

## 178.

ON THE A POSTERIORI DEMONSTRATION OF THE PORISM OF  
THE IN-AND-CIRCUMSCRIBED TRIANGLE.

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IN my former paper "On the Porism of the In-and-circumscribed Triangle" (*Journal*, t. I. p. 344 [175]) the two porisms (the homographic and the allographic) were established *à priori*, i.e. by means of an investigation of the order of the curve enveloped by the third side of the triangle. I propose in the present paper to give the *à posteriori* demonstration of these two porisms; first according to Poncelet, and then in a form not involving (as do his demonstrations) the principle of projections. My objection to the employment of the principle may be stated as follows: viz. that in a systematic development of the subject, the theorems relating to a particular case and which are by the principle in question extended to the general case, are not in anywise more simple or easier to demonstrate than are the theorems for the general case; and, consequently that the circuitry of the method can and ought to be avoided.

The porism (homographic) of the in-and-circumscribed triangle, viz.

If a triangle be inscribed in a conic, and two of the sides envelope conics having double contact with the circumscribed conic, then will the third side envelope a conic having double contact with the circumscribed conic.

The following is Poncelet's demonstration, the Nos. are those of the *Traité des Propriétés Projectives* [Paris, 1822]:

No. 431. If a triangle be inscribed in a circle and two of the sides are parallel to given lines, then the third side envelopes a concentric circle.

This is evident, for, the angle in the segment subtended by the third side being constant, the length of the third side is constant; hence, the length of the perpendicular from the centre upon the third side is also constant, and the third side envelopes a concentric circle.

Hence, by the principle of projections:

If a triangle be inscribed in a conic and two of the sides pass through given points, the remaining side envelopes a conic having double contact with the circumscribed conic, the line through the two points being the chord of contact.

No. 434. Conversely, if there be a triangle inscribed in a conic and the first side envelope a conic having double contact with the circumscribed conic, and the second side pass through a fixed point in the chord of contact, then will the third side also pass through a fixed point in the chord of contact.

No. 437. In particular, if there be a triangle inscribed in a conic and two of the sides pass through fixed points, then will the third side pass through a fixed point, viz. the point forming with the other two points a conjugate system.

No. 439. It follows that:

If there be a triangle inscribed in a conic and the first side passes through a fixed point, and the second side envelopes a conic having double contact with the circumscribed conic, then will the third side envelope a conic having double contact with the circumscribed conic.

For the chord of contact meets the polar of the fixed point with respect to the circumscribed conic in a point; the line joining this point with the third angle (i.e. the angle opposite the third side) of the triangle meets the conic in a variable point; and joining this variable point with the first and second angles of the triangle we have a new triangle; two of the sides of this new triangle (by Nos. 434 and 437) pass through fixed points; hence the remaining side, i.e. the third side of the original triangle, touches a conic having double contact with the circumscribed conic.

We have thus passed from the case of the two sides passing through fixed points to that of one of the two sides enveloping a conic having double contact with the given conic and the other of them passing through a fixed point; and, by a repetition of the reasoning, Poncelet passes to the general case, viz.

If there be a triangle inscribed in a conic, and two of the sides envelope conics having double contact with the circumscribed conic, then will the third side envelope a conic having double contact with the circumscribed conic.

But it is somewhat more simple to omit the intermediate case of a conic and point, and pass directly, by the reasoning of No. 439, from the case of two points to that of two conics.

In fact, considering the point of intersection of the two chords of contact, the line joining this point with the third angle of the triangle meets the conic in a variable point, and joining this variable point with the first and second angles of the triangle we have a new triangle: two of the sides of this new triangle (by No. 434) pass through fixed points; hence the remaining side, i.e. the third side of the original triangle, envelopes a conic having double contact with the circumscribed conic; and the general case is thus established.

The porism (allographic) of the in-and-circumscribed triangle, viz.

If a triangle be inscribed in a conic and two of the sides envelope conics meeting the circumscribed conic in the same four points, then the third side will touch a conic meeting the circumscribed conic in the four points.

The following is Poncelet's demonstration :

No. 433. In the particular case of the homographic porism, viz.—that in which two of the sides of the triangle pass through fixed points and the remaining side envelopes a conic having double contact with the circumscribed conic—it is easy to see that the lines joining the angles of the triangle with the two fixed points and with the point of contact on the third side, meet in a point; this follows at once by the principle of projection from the case in No. 431, viz. the case of a triangle inscribed in a circle when two of the sides are parallel to given lines and the third side touches a concentric circle. Hence,

No. 531. If there be a triangle inscribed in a conic, and two of the sides envelope fixed curves, and the third side envelopes a certain curve; the lines joining the angles of the triangle with the points of contact meet in a point.

In fact, attending only to the infinitesimal variation of the position of the triangle, the curves enveloped by the first and second sides may be replaced by the points of contact on these sides, and the curve enveloped by the third side may be replaced by a conic having double contact with the circumscribed conic, and the general case thus follows at once from the particular one.

Nos. 162 and 163. LEMMA<sup>(1)</sup>. If, on the sides of a triangle  $ABC$ , there are taken any three points  $L, M, N$  in the same line, and the harmonics  $A', B', C'$  of these points (i.e. the harmonic of each point with respect to the two vertices on the same side of the triangle), then the lines  $AA', BB', CC'$  will meet in a point; and, moreover, if  $A'L, B'M, C'N$  are bisected in  $F, G, H$  (or, what is the same thing, if  $\overline{FA}^2 = \overline{FB} \cdot \overline{FC}$ ,  $\overline{GB}^2 = \overline{GC} \cdot \overline{GA}$ ,  $\overline{HC}^2 = \overline{HA} \cdot \overline{HB}$ ), then will the three points  $F, G, H$  lie in a line. This is, in fact, the theorem No. 164,—In any complete quadrilateral the middle points of the three diagonals lie in a line.

It is now easy to prove a particular case of the allographic porism, viz.

No. 531. If there be a triangle inscribed in a circle, such that two of the sides envelope circles having a common secant (real or ideal) with the circumscribed circle; then will the third side envelope a circle having the same common secant with the circumscribed circle.

For if the triangle be  $ABC$ , and the points of contact of the sides  $CB, CA$  with the enveloped circles and the point of contact of the side  $AB$  with the enveloped curve, be  $A', B', C'$ ; if moreover the points of intersection of the circumscribed circle and the two enveloped circles be  $M, N$ , and the common secant  $MN$  meet the sides

<sup>1</sup> I have not thought it necessary to give the figures; they can be supplied without difficulty.

of the triangle in  $F, G, H$ ; then  $F, G, H$  and  $A', B', C'$  are points on the sides of the triangle  $ABC$ , such that  $F, G, H$  lie in a line, and  $AA', BB', CC'$  meet in a point. And by a property of the circle

$$\overline{FA'}^2 = FM \cdot FN = FB \cdot FC,$$

$$\overline{GB'}^2 = GM \cdot GN = GC \cdot GA;$$

whence by the lemma (or rather its converse)  $\overline{HC'}^2 = HA \cdot HB$  and by a property of the circle  $HA \cdot HB = HM \cdot HN$ ; and therefore,  $\overline{HC'}^2 = HM \cdot HN$ , a property which can only belong to a circle having, with the other circles, the common secant  $MN$ : the particular case is thus demonstrated. And the principle of projections leads at once to the general case of the allographic porism.

To exhibit the demonstrations in a form independent of the principle of projections, it will be convenient to enunciate the following three lemmas: the first of them being, in fact, the theorem contained in No. 434, as generalised by No. 531; the second of them a theorem connected with and including the properties of the circle assumed in Poncelet's demonstration of the allographic porism; and the third of them a theorem derivable by the principle of projections from the theorem in Nos. 162 and 163.

LEMMA I. If there be a triangle inscribed in a conic, such that two of the sides envelope given curves and the third side envelopes a curve; then the lines joining the angles of the triangle with the points of contact of the opposite sides meet in a point.

LEMMA II. If there be three conics meeting in the same four points, then any line meets the conic in six points forming a system in involution.

COROLL. 1. If the line be a tangent to one of the conics, then the point of contact is the double or sibi-conjugate point of the involution formed by the points of intersection with the other two conics. And conversely if the curve enveloped by the line is not given, but the preceding property holds for all positions of the tangent line; then the curve enveloped by such line is a conic passing through the points of intersection of the two given conics.

COROLL. 2. If one of the conics be a pair of coincident lines, then the other two conics are conics having double contact, with the line in question for their chord of contact; any line meets the chord of contact in a point which is a double or sibi-conjugate point of the involution formed by the points of intersection with the other two conics; and if the line be a tangent to one of the conics, then the point of contact and the point of intersection with the chord of contact are harmonics with respect to the points of intersection with the other conic. And conversely if every tangent of a curve intersect a line and conic in such manner that the point of contact and the point of intersection with the line are harmonics with respect to the points of intersection with the conic; then the curve is a conic having double contact with the given conic, and the line in question is the chord of contact.

The third lemma is a theorem (first explicitly stated, so far as I am aware, by Steiner, *Lehrsätze* 24 and 25, *Crelle*, t. III. [1828], p. 212, and demonstrated by Bauer, t. XIX. [1839], p. 227) which, in a note in the *Phil. Mag.*, Augt. 1853 [118], I have called the *Theorem of the harmonic relation of two lines with respect to a quadrilateral*.

LEMMA III. If on each of the diagonals of a quadrilateral there be taken two points harmonically related with respect to the angles upon this diagonal; then if three of the points lie in a line, the other three points will also lie in a line: the two lines are said to be harmonically related with respect to the quadrilateral.

The relation may be exhibited under a different form. The three diagonals of the quadrilateral form a triangle, the sides of which contain the six angles of the quadrilateral; and considering only three of the six angles (one angle on each diagonal) these three angles are points which either lie in a line, or else are such that the lines joining them with the opposite angles of the triangle meet in a point. Each of the three points is, with respect to the involution formed by the two angles of the triangle and the two points harmonically related thereto, a double or sibi-conjugate point, and we have thus a theorem of the harmonic relation of two lines to a triangle and line, or else to a triangle and point, viz. *Theorem*, If on the sides of a *triangle* there be taken three points which either lie in a *line* or else are such that the lines joining them with the opposite angles of the triangle meet in a *point*; and if on each side of the triangle there be taken two points forming with the two angles on the same side an involution having the first-mentioned point on the same side for a double or sibi-conjugate point; then if three of the six points lie in a *line*, the other three of the six points will also lie in a *line*; the two lines are said to be harmonically related to the triangle and line, or (as the case may be) to the triangle and point.

The proof of the two porisms is by the preceding lemmas rendered very simple.

Demonstration of the homographic porism.

First, the particular case, where two of the sides pass through fixed points. Lemma I. gives the construction of the point of contact on the third side, and the figure shows that the point of contact and the point in which the third side is intersected by the line through the two given points are harmonics with respect to the points of intersection of the third side with the circumscribed conic. Hence, (Lemma II. Coroll. 2) the curve touched by the third side is a conic having double contact with the circumscribed conic, and the chord of contact is the line joining the two given points; and conversely if one of the sides touch a conic having double contact with the circumscribed conic and another of the sides passes through a fixed point on the chord of contact, then the third side will also pass through a fixed point on the chord of contact. The general case is deduced from the particular one precisely as before, viz. where two of the sides touch conics having double contact with the circumscribed conic, then considering the point of intersection of the two chords of contact, the line joining this point with the third angle of the triangle meets the circumscribed conic in a variable point, and joining this variable point with the first

and second angles of the triangle, we have a new triangle, two of the sides of which (by the converse of the particular case) pass through fixed points: hence the remaining side, i.e. the third side of the original triangle, touches a conic having double contact with the circumscribed conic.

Demonstration of the allographic porism.

Let  $ABC$  be the triangle,  $A', B', C'$  the points of contact on the three sides, then by Lemma I. the lines  $AA', BB', CC'$  meet in a point. Take a pair of lines passing through the points of intersection of the circumscribed conic with the two given conics enveloped by the sides  $CA, CB$ , and let one of these lines meet the sides of the triangle in the points  $F, G, H$ , and the other of them meet the sides of the triangle in the points  $F', G', H'$ . Then considering the following three conics, viz. the last-mentioned pair of lines, the circumscribed conic, and the conic enveloped by the side  $CA$ ; these are conics passing through the same four points, and the side  $CA$  is a tangent to one of them: hence by Lemma II. Coroll. 1,  $G, G', C, A$  will be an involution having the point  $B'$  for a double or sibi-conjugate point, and similarly  $F, F', G, B$  are an involution having the point  $A'$  for a double or sibi-conjugate point. It follows from Lemma III. that  $H, H', A, B$  will be an involution having  $C'$  for a double or sibi-conjugate point. Hence by Lemma II. Coroll. 1 (the two given conics being the before-mentioned pair of lines and the circumscribed conic) the curve enveloped by the side  $AB$  will be a conic passing through the points of intersection of the pair of lines and the circumscribed conic, or, what is the same thing, the points of intersection of the circumscribed conic and the conics enveloped by the other two sides.

2, *Stone Buildings*, Oct. 2, 1856.