0	+ 80·6	+ 6·7	+ 330·8	8·2206
Sept. 22.5	−10.5	+ 183·6	−0·5	+ 80·1	+ 6·1	+ 336·9	8·4152
Nov. 11.5	−11·1	+ 172·7	0·0	+ 80·1	+ 5·5	+ 342·4	8·6000
Dec. 31.5	−11·5	+ 161·2	+0·5	+ 80·6	+ 4·8	+ 347·2	8·7760
1820 Feb. 19.5	−11·6	+ 149·6	+1·0	+ 81·6	+ 4·1	+ 351·3	8·8439
Apr. 9.5	−11·5	+ 138·1	+1·4	+ 83·0	+ 3·3	+ 354·6	9·1031
May 29.5	−11.2	+ 126·9	+1·8	+ 84·8	+ 2·4	+ 357·0	9·2540
July 18.5	−10·7	+ 116·2	+2·1	+ 86·9	+ 1·5	+ 358·5	9·3955
Sept. 6.5	−10·0	+ 106·2	+2·4	+ 89·3	+ 0·4	+ 358·9	9·5266
Oct. 26.5	− 9·1	+ 97·1	+2·5	+ 91·8	− 0·7	+ 358·2	9·6476
Dec. 15.5	− 8·2	+ 88·9	+2·6	+ 94·4	− 1·9	+ 356·3	9·7584
1821 Feb. 3.5	− 7·1	+ 81·8	+2·6	+ 97·0	− 3·1	+ 353·2	9·8582
March 25.5	− 6·0	+ 75·8	+2·4	+ 99·4	− 4·4	+ 348·8	9·9468
May 14.5	− 4·9	+ 70·9	+2·2	+ 101·6	− 5·6	+ 343.2	10·0228
July 3.5	− 3·8	+ 67·1	+1·9	+ 103·5	− 6·9	+ 336·3	10·0866
Aug. 22.5	− 2·7	+ 64·4	+1·5	+ 105·0	− 8·1	+ 328·2	10·1380
Oct. 11.5	− 1.7	+ 62·7	+1·0	+ 106·0	− 9·3	+ 318·9	10·1780
Nov. 30.5	− 0.7	+ 62·0	+0·5	+ 106·5	−10·5	+ 308·4	10·2044

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1825 Sept. 30.5 — 1817 July 14.5

	$d\delta\pi$	$'f$	$\lambda d\delta n$	$'f$	$''f$	P	$'f$
1817 May 25.5	+83″·9	— 575″·7	+ 6″·003	+84″·175	— 2975″·149	—72″·6	+1191″·5
July 14.5	+63·7	— 512·0	+ 4·648	+88·823	— 2890·974	—52·0	+1139·5
Sept. 2.5	+47·0	— 465·0	+ 3·388	+92·211	— 2802·151	—35·2	+1104·3
Oct. 22.5	+33·2	— 431·8	+ 2·211	+94·422	— 2709·940	—21·9	+1082·4
Dec. 11.5	+22·4	— 409·4	+ 1·119	+95·541	— 2615·518	—12·0	+1070·4
1818 Jan. 30.5	+14·6	— 394·8	+ 0·112	+95·653	— 2519·977	— 5·4	+1065·0
March 21.5	+ 9·8	— 385·0	— 0·778	+94·875	— 2424·324	— 2·1	+1062·9
May 10.5	+ 7·7	— 377·3	— 1·506	+93·369	— 2329·449	— 1·8	+1061·1
June 29.5	+ 7·8	— 369·5	— 2·036	+91·333	— 2236·080	— 3·8	+1057·3
Aug. 18.5	+ 9·3	— 360·2	— 2·360	+88·973	— 2144·747	— 7·2	+1050·1
Oct. 7.5	+11·1	— 349·1	— 2·506	+86·467	— 2055·774	—11·0	+1039·1
Nov. 26.5	+12·8	— 336·3	— 2·525	+83·942	— 1969·307	—14·6	+1024·5
1819 Jan. 15.5	+14·3	— 322·0	— 2·470	+81·472	— 1885·365	—17·8	+1006·7
March 6.5	+15·4	— 306·6	— 2·380	+79·092	— 1803·893	—20·7	+ 986·0
Apr. 25.5	+16·4	— 290·2	— 2·283	+76·809	— 1724·801	—23·3	+ 962·7
June 14.5	+17·4	— 272·8	— 2·192	+74·617	— 1647·992	—25·8	+ 936·9
Aug. 3.5	+18·4	— 254·4	— 2·113	+72·504	— 1573·375	—28·2	+ 908·7
Sept. 22.5	+19·5	— 234·9	— 2·048	+70·456	— 1500·871	—30·7	+ 878·0
Nov. 11.5	+20·8	— 214·1	— 1·996	+68·460	— 1430·415	—33·2	+ 844·8
Dec. 31·5	+22·1	— 192·0	— 1·954	+66·506	— 1361·955	—35·8	+ 809·0
1820 Feb. 19.5	+23·5	— 168·5	— 1·922	+64·584	— 1295·449	—38·3	+ 770·7
Apr. 9.5	+24·9	— 143·6	— 1·896	+62·688	— 1230·865	—40·8	+ 729·9
May 29.5	+26·3	— 117.3	— 1·875	+60·813	— 1168·177	—43·2	+ 686·7
July 18.5	+27·6	— 89·7	— 1·856	+58·957	— 1107.364	—45·4	+ 641·3
Sept. 6.5	+28·8	— 60·9	— 1·838	+57·119	— 1048·407	—47·4	+ 593·9
Oct. 26.5	+29·8	— 31·1	— 1·821	+55·298	— 991·288	—49·1	+ 544·8
Dec. 15.5	+30·5	— 0·6	— 1·803	+53·495	— 935·990	—50·5	+ 494·3
1821 Feb. 3.5	+31·1	+ 30·5	— 1·783	+51·712	— 802·495	—51·6	+ 442·7
March 25.5	+31·3	+ 61·8	— 1·762	+49·950	— 830·783	—52·2	+ 390·5
May 14.5	+31·2	+ 93·0	— 1·739	+48·211	— 780·833	—52·5	+ 338·0
July 3.5	+30·9	+ 123·9	— 1·715	+46·496	— 732·622	—52·4	+ 285·6
Aug. 22.5	+30·1	+ 154·0	— 1·689	+44·807	— 686·126	—51·7	+ 233·9
Oct. 11.5	+29·1	+ 183·1	— 1·661	+43·106	— 641·319	—50·7	+ 183·2
Nov. 30.5	+27·7	+ 210·8	— 1·633	+41·513	— 598·173	—49.2	+ 134·0

	$d\delta\Omega$	'f	$d\delta i$	'f	$d\delta\varphi$	'f	Δ
1821 Nov. 30.5	— 0″·7	+62″·0	+0″·5	+ 106″·5	—10″·5	+ 308″·4	10·2044
1822 Jan. 19.5	+ 0·1	+62·1	—0·1	+ 106·4	—11·6	+ 296·8	10·2180
March 10.5	+ 0·8	+62·9	—0·7	+ 105·7	—12·6	+ 284·2	10·2190
Apr. 29.5	+ 1·3	+64·2	—1·3	+ 104·4	—13·5	+ 270·7	10·2075
June 18·5	+ 1·7	+65·9	—2·0	+ 102·4	—14·2	+ 256·5	10·1837
Aug. 7.5	+ 2·0	+67·9	—2·6	+ 99·8	—14·9	+ 241·6	10·1475
Sept. 26.5	+ 2·1	+70·0	—3·2	+ 96·6	—15·4	+ 226·2	10·0996
Nov. 15.5	+ 2·1	+72·1	—3·8	+ 92·8	—15·8	+ 210·4	10·0407
1823 Jan. 4.5	+ 1·9	+74·0	—4·3	+ 88·5	—16·1	+ 194·3	9·9708
Feb. 23·5	+ 1·5	+75·5	—4·8	+ 83·7	—16·2	+ 178·1	9·8908
Apr. 14.5	+ 1·1	+76·6	—5·2	+ 78·5	—16·2	+ 161·9	9·8012
June 3.5	+ 0·6	+77·2	—5·5	+ 73·0	—16·0	+ 145·9	9·7028
July 23.5	— 0·1	+77·1	—5·8	+ 67·2	—15·7	+ 130·2	9·5970
Sept 11.5	— 0·8	+76·3	—6·0	+ 61·2	—15·3	+ 114·9	9·4845
Oct. 31.5	— 1·5	+74·8	—6·1	+ 55·1	—14·7	+ 100·2	9·3655
Dec 20·5	— 2·3	+72·5	—6·1	+ 49·0	—14·0	+ 86·2	9·2412
1824 Feb. 8.5	— 3·0	+69·5	—6·0	+ 43·0	—13·2	+ 73·0	9·1142
March 29.5	— 3·8	+65·7	—5·8	+ 37·2	—12·3	+ 60·7	8·9852
May 18.5	— 4·5	+61·2	—5·6	+ 31·6	—11·3	+ 49·4	8·8558
July 7.5	— 5·1	+56·1	—5·3	+ 26·3	—10·2	+ 39·2	8·7276
Aug. 26.5	— 5·7	+50·4	—4·9	+ 21·4	— 9·1	+ 30·1	8·6018
Oct. 15.5	— 6·2	+44·2	—4·5	+ 16·9	— 7·9	+ 22·2	8·4806
Dec. 4.5	— 6·5	+37·7	—4·0	+ 12·9	— 6·7	+ 15·5	8·3660
1825 Jan 23.5	— 6·8	+30·9	—3·5	+ 9·4	— 5·4	+ 10·1	8·2592
March 14.5	— 7·0	+23.9	—3·0	+ 6·4	— 4·2	+ 5·9	8·1632
May 3.5	— 7·0	+16·9	—2·5	+ 3·9	— 3·0	+ 2·9	8·0787
June 22.5	— 6·9	+10·0	—2·0	+ 1·9	— 1·9	+ 1·0	8·0074
Aug. 11.5	— 6·8	+ 3·2	—1·4	+ 0·5	— 0·9	+ 0·1	7·9506
Sept. 30.5	— 6·4	— 3·2	—0·9	— 0·4	— 0·1	0·0	7·9078
Nov. 19·5	— 6·0		—0·4		+ 0·6		7·8793

	$d\delta\pi$	$'f$	$\lambda d\delta n$	$'f$	$''f$	P	$'f$
1821 Nov. 30.5	+27″7	+ 210″8	— 1″633	+41″513	—598″173	—49″2	+ 134″0
1822 Jan. 19.5	+26·1	+ 236·9	— 1·604	+39·909	—556·660	—47·3	+ 86·7
March 10.5	+24·1	+ 261·0	— 1·575	+38·334	—516·751	—45·0	+ 41·7
Apr. 29·5	+21·8	+ 282·8	— 1·547	+36·787	—478·417	—42·3	— 0·6
June 18·5	+19·3	+ 302·1	— 1·521	+35·266	—441·630	—39·2	— 39·8
Aug. 7.5	+16·6	+ 318·7	— 1·496	+33·770	—406·364	—35·9	— 75·7
Sept. 26.5	+13·7	+ 322·4	— 1·473	+32·297	—372·594	—32·3	— 108·0
Nov. 15.5	+10·7	+ 343·1	— 1·453	+30·844	—340·297	—28·5	— 136·5
1823 Jan. 4.5	+ 7·5	+ 350·6	— 1·437	+29·407	—309·453	—24·5	— 161·0
Feb. 23.5	+ 4·3	+ 354·9	— 1·425	+27·982	—280·046	—20·5	— 181·5
Apr. 14.5	+ 1·0	+ 355·9	— 1·416	+26·566	—252·064	—16·3	— 197·8
June 3.5	— 2·2	+ 353·7	— 1·411	+25·155	—225·498	—12·1	— 209·9
July 23·5	— 5·5	+ 348·2	— 1·411	+23·744	—200·343	— 8.0	— 217·9
Sept. 11.5	— 8·6	+ 339·6	— 1·416	+22·328	—176·599	— 4.0	— 221·9
Oct. 31.5	—11·6	+ 328·0	— 1·425	+20·903	—154·271	— 0·1	— 222·0
Dec. 20.5	—14·5	+ 313·5	— 1·438	+19·465	—133·368	+ 3·6	— 218·4
1824 Feb. 8.5	—17·2	+ 296·3	— 1·455	+18·010	—113·903	+ 7·1	— 211·3
March 29.5	—19·7	+ 276·6	— 1·475	+16·535	— 95·893	+10·3	— 201·0
May 18.5	—21·9	+ 254·7	— 1·498	+15·037	— 79·358	+13·2	— 187·8
July 7.5	—23·8	+ 230·9	— 1·523	+13·514	— 64·321	+15·7	— 172·1
Aug. 26.5	—25·4	+ 205·5	— 1·548	+11·966	— 50·807	+17·8	— 154·3
Oct. 15.5	—26·7	+ 178·8	— 1·573	+10·393	— 38·841	+19·5	— 134·8
Dec. 4.5	—27·6	+ 151·2	— 1·595	+ 8·798	— 28·448	+20·7	— 114·1
1825 Jan. 23.5	—28·2	+ 123·0	— 1·613	+ 7·185	— 19·650	+21·5	— 92·6
March 14.5	—28·4	+ 94·6	— 1·624	+ 5·561	— 12·465	+21·7	— 70·9
May 3.5	—28·1	+ 66·5	— 1·623	+ 3·938	— 6·904	+21·5	— 49·4
June 22.5	—27·5	+ 39·0	— 1·607	+ 2·331	— 2·966	+20·8	— 28·6
Aug. 11.5	—26·4	+ 12·6	— 1·571	+ 0·760	— 0·635	+19·5	— 9·1
Sept. 30.5	—25·0	— 12·4	— 1·506	— 0·746	+ 0·125	+17·8	+ 8·7
Nov. 19.5	—23·1		— 1·402		— 0·621	+15·7	

SATURN

1825 Sept. 30.5 — 1817 July 14.5

	$d\delta\Omega$	$'f$	$d\delta i$	$'f$	$d\delta\varphi$	$'f$	Δ
1817 May 25·5	— 0″5	+14″1	— 0″1	+ 0″3	+ 0″4	+27″0	8.44
July 14.5	— 0·6	+13·5	— 0·1	+ 0·2	+ 0·4	+27·4	8·16
Sept. 2.5	— 0·7	+12·8	0·0	+ 0·2	+ 0·5	+27·9	7·91
Oct. 22.5	— 0·8	+12·0	0·0	+ 0·2	+ 0·4	+28·3	7·70
Dec. 11.5	— 0·8	+11·2	+ 0·1	+ 0·3	+ 0·2	+28·5	7·55
1818 Jan. 30.5	— 0·7	+10·5	+ 0·2	+ 0·5	0·0	+28·5	7·46
March 21.5	— 0·5	+10·0	+ 0·2	+ 0·7	— 0·4	+28·1	7·48
May 10.5	— 0·3	+ 9·7	+ 0·2	+ 0·9	— 0·7	+27·4	7·62
June 29.5	— 0·1	+ 9·6	+ 0·2	+ 1·1	— 0·9	+26·5	7·87
Aug. 18.5	0·0	+ 9·6	+ 0·2	+ 1·3	— 0·8	+25·7	8·22
Oct. 7.5	+ 0·1	+ 9·7	+ 0·1	+ 1·4	— 0·7	+25·0	8·61
Nov. 26.5	+ 0·1	+ 9·8	0·0	+ 1·4	— 0·5	+24·5	9·08
1819 Jan. 15.5	0·0	+ 9·8	0·0	+ 1·4	— 0·3	+24·2	9·54
March 6.5	0·0	+ 9·8	0·0	+ 1·4	— 0·2	+24·0	10·00
Apr. 25.5	— 0·1	+ 9·7	0·0	+ 1·4	— 0·1	+23·9	10·45
June 14.5	— 0·2	+ 9·5	0·0	+ 1·4	— 0·1	+23·8	10·85
Aug. 3.5	— 0·3	+ 9·2	0·0	+ 1·4	— 0·1	+23·7	11·20
Sept. 22.5	— 0·3	+ 8·9	0·0	+ 1·4	— 0·1	+23·6	11·55
Nov. 11.5	— 0·3	+ 8·6	0·0	+ 1·4	— 0·1	+23·5	11·85
Dec. 31.5	— 0·4	+ 8·2	0·0	+ 1·4	— 0·2	+23·3	12·13
1820 Feb. 19.5	— 0·4	+ 7·8	0·0	+ 1·4	— 0·2	+23·1	12·40
Apr. 9·5	— 0·4	+ 7·4	+ 0·1	+ 1·5	— 0·3	+22·8	12·68
May 29.5	— 0·4	+ 7·0	+ 0·1	+ 1·6	— 0·3	+22·5	12·95
July 18.5	— 0·4	+ 6·6	+ 0·1	+ 1·7	— 0·3	+22·2	13·20
Sept. 6.5	— 0·4	+ 6·2	+ 0·1	+ 1·8	— 0·4	+21·8	13·40
Oct. 26.5	— 0·4	+ 5·8	+ 0·1	+ 1·9	— 0·4	+21·4	13·55
Dec. 15.5	— 0·3	+ 5·5	+ 0·1	+ 2·0	— 0·5	+20·9	13·70
1821 Feb. 3.5	— 0·3	+ 5·2	+ 0·1	+ 2·1	— 0·5	+20·4	13·85
March 25.5	— 0·3	+ 4·9	+ 0·1	+ 2·2	— 0·5	+19·9	14·00
May 14.5	— 0·3	+ 4·6	+ 0·1	+ 2.3	— 0·6	+19·3	14·15
July 3.5	— 0·2	+ 4·4	+ 0·1	+ 2·4	— 0·6	+18·7	14·30
Aug. 22.5	— 0·2	+ 4·2	+ 0·1	+ 2·5	— 0·6	+18·1	14·40
Oct. 11.5	— 0·2	+ 4·0	+ 0·1	+ 2·6	— 0·6	+17·5	14·50
Nov. 30.5	— 0·1	+ 3·9	+ 0·1	+ 2·7	— 0·7	+16·8	14·55

SATURN

1825 Sept. 30.5 — 1817 July 14.5

	$d\delta\pi$	$'f$	$\lambda d\delta n$	$'f$	$''f$	P	$'f$
1817 May 25·5	— 2″0	—59″7	— 0″184	+ 1″845	—14″232	+ 2″2	+ 115″1
July 14·5	— 2·2	—61·9	— 0·201	+ 1·644	—12·387	+ 2·2	+ 117·3
Sept. 2.5	— 2·3	—64·2	— 0·205	+ 1·439	—10·743	+ 2·0	+ 119·3
Oct. 22.5	— 2·2	—66·4	— 0·181	+ 1·258	— 9·304	+ 1·6	+ 120·9
Dec. 11.5	— 1·9	—68·3	— 0·128	+ 1·130	— 8·046	+ 1·1	+ 122·0
1819 Jan. 30.5	— 1·5	—69·8	— 0·041	+ 1·089	— 6·916	+ 0·6	+ 122·6
March 21.5	— 1·1	—70·9	+ 0·064	+ 1·153	— 5·827	+ 0·3	+ 122·9
May 10.5	— 0·9	—71·8	+ 0·157	+ 1·310	— 4·674	+ 0·2	+ 123·1
June 29.5	— 0·7	—72·5	+ 0·209	+ 1·519	— 3·364	+ 0·4	+ 123·5
Aug. 18.5	— 0·6	—73·1	+ 0·210	+ 1·729	— 1·845	+ 0·6	+ 124·1
Oct. 7.5	— 0·4	—73·5	+ 0·172	+ 1·901	— 0·116	+ 0·6	+ 124·7
Nov. 26.5	— 0·2	—73·7	+ 0·116	+ 2·017	+ 1·785	+ 0·5	+ 125·2
1819 Jan. 15.5	+ 0·2	—73·5	+ 0·053	+ 2·070	+ 3·802	+ 0·2	+ 125·4
March 6.5	+ 0·7	—72·8	— 0·004	+ 2·066	+ 5·872	— 0·2	+ 125·2
Apr. 25.5	+ 1·1	—71·7	— 0·049	+ 2·017	+ 7·938	— 0·6	+ 124·6
June 14.5	+ 1·4	—70·3	— 0·080	+ 1·937	+ 9·955	— 1·0	+ 123·6
Aug. 3.5	+ 1·6	—68·7	— 0·103	+ 1·834	+11·892	— 1·3	+ 122·3
Sept. 22.5	+ 1·9	—66·8	— 0·122	+ 1·712	+13·726	— 1·6	+ 120·7
Nov. 11.5	+ 2·0	—64·8	— 0·134	+ 1·578	+15·438	— 1·9	+ 118·8
Dec. 31.5	+ 2·2	—62·6	— 0·144	+ 1·434	+17·016	— 2·1	+ 116·7
1820 Feb. 19.5	+ 2·2	—60·4	— 0·150	+ 1·284	+18·450	— 2·4	+ 114·3
Apr. 9.5	+ 2·3	—58·1	— 0·151	+ 1·133	+19·734	— 2·5	+ 111·8
May 29.5	+ 2·3	—55·8	— 0·152	+ 0·981	+20·867	— 2·7	+ 109·1
July 18.5	+ 2·3	—53·5	— 0·151	+ 0·830	+21·848	— 2·8	+ 106·3
Sept. 6.5	+ 2·3	—51·2	— 0·149	+ 0·681	+22·678	— 2·9	+ 103·4
Oct. 26.5	+ 2·3	—48·9	— 0·147	+ 0·534	+23·359	— 3·0	+ 100·4
Dec. 15.5	+ 2·2	—46·7	— 0·143	+ 0·391	+23·893	— 3·1	+ 97·3
1821 Feb· 3.5	+ 2·2	—44·5	— 0·139	+ 0·252	+24·284	— 3·1	+ 94·2
March 25.5	+ 2·1	—42·4	— 0·134	+ 0·118	+24·536	— 3·2	+ 91·0
May 14.5	+ 2·1	—40·3	— 0·128	— 0·010	+24·654	— 3·2	+ 87·8
July 3.5	+ 2·0	—38·3	— 0·123	— 0·133	+24·644	— 3·2	+ 84·6
Aug. 22.5	+ 1·9	—36·4	— 0·117	— 0·250	+24·511	— 3·2	+ 81·4
Oct. 11.5	+ 1·9	—34·5	— 0·111	— 0·361	+24·261	— 3·2	+ 78·2
Nov. 30.5	+ 1·8	—32·7	— 0·105	— 0·466	+23·900	— 3·2	+ 75·0

	$d\delta\Omega$	$'f$	$d\delta i$	$'f$	$d\delta\varphi$	$'f$	Δ
1821 Nov. 30.5	− 0″·1	+ 3″·9	+ 0″·1	+ 2″·7	− 0″·7	+16″·8	14·55
1822 Jan. 19.5	− 0·1	+ 3·8	+ 0·1	+ 2·8	− 0·7	+16·1	14·60
March 10.5	− 0·1	+ 3·7	+ 0·1	+ 2·9	− 0·7	+15.4	14·70
Apr. 29.5	− 0·1	+ 3·6	+ 0·1	+ 3·0	− 0·7	+14·7	14·70
June 18.5	0·0	+ 3·6	0·0	+ 3·0	− 0·7	+14·0	14·75
Aug. 7.5	0·0	+ 3·6	0·0	+ 3·0	− 0·7	+13·3	14·80
Sept. 26.5	0·0	+ 3·6	0·0	+ 3·0	− 0·7	+12·6	14·80
Nov. 15.5	0·0	+ 3·6	0·0	+ 3·0	− 0·7	+11·9	14·80
1823 Jan. 4.5	0·0	+ 3·6	0·0	+ 3·0	− 0·7	+11·2	14·75
Feb. 23.5	0·0	+ 3·6	− 0·1	+ 2·9	− 0·7	+10·5	14·70
Apr. 14.5	0·0	+ 3·6	− 0·1	+ 2·8	− 0·7	+ 9·8	14·70
June 3.5	0·0	+ 3·6	− 0·1	+ 2·7	− 0·7	+ 9·1	14·70
July 23.5	0·0	+ 3·6	− 0·1	+ 2·6	− 0·7	+ 8·4	14·60
Sept. 11.5	0·0	+ 3·6	− 0·1	+ 2·5	− 0·7	+ 7·7	14·55
Oct. 31.5	0·0	+ 3·6	− 0·1	+ 2·4	− 0·6	+ 7·1	14·45
Dec. 20.5	− 0·1	+ 3·5	− 0·2	+ 2·2	− 0·6	+ 6·5	14·40
1824 Feb. 8.5	− 0·1	+ 3·4	− 0·2	+ 2·0	− 0·6	+ 5·9	14·30
March 29.5	− 0·1	+ 3·3	− 0·2	+ 1·8	− 0·6	+ 5·3	14·15
May 18.5	− 0·2	+ 3·1	− 0·2	+ 1·6	− 0·5	+ 4·8	14·00
July 7.5	− 0·2	+ 2·9	− 0·2	+ 1·4	− 0·5	+ 4·3	13·90
Aug. 26.5	− 0·2	+ 2·7	− 0·2	+ 1·2	− 0·5	+ 3·8	13·70
Oct. 15.5	− 0·3	+ 2·4	− 0·2	+ 1·0	− 0·5	+ 3·3	13·60
Dec. 4.5	− 0·3	+ 2·1	− 0·2	+ 0·8	− 0·5	+ 2·8	13·40
1825 Jan. 23.5	− 0·3	+ 1·8	− 0·2	+ 0·6	− 0·5	+ 2·3	13·20
March 14.5	− 0·4	+ 1·4	− 0·2	+ 0·4	− 0·5	+ 1·8	13·00
May 3.5	− 0·4	+ 1·0	− 0·1	+ 0·3	− 0·5	+ 1·3	12·80
June 22.5	− 0·4	+ 0·6	− 0·1	+ 0·2	− 0·5	+ 0·8	12·50
Aug. 11.5	− 0·4	+ 0·2	− 0·1	+ 0·1	− 0·5	+ 0·3	12·20
Sept. 30.5	− 0·4	− 0·2	− 0·1	0·0	− 0·6	− 0·3	11·90
Nov. 19.5	− 0·4		0·0		− 0·6		11·60

	dδπ	'f	λdδn	'f	''f	P	'f
1821 Nov. 30.5	+ 1″8	—32″7	— 0″105	— 0″466	+23″900	— 3″2	+75″0
1822 Jan. 19.5	+ 1·7	—31·0	— 0·099	— 0·565	+23·434	— 3·2	+71·8
March 10·5	+ 1·6	—29·4	— 0·092	— 0·657	+22·869	— 3·2	+68·6
Apr· 29·5	+ 1·6	—27·8	— 0·085	— 0·742	+22·212	— 3·2	+65·4
June 18·5	+ 1·5	—26·3	— 0·078	— 0·820	+21·470	— 3·2	+62·2
Aug. 7.5	+ 1·4	—24·9	— 0·070	— 0·890	+20·650	— 3·1	+59·1
Sept. 26·5	+ 1·4	—23·5	— 0·063	— 0·953	+19·760	— 3·1	+56·0
Nov. 15.5	+ 1·3	—22·2	— 0·055	— 1·008	+18·807	— 3·1	+52·9
1823 Jan. 4.5	+ 1·2	—21·0	— 0·048	— 1·056	+17·799	— 3·0	+49·9
Feb. 23.5	+ 1·2	—19·8	— 0·040	— 1·096	+16·743	— 3·0	+46·9
Apr. 14.5	+ 1·1	—18·7	— 0·032	— 1·128	+15·647	— 2·9	+44·0
June 3.5	+ 1·1	—17·6	— 0·023	— 1·151	+14·519	— 2·9	+41·1
July 23.5	+ 1·0	—16·6	— 0·015	— 1·166	+13·368	— 2·8	+38·3
Sept. 11.5	+ 1·0	—15·6	— 0·006	— 1·172	+12·202	— 2·8	+35·5
Oct. 31·5	+ 1·0	—14·6	+ 0·004	— 1·168	+11·030	— 2·8	+32·7
Dec. 20.5	+ 0·9	—13·7	+ 0·013	— 1·155	+ 9·862	— 2·7	+30·0
1824 Feb. 8.5	+ 0·9	—12·8	+ 0·023	— 1·132	+ 8·707	— 2·6	+27·4
March 29.5	+ 0·9	—11·9	+ 0·034	— 1·098	+ 7·575	— 2·6	+24·8
May 18.5	+ 0·9	—11·0	+ 0·044	— 1·054	+ 6·477	— 2·6	+22·2
July 7.5	+ 0·9	—10·1	+ 0·056	— 0·998	+ 5·423	— 2·5	+19·7
Aug. 26.5	+ 1·0	— 9·1	+ 0·068	— 0·930	+ 4·425	— 2·5	+17·2
Oct. 15.5	+ 1·0	— 8·1	+ 0·080	— 0·850	+ 3·495	— 2·5	+14·7
Dec. 4.5	+ 1·1	— 7·0	+ 0·093	— 0·757	+ 2·645	— 2·4	+12·3
1825 Jan. 23.5	+ 1·1	— 5·9	+ 0·107	— 0·650	+ 1·888	— 2·4	+ 9·9
March 14.5	+ 1·2	— 4·7	+ 0·121	— 0·529	+ 1·238	— 2·2	+ 7·6
May 3.5	+ 1·3	— 3·4	+ 0·136	— 0·393	+ 0·709	— 2·3	+ 5·3
June 22.5	+ 1·3	— 2·1	+ 0·148	— 0·245	+ 0·316	— 2·2	+ 3·1
Aug. 11.5	+ 1·4	— 0·7	+ 0·160	— 0·085	+ 0·071	— 2·1	+ 1·0
Sept. 30.5	+ 1·4	+ 0·7	+ 0·172	+ 0·087	— 0·014	— 2·0	— 1·0
Nov. 19.5	+ 1·5		+ 0·184		+ 0·073	— 1·8	

Wacław Sierpiński

Sur les bases dénombrables de la famille de tous les ensembles linéaires dénombrables

Présenté dans la séance du 18 novembre 1949.

On dit qu'une famille Φ d'ensembles admet une *base dénombrable,* s'il existe une suite infinie $E_1, E_2, \ldots$ d'ensembles (appartenant à Φ ou non), telle que tout ensemble de la famille Φ est limite d'une suite infinie d'ensembles extraits de la suite $E_1, E_2, \ldots$.

En admettant l'hypothèse du continu M. Mazur a démontré qu'il existe une base dénombrable pour la famille Φ de tous les ensembles linéaires dénombrables [1]).

Le but de cette Note est de démontrer (sans faire appel à l'hypothèse du continu) *qu'il n'existe pour cette famille Φ aucune base dénombrable formée d'ensembles linéaires mesurables L.* A ce but il suffira évidemment de démontrer ce

Théorème. *$E_1, E_2, \ldots$ étant une suite infinie d'ensembles linéaires mesurables L, il existe un ensemble linéaire dénombrable D qui n'est pas limite d'aucune suite extraite de la suite $E_1, E_2, \ldots$.*

Démonstration. Soit $E_1, E_2, \ldots$ une suite infinie donnée d'ensembles linéaires mesurables L. Soit n un nombre naturel donné et désignons par I l'intervalle (fermé) $(0, 1)$. L'ensemble IE_n étant mesurable, il existe, comme on sait, deux ensembles fermés P_n et Q_n, tels que $P_n \subset IE_n$, $Q_n \subset I - E_n$, $\operatorname{mes} P_n > \operatorname{mes} IE_n - 2^{-n-2}$, $\operatorname{mes} Q_n > \operatorname{mes}(I - E_n) - 2^{-n-2}$, donc (vu que $P_n Q_n = 0$ et que $\operatorname{mes} IE_n + \operatorname{mes}(I - E_n) = \operatorname{mes} I = 1$): $\operatorname{mes}(P_n + Q_n) > 1 - 2^{-n-1}$.

Posons

$$F = \prod_{n=1}^{\infty} (P_n + Q_n) \tag{1}$$

— ce sera un ensemble fermé de mesure $> 1 - \sum_{n=1}^{\infty} 2^{-n-1} = \frac{1}{2}$, et il existe un ensemble parfait $P \subset F$.

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[1]) S. Mazur, *C. R. Soc. Sc. Varsovie* 31 (séance du 18 octobre 1938); aussi W. Sierpiński, *Fund. Math.* 31, p. 259.

Soit D un ensemble dénombrable contenu dans P et dense dans P. Je dis que D n'est pas limite d'aucune suite infinie extraite de la suite $E_1, E_2, \ldots$.

Admettons, en effet, qu'il existe une suite infinie de nombres naturels, $n_1, n_2, \ldots$, telle que

$$(2) \qquad D = \lim_{k=\infty} E_{n_k};$$

il en résulte, comme on sait, que

$$(3) \qquad P - D = \lim_{k=\infty} (P - E_{n_k}).$$

Or, d'après $P \subset F$ et (1) on a $P \subset P_n + Q_n$ pour $n = 1, 2, \ldots$ et, comme $P_n \subset IE_n$, on en trouve $P - E_n \subset PQ_n$ et, vu que, d'autre part, $PQ_n \subset P(I - E_n) = P - E_n$, on a

$$(4) \qquad P - E_n = PQ_n \quad \text{pour} \quad n = 1, 2, \ldots.$$

D'après (4), la formule (3) donne

$$P - D = \lim_{k=\infty} PQ_{n_k}.$$

Or, les ensembles PQ_{n_k} étant fermés, l'ensemble $P - D$ est un F_σ, ce qui est impossible, P étant parfait et D étant dénombrable, dense dans P.

Notre théorème est ainsi démontré.

Il est à remarquer qu'on peut pareillement démontrer *qu'il n'existe pour notre famille Φ aucune base dénombrable formée d'ensembles linéaires jouissants de la propriété de Baire (au sens large).*

En effet, on démontre sans peine que, $E_1, E_2, \ldots$ étant une suite infinie d'ensembles linéaires jouissants de la propriété de Baire au sens large (c. à d. relativement à la droite), il existe un ensemble linéaire K de 1^re^ catégorie sur le complémentaire duquel (par rapport à la droite) chacun des ensembles $E_1, E_2, \ldots$ est ouvert (ou vide). L'ensemble K étant de 1^re^ catégorie, il existe un ensemble parfait P, tel que $PK = 0$. Les ensembles $P - E_n$ $(n = 1, 2, \ldots)$ sont donc fermés et la formule (3) (qui résulte de (2)) prouve que $P - D$ est un F_σ, ce qui est impossible pour D dénombrable dense dans P.

Wacław Sierpiński

O bazach przeliczalnych rodziny wszystkich zbiorów liniowych przeliczalnych

Komunikat wygłoszony na posiedzeniu w dniu 18 listopada 1949 r.

Streszczenie

S. Mazur dowiódł przy pomocy hipotezy continuum, że istnieje baza przeliczalna dla rodziny wszystkich zbiorów liniowych przeliczalnych. Autor zaś dowodzi (bez użycia hipotezy continuum), że baza taka nie może się składać ze zbiorów liniowych mierzalnych (L).

Michał Kamieński

Researches on the origin of the Comet Wolf I
Part V—C

Influence of Jupiter and Saturn on the motion of the Comet during the period

1817 July 14.5 — 1809 Apr. 27.5

Mémoire présenté à la séance du 18 novembre 1949.

1. The system of elements P_{-14} deduced in my previous works was taken as a basis for the researches given below:

1817 July 14.5 Greenwich Mean Time

$$P_{-14} \ldots \left.\begin{array}{ll} M = 320^\circ 29' 52''.8 & \Omega = 212^\circ 43' 12''.7 \\ \varphi = 24^\circ 9' 0''.9 & \pi = 12^\circ 51' 31''.4 \\ n = 429''.1447 & i = 26^\circ 55' 56''.6 \end{array}\right\} 1950.0$$

With the help of this system, the perturbations in the comet's motion, due to Jupiter and Saturn during the period 1817 — 1809 were carried on. The actual computations were performed by applying the method of variation of arbitrary constants. The systems of elements were changed every 50 or 25 days according to the distances of the comet from Jupiter. These distances varied from 1.91 to 5.14; consequently they were comparatively great.

After the integration of the differentials of perturbations given in the Tables below, the author obtained the following results:

1817 July 14.5 — 1809 April 27.5			
	Jupiter	Saturn	Total
δM	+ 14713″.5	— 23″.2	+ 244′50″.3
$\delta\varphi$	— 1747 .6	— 34 .0	— 29 41 .6
$\delta\Omega$	+ 544 .2	+ 14 .1	+ 9 18 .3
$\delta\pi$	— 873 .9	+ 38 .7	— 13 55 .2
δi	— 448 .3	— 6 .3	— 7 34 .6
δn	— 6″.7674	— 0″.0234	— 6″.7908

Adding these results to the system P_{-14} of elements, the author deduced the following perturbed system:

1809 April 27.5 Greenwich Mean Time

$$P_{-15} \ldots \left.\begin{array}{ll} M = 326^{\circ}57'29''.0 & \Omega = 212^{\circ}52'31''.0 \\ \varphi = 23^{\circ}39'19''.3 & \pi = 12^{\circ}37'36''.2 \\ n = 422''.3539 & i = 26^{\circ}48'22''.0 \end{array}\right\} 1950.0$$

2. With the help of the system P_{-14} for the moment 1817 July 14.5 the author, taking still into consideration further perturbations due to Jupiter and Saturn, computed the following exact distances of the comet from Jupiter:

1815 Oct.	3.5	$\triangle = 1.9714$	1816 Jan.	11.5	$\triangle = 1.9123$
1815 Oct.	28.5	$\triangle = 1.9455$	1816 Feb.	5.5	$\triangle = 1.9180$
1815 Nov.	22.5	$\triangle = 1.9265$	1816 March	1.5	$\triangle = 1.9323$
1815 Dec.	17.5	$\triangle = 1.9152$	1816 March	26.5	$\triangle = 1.9556$

As can easily be seen, in the period 1815—1816 the comet wandered far enough from Jupiter, and there is no reason for its engendering just then the comet Barnard 1892 V.

Thus, our exact investigations into the motion of the Comet Wolf I, comprising the period 1942 — 1809, confirmed entirely our views given previously in Part II of our „Researches on the origin of the Comet Wolf I".

3. In Part I of the author's „Researches", which appeared in the Bulletin de l'Académie Polonaise des Sciences et des Lettres in spring 1939 and was reprinted in the „Reprint of the Astronomical Observatory of the Warsaw University Nr 45", the scheme of further investigations of a problem on the origin of the Comet Wolf I was outlined. But, unfortunately, all the store of these reprints perished in fire in August 1944 when the Warsaw Observatory was burned down by the Germans...

According to this scheme, the perturbations in the comet's motion due to Jupiter and Saturn for the period 1884 — 1800 ought to be computed. Towards the end of July 1944 the computations were carried on for the period 1884—1804. A part of them was saved during our mournful exodus from the burning town. The rest (1809—1804) remained at the Observatory and perished. It was computed anew in Cracow in 1945, together with the data for the next period (1804—1800).

4. The author's researches concerning the motion of the Comet Wolf I which he began as early as in 1908 and performed with many interruptions up to the present time, may be partially summarized as follows:

A. The exact perturbations for the period 1884 — 1942 due to Venus, the Earth, Mars, Jupiter, Saturn and Uranus were computed.

The perturbations due to Mercury and to Neptune for the period 1884—1919 were computed as well, but perished in fire before they were published. They were computed anew and published in the Bulletin de l'Académie Polonaise des Sciences et des Lettres in 1948—1949.

All the observations of the comet during the period 1884—1942 were elaborated very accurately. They agree very well with the theory.

B. The perturbations due to Jupiter and Saturn for the period 1942 — 1950 were computed exactly and a research-ephemeris of the comet for its next reapparition in 1950 was prepared.

C. The exact perturbations for the periods 1884—1809—1800 due to Jupiter and Saturn were performed. It was positively proved that the comet did not approach closely to Jupiter in 1815—1816, as can be easily seen from the numbers given above in § 2.

The minimum distances of the comet from Jupiter in the period 1800—1950 were:

1816 Jan.	7.5	$\triangle = 1.9122$
1839 Nov.	13.1	$\triangle = 0.5382$
1875 June	8.7	$\triangle = 0.1180$
1922 Sept.	27.1	$\triangle = 0.1247$

Cracow, Sept. 1, 1949.

JUPITER

1817 July 14.5 — 1809 Apr. 27.5

	$d\delta\Omega$	'f	$d\delta i$	'f	$d\delta\varphi$	'f	Δ
1809 March 8.5	+ 0·7″	+ 544·3″	+ 0·1″	— 448·3″	+ 7·7″	—1752·2″	5·1402
Apr. 27.5	— 0·4	+ 543·9	0·0	— 448·3	+ 9·6	—1742·6	4·8275
June 16.5	— 1·6	+ 542·3	+ 0·1	— 448·2	+13·0	—1729·6	4·4834
Aug. 5.5	— 2·5	+ 539·8	+ 0·4	— 447·8	+18·6	—1711·0	4·1120
Sept. 24.5	— 3·0	+ 536·8	+ 0·8	— 447·0	+26·4	—1684·6	3·7229
Nov. 13.5	— 2·8	+ 534·0	+ 1·3	— 445·7	+35·8	—1648·8	3·3335
1810 Jan. 2.5	— 1·7	+ 532·3	+ 1·4	— 444·3	+43·8	—1605·0	2·9720
Feb. 21.5	— 0·3	+ 532·0	+ 0·5	— 443·8	+44·5	—1560·5	2·6729
Apr. 12.5	0·0	+ 532·0	— 1·7	— 445·5	+32·3	—1528.2	2·4743
June 1·5	— 2·5	+ 529·5	— 4·5	— 450·0	+10·0	—1518·2	2·3998
July 21.5	— 7·3	+ 522·2	— 6·5	— 456·5	—10·1	—1528·3	2·4459
Sept. 9.5	—12·0	+ 510·2	— 6·8	— 463·3	—20·1	—1548·4	2·5830
Oct. 29.5	—15·1	+ 495·1	— 5·9	— 469·2	—21·8	—1570·2	2·7743
Dec. 18.5	—16·4	+ 478·7	— 4·5	— 473·7	—19·5	—1589·7	2·9907
1811 Feb. 6.5	—16·4	+ 462·3	— 3·2	— 476·9	—16·2	—1605·9	3·2112
March 28.5	—15·8	+ 446·5	— 2·0	— 478·9	—13·3	—1619·2	3·4231
May 17.5	—14·8	+ 431·7	— 1·0	— 479·9	—11·0	—1630·2	3·6185
July 6.5	—13·7	+ 418·0	— 0·3	— 480·2	— 9·3	—1639·5	3·7938
Aug. 25.5	—12·5	+ 405·5	+ 0·3	— 479·9	— 7·9	—1647·4	3·9477
Oct. 14.5	—11·6	+ 393·9	+ 0·8	— 479·1	— 6·9	—1654·3	4·0791
Dec. 3.5	—10·7	+ 383·2	+ 1·1	— 478·0	— 6·0	—1660·3	4·1874
1812 Jan. 22.5	—10·0	+ 373·2	+ 1·4	— 476·6	— 5·3	—1665·6	4·2737
March 12.5	— 9·6	+ 363·6	+ 1·7	— 474·9	— 4·5	—1670·1	4·3384
May 1.5	— 9·3	+ 354·3	+ 2·0	— 472·9	— 3·7	—1673·8	4·3832
June 20.5	— 9·2	+ 345·1	+ 2·3	— 470·6	— 2·8	—1676·6	4·4087
Aug. 9.5	— 9·4	+ 335·7	+ 2·7	— 467·9	— 1·7	—1678·3	4·4154
Sept. 28.5	— 9·6	+ 326·1	+ 3·2	— 464·7	— 0·5	—1678·8	4·4041
Nov. 17.5	—10·1	+ 316·0	+ 3·8	— 460·9	+ 0·9	—1677·9	4·3767
1813 Jan. 6.5	—10·6	+ 305·4	+ 4·5	— 456·4	+ 2·5	—1675·4	4·3337
Feb. 25.5	—11·3	+ 294·1	+ 5·3	— 451·1	+ 4·3	—1671·1	4·2759
Apr. 16.5	—12·1	+ 282·0	+ 6·3	— 444·8	+ 6·3	—1664·8	4·2044
June 5.5	—12·9	+ 269·1	+ 7·4	— 437·4	+ 8·7	—1956·1	4·1202
July 25.5	—13·7	+ 255·4	+ 8·8	— 428·6	+11·3	—1644·8	4·0240
Sept. 13.5	—14·5	+ 240·9	+10·3	— 418·3	+14·3	—1630·5	3·9170

JUPITER

1817 July 14.5 — 1809 Apr. 27.5

	$d\delta\pi$	$'f$	$\lambda d\delta n$	$'f$	$''f$	P	$'f$
1809 March 8.5	+ 9″1	— 875″6	— 0″670	—337″685	+15643″473	+ 5″5	— 597″4
Apr. 27.5	+ 2·2	— 873·4	— 1·540	—339·225	+15305·788	+ 11·5	— 585·9
June 16.5	— 6·3	— 879·7	— 2·752	—341·977	+14966·569	+ 18·2	— 567·7
Aug. 5·5	— 15·8	— 895·5	— 4·408	—346·385	+14624·586	+ 24·8	— 542·9
Sept. 24.5	— 25·8	— 921·3	— 6·501	—352·886	+14278·201	+ 29·4	— 513·5
Nov. 13.5	— 35·1	— 956·4	— 8·813	—361·699	+13925·315	+ 29·8	— 483·7
1810 Jan. 2.5	— 43·8	—1000·2	—10·556	—372·255	+13563·616	+ 20·6	— 463·1
Feb. 21.5	— 55·3	—1055·5	—10.205	—382·460	+13191·361	+ 17·2	— 445·9
Apr. 12.5	— 75·4	—1130.9	— 6·249	—388·709	+12808·901	+ 18·7	— 427·2
June 1.5	— 103·9	—1234·8	+ 0·490	—388·219	+12420·192	+ 38·6	— 388·6
July 21.5	— 126·5	—1361·3	+ 6·642	—381·577	+12031·973	+ 69·0	— 319·6
Sept. 9.5	— 137·0	—1498·3	+ 9·863	—371·714	+11650·396	+ 93·5	— 226·1
Oct. 29.5	— 131·7	—1630·0	+10·462	—361·252	+11278·682	+105·0	— 121·1
Dec. 18.5	— 118·8	—1748·8	+ 9·691	—351·561	+10917·430	+106·3	— 14·8
1811 Feb. 6.5	— 103·8	—1852·6	+ 8·467	—343·094	+11565·869	+102·1	+ 87·3
March 28.5	— 89·7	—1942·3	+ 7·232	—335·862	+10222·775	+ 96·0	+ 183·3
May 17.5	— 77·8	—2020·1	+ 6·151	—329·711	+ 9886·913	+ 89·9	+ 273·2
July 6.5	— 68·2	—2088·3	+ 5·252	—324·459	+ 9557·202	+ 84·6	+ 357·8
Aug. 25.5	— 60·8	—2149·1	+ 4·541	—319·918	+ 9232·743	+ 80·6	+ 438·4
Oct. 14.5	— 55·2	—2204·3	+ 3·973	—315·945	+ 8912·825	+ 77·8	+ 516·2
Dec. 3.5	— 51·2	—2255·5	+ 3·524	—312·421	+ 8596·880	+ 76·0	+ 592·2
1812 Jan. 22.5	— 48·6	—2304·1	+ 3·173	—309·248	+ 8284·459	+ 75·2	+ 667·4
March 12.5	— 47·0	—2351·1	+ 2·899	—306·349	+ 7975·211	+ 75·3	+ 742·7
May 1.5	— 46·2	—2397·3	+ 2·686	—303·663	+ 7668·862	+ 76·1	+ 818·8
June 20·5	— 46·0	—2443·3	+ 2·523	—301·140	+ 7365·199	+ 77·4	+ 896·2
Aug. 9.5	— 46·3	—2489·6	+ 2·400	—298·740	+ 7064·059	+ 79·2	+ 975·4
Sept. 28.5	— 47·1	—2536·7	+ 2·313	—296·427	+ 6765·319	+ 81·3	+1056·7
Nov. 17.5	— 48·1	—2584.8	+ 2·254	—294·173	+ 6468·892	+ 83·7	+1140·4
1813 Jan. 6.5	— 49·2	—2634·0	+ 2·219	—291·954	+ 6174·719	+ 86·3	+1226·7
Feb. 25.5	— 50·4	—2684·4	+ 2·209	—289·745	+ 5882·765	+ 89·1	+1315·8
Apr. 16.5	— 51·7	—2736·1	+ 2·222	—287·523	+ 5593·020	+ 91·9	+1407·7
June 5.5	— 52·8	—2788·9	+ 2·257	—285·266	+ 5305·497	+ 94·7	+1502·4
July 25.5	— 53·7	—2842·6	+ 2·317	—282·949	+ 5020·231	+ 97·3	+1599·7
Sept. 13.5	— 54·4	—2897·0	+ 2·404	—280·545	+ 4737·282	+ 99·9	+1699·6

	$d\delta\Omega$	'f	$d\delta i$	'f	$d\delta\varphi$	'f	Δ
1813 Sept. 13.5	—14″·5	+ 240″·9	+10″·3	— 418″·3	+14″·3	—1630″·5	3.9170
Nov. 2.5	—15·2	+ 225·7	+12·1	— 406.2	+17·6	—1612·9	3·8009
1813 Dec. 22.5	—15·8		+14·2		+21·4		3·6758
1813 Sept. 13.5	— 7·3	+ 244·6	+ 5·2	— 421·0	+ 7·1	—1634·2	3·9170
Oct. 8.5	— 7·4	+ 237·2	+ 5·6	— 415·4	+ 7.9	—1626·3	3·8601
Nov. 2.5	— 7·6	+ 229·6	+ 6·1	— 409·3	+ 8·8	—1617·5	3·8009
Nov. 27.5	— 7·8	+ 221·8	+ 6·6	— 402·7	+ 9·7	—1607·8	3·7394
Dec. 22.5	— 7·9	+ 213·9	+ 7·1	— 395·6	+10·7	—1597·1	3·6758
1814 Jan. 16.5	— 8·0	+ 205·9	+ 7·6	— 388·0	+11·8	—1585·3	3·6105
Feb. 10.5	— 8·1	+ 197·8	+ 8·2	— 379·8	+12·9	—1572·4	3·5434
March 7.5	— 8·2	+ 189·6	+ 8·9	— 370·9	+14·1	—1558·3	3·4747
Apr. 1.5	— 8·2	+ 181·4	+ 9·5	— 361·4	+15·3	—1543·0	3·4043
Apr. 26.5	— 8·2	+ 173·2	+10·2	— 351·2	+16·7	—1526·3	3·3325
May 21.5	— 8·1	+ 165·1	+11·0	— 340·2	+18·2	—1508·1	3·2597
June 15.5	— 8·0	+ 157·1	+11·7	— 328·5	+19·8	—1488·3	3·1861
July 10.5	— 7·8	+ 149·3	+12·5	— 316·0	+21·5	—1466·8	3·1118
Aug. 4.5	— 7·6	+ 141·7	+13·4	— 302·6	+23·3	—1443·5	3·0374
Aug. 29.5	— 7·2	+ 134·5	+14·2	— 288·4	+25·2	—1418·3	2·9623
Sept. 23.5	— 6·8	+ 127·7	+15·1	— 273·3	+27·3	—1391·0	2·8870
Oct. 18.5	— 6·3	+ 121·4	+16·0	— 257·3	+29·5	—1361·5	2·8118
Nov. 12.5	— 5·7	+ 115·7	+16·9	— 240·4	+31·9	—1329·6	2·7371
Dec. 7.5	— 5·1	+ 110·6	+17·7	— 222·7	+34·4	—1295·2	2·6628
1815 Jan. 1.5	— 4·3	+ 106·3	+18·6	— 204·1	+37·1	—1258·1	2·5895
Jan. 26.5	— 3·4	+ 102·9	+19·4	— 184·7	+39·9	—1218·2	2·5180
Febr. 20.5	— 2·3	+ 100·6	+20·1	— 164·6	+42·9	—1175·3	2·4484
March 17.5	— 1·2	+ 99·4	+20·7	— 143·9	+45·9	—1129·4	2·3804
Apr. 11.5	— 0·1	+ 99·3	+21·2	— 122·7	+49·1	—1080·3	2·3148
May 6.5	+ 1·2	+ 100·5	+21·5	— 101·2	+52·2	—1028·1	2·2522
May 31.5	+ 2·4	+ 102·9	+21·6	— 79·6	+55·4	— 972·7	2·1927
June 25.5	+ 3·7	+ 106·6	+21·4	— 58·2	+58·3	— 914·4	2·1379
July 20.5	+ 4·9	+ 111·5	+20·9	— 37·3	+61·0	— 853·4	2·0882
Aug. 14.5	+ 5·9	+ 117·4	+20·0	— 17·3	+63·3	— 790·1	2·0434

	$d\delta\pi$	'f	$\lambda d\delta n$	'f	''f	P	'f
1813 Sept. 13.5	— 54″4	—2897″0	+ 2″404	—280″545	+4737″282	+ 99″9	+1699″6
Nov. 2.5	— 54·7	—2951·7	+ 2·521	—278·024	+4456·737	+102·1	+1801·7
1813 Dec. 22.5	— 54·4		— 2·675		+4178·713	+104·0	
1813 Sept. 13.5	— 27·2	—2883·5	+ 0·601	—140·577	+4737·434	+ 49·9	+1674·5
Oct. 8.5	— 27·3	—2910·8	+ 0·615	—139·962	+4596·857	+ 50·5	+1725·0
Nov. 2.5	— 27·3	—2938·1	+ 0·630	—139·332	+4456·895	+ 51·1	+1776·1
Nov. 27.5	— 27·3	—2965·4	+ 0·648	—138·684	+4317·563	+ 51·6	+1827·7
Dec. 22.5	— 27·2	—2992·6	+ 0·669	—138·015	+4178·879	+ 52·0	+1879·7
1814 Jan. 16.5	— 27·0	—3019·6	+ 0·691	—137·324	+4040·864	+ 52·3	+1932·0
Feb. 10.5	— 26·7	—3046·3	+ 0·718	—136·606	+3903·540	+ 52·6	+1984·6
March 7.5	— 26·3	—3072·6	+ 0·748	—135·858	+3766·934	+ 52·8	+2037·4
Apr. 1.5	— 25·7	—3098·3	+ 0·782	—135·076	+3631·076	+ 52·9	+2090·3
Apr. 26.5	— 25·0	—3123·3	+ 0·820	—134·256	+3496·000	+ 52·8	+2143·1
May 21.5	— 24·0	—3147·3	+ 0·864	—133·392	+3361·744	+ 52·6	+2195·7
June 15.5	— 23·0	—3170·3	+ 0·912	—132·480	+3228·352	+ 52·2	+2247·9
July 10.5	— 21·7	—3192·0	+ 0·967	—131·513	+3095·872	+ 51·6	+2299·5
Arg. 4·5	— 20·1	—3212·1	+ 1·029	—130·484	+2964·359	+ 50·7	+2350·2
Aug. 29.5	— 18·1	—3230·2	+ 1·099	—129·385	+2833·875	+ 49·5	+2399·7
Sept. 23.5	— 15·8	—3246·0	+ 1·178	—128·207	+2704·490	+ 48·0	+2447·7
Oct. 18.5	— 13·0	—3259·0	+ 1·267	—126·940	+2576·283	+ 46·0	+2493·7
Nov. 12.5	— 9·8	—3268·8	+ 1·368	—125·572	+2449·343	+ 43·5	+2537·2
Dec. 7.5	— 6·0	—3274·8	+ 1·482	—124·090	+2323·771	+ 40·5	+2577·7
1815 Jan. 1.5	— 1·6	—3276·4	+ 1·609	—122·481	+2199·681	+ 36·8	+2614·5
Jan. 26.5	+ 3·6	—3272·8	+ 1·753	—120·728	+2077·200	+ 32·3	+2646·8
Feb. 20.5	+ 9·5	—3263·3	+ 1·912	—118·816	+1956·472	+ 27·0	+2673·8
March 17.5	+ 16·2	—3247·1	+ 2·091	—116·725	+1837·656	+ 20·8	+2694·6
Apr. 11.5	+ 23·9	—3223·2	+ 2·289	—114·436	+1720·931	+ 13·4	+2708·0
May 6.5	+ 32·7	—3190·5	+ 2·507	—111·929	+1606·495	+ 4·9	+2712·9
May 31.5	+ 42·4	—3148·1	+ 2·746	—109·183	+1494·566	— 4·8	+2708·1
June 25.5	+ 53·1	—3095·0	+ 3·000	—106·183	+1385·383	— 15·8	+2692·3
July 20.5	+ 64·7	—3030·3	+ 3·268	—102·915	+1279·200	— 28·0	+2664·3
Aug. 14.5	+ 76·7	—2953·6	+ 3·546	— 99·369	+1176·285	— 41·2	+2623·1

	$d\delta\Omega$	$'f$	$d\delta i$	$'f$	$d\delta\varphi$	$'f$	Δ
1815 Aug. 14.5	+ 5″·9	+ 117″·4	+20″·0	−17″·3	+63″·3	− 790″·1	2·0434
Sept. 8.5	+ 6·7	+ 124·1	+18·8	+ 1·5	+65·1	− 725·0	2·0042
Oct. 3.5	+ 7·3	+ 131·4	+17·2	+18·7	+66·2	− 658·8	1·9714
Oct. 28.5	+ 7·4	+ 138·8	+15·2	+33·9	+66·5	− 592·3	1·9455
Nov. 22.5	+ 7·2	+ 146·0	+12·9	+46·8	+66·0	− 526·3	1·9265
Dec. 17.5	+ 6·6	+ 152·6	+10·4	+57·2	+64·4	− 461·9	1·9152
1816 Jan. 11.5	+ 5·5	+ 158·1	+ 7·8	+65·0	+62·0	− 399·9	1·9123
Feb. 5.5	+ 4·0	+ 162·1	+ 5·1	+70·1	+58·6	− 341·3	1·9180
March 1.5	+ 2·2	+ 164·3	+ 2·6	+72·7	+54·4	− 286·9	1·9323
March 26.5	+ 0·2	+ 164·5	+ 0·2	+72·9	+49·7	− 237·2	1·9556
Apr. 20.5	− 1·8	+ 162·7	− 1·8	+71·1	+44·6	− 192·6	1·9874
May 15.5	− 3·9	+ 158·8	− 3·4	+67·7	+39·3	− 153·3	2·0277
June 9.5	− 5·8	+ 153·0	− 4·7	+63·0	+34·0	− 119·3	2·0756
July 4.5	− 7·6	+ 145·4	− 5·6	+57·4	+29·0	− 90·3	2·1310
July 29.5	− 9·0	+ 136·4	− 6·1	+51·3	+24·2	− 66·1	2·1941
Aug. 23.5	−10·2	+ 126·2	− 6·4	+44·9	+19·8	− 46·3	2·2648
Sept. 17.5	−11·1	+ 115·1	− 6·3	+38·6	+15·9	− 30·4	2·3427
Oct. 12.5	−11·6	+ 103·5	− 6·1	+32·5	+12·4	− 18·0	2·4272
Nov. 6.5	−11·9	+ 91·6	− 5·7	+26·8	+ 9·4	− 8·6	2·5177
Dec. 1.5	−11·9	+ 79·7	− 5·2	+21·6	+ 6·8	− 1·8	2·6142
Dec. 26.5	−11·5	+ 68·2	− 4·5	+17·1	+ 4·7	+ 2·9	2·7161
1817 Jan. 20.5	−11·4	+ 56·8	− 4·0	+13·1	+ 2·8	+ 5·7	2·8236
Feb. 14.5	−10·8	+ 46·0	− 3·4	+ 9·7	+ 1·4	+ 7·1	2·9364
March 11.5	−10·2	+ 35·8	− 2·9	+ 6·8	+ 0·2	+ 7·3	3·0546
Apr. 5.5	− 9·4	+ 26·4	− 2·3	+ 4·5	− 0·7	+ 6·6	3·1779
Apr. 30.5	− 8·6	+ 17·8	− 1·8	+ 2·7	− 1·4	+ 5·2	3·3065
May 25.5	− 7·8	+ 10·0	− 1·4	+ 1·3	− 1·9	+ 3·3	3·4401
June 19.5	− 6·9	+ 3·1	− 1·0	+ 0·3	− 2·2	+ 1·1	3·5788
July 14.5	− 6·0	− 2·9	− 0·6	− 0·3	− 2·3	− 1·2	3·7225
Aug. 8.5	− 5·1	− 8·0	− 0·4	− 0·7	− 2·3	− 3·5	3·8710
1817 Sept. 2.5	− 4·3		− 0·1		− 2·1		4·0246

	$d\delta\pi$	'f	$\lambda d\delta n$	'f	''f	P	'f
1815 Aug. 14.5	+ 76·7	−2953·6	+ 3·546	−99·369	+1176·285	− 41·2	+2623·1
Sept. 8.5	+ 89·8	−2863·8	+ 3·830	−95·539	+1076·916	− 55·3	+2567·8
Oct. 3.5	+ 102·7	−2761·1	+ 4·108	−91·431	+ 981·377	− 69·9	+2497·9
Oct. 28.5	+ 115·2	−2645·9	+ 4·370	−87·061	+ 889·946	− 84·6	+2413·3
Nov. 22.5	+ 127·1	−2518·8	+ 4·610	−82·451	+ 802·885	− 98·9	+2314·4
Dec. 17.5	+ 137·6	−2381·2	+ 4·813	−77·638	+ 720·434	−112·1	+2202·3
1816 Jan. 11.5	+ 146·4	−2234.8	+ 4·968	−72·670	+ 642·796	−123·8	+2078·5
Feb. 5.5	+ 153·0	−2081·8	+ 5·068	−67·602	+ 570·126	−133·3	+1945·2
March 1.5	+ 157·2	−1924·6	+ 5·109	−62·493	+ 502·524	−140·3	+1804·9
March 26.5	+ 158·8	−1765·8	+ 5·087	−57·406	+ 440·031	−144·6	+1660·3
Apr. 20.5	+ 157·9	−1607·9	+ 5·007	−52.399	+ 382·625	−146·2	+1514·1
May 15.5	+ 154·8	−1453·1	+ 4·874	−47·525	+ 330·226	−145·1	+1369·0
June 9.5	+ 149·7	−1303·4	+ 4·701	−42·824	+ 282·701	−141·9	+1227·1
July 4.5	+ 143·2	−1160·2	+ 4·499	−38·325	+ 239·877	−136·7	+1090·4
July 29.5	+ 135·5	−1024·7	+ 4·263	−34·062	+ 201·552	−130·1	+ 960·3
Aug. 23.5	+ 127·0	− 897·7	+ 4·013	−30·049	+ 167·490	−122·3	+ 838·0
Sept. 17.5	+ 118·0	− 779·7	+ 3·752	−26·297	+ 137·441	−113·7	+ 724·3
Oct. 12.5	+ 108·8	− 670·9	+ 3·492	−22·805	+ 111·144	−104·8	+ 619·5
Nov. 6.5	+ 99·8	− 571·1	+ 3·233	−19·572	+ 88·339	− 95·8	+ 523·7
Dec. 1.5	+ 91·0	− 480·1	+ 2·982	−16·590	+ 68·767	− 86·9	+ 436·8
Dec. 26.5	+ 82·6	− 397·5	+ 2·740	−13·850	+ 52·177	− 78·4	+ 358·4
1817 Jan. 20.5	+ 74·6	− 322·9	+ 2·508	−11·342	+ 38·327	− 70·1	+ 288·3
Feb. 14.5	+ 67·1	− 255·8	+ 2·286	− 9·056	+ 26·985	− 62·4	+ 225·9
March 11.5	+ 60·1	− 195·7	+ 2·075	− 6·981	+ 17·929	− 55·1	+ 170·8
Apr. 5.5	+ 53·5	− 142·2	+ 1·875	− 5·106	+ 10·948	− 48·3	+ 122·5
Apr. 30.5	+ 47·4	− 94·8	+ 1·683	− 3·423	+ 5·842	− 42·0	+ 80·5
May 25.5	+ 41·8	− 53·0	+ 1·501	− 1·922	+ 2·419	− 36·2	+ 44·3
June 19·5	+ 36·7	− 16·3	+ 1·328	− 0·594	+ 0·497	− 30·9	+ 13·4
July 14.5	+ 31·8	+ 15·5	+ 1·161	+ 0·567	− 0·097	− 26·0	− 12·6
Aug. 8.5	+ 27·5	+ 43·0	+ 1·001	+ 1·568	+ 0·470	− 21·6	− 34·2
1817 Sept. 2.5	+ 23·5		+ 0·847		+ 2·058	− 17·6	

SATURN

1817 July 14.5 — 1809 Apr. 27.5

	dδΩ	'f	dδi	'f	dδφ	'f	Δ
1809 March 8.5	— 0″·2	+14″·2	0″·0	— 6″·3	— 0″·2	—33″·9	8·45
Apr. 27.5	— 0·1	+14·1	0·0	— 6·3	— 0·1	—34·0	8·89
June 16.5	— 0·1	+14·0	0·0	— 6·3	0·0	34·0	9·40
Aug. 5.5	0·0	+14·0	0·0	— 6·3	+ 0·2	—33·8	9·89
Sept. 24.5	0·0	+14·0	0·0	— 6·3	+ 0·3	—33·5	10·40
Nov. 13.5	+ 0·1	+14·1	0·0	— 6·3	+ 0·5	—33·0	10·90
1810 Jan. 2.5	+ 0·1	+14·2	0·0	— 6·3	+ 0·6	—32·4	11·40
Feb. 21.5	0·0	+14·2	— 0·1	— 6·4	+ 0·6	—31·8	11·85
Apr. 12.5	0·0	+14·2	— 0·1	— 6·5	+ 0·6	—31·2	12·15
June 1.5	0·0	+14·2	— 0.1	— 6·6	+ 0.5	—30·7	12·40
July 21.5	— 0·1	+14·1	— 0·1	— 6·7	+ 0·4	—30·3	12·60
Sept. 9.5	— 0·2	+13·9	— 0·1	— 6·8	+ 0·4	—29·9	12·80
Oct. 29.5	— 0·2	+13·7	— 0·1	— 6·9	+ 0·3	—29·6	13·00
Dec. 18.5	— 0·3	+13·4	— 0·1	— 7·0	+ 0·3	—29·3	13·10
1811 Feb. 6.5	— 0·3	+13·1	— 0·1	— 7·1	+ 0·2	—29·1	13·17
March 28.5	— 0·4	+12·7	0·0	— 7·1	+ 0·3	—28·8	13·20
May 17.5	— 0·4	+12·3	0·0	— 7·1	+ 0·3	—28·5	13·20
July 6.5	— 0·5	+11·8	0·0	— 7·1	+ 0·3	—28·2	13·25
Aug. 25.5	— 0·5	+11·3	0·0	— 7·1	+ 0·4	—27·8	13·30
Oct. 14.5	— 0·5	+10·8	0·0	— 7·1	+ 0·4	—27·4	13·30
Dec. 3.5	— 0·5	+10·3	+ 0·1	— 7·0	+ 0·4	—27·0	13·30
1812 Jan. 22.5	— 0·5	+ 9·8	+ 0·1	— 6·9	+ 0·5	—26·5	13·30
March 12.5	— 0·6	+ 9·2	+ 0·1	— 6·8	+ 0·5	—26·0	13·30
May 1.5	— 0.6	+ 8·6	+ 0·1	— 6·7	+ 0·6	—25·4	13·25
June 20.5	— 0.6	+ 8·0	+ 0·1	— 6·6	+ 0·6	—24·8	13·20
Aug. 9.5	— 0·6	+ 7·4	+ 0·2	— 6·4	+ 0·7	—24·1	13·20
Sept. 28.5	— 0·6	+ 6·8	+ 0·2	— 6·2	+ 0·7	—23·4	13·20
Nov. 17.5	+ 0·6	+ 6·2	+ 0·2	— 6·0	+ 0·7	—22·7	13·15
1813 Jan. 6.5	— 0·5	+ 5·7	+ 0·2	— 5·8	+ 0·8	—21·9	13·12
Feb. 25.5	— 0.5	+ 5·2	+ 0·2	— 5·6	+ 0·8	—21·1	13·10
Apr. 16.5	— 0.5	+ 4·7	+ 0·3	— 5·3	+ 0·8	—20·3	13·02
June 5.5	— 0·5	+ 4·2	+ 0·3	— 5·0	+ 0·8	—19·5	12·95
July 25.5	— 0·4	+ 3·8	+ 0·3	— 4·7	+ 0·8	—18·7	12·90
Sept. 13.5	— 0·4	+ 3·4	+ 0·3	— 4·4	+ 0·9	—17·8	12·85

SATURN

1817 July 14.5 — 1809 Apr. 27.5

	$d\delta\pi$	'f	$\lambda d\delta n$	'f	"f	P	'f
1809 March 8.5	+ 2·3	+ 37·8	+ 0·164	— 1·230	+30·009	— 1·9	— 51·3
Apr. 27.5	+ 1·7	+ 39·5	+ 0·106	— 1·124	+28·779	— 1·2	— 52·5
June 16.5	+ 1·0	+ 40·5	+ 0·041	— 1·083	+27·655	— 0·5	— 53·0
Aug. 5.5	+ 0·5	+ 41·0	— 0·016	— 1·099	+26·572	— 0·1	— 53·1
Sept. 24·5	+ 0·2	+ 41·2	— 0·065	— 1·164	+25·473	+ 0·1	— 53·0
Nov. 13.5	0·0	+ 41·2	— 0·104	— 1·268	+24·309	+ 0·2	— 52·8
1810 Jan. 2.5	— 0·1	+ 41·1	— 0·135	— 1·403	+23·041	+ 0·1	— 52·7
Feb. 21.5	— 0·1	+ 41·0	— 0·140	— 1·543	+21·638	0·0	— 52·7
Apr. 12.5	0·0	+ 41·0	— 0·126	— 1·669	+20·095	— 0·2	— 52·9
June 1.5	— 0·1	+ 40·9	— 0·106	— 1·775	+18·426	— 0·3	— 53·2
July 21.5	— 0·2	+ 40·7	— 0·082	— 1·857	+16·651	— 0·3	— 53·5
Sept. 9.5	— 0·4	+ 40·3	— 0·056	— 1·913	+14·794	— 0·2	— 53·7
Oct. 29.5	— 0·6	+ 39·7	— 0·025	— 1·938	+12·881	+ 0·1	— 53·6
Dec. 18.5	— 0·7	+ 39·0	— 0·001	— 1·939	+10·943	+ 0·1	— 53·5
1811 Feb. 6.5	— 0·9	+ 38·1	+ 0·020	— 1·919	+ 9·004	+ 0·3	— 53·2
March 28.5	— 1·1	+ 37·0	+ 0·037	— 1·882	+ 7·085	+ 0·4	— 52·8
May 17.5	— 1·3	+ 35·7	+ 0·051	— 1·831	+ 5·203	+ 0·6	— 52·2
July 6.5	— 1·4	+ 34·3	+ 0·063	— 1·768	+ 3·372	+ 0·7	— 51·5
Aug. 25.5	— 1·5	+ 32·8	+ 0·073	— 1·695	+ 1·604	+ 0·9	— 50·6
Oct. 14.5	— 1·5	+ 31·3	+ 0·080	— 1·615	— 0·091	+ 1·0	— 49·6
Dec. 3.5	— 1·6	+ 29·7	+ 0·086	— 1·529	— 1·706	+ 1·0	— 48·6
1812 Jan. 22.5	— 1·6	+ 28·1	+ 0·091	— 1·438	— 3·235	+ 1·1	— 47·5
March 12.5	— 1·6	+ 26·5	+ 0·094	— 1·344	— 4·673	+ 1·2	— 46·3
May 1.5	— 1·6	+ 24·9	+ 0·097	— 1·247	— 6·017	+ 1·2	— 45·1
June 20.5	— 1·6	+ 23·3	+ 0·100	— 1·147	— 7·264	+ 1·2	— 43·9
Aug. 9.5	— 1·5	+ 21·8	+ 0·100	— 1·047	— 8·411	+ 1·3	— 42·6
Sept. 28.5	— 1·5	+ 20·3	+ 0·101	— 0·946	— 9·458	+ 1·3	— 41·3
Nov. 17.5	— 1·4	+ 18·9	+ 0·102	— 0·844	—10·404	+ 1·3	— 40·0
1813 Jan. 6.5	— 1·3	+ 17·6	+ 0·101	— 0·743	—11·248	+ 1·3	— 38·7
Feb. 25.5	— 1·2	+ 16·4	+ 0·101	— 0·642	—11·991	+ 1·2	— 37·5
Apr. 16.5	— 1·2	+ 15·2	+ 0·100	— 0·542	—12·633	+ 1·2	— 36·3
June 5.5	— 1·1	+ 14·1	+ 0·099	— 0·443	—13·175	+ 1·2	— 35·1
July 25.5	— 1·0	+ 13·1	+ 0·097	— 0·346	—13·618	+ 1·2	— 33·9
Sept. 13.5	— 0·9	+ 12·2	+ 0·096	— 0·250	—13·964	+ 1·1	— 32·8

	dδΩ	'f	dδi	'f	dδφ	'f	Δ
1813 Sept. 13.5	— 0″4	+ 3″4	+ 0″3	— 4″4	+ 0″9	— 17″8	12·85
Nov. 2.5	— 0·4	+ 3·0	+ 0·3	— 4·1	+ 0·9	— 16·9	12·78
Dec. 22.5	— 0·4	+ 2·6	+ 0·3	— 3·8	+ 0·9	— 16·0	12·70
1814 Feb. 10.5	— 0·3	+ 2·3	+ 0·3	— 3·5	+ 0·9	— 15·1	12·65
Apr. 1.5	— 0·3	+ 2·0	+ 0·3	— 3·2	+ 0·9	— 14·2	12·60
May 21.5	— 0·2	+ 1·8	+ 0·3	— 2·9	+ 0·9	— 13·3	12·50
July 10·5	— 0·2	+ 1·6	+ 0·3	— 2·6	+ 0·9	— 12·4	12·40
Aug. 29.5	— 0·2	+ 1·4	+ 0·3	— 2·3	+ 0·9	— 11·5	12·30
Oct. 18.5	— 0·1	+ 1·3	+ 0·3	— 2·0	+ 0·8	— 10·7	12·20
Dec. 7.5	— 0·1	+ 1·2	+ 0·3	— 1·7	+ 0·8	— 9·9	12·10
1815 Jan. 26.5	— 0·1	+ 1·1	+ 0·3	— 1·4	+ 0·8	— 9·1	12·00
March 17.5	0·0	+ 1·1	+ 0·3	— 1·1	+ 0·8	— 8·3	11·90
May 6.5	0·0	+ 1·1	+ 0·3	— 0·8	+ 0·7	— 7·6	11·70
June 25.5	0·0	+ 1·1	+ 0·3	— 0·5	+ 0·7	— 6·9	11·60
Aug. 14.5	+ 0·1	+ 1·2	+ 0·2	— 0·3	+ 0·7	— 6·2	11·40
Oct. 3.5	+ 0·1	+ 1·3	+ 0·2	— 0·1	+ 0·6	— 5·6	11·25
Nov. 22.5	+ 0·1	+ 1·4	+ 0·2	+ 0·1	+ 0·6	— 5·0	11·10
1816 Jan. 11·5	+ 0·1	+ 1·5	+ 0·1	+ 0·2	+ 0·6	— 4·4	10·90
March 1.5	+ 0·1	+ 1·6	+ 0·1	+ 0·3	+ 0·5	— 3·9	10·70
Apr. 20.5	+ 0·1	+ 1·7	+ 0·1	+ 0·4	+ 0·5	— 3·4	10·50
June 9.5	+ 0·1	+ 1·8	+ 0·1	+ 0·5	+ 0·4	— 3·0	10·30
July 29.5	0·0	+ 1·8	0·0	+ 0·5	+ 0·4	— 2·6	10·06
Sept. 17.5	0·0	+ 1·8	0·0	+ 0·5	+ 0·4	— 2·2	9·81
Nov. 6.5	— 0·1	+ 1·7	0·0	+ 0·5	+ 0·4	— 1·8	9·55
Dec. 26.5	— 0·2	+ 1·5	— 0·1	+ 0·4	+ 0·4	— 1·4	9·28
1817 Feb. 14.5	— 0·3	+ 1·2	— 0·1	+ 0·3	+ 0·4	— 1·0	9·00
Apr. 5.5	— 0·4	+ 0·8	— 0·1	+ 0·2	+ 0·4	— 0·6	8·72
May 25.5	— 0·5	+ 0·3	— 0·1	+ 0·1	+ 0·4	— 0·2	8·44
July 14.5	— 0·6	— 0·3	— 0·1	0·0	+ 0·4	+ 0·2	8·16
Sept. 2.5	— 0·7		0·0		+ 0·5		7·91

	$d\delta\pi$	'f	$\lambda d\delta n$	'f	"f	P	'f
1813 Sept. 13.5	— 0″9	+ 12″2	+ 0″096	— 0″250	—13″964	+ 1″1	— 32″8
Nov. 2.5	— 0·8	+ 11·4	+ 0·095	— 0·155	—14.214	+ 1·1	— 31·7
Dec. 22.5	— 0·7	+ 10·7	+ 0·094	— 0·061	—14·369	+ 1·0	— 30·7
1814 Feb. 10.5	— 0·6	+ 10·1	+ 0·091	+ 0·030	—14·430	+ 1·1	— 29·6
Apr. 1.5	— 0·5	+ 9·6	+ 0·089	+ 0·119	—14·400	+ 1·1	— 28·5
May 21.5	— 0·4	+ 9·2	+ 0·086	+ 0·205	—14·281	+ 1·0	— 27·5
July 10.5	— 0·3	+ 8·9	+ 0·083	+ 0·288	—14·076	+ 1·0	— 26·5
Aug. 29.5	— 0·2	+ 8·7	+ 0·080	+ 0·368	—13·788	+ 0·9	— 25·6
Oct. 18.5	— 0·1	+ 8·6	+ 0·077	+ 0·445	—13·420	+ 0·8	— 24·8
Dec. 7·5	0·0	+ 8·6	+ 0·074	+ 0·519	—12·975	+ 0·8	— 24·0
1815 Jan. 26.5	0·0	+ 8·6	+ 0·069	+ 0·588	—12·456	+ 0·8	— 23·2
March 17.5	+ 0·1	+ 8·7	+ 0·064	+ 0·652	—11·868	+ 0·8	— 22·4
May 6.5	+ 0·1	+ 8·8	+ 0·058	+ 0·710	—11·216	+ 0·8	— 21·6
June 25.5	+ 0.1	+ 8·9	+ 0·052	+ 0·762	—10·506	+ 0·8	— 20·8
Aug. 14.5	+ 0·2	+ 9·1	+ 0·045	+ 0·807	— 9·744	+ 0·8	— 20·0
Oct. 3.5	+ 0·2	+ 9·3	+ 0·038	+ 0·845	— 8·937	+ 0·8	— 19·2
Nov. 22.5	+ 0·1	+ 9·4	+ 0·028	+ 0·873	— 8·092	+ 0·9	— 18·3
1816 Jan. 11.5	+ 0·1	+ 9·5	+ 0·018	+ 0·891	— 7·219	+ 0·9	— 17·4
March 1.5	0·0	+ 9·5	+ 0·006	+ 0·897	— 6·328	+ 1·0	— 16·4
Apr. 20.5	— 0·1	+ 9·4	— 0·007	+ 0·890	— 5·431	+ 1·1	— 15·3
June 9.5	— 0·2	+ 9·2	— 0·022	+ 0·868	— 4.541	+ 1·2	— 14·1
July 29.5	— 0·4	+ 8·8	— 0·040	+ 0·828	— 3·673	+ 1·4	— 12·7
Sept. 17.5	— 0·6	+ 8·2	— 0·060	+ 0·768	— 2·845	+ 1·5	— 11·2
Nov. 6.5	— 0·8	+ 7·4	— 0·082	+ 0·686	— 2·077	+ 1·7	— 9·5
Dec. 26.5	— 1·1	+ 6·3	— 0·107	+ 0·579	— 1·391	+ 1·9	— 7·6
1817 Feb. 14.5	— 1·4	+ 4·9	— 0·134	+ 0·445	— 0·812	+ 2·1	— 5·5
Apr, 5.5	— 1·8	+ 3·1	— 0·161	+ 0·284	— 0·367	+ 2·2	— 3·3
May 25.5	— 2·0	+ 1·1	— 0·184	+ 0·100	— 0·083	+ 2·2	— 1·1
July 14.5	— 2·2	— 1·1	— 0·201	— 0·101	+ 0·017	+ 2·2	+ 1·1
Sept. 2.5	— 2·3		— 0·205		— 0·084	+ 2·0	

P o s i e d z e n i e

z dnia 16 grudnia 1949 r.

T a d e u s z W a ż e w s k i

Sur certaines conditions de coïncidence asymptotique des intégrales des deux systèmes d'équations différentielles

Note présentée à la séance du 16 décembre 1949.

§ 1. Soit

$$\dot{w} = \frac{dw(t)}{dt} = W(t, w) \qquad \text{(système } S\text{)}$$

un système de n équations différentielles écrit sous forme vectorielle, la variable t et le vecteur w étant réels. Nous dirons que ce système possède *la propriété P* dans Ω lorsque 1⁰) $W(t, w)$ est défini et continu dans Ω, 2⁰) Ω est un ensemble ouvert et 3⁰) par chaque point de Ω passe une intégrale unique du système (S).

Admettons la propriété P pour le système (S) envisagé dans Ω, Nous entendrons par intégrale de (S) chaque fonction $w(t)$ remplissant (S) ou bien le diagramme d'une telle fonction. L'intégrale issue de $M \epsilon \Omega$ sera désignée par $I(M)$, sa partie composée de M et des points situés à droite de M sera désignée par $Demi_+I(M)$ (demi-intégrale droite issue de M). Les $I(M)$ et $Demi\, I_+(M)$ seront, par hypothèse, saturées c.-à-d. prolongées jusqu'à la frontière de Ω.

Si M varie dans $A \subset \Omega$ alors $I(M)$ et $Demi_+I(M)$ engendront les ensembles $\underset{(S)}{Z}(A)$ et $\underset{(S)}{\overset{+}{Z}}(A)$ dites zone d'émission de A et sa zone d'émission droite.

On dira qu'une suite d'ensembles $B_n \subset \Omega$ tend vers le côté droit de la frontière de Ω (*front*(Ω)), c.-à-d. que

$$B_n \xrightarrow[(S)]{+} front\,(\Omega)$$

lorsque $\underset{(S)}{\overset{+}{Z}}(B_n)$ tend vers front (Ω).

§ 2. Soient $\{C_n\}$ et $\{D_n\}$ deux suites infinies d'ensembles avec $C_{n+1} \subset C_n$, $D_{n+1} \subset D_n$. Nous dirons que $\{C_n\}$ est contenue dans $\{D_n\}$ c.-à-d. $\{C_n\} \subset \{D_n\}$ lorsque chaque D_i contient presque tous les C_n.

Si $\{C_n\} \subset \{D_n\}$ et $\{D_n\} \subset \{C_n\}$ les $\{C_n\}$ et $\{D_n\}$ seront considérés comme équivalents. La famille de toutes les suites $\{D_n\}$ équivalentes à $\{C_n\}$ sera désignée par $\{\{C_n\}\}$ [1]).

§ 3. Soit $\overset{o}{w}$ une intégrale de (S), $M_n \in \overset{o}{w}$, $M_n \xrightarrow[(S)]{+} front\,(\Omega)$. La famille $\{\{\underset{(S)}{\overset{+}{Z}}(M_n)\}\}$ sera dite extrémité asymptotique droite intérieure de $\overset{o}{w}$ et sera désignée par

$$extr(\overset{o}{w}). \tag{1}$$

M_n ayant la même signification, désignons par $V_n \subset \Omega$ un voisinage de M_n, tel que

$$V_n \xrightarrow[(S)]{+} front\,(\Omega), \quad \underset{(S)}{\overset{+}{Z}}(V_{n+1}) \subset \underset{(S)}{\overset{+}{Z}}(V_n), \quad \prod_{n=1}^{\infty} (\underset{(S)}{Z}(V_n)) = \overset{o}{w}.$$

La famille (c.-à-d. le „soma") $\{\{\underset{(S)}{\overset{+}{Z}}(V_n)\}\}$ sera dite extrémité asymptotique droite extérieure de $\overset{o}{w}$ ou, tout court, extrémité droite de $\overset{o}{w}$. Elle sera désignée par

$$\underset{(S)}{\overset{+}{Extr}}\,(\overset{o}{w}).$$

Cette notion dépend de $\overset{o}{w}$ et de (S), tandis que (1) dépend exclusivement de $\overset{o}{w}$.

[1]) La notion de $\{\{C_n\}\}$ se rattache à la notion du „soma" de M. Carathéodory (*Entwurf fur eine Algebraisierung des Integralbegriffes* Sitz. Ber. der math.-naturwiss. Abt. d. Bayerichen Akad. d. Wiss. zu München, 1938 p. 27—67) et à la notion du „Ende" et du „Primende" du même auteur.

§ 4. Soit

$$\dot{u} = U(t, u) \qquad (\textit{système } T)$$

un autre système. Nous avons le suivant:

Théorème 1. *Soient* $\overset{\circ}{u}$, u_n, $\overset{\circ}{w}$, w_n *les intégrales des systèmes* (T) *et* (S) *jouissant dans* Ω *de la propriété* P. *Si* [2])

$$\overset{+}{extr}(w_n) \subset \underset{(T)}{\overset{+}{Extr}}\, u_n, \quad u_n \to \overset{\circ}{u}, \quad \underset{(T)}{\overset{+}{Extr}}\, \overset{\circ}{u} \subset \underset{(S)}{\overset{+}{Extr}}\, \overset{\circ}{w}$$

alors $w_n \to w_0$.

Remarque. Si $\underset{(T)}{\overset{+}{Extr}}\, u = \underset{S}{\overset{+}{Extr}}\, w$, alors on dira que les intégrales u et w, des systèmes (S) et (T) coïncident asymptotiquement du côté droit et nous écrirons $u \underset{S,T}{\overset{+}{=}} w$ ou, tout court, $u \overset{+}{=} w$. Du théorème précédent résulte le théorème [3]):

Si $u_n \overset{+}{=} w_n$, $\overset{\circ}{u} \overset{+}{=} \overset{\circ}{w}$, $u_n \to \overset{\circ}{u}$ alors $w_n \to \overset{\circ}{w}$.

§ 5. Chaque système (S) jouissant de la propriété P, envisagé dans un voisinage d'une intégrale w, peut être transformé en un système „banal".

$$\dot{y}_i = 0, \quad (i = 1, \ldots, n). \qquad (\text{système } B)$$

La coïncidence asymptotique des intégrales de (T) et (B) mérite donc un interêt spécial.

§ 6. Considérons, sur le plan à deux dimensions, une seule équation

$$\dot{\alpha} = \vartheta(\alpha, t). \qquad (\text{équation } \Theta)$$

Nous dirons qu'une telle équation jouit dans la bande E, $(0 < \alpha < s,\ -\infty < t + \infty)$ de la propriété Q lorsque 1⁰) elle jouit de la propriété P dans E, 2⁰) chaque intégrale $\delta = c$ de l'équation banale $\dot{\delta} = 0$ envisagée dans E coïncide asymptotiquement du côté droit avec une intégrale de Θ.

²) $\{\{C_n\}\} \subset \{\{D_n\}\}$ *veut dire que* $\{C_n\} \subset \{D_n\}$.

³) J'ai communiqué ce théorème et la notion de la coïncidence asymptotique sous une autre forme le 24.VI.1949 à la séance de l'Acad. Pol. des Sciences et des Lettres.

Nous désignerons par $\alpha(t, k, +\infty)$ l'intégrale de (Θ) qui coïncide asymptotiquement (du côté droit) avec l'intégrale $\delta = k$ de l'équation $\dot{\delta} = 0$. Il est clair ce que veut dire que l'équation

$$\dot{\beta} = \gamma(\beta, t) \qquad \text{(équation } \Gamma)$$

jouit de la propriété Q dans la bande E. La fonction $\beta(t, k, +\infty)$ aura une signification analogue à $\alpha(t, k, +\infty)$.

Théorème 2. *Prémisses:* 1°) *Les équations* Θ *et* Γ *jouissent de la propriété* Q *dans la bande* E, 2°) *le système*

$$\dot{x}_i = f_i(t, x_1, \ldots, x_n), \qquad (i = 1, \ldots, n) \qquad \textit{(système } R)$$

jouit de la propriété P *dans l'espace tout entier*, 3°) *On pose* $r = \sqrt{\sum_{i=1}^{n} (x_i - \overset{\circ}{x}_i)^2}$, 4°) *Il existe deux suites de nombres* $\{k_n\}$, $\{T_n\}$, *tels que*

$$0 < k_n < s, \; k_n \to 0, \; T_n \to +\infty$$

$$\sum_i (x_i - \overset{\circ}{x}_i) f_i - r\,\alpha(t, k_n, +\infty) > 0$$

lorsque

$$r = \alpha(t, k_n, +\infty), \; t \geqslant T_n$$

et

$$\sum_i (x_i - \overset{\circ}{x}_i) f_i - r\,\beta(t, k_n, +\infty) < 0$$

lorsque

$$r = \beta(t, k_n, +\infty), \; t \geqslant T_n.$$

4°) *Il existe au plus une intégrale* $x_i(t)$ *du système* (R), *telle que* $x_i(t) \to \overset{\circ}{y}_i$ *lorsque* $t \to +\infty$.

Thèse. Il existe une intégrale (unique) $x_i = \lambda_i(t)$ *du système* (R) *qui coïncide asymptotiquement du côté droit avec l'intégrale* $y_i = \overset{\circ}{x}_i$ *du système banal* (B).

La démonstration s'appuie sur le:

Lemme 1. *Prémisses:* 1°) $\varphi(t)$ *et* $\psi(t)$ *sont de classe* C^1 *et* $\varphi(t) < \psi(t)$ *pour* $t \geqslant T$, 2°) *l'ensemble* H *est défini par les inégalités* $r \leqslant \varphi(t)$, $t \geqslant T$ *et l'ensemble* K *est défini par* $r \leqslant \psi(t)$,

$t \geqslant T$, 3⁰) *l'ensemble* h *est défini par* $r = \varphi(t)$, $t \geqslant T$, *et l'ensemble* k *par* $r = \psi(t)$, $t \geqslant T$, 4⁰) *les sections de* H *et* K *par le plan* $t = T$ *sont désigneés par* h^* *et* k^*, 5⁰) *en posant*

$$\xi = \frac{1}{r} \sum_i (x_i - \overset{\circ}{x}_i) f_i$$

on a

$$\xi < \frac{d\varphi}{dt} \quad sur \quad h \quad et \quad \xi > \frac{d\psi}{dt} \quad sur \quad k.$$

Thèse. Il existe un ensemble e^*, *tel que* $h^* \subset e^* \subset k^*$, $H \subset \underset{(R)}{\overset{+}{Z}}(e^*) \subset K$.

La démonstration de ce lemme s'appuie sur un principe topologique que j'ai indiqué précédemment [4]).

On applique ce principe à l'ensemble $\varphi(t) \leqslant r \leqslant \psi(t)$, $t \geqslant T$.

§ 7. **Exemple.** Supposons que $\lambda(z)$ soit continue pour $0 < z < c$, que $\int_0^c |\lambda'(kz)|\, dz < +\infty$ pour toute $k = k(z)$ continue et borneé, que $\int_0^c \left|\frac{\lambda(z)}{z}\right| dz < +\infty$ et que $-\infty < b < 0$.

En posant $\vartheta(\alpha, t) = \lambda(e^{\alpha + bt})$ on peut démontrer que l'équation Θ du § 6 possède la propriété Q.

Il en résulte que chaque intégrale de l'équation $\dot{x} = bx$ coïncide asymptotiquement du côté droit avec une intégrale de l'équation $\dot{x} = [b + \lambda(x)]\, x$. Il suffit, en effet, de soumettre ces équations à la transformation $x = e^{\alpha + bt}$, $t = t$.

§ 8. En remplaçant les ensembles h^* et k^* du Lemme 1 par des ensembles jouissant un rôle topologique analogue, on peut mettre le Théorème 2 sous une forme plus générale. L'exemple précédent fait entrevoir une autre direction de la généralisation possible.

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[4]) *Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires* (Ann. Soc. Polon. d. Math. t. 20, p. 279—313). Cf. aussi une note dans les Rendiconti dei Lincei (août 1947. Serie VIII, t. III, p. 210.)

Tadeusz Ważewski

O pewnych warunkach koincydencji asymptotycznej całek dwu układów równań różniczkowych

Komunikat wygłoszony na posiedzeniu w dniu 16 grudnia 1949 r.

Streszczenie

Autor nadaje nową postać wprowadzonemu przezeń poprzednio (na posiedzeniu P.A.U. z dnia 24.VI.1949) pojęciu koincydencji asymptotycznej całek dwu układów równań różniczkowych. W tym celu wprowadza pojęcie asymptotycznych końców całek, nawiązując do teorii „końców" (Ende) Caratheodory'ego. Autor podaje warunek wystarczający na to, aby ciągłemu przemieszczeniu całki jednego układu równań odpowiadała ciągła zmiana związanej z nią asymptotycznie całki drugiego układu. Autor wskazuje w końcu wystarczający warunek koincydencji całek układu z całkami tzw. układu banalnego. Warunek ten polega na porównaniu całek układu z całkami jednego równania. Twierdzenia swe ilustruje na prostym przykładzie.

Włodzimierz Parachoniak

Tortońska facja tufitowa między Bochnią a Tarnowem

Komunikat przedstawiony przez T. Wojno na posiedzeniu dnia 16 grudnia 1949 r.

Autor zajął się zagadnieniem występowania tufitów na Przedgórzu Karpat między Bochnią a Tarnowem (w utworach tortońskich) oraz możliwością wykorzystania ich dla celów stratygrafii i tektoniki wymienionego obszaru. Dotychczas przypuszczano istnienie jednego zasadniczego horyzontu, odpowiadającego jednemu z trzech horyzontów tufitowych stwierdzanych w Siedmiogrodzie, a występującego tam na granicy tortonu i sarmatu.

Autor na podstawie wyników analiz petrograficznych, wykonanych na 29 próbkach tufitów (zebranych w 18 punktach), przeprowadza korelację tych utworów z tufitami występującymi na Przedgórzu Karpat Wschodnich. Następnie wykazuje zgodność składu chemicznego szkliwa tufitu z Chodowic obok Stryja (występującego w tzw. warstwach daszawskich, wieku górno-tortońskiego) oraz szkliwa wyodrębnionego z tufitu z Bochni (występującego w warstwach chodenickich wieku dolno-tortońskiego), udowadniając tym wspólną genezę obu szkliw, wywodzących się prawdopodobnie z węgierskich ryolitów plagioklazowych.

Szczegółowe obserwacje terenowe nad występowaniem badanych tufitów, z uwzględnieniem badań ich struktury (uziarnienie) mogą mieć znaczenie dla rozwiązywania zagadnień tektoniki. Autor wykazał to na przykładzie opracowania odkrywki w Chodenicach, stwierdzając, że występująca w odkrywce seria tufitów stanowi serię odwróconą północnego skrzydła obalonej ku północy antykliny bocheńskiej.

Na podstawie zestawienia 16 analiz ciężkich minerałów, wyodrębnionych z tufitów, autor wykazuje zróżnicowanie w ich jakościowym składzie, co może mieć znaczenie dla celów korelacyjnych.

Michał Kamieński

Researches on the origin of the Comet Wolf I
Part VI

Heliocentric perturbations due to Jupiter and Saturn in the motion of the Comet Wolf I during the period

1809 April 27.5 — 1800 January 4.5

Mémoire présenté à la séance du 16 décembre 1949.

1. With the present article ends the first series of the author's investigations on the origin of the Comet Wolf I for the period 1884 — 1800. It refers to the influence of the two most powerful planets of the solar system, i. e. of Jupiter and Saturn, on the motion of the comet during the space of time 1809—1800.

The distances of the comet from Jupiter were in this period rather large, varying from 9.19 in the beginning of 1800 and diminishing to 4.11 in the second half of 1809. Hence it follows that the perturbations due to Jupiter ought not to be considerable for the said period. They were computed, as well as those for Saturn, by the method of variation of arbitrary constants. Owing to the large distances of the comet from Jupiter and Saturn, the elements could be changed every 50 days, what appeared to be quite sufficient for our purposes.

The perturbations due to Venus, the Earth, Mars and Uranus are small and can hardly influence general character of the comet's motion in the given period. Consequently, they were neglected.

The article was written twice: in Warsaw in 1944 and in Cracow in the summer of 1945, because the first manuscript had perished by fire when the Warsaw Observatory was being destroyed in August 1944.

2. The system P_{-15} of elements for 1809 April 27.5 Gr. Mean Time deduced in Part V—C of the author's „Researches ot the origin of the Comet Wolf I" was taken as a basis for computations:

1809 April 27.5 Greenwich Mean Time

$$P_{-15} \ldots \left.\begin{array}{ll} M = 326^{\circ}57'29''.0 & \Omega = 212^{\circ}52'31''.0 \\ \varphi = 23^{\circ}39'19''.3 & \pi = 12^{\circ}37'36''.2 \\ n = 422''3539 & i = 26^{\circ}48'22''.0 \end{array}\right\} 1950.0$$

With this system, the retrograde perturbations in the motion of the comet up to 1800 January 4.5 were computed. The differentials of perturbations are given in the Tables below. After their integration the author obtained:

1809 April 27.5 — 1800 January 4.5			
	Jupiter	Saturn	Total
δM	+ 170″.6	+ 548″.2	+ 718″.8
$\delta\varphi$	+ 63″.6	+ 101″.3	+ 164″.9
$\delta\Omega$	+ 46″.8	+ 60″.8	+ 107″.6
$\delta\pi$	+ 283″.7	— 303″.2	— 19″.5
δi	— 75″.2	— 27″.7	— 102″.9
δn	— 1″.0523	+ 0″.0688	— 0″.9835

Adding these totals to the system P_{-15} the author derived the following perturbed system P_{-16} of elements:

1800 January 4.5 Greenwich Mean Time

$$P_{-16} \ldots \left.\begin{array}{ll} M = 288^{\circ}16'4''.5 & \Omega = 212^{\circ}54'18''.6 \\ \varphi = 23^{\circ}42'4''.2 & \pi = 12^{\circ}37'16''.7 \\ n = 421''.3704 & i = 26^{\circ}46'39''.1 \end{array}\right\} 1950.0$$

3. The motion of this comet was investigated by the author in the period of 150 years, i. e. from the beginning of 1800 to the end of 1950. The observations, however, extend for the period 1884 — 1942 only. Since the theory of the comet agrees with the observations rather well, the author considers the extension of the retrograde computation of perturbations at least up to 1800 as being quite possible. It seems even feasible to prolong these computations for some decennials back, possibly to the middle of the XVIII century, although with decreasing exactitude.

In this article and in previous articles of the author, it was stated positively that the orbits of the comet and of Jupiter d i d n o t i n t e r s e c t in the said period, since the distance $\triangle$ m i n i m u m of these bodies was

$$\triangle = 0.1180 \text{ for 1875 June 8.7.}$$

Consequently, the comet could not be engendered as a product of ejection of matter from the surface of Jupiter in the beginning of the XIX century.

The author takes the liberty to express the opinion that the problem of comet's origin, particularly of short periodic comets, might be solved on the same lines as was done by Prof. H. Alfvén in his marvellous works concerning the origin of our planets (v. On the Cosmogony of the Solar System, Parts I—II—III, 1942—1945).

Warsaw, July 1944 — Cracow, November 1949.

JUPITER

1809 April 27.5 — 1800 Jan. 4.5

	$d\delta\Omega$	'f	$d\delta i$	'f	$d\delta\varphi$	'f	Δ
	″	″	″	″	″	″	
1800 Jan. 4.5	− 6·8	+ 50·2	− 3·4	− 73·5	− 8·1	+ 67·7	9·1865
Feb. 23.5	− 7·3	+ 43·4	− 3·1	− 76·9	− 7·7	+ 59·6	9·0296
Apr. 14.5	− 7·8	+ 36·1	− 2·7	− 80·0	− 7·3	+ 51·9	8·8706
June 3.5	− 8·1	+ 28·3	− 2·2	− 82·7	− 6·9	+ 44·6	8·7102
July 23.5	− 8·2	+ 20·2	− 1·7	− 84·9	− 6·6	+ 37·7	8·5490
Sept. 11.5	− 8·1	+ 12·0	− 1·1	− 86·6	− 6·3	+ 31·1	8·3874
Oct. 31.5	− 7·8	+ 3·9	− 0·5	− 87·7	− 6·2	+ 24·8	8·2243
Dec. 20.5	− 7·3	− 3·9	+ 0·1	− 88·2	− 6·4	+ 18·6	8·0588
1801 Feb. 8.5	− 6·5	− 11·2	+ 0·6	− 88·1	− 6·7	+ 12·2	7·8893
March 30.5	− 5·9	− 17·7	+ 1·2	− 87·5	− 7·3	+ 5·5	7·7122
May 19.5	− 4·4	− 23·6	+ 1·5	− 86·3	− 8·0	− 1·8	7·5256
July 8.5	− 3·1	− 28·0	+ 1·7	− 84·8	− 8·8	− 9.8	7·3265
Aug. 27.5	− 1·8	− 31·1	+ 1·9	− 83·1	− 9·3	− 18·6	7·1138
Oct. 16.5	− 0·7	− 32·9	+ 1·8	− 81·2	− 9·6	− 27·9	6·8908
Dec. 5.5	+ 0·3	− 33·6	+ 1·6	− 79·4	− 9·4	− 37·5	6·6612
1802 Jan. 24.5	+ 1·0	− 33·3	+ 1·3	− 77·8	− 8·9	− 46·9	6·4307
March 15.5	+ 1·4	− 32·3	+ 1·0	− 76·5	− 8·3	− 55·8	6·2083
May 4.5	+ 1·4	− 30·9	+ 0·7	− 75·5	− 7·8	− 64·1	5·9981
June 23.5	+ 1·2	− 29·5	+ 0·4	− 74·8	− 7·4	− 71·9	5·8036
Aug. 12.5	+ 0·7	− 28·3	+ 0·2	− 74·4	− 7·3	− 79·3	5·6270
Oct. 1.5	+ 0·2	− 27·6	0·0	− 74·2	− 7·5	− 86·6	5·4699
Nov. 20.5	− 0·4	− 27·4	− 0·1	− 74·2	− 7·9	− 94·1	5·3324
1803 Jan. 9.5	− 1·0	− 27·8	− 0·1	− 74·3	− 8·5	− 102·0	5·2138
Feb. 28.5	− 1·5	− 28·8	0·0	− 74·4	− 9·3	− 110·5	5·1140
Apr. 19.5	− 1·8	− 30·3	+ 0·1	− 74·4	− 10·1	− 119·8	5·0325
June 8.5	− 1·9	− 32·1	+ 0·2	− 74·3	− 10·9	− 129·9	4·9683
July 28·5	− 1·9	− 34·0	+ 0·2	− 74·1	− 11·7	− 140·8	4·9205
Sept. 16.5	− 1·7	− 35·9	+ 0·3	− 73·9	− 12·2	− 152·5	4·8891
Nov. 5.5	− 1·4	− 37·6	+ 0·3	− 73·6	− 12·6	− 164·7	4·8731
Dec. 25.5	− 1·1	− 39·0	+ 0·2	− 73·3	− 12·8	− 177·3	4·8719
1804 Feb. 13.5	− 0·7	− 40·1	+ 0·2	− 73·1	− 12·7	− 190·1	4·8845
Apr. 3.5	− 0·3	− 40·8	+ 0·1	− 72·9	− 12·2	− 202·8	4·9104
May 23.5	0·0	− 41·1	0·0	− 72·8	− 11·5	− 215·0	4·9488
		− 41·1		− 72·8		− 226·5	

JUPITER

1809 April 27.5 — 1800 Jan. 4.5

	$d\delta\pi$	'f	$\lambda d\delta n$	'f	''f	P	'f
1800 Jan. 4.5	−1″·9	+284″·5	+0″·357	−52″·789	+1480″·466	−15″·2	−1302″·1
Feb. 23.5	−3·0	+282·6	+0·395	−52·432	+1428·034	−13·1	−1317·3
Apr. 14.5	−3·8	+279·6	+0·446	−52·037	+1375·997	−11·3	−1330·4
June 3.5	−4·2	+275·8	+0·514	−51·591	+1324·406	−9·8	−1341·7
July 23.5	−4·3	+271·6	+0·603	−51·077	+1273·329	−8·7	−1351·5
Sept. 11.5	−4·0	+267·3	+0·723	−50·474	+1222·855	−7·9	−1360·2
Oct. 31.5	−3·4	+263·3	+0·867	−49·751	+1173·104	−7.4	−1368·1
Dec. 20.5	−2·7	+259·9	+1·049	−48·884	+1124·220	−7·0	−1375·5
1801 Feb. 8.5	−1·9	+257·2	+1·267	−47·835	+1076·385	−6·5	−1382·5
March 30.5	−1·3	+255·3	+1·513	−46·568	+1029·817	−5·7	−1389·0
May 19.5	−1·4	+254·0	+1·768	−45·055	+984·762	−4·4	−1394·7
July 8.5	−2·0	+252·6	+2·003	−43·287	+941·475	−2·3	−1399·1
Aug. 27.5	−3·1	+250·6	+2·180	−41·284	+900·191	+0·5	−1401·4
Oct. 16.5	−4·6	+247·5	+2·267	−39·104	+861·087	+3·7	−1400·9
Dec. 5.5	−5·9	+242·9	+2·250	−36·837	+824·250	+6·8	−1397·2
1802 Jan. 24.5	−6·6	+237·0	+2·137	−34·587	+789·663	+9·4	−1390·4
March 15.5	−6·6	+230·4	+1·958	−32·450	+757·213	+11·2	−1381·0
May 4.5	−5·8	+223.8	+1·742	−30·492	+726·721	+12·3	−1369·8
June 23.5	−4·4	+218·0	+1·507	−28·750	+697·971	+12·9	−1357·5
Aug. 12.5	−2·7	+213·6	+1·279	−27·243	+670·728	+13·1	−1344·6
Oct. 1.5	−1·0	+210·9	+1·066	−25·964	+644·764	+13·3	−1331·5
Nov. 20.5	+0·5	+209·9	+0·876	−24·898	+619·866	+13·7	−1318·2
1803 Jan. 9.5	+1·6	+210·4	+0·709	−24·022	+595·844	+14·5	−1304·5
Feb. 28.5	+2·3	+212·0	+0·568	−23·313	+572·531	+15·7	−1290·0
Apr. 19.5	+2·4	+214·3	+0·452	−22·745	+549·786	+17·6	−1274·3
June 8.5	+1·8	+216·7	+0·360	−22·293	+527·493	+19·9	−1256·7
Jnly 28.5	+0·6	+218·5	+0·290	−21·933	+505·560	+22·8	−1236·8
Sept. 16.5	−1·1	+219·1	+0·241	−21·643	+483·917	+26·3	−1214·0
Nov. 5.5	−3·4	+218·0	+0·209	−21·402	+462·515	+30·1	−1187·7
Dec. 25.5	−6·1	+214·6	+0·194	−21·193	+441·322	+34·2	−1157·6
1804 Feb. 13.5	−9·1	+208·5	+0·192	−20·998	+420·323	+38·3	−1123·4
Apr. 3.5	−12·2	+199·4	+0·202	−20·807	+399·516	+42·5	−1085·1
May 23.5	−15·3	+187·2	+0·220	−20·605	+378·911	+46·5	−1042·6
		+171·9		−20·385			−996·1

	$d\delta\Omega$	'f	$d\delta i$	'f	$d\delta\varphi$	'f	Δ
1804 May 23.5	0″·0	— 41″·1	0″·0	— 72″·8	— 11″·5	— 226″·5	4·9488
July 12.5	+ 0·3	— 40·8	— 0·1	— 72·9	— 10·6	— 237·1	4·9988
Aug. 31.5	+ 0·4	— 40·4	— 0·2	— 73·1	— 9·3	— 246·4	5·0585
Oct. 20.5	+ 0·4	— 40·0	— 0·2	— 73·3	— 7·9	— 254·3	5·1274
Dec. 9.5	+ 0·4	— 39·6	— 0·2	— 73·5	— 6·4	— 260·7	5·2053
1805 Jan. 28.5	+ 0·2	— 39·4	— 0·1	— 73·6	— 4·7	— 265·4	5·2902
March 19.5	0·0	— 39·4	0·0	— 73·6	— 3·0	— 268·4	5·3814
May 8.5	— 0·3	— 39·7	+ 0·2	— 73·4	— 1·2	— 269·6	5·4770
June 27.5	— 0·5	— 40·2	+ 0·4	— 73·0	+ 0·7	— 268·9	5·5759
Aug. 16.5	— 0·8	— 41·0	+ 0·7	— 72·3	+ 2·3	— 266·6	5·6770
Oct. 5.5	— 1·0	— 42·0	+ 1·1	— 71·2	+ 4·0	— 262·6	5·7791
Nov. 24.5	— 1·2	— 43·2	+ 1·5	— 69·7	+ 5·5	— 257·1	5·8814
1806 Jan. 13.5	— 1·3	— 44·5	+ 1·9	— 67·8	+ 7·0	— 250·1	5·9829
March 4.5	— 1·3	— 45·8	+ 2·3	— 65·5	+ 8·3	— 241·8	6·0823
Apr. 23.5	— 1·3	— 47·1	+ 2·7	— 62·8	+ 9·5	— 232·3	6·1774
June 12.5	— 1·1	— 48·2	+ 3·1	— 59·7	+ 10·6	— 221·7	6·2676
Aug. 1.5	— 0·8	— 49·0	+ 3·4	— 56·3	+ 11·5	— 210·2	6·3521
Sept. 20.5	— 0·5	— 49·5	+ 3·8	— 52·5	+ 12·2	— 198·0	6·4296
Nov. 9.5	— 0·1	— 49·6	+ 4·1	— 48·4	+ 12·8	— 185·2	6·4986
Dec. 29.5	+ 0·4	— 49·2	+ 4·3	— 44·1	+ 13·2	— 172·0	6·5583
1807 Feb. 17.5	+ 1·0	— 48·2	+ 4·4	— 39·7	+ 13·5	— 158·5	6·6072
Apr. 8.5	+ 1·5	— 46·7	+ 4·5	— 35·2	+ 13·6	— 144·9	6·6440
May 28.5	+ 2·1	— 44·6	+ 4·5	— 30·7	+ 13·5	— 131·4	6·6679
July 17.5	+ 2·7	— 41·9	+ 4·4	— 26·3	+ 13·3	— 118·1	6·6773
Sept. 5.5	+ 3·3	— 38·6	+ 4·3	— 22·0	+ 13·0	— 105·1	6·6708
Oct. 25.5	+ 3·8	— 34·8	+ 4·0	— 18·0	+ 12·5	— 92·6	6·6469
Dec. 14.5	+ 4·2	— 30·6	+ 3·7	— 14·3	+ 11·9	— 80·7	6·6042
1808 Feb. 2.5	+ 4·4	— 26·2	+ 3·4	— 10·9	+ 11·2	— 69·5	6·5426
March 23.5	+ 4·6	— 21·6	+ 2·9	— 8·0	+ 10·4	— 59·1	6·4595
May 12.5	+ 4·6	— 17·0	+ 2·5	— 5·5	+ 9·5	— 49·6	6·3536
July 1.5	+ 4·4	— 12·6	+ 2·0	— 3·5	+ 8·6	— 41·0	6·2237
Aug. 20.5	+ 4·0	— 8·6	+ 1·5	— 2·0	+ 7·8	— 33·2	6·0674
Oct. 9.5	+ 3·5	— 5·1	+ 1·0	— 1·0	+ 7·2	— 26·0	5·8819
Nov. 28.5	+ 2·7	— 2·4	+ 0·6	— 0·4	+ 6·8	— 19·2	5·6667
1809 Jan. 17.5	+ 1·8	— 0·6	+ 0·3	— 0·1	+ 6·9	— 12·3	5·4195

	$d\delta\pi$	$'f$	$\lambda\delta dn$	$'f$	$''f$	P	$'f$
1804 May 23.5	− 15″·3	+ 171″·9	+ 0″·220	−20″·385	+378″·911	+ 46″·5	− 996″·1
July 12.5	− 18·3	+ 153·6	+ 0·245	−20·140	+358·526	+ 50·1	− 946·0
Aug. 31.5	− 21·0	+ 132·6	+ 0·273	−19·867	+338·386	+ 53·3	− 892·7
Oct. 20·5	− 23·4	+ 109·2	+ 0·305	−19·562	+318·519	+ 56·0	− 836·7
Dec. 9.5	− 25·4	+ 83·8	+ 0·338	−19·224	+298·957	+ 58·0	− 778·7
1805 Jan. 28.5	− 27·0	+ 56·8	+ 0·371	−18·853	+279·733	+ 59·4	− 719·3
March 19.5	− 28·0	+ 28·8	+ 0·404	−18·449	+260·880	+ 60·2	− 659·1
May 8·5	− 28·5	+ 0·3	+ 0·436	−18·013	+242·431	+ 60·3	− 598·8
June 27.5	− 28·6	− 28·3	+ 0·466	−17·547	+224·418	+ 59·8	− 539·0
Aug. 16.5	− 28·1	− 56·4	+ 0·493	−17·054	+206·871	+ 58·7	− 480·3
Oct. 5.5	− 27·2	− 83·6	+ 0·521	−16·533	+189·817	+ 57·1	− 423·2
Nov. 24.5	− 25·8	− 109·4	+ 0·547	−15·986	+173·284	+ 55·0	− 368·2
1806 Jan. 13.5	− 24·1	− 133·5	+ 0·573	−15·413	+157·298	+ 52·5	− 315·7
March 4.5	− 22·0	− 155·5	+ 0·599	−14·814	+141·885	+ 49·5	− 266·2
Apr. 23.5	− 19·6	− 175·1	+ 0·625	−14·189	+127·071	+ 46·2	− 220·0
June 12.5	− 16·9	− 192·0	+ 0·651	−13·528	+112·882	+ 42·6	− 177·4
Aug. 1.5	− 14·0	− 206·0	+ 0·678	−12·860	+ 99·344	+ 38·7	− 138·7
Sept. 20.5	− 10·9	− 216·9	+ 0·709	−12·151	+ 86·484	+ 34·7	− 104·0
Nov. 9.5	− 7·7	− 224·6	+ 0·739	−11·412	+ 74·333	+ 30·5	− 73·5
Dec. 29.5	− 4·2	− 228·8	+ 0·772	−10·640	+ 62·921	+ 26·3	− 47·2
1807 Feb. 17.5	− 1·1	− 229·9	+ 0·806	− 9·834	+ 52·281	+ 22·0	− 25·2
Apr. 8.5	+ 2·3	− 227·6	+ 0·841	− 8·993	+ 42·447	+ 17·8	− 7·4
May 28.5	+ 5·6	− 222·0	+ 0·877	− 8·116	+ 33·454	+ 13·7	+ 6·3
July 17.5	+ 8·7	− 213·3	+ 0·913	− 7·203	+ 25·338	+ 9·8	+ 16·1
Sept. 5.5	+ 11·7	− 201·6	+ 0·947	− 6·256	+ 18·135	+ 6·1	+ 22·2
Oct. 25.5	+ 14·5	− 187·1	+ 0·977	− 5·279	+ 11·879	+ 2·7	+ 24·9
Dec. 14.5	+ 17·0	− 170·1	+ 1·000	− 4·279	+ 6·600	− 0·3	+ 24·6
1808 Feb. 2.5	+ 19·1	− 151·0	+ 1·013	− 3·266	+ 2·321	− 2·9	+ 21·7
March 23.5	+ 20·8	− 130·2	+ 1·009	− 2·257	− 0·945	− 4·9	+ 16·8
May 12.5	+ 21·9	− 108·3	+ 0·982	− 1·275	− 3·202	− 6·3	+ 10·5
July 1.5	+ 22·4	− 85·9	+ 0·921	− 0·354	− 4·477	− 6·9	+ 3·6
Aug. 20.5	+ 22·0	− 63·9	+ 0·792	+ 0·438	− 4·831	− 6·6	− 3·0
Oct. 9.5	+ 20·6	− 43·3	+ 0·629	+ 1·067	− 4·393	− 5·3	− 8·3
Nov. 28.5	+ 18·2	− 25·1	+ 0·351	+ 1·418	− 3·326	− 3·0	− 11·3
1809 Jan. 17.5	+ 14·3	− 10·8	− 0·065	+ 1·353	− 1·908	+ 0·6	− 10·7

	dδΩ	'f	dδi	'f	dδφ	'f	Δ
1809 Jan. 17.5	+ 1″·8	— 0″·6	+ 0″·3	— 0″·1	+ 6″·9	— 12″·3	5·4195
March 8.5	+ 0·7	+ 0·1	+ 0·1	0·0	+ 7·7	— 4·6	5·1402
April 27.5	— 0·4	— 0·3	0·0	0·0	+ 9·6	+ 5·0	4·8280
June 16.5	— 1·6	— 1·9	+ 0·1	+ 0·1	+ 13·0	+ 18·0	4·4841
Aug. 5.5	— 2·5		+ 0·4		+ 18·6		4·1131

SATURN

1809 April 27.5 — 1800 Jan. 4.5

	dδΩ	'f	dδi	'f	dδφ	'f	Δ
1800 Jan. 4.5	— 0″·5	+ 61″·0	— 0″·3	— 27″·6	— 0″·4	+ 101″·5	11·908
Feb. 23.5	— 0·6	+ 60·5	— 0·2	— 27·9	— 0·3	+ 101·1	11·912
Apr. 14.5	— 0·6	+ 59.9	— 0·2	— 28·1	— 0·2	+ 100·8	11·916
June 3.5	— 0·6	+ 59·3	— 0·2	— 28·3	— 0·2	+ 100·6	11·919
July 23.5	— 0·6	+ 58·7	— 0·1	— 28·5	— 0·2	+ 100·4	11·916
Sept. 11.5	— 0·6	+ 58·1	— 0·1	— 28·6	— 0·1	+ 100·2	11·908
Oct. 31.5	— 0·6	+ 57·5	0·0	— 28·7	— 0·1	+ 100·1	11·888
Dec. 20.5	— 0·5	+ 56·9	0·0	— 28·7	— 0·1	+ 100·0	11·852
1801 Feb. 8.5	— 0·5	+ 56·4	0·0	— 28·7	— 0·2	+ 99·9	11·790
March 30.5	— 0·4	+ 55·9	+ 0·1	— 28·7	— 0·3	+ 99·7	11·703
May 19.5	— 0·3	+ 55·5	+ 0·1	— 28·6	— 0·4	+ 99·4	11·568
July 8.5	— 0·2	+ 55·2	+ 0·1	— 28·5	— 0·5	+ 99·0	11·405
Aug. 27.5	— 0·1	+ 55·0	+ 0·1	— 28·4	— 0·6	+ 98·5	11·188
Oct. 16.5	0·0	+ 54·9	+ 0·1	— 28·3	— 0·6	+ 97·9	10·927
Dec. 5.5	0·0	+ 54·9	+ 0·1	— 28·2	— 0·6	+ 97·3	10·628
1802 Jan. 24.5	+ 0·1	+ 54·9	+ 0·1	— 28·1	— 0·6	+ 96·7	10·301
March 15.5	+ 0·1	+ 55·0	+ 0·1	— 28·0	— 0·5	+ 96·1	9·963
May 4.5	+ 0·1	+ 55·1	0·0	— 27·9	— 0·4	+ 95·6	9·616
June 23.5	0·0	+ 55·2	0·0	— 27·9	— 0·4	+ 95·2	9·271
Aug. 12.5	— 0·1	+ 55·2	0·0	— 27·9	— 0·3	+ 94·8	8·937
Oct. 1.5	— 0·2	+ 55·1	0·0	— 27·9	— 0·3	+ 94·5	8·613
Nov. 20.5	— 0·4	+ 54·9	0·0	— 27·9	— 0·4	+ 94·2	8·299
		+ 54·5		— 27·9		+ 93·8	

	$d\delta\pi$	'f	$\lambda d\delta n$	'f	''f	P	'f
1809 Jan. 17.5	+ 14''3	— 10''8	— 0''065	+ 1''353	— 1''908	+ 0''6	— 10''7
March 8.5	+ 9·1	— 1·7	— 0·670	+ 0·683	— 0·555	+ 5·5	— 5·2
April 27.5	+ 2·2	+ 0·5	— 1·540	— 0·857	+ 0·128	+ 11·5	+ 6·3
June 16.5	— 6·3	— 5·8	— 2·752	— 3·609	— 0·729	+ 18·2	+ 24·5
Aug. 5.5	— 15·8		— 4·408		— 4·338	+ 24·8	

SATURN

1809 April 27.5 — 1800 Jan. 4.5

	$d\delta\pi$	'f	$\lambda d\delta n$	'f	''f	P	'f
1800 Jan. 4.5	— 2''2	— 302''1	— 0''123	+ 3''500	+ 29''321	+ 1''7	+ 518''0
Feb. 23.5	— 2·1	— 304·3	— 0·117	+ 3·377	+ 32·698	+ 1·6	+ 519·7
Apr. 14.5	— 2·0	— 306·4	— 0·110	+ 3·260	+ 35·958	+ 1·5	+ 521·3
June 3·5	— 1·9	— 308·4	— 0·100	+ 3·150	+ 39·108	+ 1·3	+ 522·8
July 23.5	— 1·7	— 310·3	— 0·087	+ 3·050	+ 42·158	+ 1·1	+ 524·1
Sept. 11·5	— 1·6	— 312·0	— 0·071	+ 2·963	+ 45·121	+ 0·9	+ 525·2
Oct. 31·5	— 1·3	— 313·6	— 0·051	+ 2·892	+ 48·013	+ 0·7	+ 526·1
Dec. 20.5	— 1·1	— 314·9	— 0·026	+ 2·841	+ 50·824	+ 0·4	+ 526·8
1801 Feb. 8.5	— 0·9	— 316·0	+ 0·003	+ 2·815	+ 53·669	+ 0·2	+ 527·2
March 30.5	— 0·7	— 316·9	+ 0·036	+ 2·818	+ 56·487	+ 0·1	+ 527·4
May 19.5	— 0·5	— 317·6	+ 0·071	+ 2·854	+ 59·341	0·0	+ 527·5
July 8.5	— 0·4	— 318·1	+ 0·105	+ 2·925	+ 62·266	0·0	+ 527·5
Aug. 27.5	— 0·4	— 318·5	+ 0·133	+ 3·030	+ 65·296	+ 0·1	+ 527·5
Oct. 16.5	— 0·4	— 318·9	+ 0·148	+ 3·163	+ 68·459	+ 0·3	+ 527·6
Dec. 5.5	— 0·4	— 319·3	+ 0·150	+ 3·311	+ 71·770	+ 0·5	+ 527·9
1802 Jan. 24.5	— 0·4	— 319·7	+ 0·137	+ 3·461	+ 75·231	+ 0·6	+ 528·4
March 15.5	— 0·3	— 320·1	+ 0·113	+ 3·598	+ 78·829	+ 0·6	+ 529·0
May 4.5	0·0	— 320·4	+ 0·081	+ 3·711	+ 82·540	+ 0·5	+ 529·6
June 23.5	+ 0·3	— 320·4	+ 0·044	+ 3·702	+ 86·332	+ 0·3	+ 530·1
Aug. 12.5	+ 0·8	— 320·1	+ 0·005	+ 3·836	+ 90·168	0·0	+ 530·4
Oct. 1.5	+ 1·3	— 319·3	— 0·036	+ 3·841	+ 94·009	— 0·4	+ 530·4
Nov. 20.5	+ 1·8	— 318·0	— 0·076	+ 3·805	+ 97·814	— 0·9	+ 530·0
		— 316·2		+ 3·729			+ 529·1

	dδΩ	'f	dδi	'f	dδφ	'f	Δ
1802 Nov. 20·5	− 0″·4	+ 54″·5	0″·0	− 27″·9	− 0″·4	+ 93″·8	8·299
1803 Jan. 9.5	− 0·6	+ 53·9	0·0	− 27·9	− 0·4	+ 93·4	7·999
Feb. 28.5	− 0·8	+ 53·1	0·0	− 27·9	− 0·5	+ 92·9	7·714
Apr. 19.5	− 1·0	+ 52·1	0·0	− 27·9	− 0·6	+ 92·3	7·443
June 8.5	− 1·2	+ 50·9	+ 0·1	− 27·8	− 0·8	+ 91·5	7·185
July 28.5	− 1·4	+ 49·5	+ 0·2	− 27·6	− 1·0	+ 90·5	6·938
Sept. 16.5	− 1·6	+ 47·9	+ 0·3	− 27·3	− 1·2	+ 89·3	6·704
Nov. 5.5	− 1·9	+ 46·0	+ 0·4	− 26·9	− 1·4	+ 87·9	6·482
Dec. 25.5	− 2·1	+ 43·9	+ 0·5	− 26·4	− 1·7	+ 86·2	6·272
1804 Feb. 13.5	− 2·2	+ 41·7	+ 0·6	− 25·8	− 1·9	+ 84·3	6·071
Apr. 3.5	− 2·4	+ 39·3	+ 0·7	− 25·1	− 2·2	+ 82·1	5·878
May 23.5	− 2·6	+ 36·7	+ 0·9	− 24·2	− 2·5	+ 79·6	5·695
July 12.5	− 2·7	+ 34·0	+ 1·0	− 23·2	− 2·8	+ 76·8	5·521
Aug. 31.5	− 2·8	+ 31·2	+ 1·2	− 22·0	− 3·1	+ 73·7	5·355
Oct. 20.5	− 2·8	+ 28·4	+ 1·4	− 20·6	− 3·4	+ 70·3	5·199
Dec. 9.5	− 2·8	+ 25·6	+ 1·5	− 19·1	− 3·7	+ 66·6	5·053
1805 Jan. 28.5	− 2·8	+ 22·8	+ 1·6	− 17·5	− 4·0	+ 62·6	4·916
March 19.5	− 2·7	+ 20·1	+ 1·8	− 15·7	− 4·2	+ 58·4	4·788
May 8.5	− 2·5	+ 17·6	+ 1·9	− 13·8	− 4·4	+ 54·0	4·668
June 27.5	− 2·4	+ 15·2	+ 2·0	− 11·8	− 4·6	+ 49·4	4·558
Aug. 16.5	− 2·1	+ 13·1	+ 2·0	− 9·8	− 4·7	+ 44·7	4·460
Oct. 5.5	− 1·9	+ 11·2	+ 2·0	− 7·8	− 4·8	+ 39·9	4·373
Nov. 24.5	− 1·6	+ 9·6	+ 2·0	− 5·8	− 4·7	+ 35·2	4·300
1806 Jan. 13.5	− 1·3	+ 8·3	+ 1·9	− 3·9	− 4·6	+ 30·6	4·241
March 4.5	− 1·0	+ 7·3	+ 1·7	− 2·2	− 4·4	+ 26·2	4·197
Apr. 23.5	− 0·7	+ 6·6	+ 1·5	− 0·7	− 4·2	+ 22·0	4·169
June 12.5	− 0·5	+ 6·1	+ 1·3	+ 0·6	− 3·8	+ 18·2	4·158
Aug. 1.5	− 0·3	+ 5·8	+ 1·1	+ 1·7	− 3·4	+ 14·8	4·165
Sept. 20.5	− 0·1	+ 5·7	+ 0·8	+ 2·5	− 2·9	+ 11·9	4·192
Nov. 9.5	0·0	+ 5·7	+ 0·6	+ 3·1	− 2·5	+ 9·4	4·238
Dec. 29.5	0·0	+ 5·7	+ 0·3	+ 3·4	− 2·0	+ 7·4	4·305
1807 Feb. 17.5	0·0	+ 5·7	+ 0·1	+ 3·5	− 1·6	+ 5·8	4·392
Apr. 8.5	0·0	+ 5·7	− 0·1	+ 3·4	− 1·2	+ 4·6	4·500
May 28.5	− 0·1	+ 5·6	− 0·2	+ 3·2	− 0·9	+ 3·7	4·630
July 17.5	− 0·2	+ 5·4	− 0·3	+ 2·9	− 0·6	+ 3·1	4·784
Sept. 5.5	− 0·3	+ 5·1	− 0·4	+ 2·5	− 0·4	+ 2·7	4·961

	$d\delta\pi$	$'f$	$\lambda d\delta n$	$'f$	$''f$	P	$'f$
1802 Nov. 20.5	+ 1″·8	− 316″·2	− 0″·076	+ 3″·729	+ 97″·814	− 0″·9	+ 529″·1
1803 Jan. 9.5	+ 2·3	− 313·9	− 0·116	+ 3·613	+ 101·543	− 1·4	+ 527·7
Feb. 28.5	+ 2·8	− 311·1	− 0·156	+ 3·457	+ 105·156	− 2·0	+ 525·7
Apr. 19.5	+ 3·4	− 307·7	− 0·193	+ 3·264	+ 108·613	− 2·6	+ 523·1
June 8.5	+ 3·9	− 303·8	− 0·230	+ 3·034	+ 111·877	− 3·2	+ 519·9
July 28.5	+ 4·4	− 299·4	− 0·267	+ 2·767	+ 114·911	− 3·9	+ 516·0
Sept. 16.5	+ 4·8	− 294·6	− 0·302	+ 2·465	+ 117·678	− 4·5	+ 511·5
Nov. 5.5	+ 5·3	− 289·3	− 0·335	+ 2·130	+ 120·143	− 5·2	+ 506·3
Dec. 25.5	+ 5·7	− 283·6	− 0·368	+ 1·762	+ 122·273	− 5·9	+ 500·4
1804 Feb. 13.5	+ 6·0	− 277·6	− 0·400	+ 1·362	+ 124·035	− 6·6	+ 493·8
Apr. 3.5	+ 6·4	− 271·2	− 0·431	+ 0·931	+ 125·397	− 7·3	+ 486·5
May 23.5	+ 6·7	− 264·5	− 0·460	+ 0·471	+ 126·328	− 8·1	+ 478·4
July 12.5	+ 7·0	− 257·5	− 0·487	− 0·016	+ 126·799	− 8·9	+ 469·5
Aug. 31.5	+ 7·2	− 250·3	− 0·512	− 0·528	+ 126·783	− 9·7	+ 459·8
Oct. 30.5	+ 7·4	− 242·9	− 0·534	− 1·062	+ 126·255	− 10·5	+ 449·3
Dec. 9.5	+ 7·6	− 235·3	− 0·552	− 1·614	+ 125·193	− 11·3	+ 438·0
1805 Jan. 28.5	+ 7·7	− 227·6	− 0·564	− 2·178	+ 123·579	− 12·2	+ 425·8
March 19.5	+ 7·8	− 219·8	− 0·572	− 2·750	+ 121·401	− 13·1	+ 412·7
May 8.5	+ 7·9	− 211·9	− 0·572	− 3·322	+ 118·651	− 14·0	+ 398·7
June 27.5	+ 8·0	− 203·9	− 0·564	− 3·886	+ 115·329	− 15·0	+ 383·7
Aug. 16.5	+ 8·0	− 195·9	− 0·546	− 4·432	+ 111·443	− 15·9	+ 367·8
Oct. 5.5	+ 8·0	− 187·9	− 0·516	− 4·948	+ 107·011	− 16·9	+ 350·9
Nov. 24.5	+ 8·0	− 179·9	− 0·475	− 5·423	+ 102·063	− 17·8	+ 333·1
1806 Jan. 13.5	+ 8·1	− 171·8	− 0·421	− 5·844	+ 96·640	− 18·6	+ 314·5
March 4.5	+ 8·2	− 163·6	− 0·354	− 6·198	+ 90·796	− 19·3	+ 295·2
Apr. 23.5	+ 8·3	− 155·3	− 0·276	− 6·474	+ 84·598	− 20·0	+ 275·2
June 12.5	+ 8·4	− 146·9	− 0·187	− 6·661	+ 78·124	− 20·4	+ 254·8
Aug. 1.5	+ 8·6	− 138·3	− 0·092	− 6·753	+ 71·463	− 20·7	+ 234·1
Sept. 20.5	+ 8·8	− 129·5	+ 0·007	− 6·746	+ 64·710	− 20·8	+ 213·3
Nov. 9.5	+ 8·9	− 120·6	+ 0·105	− 6·641	+ 57·964	− 20·6	+ 192·7
Dec. 29.5	+ 9·1	− 111·5	+ 0·199	− 6·442	+ 51·323	− 20·2	+ 172·5
1807 Feb. 17.5	+ 9·2	− 102·3	+ 0·286	− 6·156	+ 44·881	− 19·6	+ 152·9
Apr. 8.5	+ 9·3	− 93·0	+ 0·361	− 5·795	+ 38·725	− 18·8	+ 134·1
May 28.5	+ 9·3	− 83·7	+ 0·424	− 5·371	+ 32·930	− 17·8	+ 116·3
July 17.5	+ 9·2	− 74·5	+ 0·470	− 4·901	+ 27·559	− 16·7	+ 99·6
Sept. 5.5	+ 9·0	− 65·5	+ 0·503	− 4·398	+ 22·658	− 15·4	+ 84·2

	$d\delta\Omega$	'f	$d\delta i$	'f	$d\delta\varphi$	'f	Δ
1807 Sept. 5.5	− 0″·3	+ 5″·1	− 0″·4	+ 2″·5	− 0″·4	+ 2″·7	4·961
Oct. 25.5	− 0·4	+ 4·7	− 0·4	+ 2·1	− 0·3	+ 2·4	5·160
Dec. 14.5	− 0·5	+ 4·2	− 0·4	+ 1·7	− 0·2	+ 2·2	5·379
1808 Feb. 2.5	− 0·5	+ 3·7	− 0·4	+ 1·3	− 0·2	+ 2·0	5·625
March 23.5	− 0·6	+ 3·1	− 0·4	+ 0·9	− 0·2	+ 1·8	5·895
May 12.5	− 0·6	+ 2·5	− 0·3	+ 0·6	− 0·2	+ 1·6	6·189
July 1.5	− 0·5	+ 2·0	− 0·2	+ 0·4	− 0·2	+ 1·4	6·502
Aug. 20.5	− 0·5	+ 1·5	− 0·2	+ 0·2	− 0·3	+ 1·1	6·839
Oct. 9.5	− 0·5	+ 1·0	− 0·1	+ 0·1	− 0·3	+ 0·8	7·209
Nov. 28.5	− 0·4	+ 0·6	− 0·1	0·0	− 0·3	+ 0·5	7·598
1809 Jan. 17.5	− 0·3	+ 0·3	0·0	0·0	− 0·2	+ 0·3	8·012
March 8.5	− 0·2	+ 0·1	0·0	0·0	− 0·2	+ 0·1	8·455
April 27.5	− 0·1	0·0	0·0	0·0	− 0·1	0·0	8·914
June 16.5	− 0·1	− 0·1	0·0	0·0	0·0	0·0	9·393
Aug. 5.5	0·0		0·0		+ 0·2		9·882

	$d\delta\pi$	'f	$\lambda d\delta n$	'f	''f	P	'f
1807 Sept. 5.5	+ 9"·0	− 65"·5	+ 0"·503	− 4"·398	+ 22"·658	− 15"·4	+ 84"·2
Oct. 25.5	+ 8·8	− 56·7	+ 0·520	− 3·878	+ 18·260	− 14·1	+ 70·1
Dec. 14.5	+ 8·4	− 48·3	+ 0·524	− 3·354	+ 14·382	− 12·8	+ 57·3
1808 Feb. 2.5	+ 7·9	− 40·4	+ 0·514	− 2·840	+ 11·028	− 11·3	+ 46·0
March 23.5	+ 7·4	− 33·0	+ 0·495	− 2·345	+ 8·188	− 9·9	+ 36·1
May 12.5	+ 6·8	− 26·2	+ 0·465	− 1·880	+ 5·843	− 8·6	+ 27.5
July 1.5	+ 6·1	− 20·1	+ 0·429	− 1·451	+ 3·963	− 7·3	+ 20·2
Aug. 20.5	+ 5·4	− 14·7	+ 0·386	− 1·065	+ 2·512	− 6·1	+ 14·1
Oct. 9.5	+ 4·6	− 10·1	+ 0·336	− 0·729	+ 1·447	− 4·9	+ 9·2
Nov. 28.5	+ 3·8	− 6·3	+ 0·282	− 0·447	+ 0·718	− 3·8	+ 5·4
1809 Jan. 17.5	+ 3·1	− 3·2	+ 0·225	− 0·222	+ 0·271	− 2·8	+ 2·6
March 8.5	+ 2·3	− 0·9	+ 0·164	− 0·058	+ 0·049	− 1·9	+ 0·7
April 27.5	+ 1·7	+ 0·8	+ 0·106	+ 0·048	− 0·009	− 1·2	− 0·5
June 16.5	+ 1·0	+ 1·8	+ 0·041	+ 0·089	+ 0·039	− 0·5	− 1·0
Aug. 5.5	+ 0·5		− 0·016		+ 0·128	− 0·1	

Tadeusz Banachiewicz

A general Least Squares interpolation formula

Note présentée à la séance du 16 décembre 1949.

We assume the form of the interpolation formula to be

$$y = a_0 \varphi_0(x) + a_1 \varphi_1(x) + a_2 \varphi_2(x) + \dots + a_m \varphi_m(x), \qquad (1)$$

where $a_0, a_1, a_2, \dots a_m$ are unknown constants and $\varphi_k(x)$, $k = 0, 1, 2, \dots m$, arbitrary functions of x, supposed to be given numerically for a set of values of x_i $(i = 1, 2, \dots n)$ for which also the values of y_i are given. The constants a_k are to be determined so as to minimize the sum of squares $\sum_{i=1}^{n} (y - y_i)^2$, and so that the determination of any further term in the interpolation formula (1) could be made by correcting the already calculated coëfficients a_k, but not by calculating them anew.

We denote by $\mathbf{A}$ the array (supposed having linearly independent columns), by $\mathbf{q}$ the (triangular) inverse of the triangular square root of $\mathbf{A}^2$, by $\mathbf{Q}$ the inverse of $\mathbf{A}^2$:

$$\mathbf{A} = \begin{Bmatrix} \varphi_0(x_1) & \varphi_1(x_1) \dots \varphi_m(x_1) \\ \varphi_0(x_2) & \dots\dots\dots \varphi_m(x_2) \\ \vdots & \vdots \\ \varphi_0(x_n) & \dots\dots\dots \varphi_m(x_n) \end{Bmatrix} \qquad (\mathbf{A}^2)^{-1} = \mathbf{q} \cdot \mathbf{q} = \mathbf{Q}. \qquad (2)$$

Let $\mathbf{y}$, $\mathbf{a}$ and $\varphi(x)$ denote the columns of y_i, a_k and $\varphi_k(x)$, $\mathbf{L} = \mathbf{y} \cdot \mathbf{A}$, then

$$\mathbf{a} = \mathbf{L} : \mathbf{A}^2 = \mathbf{L} \cdot \mathbf{Q} \qquad (3)$$

$$y = \mathbf{L} \cdot \mathbf{Q} \cdot \varphi = (\mathbf{L} \cdot \tau\, \mathbf{q})(\varphi \cdot \tau\, \mathbf{q}) \qquad (4)$$

τ being the Cracovian sign of the transpose.

Putting

$$\begin{Bmatrix} \tau\, \mathbf{q} \cdot \mathbf{L} \\ \tau\, \mathbf{q} \cdot \varphi \end{Bmatrix} = \begin{Bmatrix} K_0 & K_1 & \dots K_m \\ \psi_0(x) & \psi_1(x) & \dots \psi_m(x) \end{Bmatrix} \qquad (5)$$

we obtain eventually

$$y = K_0 \psi_0(x) + K_1 \psi_1(x) + \dots + K_m \psi_m(x), \qquad (6)$$

the notations in (6) being those of Tchebycheff, who solved the problem (1859), in a quite different manner, in the case $\varphi_k(x) = x^k$. The sum of the squares of residuals v of (1) for the given set of x_i is

$$[vv] = \mathbf{y}^2 - K_0^2 - K_1^2 - \ldots - K_m^2 ; \qquad (7)$$

the weight of K_k-s is that of the equations (1), the functions $\psi_k(x)$ are normalized and mutually orthogonal.

The quantities y and $\varphi_k(x)$ in (1) can also design variates, and the above formulae solve then the statistical problem of a multi-variate regression line. For an attempt to its non-Cracovian explicit solution cf. the paper of K. Pearson, *Biometrika*, **13**, 1921, p. 296. For more details see our paper to appear in the *Acta Astronomica*.

Tadeusz Banachiewicz

Ogólny wzór interpolacyjny wg metody najmniejszych kwadratów

Komunikat przedstawiony na posiedzeniu w dniu 16 grudnia 1949 r.

Streszczenie

Autor wyznacza współczynniki wzoru interpolacyjnego (1), w założeniu, że $\varphi_k(x)$ są to zupełnie dowolne jednoznaczne funkcje x_i, czyniące zadość warunkowi liniowej niezależności kolumn krakowianu **A**. Chodzi przytym o rozwiązanie o najmniejszej sumie kwadratów odchyleń, i takie, aby dla uwzględnienia potrzebnych ewentualnie dalszych wyrazów wystarczyło poprawiać znalezione już poprzednio współczynniki, a nie trzeba było obliczać ich nanowo. Problem ten postawił był i rozwiązał w 1859 r. Czebyszew dla $\varphi_k(x) = x^k$; rozwiązanie autora jest odmienne i opiera się na rachunku krakowianowym. *Mutatis mutandis* daje ono zarazem rozwiązanie wyraźne problemu statystyki matematycznej, dotyczącego wyznaczenia współczynników wielokrotnej regresji.

Julian Tokarski and Andrew Oberc

Petrographic research in the Eocene of the so-called Regle in the Tatry Mts.

Note présentée à la séance du 16 décembre 1949.

1. Four horizons of dolomite and dolomitic limestone beds were submitted to petrographic analysis; they appear in the Eocene formation, in the so-called Regle series of the Polish Tatra, in the quarry „Pod Capkami".

2. Chemical analysis showed a quantitative differentiation of the chief constituents, proceeding consistently from bottom to top. This differentiation comprises the increase in calcium carbonate content and the decrease of that of magnesium carbonate. The clay-sand fraction, insoluble in hydrochloric acid, also showed a consistent decrease towards the top. The amounts of this fraction soluble in acid were not great (less than 1%) and similar in all horizons.

3. Qualitative and quantitative microscopic examinations allowed to represent the rocks as composed of six discerned kinds of morphological elements, i.e. carbonate rock pebbles, carbonate rock pebbles mixed with clay, remains of organic skeletons, quartz grains, loose carbonate crystals and cement.

4. Heavy mineral analysis was performed in the lower three horizons; the following minerals were found: garnet, zircone, rutile, epidote, staurolite, tourmaline, biotite and chlorite. Although they were extracted from 100 gram samples, only 33, 38, and 42 grains in all were obtained for each horizon. Therefore, no concrete conclusions may be drawn from these results, except that the Eocens beds „Pod Capkami" are characterized by a small heavy mineral content.

5. It follows from the whole of the investigations that the analysed rocks are neither dolomites nor dolomitic limestones in a genetic meaning. Their present mineral composition is the result of an amassing in the Eocene sea terrigenic detritus composed of different amounts of proper, older (Triassic) dolomites and limestones.

6. The quarry „Pod Capkami" is closed. Its place is already being covered by young forest. Technical investigations, however, performed lately to estimate their industrial value, showed their great worth as building and road material.

Julian Tokarski i Andrzej Oberc

Z petrografii eocenu partii reglowej Tatr
(Kamieniołom „pod Capkami")

Komunikat przedstawiony przez czł. T. Wojno
na posiedzeniu w dniu 16 grudnia 1949 r.

Streszczenie

1. Poddano analizie petrograficznej cztery poziomy ławic dolomitów i wapieni dolomitycznych formacji eoceńskiej w partii reglowej Tatr polskich, występujących w kamieniołomie „pod Capkami".

2. Analiza chemiczna wykazała zróżnicowanie głównych składników, postępujące stopniowo od spągu ku stropowi, polegające na wzroście węglanu wapniowego przy jednoczesnym ubytku węglanu magnezowego. W tym samym kierunku zmniejsza się stopniowo zawartość składników ilasto-piaszczystych, nierozpuszczalnych w kwasie solnym.

3. Wśród elementów morfologicznych zauważono otoczaki skał węglanowych, takież otoczaki zmieszane z iłem, szczątki szkieletów organicznych, kwarc, luźne kryształy węglanów i spoiwo.

4. Z pośród ciężkich minerałów znaleziono w trzech dolnych poziomach: granat, cyrkon, rutyl, epidot, staurolit, turmalin, biotyt i chloryt w ilościach bardzo niedużych.

5. Zbadane skały powstały z nagromadzenia w morzu eoceńskim okruchów terrygenicznych, złożonych głównie z dolomitów i wapień okresu triasowego.

6. Eksploatacja kamieniołomu, zawierającego skądinąd cenny materiał budulcowy i drogowy, została ze względu na przylegający Park Narodowy zaniechana.

Zdzisław Macierewicz, Irena Chmielewska
i Janina Świętosławska-Ścisłowska

Methoxylation of pyronones II.
Absorption spectra of pyronones and their methyl ethers

Note présentée à la séance du 16 décembre 1949.

Investigations carried out by one of us (Z. M.) on the direction of the keto-enol transformation of substitutet α . β . pyronones proved that the methylation of α' . styryl . α . γ . pyronone by means of different reagents yield a derivative of α . pyrone.

The keto-enol transformations towards α or γ configuration give rise to two different structures, possessing characteristic and different absorption maximum in ultraviolet, namely that of double unsaturated δ . lactone or γ . pyrone.

The work has been undertaken by the authors to show the relationsship between the character of the absorption-curve and the structure of pyronones.

Data on the absorption curves of α' . methyl-, α' . phenyl- and α' . β' . phenylene-pyronones and their corresponding methyl ethers are tabulated (fig. 1—3). α' . methylpyronone, as well as its methyl ether possess the absorption maximum characteristic for configuration I, thus the keto-enol transformation towards γ . carbon atom has to be accepted. This conception is in agreement with the results of other workers on the chemical properties of the discussed compounds.

The absorption curves of α' . phenylpyronone and its methyl ether are different and correspond to the γ . pyrone curve. In this case them the keto-enol transformation is believed to take place towards the γ . pyrone configuration (II).

α' . β' . phenylenepyronone, for whitch 4 . hydroxycoumarin structure is assigned show the unexpected results. Spectrum of the methyl ether correspond to that of 4 . methoxycoumarin, but the absorption curve of the unsubstituted pyronone is characteristic for the γ . chromone structure.

Explanation of the above results is difficult as yet. In order to verify the suggested hypothesis further investigations are carried on including absorption measurements of the compounds of known structure and the examination of the chemical reactions of the investigated compounds.

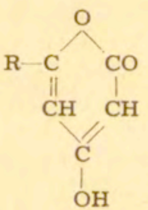
Zdzisław Macierewicz, Irena Chmielewska
i Janina Świętosławska-Ścisłowska

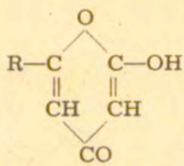
O metoksylowaniu pyrononów II. Widma absorpcyjne pyrononów i ich pochodnych metoksylowych

Komunikat przedstawiony dnia 16 grudnia 1949 r.

Streszczenie

Badania nad kierunkiem enolizacji α'. podstawionych α. γ. . pyrononów, prowadzone przez jednego z nas, wykazały, że α'. styrylo. α. γ. pyronon w wyniku metoksylowania różnymi środkami metylującymi wytwarza pochodną α. pyronu [1]). Ponieważ enolizacja w kierunku γ lub α związana jest z powstaniem dwóch różnych układów: podwójnie nienasyconego δ. laktonu (α. pyronu) I, lub też γ. pyronu II, zajęliśmy się zbadaniem, czy nie jest możliwe rozstrzygnięcie budowy pyrononów na zasadzie charakteru ich krzywej absorpcji w nadfiolecie.

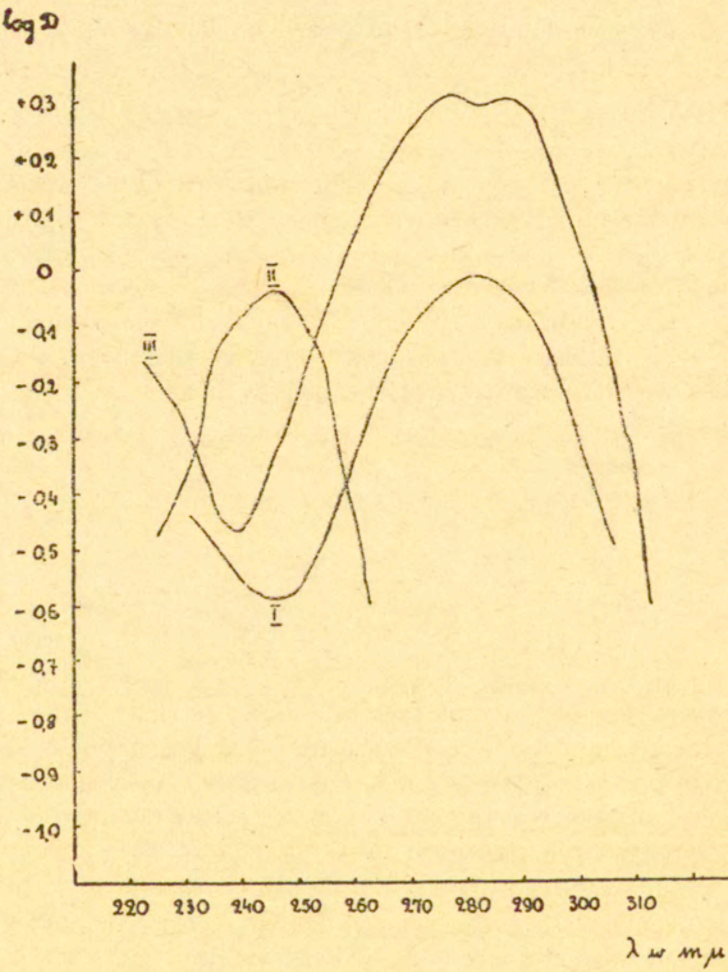

I.


II.

Podwójnie nienasycony sześcioczłonowy pierścień laktonowy, typu kwasu kumalinowego, wykazuje charakterystyczne max. absorpcji przy $\lambda = 280 - 300$ mμ, i min. przy λ około 250 mμ [2]), natomiast układowi γ. pyronu odpowiada charakterystyczne dla α. β. nienasyconych ketonów max. przy $\lambda = 230 - 250$ mμ [3]).

Przeprowadziliśmy pomiar absorpcji α'. metylo- α'. fenylo- i α'. β'. fenyleno. α. γ. pyrononów i ich eterów metylowych. Otrzymane wyniki podane są w zestawieniu i na wykresach 1—3, obok krzywych absorpcji odpowiednich związków o budowie α lub γ pyronów.

Jak wynika z badań Wittera i Stolza [4]) α' metylopyrononowi odpowiada min. absorpcji przy $\lambda = 245$ mμ i max. przy $\lambda = 280$ mμ. Ten sam typ krzywej posiada zbadany przez nas jego eter metylowy (rys. 1), należy zatem przyjąć dla α'. metylopyrononu enolizację w kierunku węgla γ (I R = CH_3), zgodnie z danymi, uzyskanymi przez innych badaczy na podstawie właściwości chemicznych tego związku [5]).



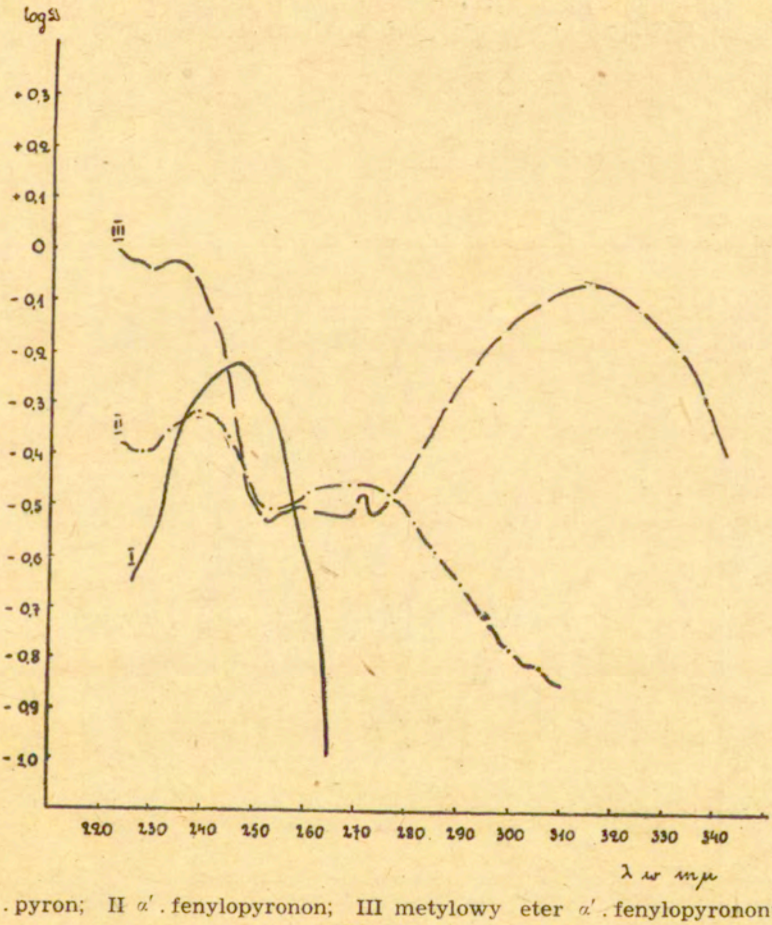
I α'. metylopyronon; II γ. pyron; III metylowy eter α'. metylopyrononu.

Krzywa γ.pyronu wg J. Am. Chem. Soc. 52, 4900 (1930),

krzywa α'.metylopyrononu wg J. Biol. Chem. 176, 455 (1948).

Rys. 1.

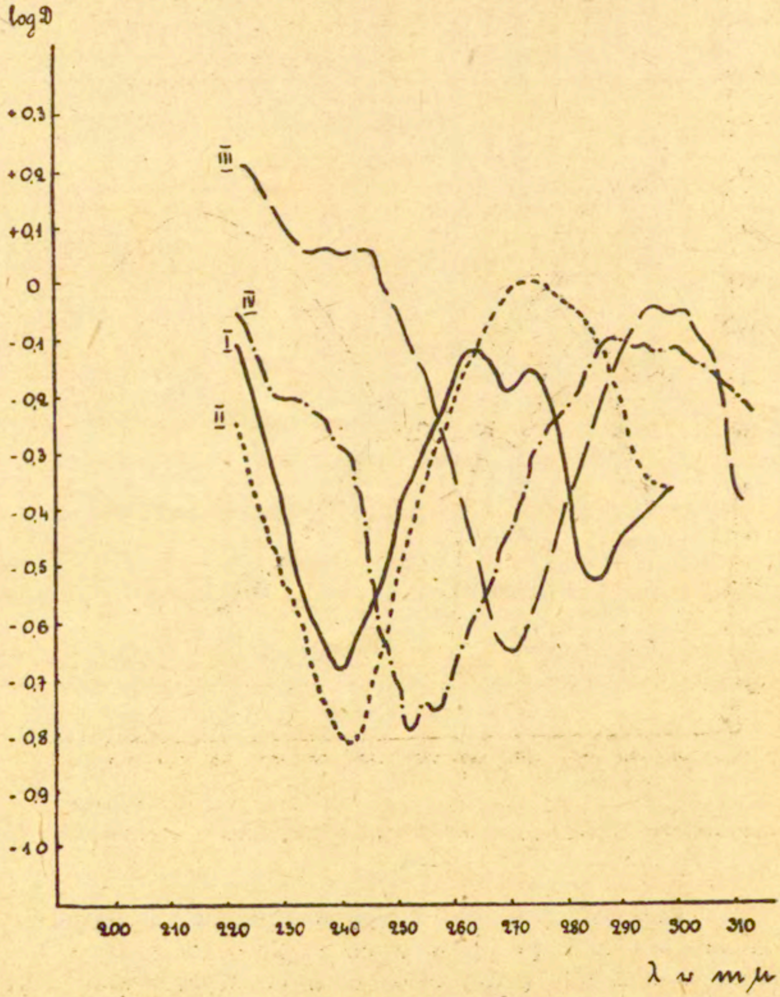
Krzywa absorpcji α'. fenylopyrononu i jego metylowego eteru [6]) (rys. 2) różni się zasadniczo od typu krzywych odpowiednich α'. metylopochodnych — oba związki wykazują max. przy $\lambda = 235 - 238$ mμ, charakterystyczne dla ugrupowania α. β. nienasyconego ketonu — w tym przypadku γ. pyronu.



I γ. pyron; II α'. fenylopyronon; III metylowy eter α'. fenylopyrononu.
Rys. 2.

Wynik ten nie jest zgodny z ogólnie przyjętym poglądem [5]), że enolizacja cyklowego ugrupowania —CO—CH_2—CO—O— zachodzi w kierunku grupy karbonylowej. Występowanie max. absorpcji przy $\lambda = 235$ mμ przemawia za tym, że enolizacja α'. fenylopyrononu odbywa się w kierunku ugrupowania laktonowego (wzór II $R = C_6H_5$).

α'.β'.fenylenopyronon, dla którego na podstawie właściwości chemicznych ustalono budowę 4.hydroksykumaryny, dał wyniki zupełnie nieoczekiwane. Jego metylowy eter wykazuje widmo absorpcyjne charakterystyczne dla podwójnie nienasyconego δ.laktonu, tzn. posiada budowę 4.metoksykumaryny, natomiast max. absorpcji niepostanowionego pyrononu przy $\lambda = 235$ mμ (rys. 3) jest charakterystyczne dla układu γ.chromonu.



I metylowy eter 4.hydroksykumaryny; II kumaryna; III chromon; IV 4.hydroksykumaryna.

Rys. 3.

Zestawienie	λ max.	λ min.
1. γ.pyron.	245	—
2. α'.metylopyronon	281	245
3. metyl. eter α'.metylopyrononu .	287	239
4. α'.fenylopyronon	238, 270	252
5. metyl. eter α'.fenylopyrononu .	235, 258, 270	252, 272
6. chromon	238, 245, 296	285
7. metyl. eter 4.hydroksykumaryny	264, 274	240, 285
8. 4.hydroksykumaryna	235, 287	252, 258
9. kumaryna	274	242

Wytłumaczenie powyższych faktów jest narazie trudne, to też prowadzone są dalsze badania, zmierzające do potwierdzenia podanych wniosków, dotyczących budowy opisanych pyrononów, zarówno na drodze chemicznej jak i przez pomiar absorpcji związków o budowie zbliżonej, a ustalonej z przebiegu ich syntezy.

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TREŚĆ — TABLE DES MATIÈRES

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