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ON THE AREA OF THE CONIC SECTION REPRESENTED BY THE GENERAL TRILINEAR EQUATION OF THE SECOND DEGREE.

[From the *Quarterly Mathematical Journal*, vol. II. (1858), pp. 248—253.]

[THE original title was "Direct Investigation of the Question discussed in the Foregoing Paper," viz. a Paper with the present title by N. M. Ferrers (now Dr Ferrers), pp. 247—248. The area S of the conic section represented by the general equation $(A, B, C, A', B', C'\between x, y, z)^2 = 1$, where the coordinates are connected by the equation $x + y + z = 1$, was by considerations founded on the form of the function found to be

$$S = \frac{2\pi (AA'^2 + BB'^2 + CC'^2 - ABC - 2A'B'C')\, \Delta}{\{A'^2 - BC + B'^2 - CA + C'^2 - AB + 2(B'C' - AA') + 2(C'A' - BB') + 2(A'B' - CC')\}^{\frac{3}{2}}},$$

where Δ is the area of the fundamental triangle: and it was remarked that a similar method might be applied to determine the area of the conic section when it is defined by the distances of its several tangents from three given points.]

The position of a point P being determined as in the foregoing paper, let α, β, γ denote in like manner the coordinates of a point O, we have

$$\alpha + \beta + \gamma = 1,$$

and consequently if ξ, η, ζ are the relative coordinates $x - \alpha, y - \beta, z - \gamma$, we have

$$\xi + \eta + \zeta = 0.$$

The expression for the distance of the two points O, P is readily obtained in terms of the relative coordinates, viz. calling this distance r, we have

$$r^2 = L\xi^2 + M\eta^2 + N\zeta^2,$$

where, if l, m, n are the sides of the triangle ABC, we have

$$L = \tfrac{1}{2}(m^2 + n^2 + l^2),$$
$$M = \tfrac{1}{2}(n^2 + l^2 - m^2),$$
$$N = \tfrac{1}{2}(l^2 + m^2 - n^2);$$

and it is to be remarked that these values give

$$MN + NL + LM = \tfrac{1}{4}(2m^2n^2 + 2n^2l^2 + 2l^2m^2 - l^4 - m^4 - n^4), = 4\Delta^2,$$

if Δ denote the area of the triangle ABC.

Consider now a conic

$$(a, b, c, f, g, h \between x, y, z)^2,$$

and suppose as usual that $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ are the inverse coefficients and that K is the discriminant, suppose also for shortness

$$P = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \between 1, 1, 1)^2.$$

The coordinates of the centre being α, β, γ, we have

$$\alpha = \frac{1}{P}(\mathfrak{A}, \mathfrak{H}, \mathfrak{G} \between 1, 1, 1),$$

$$\beta = \frac{1}{P}(\mathfrak{H}, \mathfrak{B}, \mathfrak{F} \between 1, 1, 1),$$

$$\gamma = \frac{1}{P}(\mathfrak{G}, \mathfrak{F}, \mathfrak{C} \between 1, 1, 1),$$

and writing as before ξ, η, ζ for $x-\alpha$, $y-\beta$, $z-\gamma$, so that ξ, η, ζ are the coordinates of a point P of the conic, in relation to the centre, we have x, y, z respectively equal to $\xi+\alpha$, $\eta+\beta$, $\zeta+\gamma$, and the equation of the conic gives

$$(a, \ldots \between \xi+\alpha, \eta+\beta, \zeta+\gamma)^2 = 0,$$

which may be written

$$\begin{aligned} &(a, \ldots \between \xi, \eta, \zeta)^2 \\ &+ 2(a, \ldots \between \alpha, \beta, \gamma)(\xi, \eta, \zeta) \\ &+ \ (a, \ldots \between \alpha, \beta, \gamma)^2 = 0. \end{aligned}$$

Now observing the equations

$$(a,\ h,\ g\between \alpha,\ \beta,\ \gamma) = \frac{K}{P},$$

$$(h,\ b,\ f\between \alpha,\ \beta,\ \gamma) = \frac{K}{P},$$

$$(g,\ f,\ c\between \alpha,\ \beta,\ \gamma) = \frac{K}{P},$$

we have

$$(a,\ \ldots\between \alpha,\ \beta,\ \gamma)\ (\xi,\ \eta,\ \zeta) = \frac{K}{P}(\xi+\eta+\zeta) = 0,$$

$$(a,\ \ldots\between \alpha,\ \beta,\ \gamma)^2 = \frac{K}{P}(\alpha+\beta+\gamma) = \frac{K}{P},$$

and the equation of the conic gives therefore

$$(a,\ \ldots\between \xi,\ \eta,\ \zeta)^2 + \frac{K}{P} = 0,$$

and we have as before

$$\xi+\eta+\zeta = 0.$$

To find the axes we have only to make

$$r^2, = L\xi^2 + M\eta^2 + N\zeta^2,$$

a maximum or minimum, ξ, η, ζ varying subject to the preceding two conditions; this gives

$$(a,\ h,\ g\between \xi,\ \eta,\ \zeta) + \lambda L\xi + \mu = 0,$$
$$(h,\ b,\ f\between \xi,\ \eta,\ \zeta) + \lambda M\eta + \mu = 0,$$
$$(g,\ f,\ c\between \xi,\ \eta,\ \zeta) + \lambda N\zeta + \mu = 0,$$

and multiplying by ξ, η, ζ, adding and reducing, we have

$$-\frac{K}{P} + \lambda r^2 = 0,$$

which gives

$$\lambda = \frac{K}{Pr^2}.$$

Substituting this value, and joining to the resulting three equations the equation

$$\xi+\eta+\zeta = 0,$$

we may eliminate ξ, η, ζ, μ, and the result is

$$\begin{vmatrix} a+\frac{KL}{Pr^2}, & h, & g, & 1 \\ h, & b+\frac{KM}{Pr^2}, & f, & 1 \\ g, & f, & c+\frac{KN}{Pr^2}, & 1 \\ 1, & 1, & 1 & \end{vmatrix}=0,$$

which may also be written

$$(\mathfrak{A}', \mathfrak{B}', \mathfrak{C}', \mathfrak{F}', \mathfrak{G}', \mathfrak{H}' \between 1, 1, 1)^2 = 0,$$

where $(\mathfrak{A}', \ldots)$ are what $(\mathfrak{A}, \ldots)$ become when a, b, c are changed into

$$a+\frac{KL}{Pr^2}, \quad b+\frac{KM}{Pr^2}, \quad c+\frac{KN}{Pr^2};$$

we in fact have

$$\mathfrak{A}' = \mathfrak{A} + \frac{K}{Pr^2}(bN + cM) + \frac{K^2}{P^2r^4}MN,$$

$$\vdots$$

$$\mathfrak{F}' = \mathfrak{F} - \frac{K}{Pr^2}Lf,$$

$$\vdots$$

and (observing the value of P) the result consequently is

$$P + \frac{K}{Pr^2}\{(b+c-2f)L + (c+a-2g)M + (a+b-2h)N\} + \frac{K^2}{P^2r^4}(MN + NL + LM) = 0,$$

which may also be written

$$P^3r^4 + PKr^2\{(b+c-2f)L + (c+a-2g)M + (a+b-2h)N\} + 4\Delta^2K^2 = 0.$$

Hence if r_1, r_2 are the two semiaxes, we have

$$r_1{}^2r_2{}^2 = \frac{4\Delta^2K^2}{P^3},$$

and the area is $\pi r_1 r_2$ which is equal to

$$\frac{2\pi K\Delta}{\sqrt{(P^3)}},$$

which agrees with Mr Ferrers' result.

The formula $r^2 = L\xi^2 + M\eta^2 + N\zeta^2$ which is assumed in the preceding investigation may be proved as follows:

Writing a, b, c (instead of l, m, n) for the sides of the fundamental triangle and A, B, C for the angles, the equation in question is

$$r^2 = bc \cos A \; \xi^2 + ca \cos B \, \eta^2 + ab \cos C \, \zeta^2.$$

Now writing α, β, γ for the inclinations of the line r to the sides of the triangle, we have

$$A = \beta - \gamma,$$

$$B = \gamma - \alpha,$$

$$C = \pi + \alpha - \beta.$$

Moreover taking for a moment λ, μ, ν to denote the perpendiculars from the angles on the opposite sides, we have

$$\lambda = c \sin B = b \sin C,$$

$$\mu = a \sin C = c \sin A,$$

$$\nu = b \sin A = a \sin B,$$

and

$$\xi = \frac{r \sin \alpha}{\lambda}, \quad \eta = \frac{r \sin \beta}{\mu}, \quad \zeta = \frac{r \sin \gamma}{\nu};$$

the values of ξ^2, η^2, ζ^2 consequently are

$$\frac{r^2 \sin^2 \alpha}{bc \sin B \sin C}, \; \frac{r^2 \sin^2 \beta}{ca \sin C \sin A}, \; \frac{r^2 \sin^2 \gamma}{ab \sin A \sin B},$$

and the equation to be proved becomes

$$1 = \frac{\cos A \sin^2 \alpha}{\sin B \sin C} + \frac{\cos B \sin^2 \beta}{\sin C \sin A} + \frac{\cos C \sin^2 \gamma}{\sin A \sin B},$$

or, what is the same thing,

$$\sin A \sin B \sin C = \sin A \cos A \sin^2 \alpha + \sin B \cos B \sin^2 \beta + \sin C \cos C \sin^2 \gamma,$$

or again

$$4 \sin A \sin B \sin C = \sin 2A \, (1 - \cos 2\alpha) + \sin 2B \, (1 - \cos 2\beta) + \sin 2C \, (1 - \cos 2\gamma),$$

or putting for A, B, C their values in terms of α, β, γ this is

$$\begin{aligned} -4 \sin (\beta - \gamma) \sin (\gamma - \alpha) \sin (\alpha - \beta) = \; & \sin (2\beta - 2\gamma) \, (1 - \cos 2\alpha) \\ & + \sin (2\gamma - 2\alpha) \, (1 - \cos 2\beta) \\ & + \sin (2\alpha - 2\beta) \, (1 - \cos 2\gamma), \end{aligned}$$

which is an identical equation; it is most readily proved by writing x, y, z for $\tan\alpha$, $\tan\beta$, $\tan\gamma$; the equation thus becomes

$$\frac{-4}{(1+x^2)(1+y^2)(1+z^2)}(y-z)(z-x)(x-y)$$
$$=\Sigma\,\frac{1}{(1+y^2)(1+z^2)}\{2y(1-z^2)-2z(1-y^2)\}\frac{2x^2}{1+x^2},$$

or multiplying out

$$-(y-z)(z-x)(x-y)=\Sigma\,(y-z)(1+yz)\,x^2=\Sigma x^2(y-z)+xyz\,\Sigma x\,(y-z),$$

that is

$$-(y-z)(z-x)(x-y)=\Sigma x^2(y-z) \qquad =x^2(y-z)+y^2(z-x)+z^2(x-y),$$

which is an identity.

[A different investigation of the formula $r^2=L\xi^2+M\eta^2+N\zeta^2$, by Dr Ferrers, was appended to the original Paper.]