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A MEMOIR ON THE PROBLEM OF THE ROTATION OF A SOLID BODY.

[From the *Memoirs of the Royal Astronomical Society*, vol. XXIX. (1861), pp. 307—342. Read May 11, 1860.]

THE present memoir was written for the sake of the further elaboration of the analytical theory of the Rotation of a Solid Body, upon principles similar to those of my "Memoir on the Problem of Disturbed Elliptic Motion," *Mem. R. Ast. Soc.* vol. XXVII. pp. 1—29 (1858) [212]; the like elements are adopted, and the course of the investigation corresponds precisely to that of the memoir just referred to. The formulæ for the variations of the elements in the two problems (the motion being in each case referred to a fixed plane of reference and origin of angles therein) are found to be (as it is known they should be) identical in form; an investigation, in the present memoir, of the transformation of the system to the case of a variable plane of reference and departure-point as an origin of angles in such plane, would have been a mere repetition of that contained in the former memoir, and it was therefore unnecessary to give it. A point in the present memoir to which attention may be called is the definition of the angle g (varying uniformly with the time, but used as an element) which corresponds to the mean anomaly in elliptic motion. Besides the ultimate system of formulæ for the variations of the elements in terms of the differential coefficients of the Disturbing Function with respect to the elements, it appears to me that the intermediate formulæ for the variations in terms of the differential coefficients with respect to the coordinates (which are in the ordinary investigation altogether passed over) are not without interest, and that it is possible that they might be employed with advantage in the integration of the equations of motion for the purposes of physical astronomy.

I.

In the theory of elliptic motion, where the elements are

a, the mean distance,

e, the eccentricity,

g, the mean anomaly,

ϖ, the departure of pericentre,

θ, the longitude of node,

σ, the departure of node,

ϕ, the inclination,

and if, besides, we have

n, the mean motion ($n^2a^3 =$ sum of the masses),

and Ω denote the disturbing function taken with Lagrange's sign ($\Omega = -R$, if R be the disturbing function of the *Mécanique Céleste*), then the formulæ for the variations of the elements are

$$da = \frac{2}{na}\frac{d\Omega}{dg}dt,$$

$$de = \frac{1-e^2}{na^2e}\frac{d\Omega}{dg}dt - \frac{\sqrt{1-e^2}}{na^2e}\frac{d\Omega}{d\varpi}dt,$$

$$dg = -\frac{2}{na}\frac{d\Omega}{da}dt - \frac{1-e^2}{na^2e}\frac{d\Omega}{de}dt,$$

$$d\varpi = \frac{\sqrt{1-e^2}}{na^2e}\frac{d\Omega}{de}dt,$$

$$d\phi = -\frac{\cot\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\sigma}dt - \frac{\operatorname{cosec}\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\theta}dt,$$

$$d\sigma = \frac{\cot\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\phi}dt,$$

$$d\theta = \frac{\operatorname{cosec}\phi}{na^2\sqrt{1-e^2}}\frac{d\Omega}{d\phi}dt,$$

where $\Omega = \Omega(a, e, g, \varpi, \phi, \sigma, \theta)$.

And if in these equations we write

$$h = -\frac{1}{a}(\text{sum of the masses}) = -n^2a^2,$$

$$k = na^2\sqrt{1-e^2};$$

then attending to the equation $dn = -\frac{3}{2}\frac{n}{a}da$, we have

$$dh = n^2 a\,da,$$

$$dk = \tfrac{1}{2}na\sqrt{1-e^2}\,da - \frac{na^2 e}{\sqrt{1-e^2}},$$

and thence

$$\frac{d\Omega}{da} = n^2 a\frac{d\Omega}{dh} + \tfrac{1}{2}na\sqrt{1-e^2}\frac{d\Omega}{dk},$$

$$\frac{d\Omega}{de} = -\frac{n^2 a}{\sqrt{1-e^2}}\frac{d\Omega}{dk};$$

and the formulæ are very easily transformed into

$$dh = 2n\frac{d\Omega}{dg}dt,$$

$$dg = -2n\frac{d\Omega}{dh}dt,$$

$$dk = \frac{d\Omega}{d\varpi}dt,$$

$$d\varpi = \frac{d\Omega}{dk}dt,$$

$$d\phi = -\frac{\cot\phi}{k}\frac{d\Omega}{d\sigma}dt - \frac{\operatorname{cosec}\phi}{k}\frac{d\Omega}{d\theta}dt,$$

$$d\sigma = \frac{\cot\phi}{k}\frac{d\Omega}{d\phi}dt,$$

$$d\theta = \frac{\operatorname{cosec}\phi}{k}\frac{d\Omega}{d\phi}dt,$$

where $\Omega = \Omega(h, g, k, \varpi, \phi, \sigma, \theta)$.

This is the system of formulæ which will be obtained in the sequel for the variation of the elements in the problem of rotation, the new meanings of the symbols being explained *post*, Art. IV.

And if in either of the two problems, instead of the angles ϕ, σ, θ, which refer to a fixed plane of reference and origin of angles therein, we have the angles Φ, Σ, Θ, referring to a variable plane of reference and departure-point as an origin of angles in such plane, the position of these in respect to the fixed plane of reference and origin of angles therein being determined by

θ', the longitude of node,

σ', the departure of node,

ϕ', the inclination,

and if S' denote $\Theta - \sigma'$, then the system is

$$dh = \quad 2n \frac{d\Omega}{dh} dt,$$

$$dg = - \quad 2n \frac{d\Omega}{dh} dt,$$

$$dk = \quad \frac{d\Omega}{d\varpi} dt,$$

$$d\varpi = - \quad \frac{d\Omega}{dk} dt,$$

$$d\Phi = - \quad \frac{\cot\Phi}{k}\frac{d\Omega}{d\Sigma} dt - \frac{\operatorname{cosec}\phi}{k}\frac{d\Omega}{d\Theta} dt - \quad (\cos S' d\phi' + \sin S' \sin\phi' d\theta'),$$

$$d\Sigma = \quad \frac{\cot\Phi}{k}\frac{d\Omega}{d\Phi} dt \quad + \operatorname{cosec}\Phi\,(\sin S' d\phi' - \cos S' \sin\phi' d\theta'),$$

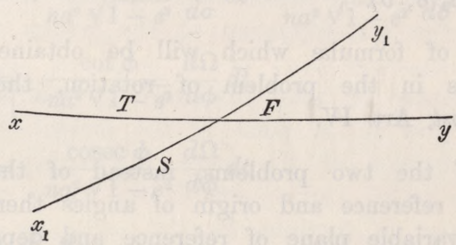
$$d\Theta = \quad \frac{\operatorname{cosec}\Phi}{k}\frac{d\Omega}{d\Phi} dt \quad + \quad \cot\Phi\,(\sin S' d\phi' - \cos S' \sin\phi' d\theta'),$$

where $\Omega = \Omega\,(h,\ g,\ k,\ \varpi,\ \Phi,\ \Sigma,\ \Theta,\ \phi',\ \sigma',\ \theta')$, or what is the same thing, $\Omega = \Omega\,(h,\ g,\ k,\ \varpi,\ \Phi,\ \Sigma,\ \Theta)$. As already remarked, it is not necessary to repeat in the present memoir the transformation to this set of formulæ.

II.

Considering, now, the problem of rotation, let the axes xyz denote axes fixed in space, and the axes $x_,y_,z_,$ denote the principal axes of the body; and if, to fix the ideas, xy is called the ecliptic and $x_,y_,$ the equator (the ecliptic xy being considered as a fixed circle of the sphere and the origin of longitudes x as a fixed point therein), then we may write

T, the longitude of node,
S, the departure of node,
F, the inclination.



And if, as usual, p, q, r are the angular velocities round the principal axes, or axes of $x_,$, $y_,$, and $z_,$, from $y_,$ to $z_,$, $z_,$ to $x_,$, and $x_,$ to $y_,$ respectively, so that

$$\begin{aligned} pdt &= -\sin S \sin F\, dT + \cos S\, dF, \\ qdt &= \cos S \sin F\, dT + \sin S\, dF, \\ rdt &= \cos F\, dT - dS; \end{aligned}$$

and if A, B, C are the moments of inertia, then the *Vis Viva* function or sum of the elements of mass, each into the half square of its velocity, is

$$\mathbf{T} = \tfrac{1}{2}(Ap^2 + Bq^2 + Cr^2);$$

and if Ω be the disturbing function (taken with Lagrange's sign), then the equations of motion

$$\frac{d}{dt}\frac{d\mathbf{T}}{dT} - \frac{d\mathbf{T}}{dT} = \frac{d\Omega}{dT}, \text{ \&c.}$$

give as usual

$$-\left[A\frac{dp}{dt}+(C-B)qr\right]\sin S\sin F+\left[B\frac{dq}{dt}+(A-C)rp\right]\cos S\sin F+\left[C\frac{dr}{dt}+(B-A)pq\right]\cos F=\frac{d\Omega}{dT},$$

$$\left[A\frac{dp}{dt}+(C-B)qr\right]\cos S+\left[B\frac{dq}{dt}+(A-C)rp\right]\sin S = \frac{d\Omega}{dF},$$

$$-\left[C\frac{dr}{dt}+(B-A)pq\right] = \frac{d\Omega}{dS},$$

or, as these equations may be written,

$$A\frac{dp}{dt} + (C-B)\,qr = -\sin S\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right) + \cos S\frac{d\Omega}{dF},$$

$$B\frac{dq}{dt} + (A-C)\,rp = \cos S\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right) - \sin S\frac{d\Omega}{dF},$$

$$C\frac{dr}{dt} + (B-A)\,pq = -\frac{d\Omega}{dS};$$

which, with the equations for p, q, r, determine the motion of the body.

III.

First, to integrate the equations of motion when the disturbing forces are neglected; we have, as usual, the integral equations

$$A\,p^2 + B\,q^2 + C\,r^2 = h,$$

$$A^2p^2 + B^2q^2 + C^2r^2 = k^2;$$

where h, k are constants of integration, viz., h is the constant of *Vis Viva*, and k is the constant of the principal area.

Moreover, if the coefficients α, β, &c. are those which belong to the transformation from xyz to $x_iy_iz_i$, viz., if in the table

	$x_{\prime}$	$y_{\prime}$	$z_{\prime}$
x	α	β	γ
y	α'	β'	γ'
z	α''	β''	γ''

the values of the coefficients are

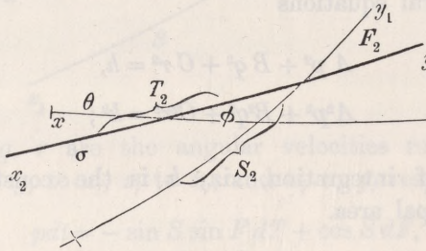
	$x_{\prime}$	$y_{\prime}$	$z_{\prime}$
x	$\cos S \cos T + \sin S \sin T \cos F$	$\sin S \cos T - \cos S \sin T \cos F$	$\sin T \sin F$
y	$\cos S \sin T - \sin S \cos T \cos F$	$\sin S \sin T + \cos S \cos T \cos F$	$-\cos T \sin F$
z	$-\sin S \sin F$	$\cos S \sin F$	$\cos F$

Then we have also, as usual, the integral equations

$$\begin{aligned} Ap\alpha \ + Bq\beta \ + Cr\gamma &= \ \ k \sin\theta \sin\phi, \\ Ap\alpha' + Bq\beta' + Cr\gamma &= -k \cos\theta \sin\phi, \\ Ap\alpha'' + Bq\beta'' + Cr\gamma'' &= \ \ k \cos\phi\,; \end{aligned}$$

where θ, ϕ are constants of integration which determine the position of the principal plane (or invariable plane, in the undisturbed motion). Considering now a new system of axes $x_2y_2z_2$, where x_2 and z_2 are in the principal plane, and x_2 is in the first instance considered as an arbitrary fixed point therein (afterwards when the plane is treated as variable, x_2 is assumed to be a departure-point), let the position of the new set of axes in reference to the axes xyz be determined by

θ, the longitude of the node,
σ, the departure of the node,
ϕ, the inclination:



so that the relation between the coordinates xyz and $x_2y_2z_2$ is given by the table

	x_2	y_2	z_2
x	a	b	c
y	a'	b'	c'
z	a''	b''	c''

where the values of the coefficients are given by

	x_2	y_2	z_2
x	$\cos\sigma\cos\theta+\sin\sigma\sin\theta\cos\phi$	$\sin\sigma\cos\theta-\cos\sigma\sin\theta\cos\phi$	$\sin\theta\sin\phi$
y	$\cos\sigma\sin\theta-\sin\sigma\cos\theta\cos\phi$	$\sin\sigma\sin\theta+\cos\sigma\cos\theta\cos\phi$	$-\cos\theta\sin\phi$
z	$\sin\sigma\sin\phi$	$\cos\sigma\sin\phi$	$\cos\phi$

and let the position of the axes $x_,y_,z_,$ in reference to the new axes $x_2y_2z_2$, be determined in a similar manner by

T_2, the longitude of node,

S_2, the departure of node,

F_2, the inclination;

so that we have the table

	$x_,$	$y_,$	$z_,$
x_2	α_2	β_2	γ_2
y_2	α_2'	β_2'	γ_2'
z_2	α_2''	β_2''	γ_2''

where the values of the coefficients are given by

	$x_{\prime}$	$y_{\prime}$	$z_{\prime}$
x_2	$\cos S_2 \cos T_2 + \sin S_2 \sin T_2 \cos F_2$	$\sin S_2 \cos T_2 - \cos S_2 \sin T_2 \cos F_2$	$\sin T_2 \sin F_2$
y_2	$\cos S_2 \sin T_2 - \sin S_2 \cos T_2 \cos F_2$	$\sin S_2 \sin T_2 + \cos S_2 \cos T_2 \cos F_2$	$-\cos T_2 \sin F_2$
z_2	$-\sin S_2 \sin F_2$	$\cos S_2 \sin F_2$	$\cos F_2$

The values of α_2, β_2, γ_2 are

$$a\alpha + a'\alpha' + a''\alpha'',$$
$$a\beta + a'\beta' + a''\beta'',$$
$$a\gamma + a'\gamma' + a''\gamma'',$$

with similar expressions for α_2', β_2', γ_2' and α_2'', β_2'', γ_2''; the last-mentioned three integrals are thus transformable into the form

$$Ap\alpha_2 + Bq\beta_2 + Cr\gamma_2 = 0,$$
$$Ap\alpha_2' + Bq\beta_2' + Cr\gamma_2' = 0,$$
$$Ap\alpha_2'' + Bq\beta_2'' + Cr\gamma_2'' = k,$$

which are, in fact, the equations which show that the plane x_2y_2 is the principal plane, or plane of maximum area.

The equations just obtained, attending to the values of α_2'', β_2'', γ_2'', give

$$Ap = -k \sin S_2 \sin F_2,$$
$$Bq = k \cos S_2 \sin F_2,$$
$$Cr = k \cos F_2;$$

and, since the expressions for p, q, r, in terms of T_2, S_2, F_2, must be similar to those in terms of T, S, F, we have

$$pdt = -\sin S_2 \sin F_2 dT_2 + \cos S_2 dF_2,$$
$$qdt = \cos S_2 \sin F_2 dT_2 + \sin S_2 dF_2,$$
$$rdt = \cos F_2 dT_2 - dS_2;$$

from which equations,

$$\sin F_2 dT_2 = (-p \sin S_2 + q \cos S_2)\, dt;$$

and substituting for $\sin S_2$, $\cos S_2$, the values $\dfrac{-Ap}{k \sin F_2}$, $\dfrac{Bq}{k \sin F_2}$, this becomes

$$k^2 \sin^2 F_2 dT_2 = k\,(Ap^2 + Bq^2)\, dt;$$

or, what is the same thing,

$$(k^2 - C^2r^2)\, dT_2 = k\,(h - Cr^2)\, dt.$$

The two equations $Ap^2+Bq^2+Cr^2=h$, $A^2p^2+B^2q^2+C^2r^2$ may be considered as determining p, q, in terms of r, and the equation $C\frac{dr}{dt}+(B-A)pq=0$, then gives

$$dt=\frac{-C\,dr}{(B-A)\,pq},$$

whence also the equation for dT_2 becomes

$$dT_2=\frac{k(h-Cr^2)}{k^2-C^2r^2}\;\frac{-C\,dr}{(B-A)\,pq}.$$

Instead of the time t, I consider a function $g=nt+\text{const.}$, n being for the present an arbitrary constant quantity which may be a function of the constants h and k; we have thus

$$dg=n\frac{-C\,dr}{(B-A)\,pq},$$

and the integral equation is

$$g=n\int\frac{-C\,dr}{(B-A)\,pq}.$$

The equation for dT_2 gives, in like manner,

$$T_2=\varpi+\int\frac{k(h-Cr^2)}{k^2-C^2r^2}\;\frac{-C\,dr}{(B-A)\,pq};$$

where it is assumed that the integrals are each of them taken from $r=r_0$, r_0 being an arbitrary constant value, say a function of h and k. The quantity g is analogous to the mean anomaly in elliptic motion, or rather it will become so when the significations of n and r_0 are fixed; it is considered as implicitly involving a constant of integration, and, consequently, no constant of integration is added to the integral: as regards T_2, the constant of integration is ϖ, which denotes the initial value (corresponding to $r=r_0$) of the angle T_2.

IV.

Recapitulating the integral equations, we have

$$\begin{aligned}
A\,p^2+B\,q^2+C\,r^2&=h,\\
A^2p^2+B^2q^2+C^2r^2&=k^2,\\
Ap&=-k\sin S_2\sin F_2,\\
Bq&=k\cos S_2\sin F_2,\\
Cr&=k\cos F_2,\\
g&=n\int\frac{-C\,dr}{(B-A)\,pq},
\end{aligned}$$

which equations give r, p, q, and thence S_2 and F_2 in terms of g and the constants h and k.

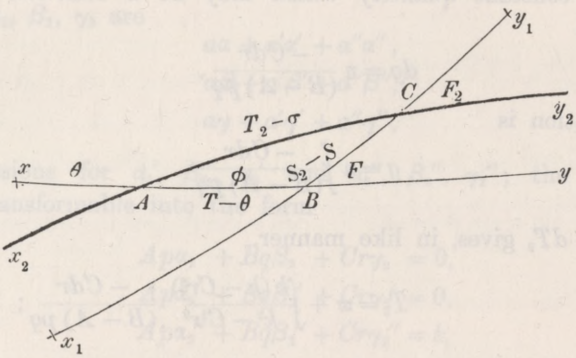
Moreover

$$T_2 = \varpi + \int \frac{k(h - Cr^2)}{k^2 - C^2r^2} \frac{-Cdr}{(B - A)pq},$$

which gives T_2 in terms of g and the constants h, k, ϖ; and then T, S, F are given in terms of T_2, S_2, F_2, and the constants θ, σ, ϕ, as follows, viz., we have a spherical triangle ABC, the sides whereof a, b, c, and the opposite angles A, B, C, are respectively,

Sides	$S_2 S$,	$T_2 - \sigma$,	$T - \theta$,
Opposite angles	ϕ,	$180^\circ - F$,	F_2,

as appears by the figure.



The above values of p, q, r, give

$$h - Cr^2 = \left(k^2 - C^2r^2\right)\left(\frac{\sin^2 S_2}{A} + \frac{\cos^2 S_2}{B}\right),$$

an equation which will be useful in the sequel.

The coefficient n, and the value r_0, which is the inferior limit of the integrals, are thus far considered as arbitrary functions of h and k. As already remarked, g (which is a variable quantity, $= nt + c$) is used as an element, and the elements are $h, k, g, \varpi, \sigma, \theta, \phi$.

As to the signification of the different elements, it is proper to remark that it is not for the purposes of the present memoir necessary to completely fix the significations of the quantities n and r_0; the only conditions imposed on these quantities in the sequel are that n shall be a function of h only, and that r_0 shall be a function of h and k, satisfying an equation of the form $r_0 = f\left(\frac{h - Cr_0^2}{k^2 - C^2r_0^2}\right)$, where f denotes an arbitrary function. The values of n and r_0 might, in accordance with these conditions, be fixed more definitively by reference to the cone rolling and sliding on the principal plane, used in the theories of Poinsot and Jacobi for the representation of the undisturbed motion, but to do this would require a further discussion of the integral equations, and it is a point which is not here entered into.

This being premised, we have (corresponding to the orbit in the theory of elliptic motion) the principal plane, with a departure point therein, the positions whereof, in respect to the fixed plane of reference, are determined by θ, the longitude of node; ϕ, the inclination; and σ, the departure of node. The precise signification of ϖ depends upon that of r_0, and the signification of r_0 being assumed to be completely determined, that of ϖ will be so likewise; ϖ is to be considered as an angle measured in the principal plane from the departure point, and determining in that plane a line (or, treating the plane as an orbit, a point) which I call the rotation pericentre, or simply the pericentre; say ϖ is the departure of the pericentre; and then g is an angle varying uniformly with the time, used for expressing in terms of the time the angle T, which determines the position, in regard to the principal plane and departure point therein, of the node of the plane of $x_{\prime}y_{\prime}$, or *equator*, upon the principal plane,—such node corresponding with the moving body in the theory of elliptic motion. We may in fact say: T, the departure of the last-mentioned node, $=\varpi$, the departure of pericentre, $+(T-\varpi)$, the rotation true anomaly of such node; $T-\varpi$ being a function of g, the rotation mean anomaly of such node. The elements are then as follows, viz.:

h, the constant of *Vis Viva*,

k, the constant of areas,

g, the rotation mean anomaly,

ϖ, the departure of rotation pericentre,

θ, the longitude of node of principal plane,

σ, the departure of ditto,

ϕ, the inclination of principal plane;

where θ, σ, ϕ determine the position of the principal plane and departure point therein, in respect to a fixed plane of reference and departure point therein; but it has been already remarked that the position of the principal plane and departure point therein might be, by the analogous quantities Θ, Σ, Φ, determined in reference to an arbitrarily varying plane of reference and departure point therein, and that the expressions for the variations of the elements would then be of the form given for this case in Art. I.

In the problem of the Rotation of the Earth, the principal plane is sensibly coincident with the plane of the equator, and on this account there is no actual physical representation of various quantities occurring in the mathematical problem. But a complete discussion of the mathematical problem does not thereby become unnecessary for the purposes of physical astronomy.

V.

If, now, the Disturbing Function is taken into account, the equations are to be integrated by the method of the variation of the elements. I use dg to denote that part of the variation of $g\,(=tdn+dc,$ if $g=nt+c)$, which depends on the variation of the constants, and in like manner for $dp,\ dq,\ dr$, &c. I assume, moreover, that x_2 is a departure point in the principal plane; this gives $d\sigma-\cos\phi\, d\theta=0$; and we see that the equations which lead to the expressions for the variations of the elements are

$$Adp=-\sin S\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt+\cos S\frac{d\Omega}{dF}\,dt,$$

$$Bdq=\ \ \cos S\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt+\sin S\frac{d\Omega}{dF}\,dt,$$

$$Cdr=-\frac{d\Omega}{dS}\,dt,$$

$$dT\ =0,$$

$$dS\ =0,$$

$$dF\ =0,$$

$$d\sigma-\cos\phi\, d\theta=0,$$

where, of course, $\Omega=\Omega\,(T,\ S,\ F)$.

The variations dh and dk are obtained from the equations

$$\tfrac{1}{2}dh=A\,pdp+B\,qdq+C\,rdr,$$
$$kdk=A^2pdp+B^2qdq+C^2rdr,$$

expressions which will presently be resumed and reduced; S_2 and F_2 may be considered as given functions of $h,\ k,\ r$, and the variations dS_2 and dF_2 can be thus obtained. The expressions for $T,\ S,\ F$, in terms of $T_2,\ S_2,\ F_2,\ \sigma,\ \theta,\ \phi$, and the equations $dT=0,\ dS=0,\ dF=0,\ d\sigma-\cos\phi\, d\theta=0$, then lead to the expressions for $d\theta,\ d\sigma,\ d\phi$, and to an equation $-dS_2+\cos F_2dT_2=0$; dT_2 is thus also given in terms of $dh,\ dk,\ dr$. And this being so, the two equations

$$g=\int\frac{-Cdr}{(B-A)\,pq},$$

$$T_2=\varpi+\int\frac{k\,(h-Cr^2)}{k^2-C^2r^2}\ \frac{-Cdr}{(B-A)\,pq},$$

lead to the expressions of $dg,\ d\varpi$, in terms of $dh,\ dk,\ dr$; the form of these expressions is simplified by partially fixing, in the manner already referred to, the significations of the quantities n and r_0, considered as functions of h and k. In this manner $\left(dh,\ dk,\ dr\right.$, being given linear functions of $\frac{d\Omega}{dT}dt,\ \frac{d\Omega}{dS}dt,\ \left.\frac{d\Omega}{dF}dt\right)$, we obtain

dh, dk, dg, $d\varpi$, $d\theta$, $d\sigma$, $d\phi$, all of them expressed in the same form; that is, in terms of the differential coefficients of the disturbing function Ω with respect to the coordinates T, S, F. We may then express the disturbing function Ω in terms of the elements, and transform the equations so as to obtain expressions for the variations of the elements in terms of the differential coefficients of Ω with respect to the elements.

VI.

Proceeding to carry out the foregoing plan, the equation

$$\tfrac{1}{2}dh = Ap\,dp + Bq\,dq + Cr\,dr,$$

putting for Adp, Bdq, Cdr, their values, gives

$$\tfrac{1}{2}dh = (-p\sin S + q\cos S)\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\,dt\right)$$
$$+ (p\cos S + q\sin S)\frac{d\Omega}{dF}\,dt - r\frac{d\Omega}{dS}\,dt,$$

which may also be written

$$\begin{aligned}\tfrac{1}{2}dh = \frac{h(h-Cr^2)}{k^2-C^2r^2}&\left\{-\sin(S_2-S)\sin F_2\frac{d\Omega}{dF}\,dt\right.\\ &\left.+\cos(S_2-S)\sin F_2\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt - \cos F_2\frac{d\Omega}{dS}\,dt\right\}\\ +\frac{k(B-A)pq}{k^2-C^2r^2}&\left\{\quad\cos(S_2-S)\sin F_2\frac{d\Omega}{dF}\,dt\right.\\ &\left.+\sin(S_2-S)\sin F_2\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt\right\}\\ +\frac{r(Ch-k^2)}{k^2-C^2r^2}&\frac{d\Omega}{dS}\,dt.\end{aligned}$$

In fact, if, in this expression, we consider first the terms involving $\frac{d\Omega}{dS}dt$, the coefficient is

$$\frac{-Cr(h-Cr^2)+r(Ch-k^2)}{k^2-C^2r^2} = \frac{-r(k^2-C^2r^2)}{k^2-C^2r^2} \text{ which is } = -r.$$

Next, the terms involving $\frac{d\Omega}{dF}dt$, the coefficient is

$$\left\{-(h-Cr^2)\sin(S_2-S)+(B-A)pq\cos(S_2-S)\right\}\frac{1}{k\sin F_2}$$
$$= \left[\left\{-(h-Cr^2)\sin S_2+(B-A)pq\cos S_2\right\}\cos S\right.$$
$$\left.+\left\{\ (h-Cr^2)\cos S_2+(B-A)pq\sin S_2\right\}\sin S\right]\frac{1}{k\sin F_2};$$

or, putting for $\sin S_2$, $\cos S_2$, their values $\dfrac{-Ap}{k\sin F_2}$, $\dfrac{Bq}{k\cos F_2}$, the coefficient is

$$\left[\left\{-(h-Cr^2)(-Ap)+(B-A)\,pq\,.\,Bq\right\}\cos S\right.$$

$$\left.+\left\{\;(h-Cr^2)\,Bq+(B-A)\,pq\,(-Ap)\right\}\sin S\right]\frac{1}{k^2\sin^2 F_2},$$

and the quantities in { } being respectively $p\,(k^2-C^2r^2)$ and $q\,(k^2-C^2r^2)$, and since $k^2-C^2r^2=k^2\sin^2 F_2$, the coefficient is $=p\cos S+q\sin S$.

In like manner, for the terms involving $\left(\operatorname{cosec} F\dfrac{d\Omega}{dT}+\cot F\dfrac{d\Omega}{dS}\right)dt$, the coefficient is

$$\left\{(h-Cr^2)\cos(S_2-S)+(B-A)\,pq\sin(S_2-S)\right\}\frac{1}{k\sin F_2}$$

$$=\left[\left\{(h-Cr^2)\cos S_2+(B-A)\,pq\sin S_2\right\}\cos S\right.$$

$$\left.+\left\{(h-Cr^2)\sin S_2-(B-A)\,pq\cos S_2\right\}\sin S\right]\frac{1}{k\sin F_2},$$

which is $=-p\sin S+q\cos S$; and the foregoing transformed expression for $\tfrac{1}{2}dh$ is thus seen to be correct.

The equation

$$kdk=A^2p\,dp+B^2q\,dq+C^2r\,dr,$$

substituting for Adp, Bdq, Cdr, their values, becomes

$$\begin{aligned} kdk= \; & Ap\left\{-\sin S\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt+\cos S\,\frac{d\Omega}{dF}\,dt\right\}\\ &+Bq\left\{\;\cos S\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt+\sin S\,\frac{d\Omega}{dF}\,dt\right\}\\ &+Cr\left\{-\frac{d\Omega}{ds}\,dt\right\} \end{aligned}$$

But, from the equations $Ap=-k\sin S_2\sin F_2$, $Bq=k\cos S_2\sin F_2$, $Cr=k\cos F_2$, we deduce

$$\begin{aligned} -Ap\sin S+Bq\cos S &= \;\;k\cos(S_2-S)\sin F_2,\\ Ap\cos S+Bq\sin S &= -k\sin(S_2-S)\sin F_2,\\ Cr &= \;\;k\cos F_2, \end{aligned}$$

and we thence find

$$dk=-\sin(S_2-S)\sin F_2\frac{d\Omega}{dF}\,dt+\cos(S_2-S)\sin F_2\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt-\cos F_2\,\frac{d\Omega}{dS}\,dt\,;$$

so that dh and dk are now determined.

VII.

The expressions for dh and dk being thus obtained, and dr being also given by the equation $Cdr = -\frac{d\Omega}{dS}\,dt$, we may now express dF_2 and dS_2 in terms of dh, dk, dr. In fact, the equations

$$Cr = k\cos F_2,$$

$$h - Cr^2 = (k^2 - C^2r^2)\left(\frac{\sin^2 S_2}{A} + \frac{\cos^2 S_2}{B}\right),$$

give immediately

$$k\sin F_2 dF_2 = -Cdr + \cos F_2 dk,$$

$$\sin S_2 \cos S_2\left(\frac{1}{A} - \frac{1}{B}\right) dS_2 = \frac{C(Ch - k^2)\,r}{(k^2 - C^2r^2)^2}\,dr + \frac{\frac{1}{2}dh}{k^2 - C^2r^2} - \frac{(h - Cr^2)\,kdk}{(k^2 - C^2r^2)^2};$$

or, multiplying by $k^2 - C^2r^2 \,(= k^2\sin^2 F_2)$, and replacing $\cos S_2 \sin F_2 \sin^2 F_2$ by $-ABpq$, and dividing, the last equation becomes

$$dS_2 = \frac{-C(Ch - k^2)\,r\,dr}{(k^2 - C^2r^2)(B - A)\,pq} - \frac{\frac{1}{2}dh}{(B - A)\,pq} + \frac{(h - Cr^2)\,kdk}{(k^2 - C^2r^2)(B - A)\,pq};$$

and thence also

$$\cos F_2\, dS_2 = \frac{-C^2(Ch - k^2)\,r^2dr}{k\,(k^2 - C^2r^2)(B - A)\,pq} - \frac{\frac{1}{2}Cr\,dh}{k\,(B - A)\,pq} + \frac{C\,(h - Cr)\,r\,dk}{(k^2 - C^2r^2)(B - A)\,pq},$$

which is the value of dT_2, as given by the equation (not yet demonstrated)

$$dT_2 = \cos S_2\, dF_2.$$

The expression for dF_2, substituting for dr and dk, their values, gives

$$\begin{aligned} k\sin F_2\, dF_2 = \;& \cos F_2 \sin F_2 \cos(S_2 - S)\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)dt \\ & - \cos F_2 \sin F_2 \sin(S_2 - S)\,\frac{d\Omega}{dF}\,dt \\ & - \cos^2 F_2 \frac{d\Omega}{dS}\,dt + \frac{d\Omega}{dS}\,dt; \end{aligned}$$

or, combining the last two terms, and reducing,

$$dF_2 = \frac{1}{k}\left\{\cos F_2 \cos(S_2 - S)\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)dt - \cos F_2 \sin(S_2 - S)\frac{d\Omega}{dF}\,dt + \sin F_2 \frac{d\Omega}{dS}\,dt\right\}.$$

We might, in like manner, transform the expression for dS_2, but it is somewhat more simple to obtain the new form by differentiation of the equation $\tan S_2 = -\frac{Ap}{Bq}$; this, in fact, gives

$$dS_2 = \frac{AB\,(pdq - qdp)}{A^2p^2 + B^2q^2} = \frac{1}{k^2\sin^2 F_2}\,(Ap\,.\,Bdq - Bq\,.\,Adp),$$

and the value of the numerator being

$$- k \sin S_2 \sin F_2 \left\{ \cos S \left(\operatorname{cosec} F \frac{d\Omega}{dT} + \cot F \frac{d\Omega}{dS} \right) dt + \sin S \frac{d\Omega}{dF} dt \right\}$$

$$- k \cos S_2 \sin F_2 \left\{ - \sin S \left(\operatorname{cosec} F \frac{d\Omega}{dT} + \cot F \frac{d\Omega}{dS} \right) dt + \cos S \frac{d\Omega}{dF} dt \right\},$$

we find

$$dS_2 = - \frac{1}{k \sin F_2} \left\{ \sin (S_2 - S) \left(\operatorname{cosec} F \frac{d\Omega}{dT} + \cot F \frac{d\Omega}{dS} \right) dt + \cos (S_2 - S) \frac{d\Omega}{dF} dt \right\};$$

the last-mentioned expressions for dF_2 and dS_2 will be used for obtaining $d\theta, d\sigma, d\phi$.

VIII.

I form now the equations

$$\begin{aligned} - \sin S \sin F \, dT + \cos S \, dF = \; & pdt - \sin S_2 \sin F_2 dT_2 + \cos S_2 dF_2 \\ & + \alpha_2 \; (- \sin \sigma \sin \phi \, d\theta + \cos \sigma \, d\phi) \\ & + \alpha_2' \; (\cos \sigma \sin \phi \, d\theta + \sin \sigma \, d\phi) \\ & + \alpha_2'' (- d\sigma + \cos \phi \, d\theta \quad), \end{aligned}$$

$$\begin{aligned} \cos S \sin F \, dT + \sin S \, dF = \; & qdt + \cos S_2 \sin F_2 dT_2 + \sin S_2 dF_2 \\ & + \beta_2 \; (- \sin \sigma \sin \phi \, d\theta + \cos \sigma \, d\phi) \\ & + \beta_2' \; (\cos \sigma \sin \phi \, d\theta + \sin \sigma \, d\phi) \\ & + \beta_2'' (- d\sigma + \cos \phi \, d\theta \quad), \end{aligned}$$

$$\begin{aligned} - dS + \cos F \, dT = \; & rdt - dS_2 + \cos F_2 dT_2 \\ & + \gamma_2 \; (- \sin \sigma \sin \phi \, d\theta + \cos \sigma \, d\phi) \\ & + \gamma_2' \; (\cos \sigma \sin \phi \, d\theta + \sin \sigma \, d\phi) \\ & + \gamma_2'' (- d\sigma + \cos \phi \, d\theta \quad); \end{aligned}$$

which will presently be useful, but which require some explanation. As the equations stand, on the left hand, dT, dS, dF, denote the entire variations of T, S, F, treating not only the constants, but also the time, as variable; but on the right hand, dT_2, dS_2, dF_2, denote the variations of T_2, S_2, F_2, depending only on the variation of the constants; if, on the left hand and the right hand respectively, dT, dS, dF, and dT_2, dS_2, dF_2, were used to denote either the entire variations of T, S, F, and T_2, S_2, F_2, or else the variations of these quantities depending only on the variations of the constants, then the terms pdt, qdt, rdt, would have to be omitted, and the equations would still be true. And it is to be noticed that, omitting the terms pdt, qdt, rdt, the equations express the relations existing between dT, dS, dF, dT_2, dS_2, dF_2, $d\theta$, $d\sigma$, $d\phi$, in virtue of the integral relations implied by the existence of the spherical triangle, the sides and angles whereof are $S_2 - S$, $T_2 - \sigma$, $T - \theta$, and ϕ, $180° - F$, F_2.

For the present purpose we are to omit the terms pdt, qdt, rdt, and to write $dT=0$, $dS=0$, $dF=0$. We have thus

$$\begin{aligned}
-(-\sin S_2 \sin F_2\, dT_2 + \cos S_2\, dF_2) &= \alpha_2\ (-\sin\sigma\sin\phi\, d\theta + \cos\sigma\, d\phi)\\
&\quad + \alpha_2'\ (\ \ \cos\sigma\sin\phi\, d\theta + \sin\sigma\, d\phi)\\
&\quad + \alpha_2''\,(-d\sigma + \cos\phi\, d\theta\qquad\qquad),\\
-(\ \ \cos S_2 \sin F_2\, dT_2 + \sin S_2\, dF_2) &= \beta_2\ (-\sin\sigma\sin\phi\, d\theta + \cos\sigma\, d\phi)\\
&\quad + \beta_2'\ (\ \ \cos\sigma\sin\phi\, d\theta + \sin\sigma\, d\phi)\\
&\quad + \beta_2''\,(-d\sigma + \cos\phi\, d\theta\qquad\qquad),\\
-(-dS_2 + \cos F_2\, dT_2) &= \gamma_2\ (-\sin\sigma\sin\phi\, d\theta + \cos\sigma\, d\phi)\\
&\quad + \gamma_2'\ (\ \ \cos\sigma\sin\phi\, d\theta + \cos\sigma\, d\phi)\\
&\quad + \gamma_2''\,(-d\sigma + \cos\phi\, d\theta\qquad\qquad).
\end{aligned}$$

Hence we have

$$\begin{aligned}
-\sin\sigma\sin\phi\, d\theta + \cos\sigma\, d\phi = &-\alpha_2\,(-\sin S_2\sin F_2\, dT_2 + \cos F_2\, dS_2)\\
&-\beta_2\,(\ \ \cos S_2 \sin F_2\, dT_2 + \sin F_2\, dS_2)\\
&-\gamma_2\,(\qquad\qquad \cos F_2\, dT_2 - dS_2\qquad),
\end{aligned}$$

which, attending to the values of α_2'', β_2'', γ_2'', may be written,

$$\begin{aligned}
-\sin\sigma\sin\phi\, d\theta + \cos\sigma\, d\phi = &-\alpha_2\,(\alpha_2''\, dT_2 + \cos S_2\, dF_2)\\
&-\beta_2\,(\beta_2''\, dT_2 + \sin S_2\, dF_2)\\
&-\gamma_2\,(\gamma_2''\, dT_2 - dS_2\qquad\quad)\\
= &-(\alpha_2 \cos S_2 + \beta_2 \sin S_2)\, dF_2 + \gamma_2 dS_2,
\end{aligned}$$

since the coefficient of dT_2 vanishes; and, in like manner,

$$\begin{aligned}
\cos\sigma\sin\phi\, d\theta + \sin\sigma\, d\phi = &-\alpha_2'\,(\alpha_2''\, dT_2 + \cos S_2\, dF_2)\\
&-\beta_2'\,(\beta_2''\, dT_2 + \sin S_2\, dF_2)\\
&-\gamma_2\,(\gamma_2''\, dT_2 - dS_2\qquad\quad)\\
= &-(\alpha_2' \cos S_2 + \beta_2' \sin S_2)\, dF_2 + \gamma_2' dS_2;
\end{aligned}$$

and so also

$$\begin{aligned}
-d\sigma + \cos\phi\, d\theta\qquad = &-\alpha_2''\,(\alpha_2''\, dT_2 + \cos S_2\, dF_2)\\
&-\beta_2''\,(\beta_2''\, dT_2 + \sin S_2\, dF_2)\\
&-\gamma_2''\,(\gamma_2''\, dT_2 - dS_2\qquad\quad)\\
= &-dT_2 - (\alpha_2'' \cos S_2 + \beta_2'' \sin S_2)\, dF_2 + \gamma_2'' dS_2.
\end{aligned}$$

But we have

$$\begin{aligned}
\alpha_2\ \cos S_2 + \beta_2\ \sin S_2 &= \cos T_2, & \gamma_2\ &= \ \ \sin T_2 \sin F_2,\\
\alpha_2'\ \cos S_2 + \beta_2'\ \sin S_2 &= \sin T_2, & \gamma_2'\ &= -\cos T_2 \sin F_2,\\
\alpha_2''\cos S_2 + \beta_2''\sin S_2 &= \ \ 0\ \ , & \gamma_2''\ &= \qquad\ \cos F_2,
\end{aligned}$$

and the foregoing equations thus become

$$\begin{aligned}
-\sin\sigma\sin\phi\,d\theta+\cos\sigma\,d\phi &= -\cos T_2\,dF_2+\sin T_2\sin F_2\,dS_2,\\
\cos\sigma\sin\phi\,d\theta+\sin\sigma\,d\phi &= -\sin T_2\,dF_2-\cos T_2\sin F_2\,dS_2,\\
-d\sigma+\cos\phi\,d\theta \qquad &= -dT_2 \qquad +\cos F_2 dS_2.
\end{aligned}$$

The last equation, making x_2 a departure point, or putting $-d\sigma+\cos\phi\,d\theta=0$, gives

$$dT_2=\cos F_2 dS_2,$$

an equation above referred to. The other two equations give

$$\begin{aligned}
d\phi &= -\cos(T_2-\sigma)\,dF_2+\sin(T_2-\sigma)\sin F_2\,dS_2,\\
\sin\phi\,d\theta &= -\sin(T_2-\sigma)\,dF_2-\cos(T_2-\sigma)\sin F_2\,dS_2,
\end{aligned}$$

which, with the equation,

$$d\sigma=\cos\phi\,d\theta,$$

give $d\theta$, $d\sigma$, $d\phi$, in terms of dS_2 and dF_2.

IX.

Proceeding to substitute for dS_2, dF_2, their values, we find

$$\begin{aligned}
k\,d\phi= &-\cos(T_2-\sigma)\left\{\cos F_2\cos(S_2-S)\left(\operatorname{cosec}F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt-\cos F_2\cos(S_2-S)\frac{d\Omega}{dF}\,dt+\sin F_2\frac{d\Omega}{dS}\,dt\right\}\\
&-\sin(T_2-\sigma)\left\{\qquad \sin(S_2-S)\left(\operatorname{cosec}F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt+\qquad\cos(S_2-S)\frac{d\Omega}{dF}\,dt\right\},
\end{aligned}$$

or reducing

$$\begin{aligned}
k\,d\phi= &-\Big[\sin(T_2-\sigma)\sin(S_2-S)+\cos(T_2-\sigma)\cos(S_2-S)\cos F_2\Big]\left(\operatorname{cosec}F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt\\
&-\Big[\sin(T_2-\sigma)\cos(S_2-S)-\cos(T_2-\sigma)\sin(S_2-S)\cos F_2\Big]\frac{d\Omega}{dF}\,dt\\
&-\Big[\qquad\qquad\qquad \cos(T_2-\sigma)\sin F_2\Big]\frac{d\Omega}{dF}\,dt\,;
\end{aligned}$$

and, in like manner,

$$\begin{aligned}
k\sin\phi\,d\theta= &-\sin(T_2-\sigma)\left\{\cos F_2\cos(S_2-S)\left(\operatorname{cosec}F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt-\cot F_2\cos(S_2-S)\frac{d\Omega}{dF}\,dt+\sin F_2\frac{d\Omega}{dS}\,dt\right\}\\
&+\cos(T_2-\sigma)\left\{\qquad \sin(S_2-S)\left(\operatorname{cosec}F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt+\qquad\cos(S_2-S)\frac{d\Omega}{dF}\,dt\right\}
\end{aligned}$$

or reducing

$$k \sin\phi \, d\theta = \left[\cos(T_2-\sigma)\sin(S_2-S) - \sin(T_2-\sigma)\cos(S_2-S)\cos F_2\right]\left(\operatorname{cosec} F \frac{d\Omega}{dT} + \cot F \frac{d\Omega}{dS}\right) dt$$

$$+ \left[\cos(T_2-\sigma)\cos(S_2-S) + \sin(T_2-\sigma)\sin(S_2-S)\cos F_2\right] \frac{d\Omega}{dF} dt$$

$$+ \left[-\sin(T_2-\sigma)\sin F_2 \right] \frac{d\Omega}{dS} dt.$$

These expressions for $d\phi$ and $d\theta$ are in a form which is convenient for some purposes, but they may be further reduced by means of the spherical triangle. In fact, in the expression for $d\phi$, the three coefficients in [ ] are respectively,

$$\sin b \sin a + \cos b \cos a \cos C = \sin B \sin A - \cos B \cos A \cos c$$

$$= \sin F \sin\phi + \cos F \cos\phi \cos(T-\theta),$$

$$\sin b \cos a - \cos b \sin a \cos C = \cos A \sin c$$

$$= \cos\phi \sin(T-\theta),$$

$$\cos b \sin C = \cos B \sin A + \sin B \cos A \cos c$$

$$= -\cos F \sin\phi + \sin F \cos\phi \cos(T-\theta);$$

and we have thus

$$k d\phi = -\left(\sin F \sin\phi + \cos F \cos\phi \cos(T-\theta)\right)\left(\operatorname{cosec} F \frac{d\Omega}{dT} + \cot F \frac{d\Omega}{dS}\right) dt$$

$$-\cos\phi \sin(T-\theta) \frac{d\Omega}{dF} dt$$

$$-\left(-\cos F \sin\phi + \sin F \cos\phi \cos(T-\theta)\right) \frac{d\Omega}{dS} dt;$$

and the right-hand side is

$$-\left(\sin F \sin\phi + \cos F \cos\phi \cos(T-\theta)\right) \operatorname{cosec} F \frac{d\Omega}{dT} dt$$

$$-\cos\phi \sin(T-\theta) \frac{d\Omega}{dF} dt$$

$$+\operatorname{cosec} F \cos\phi \cos(T-\theta) \frac{d\Omega}{dS} dt,$$

or we have finally

$$k d\phi = -\cos\phi \cos(T-\theta)\left(\cot F \frac{d\Omega}{dT} + \operatorname{cosec} F \frac{d\Omega}{dS}\right) dt$$

$$-\sin\phi \frac{d\Omega}{dT} dt$$

$$-\cos\phi \sin(T-\theta) \frac{d\Omega}{dF} dt.$$

In like manner, in the original expression for $k \sin\phi\, d\theta$, the three coefficients in [ ] are

$$\begin{aligned}
\cos b \sin a - \sin b \cos a \cos C &= \cos B \sin c \\
&= -\cos F \sin (T-\theta), \\
\cos b \cos a + \sin b \sin a \cos C &= \cos c \\
&= \cos (T-\theta), \\
-\sin b \sin C \qquad &= -\sin B \sin c \\
&= -\sin T \sin (T-\theta),
\end{aligned}$$

and thence

$$\begin{aligned}
k \sin\phi\, d\theta = &-\cos F \sin (T-\theta) \left(\operatorname{cosec} F \frac{d\Omega}{dT} + \cot F \frac{d\Omega}{dS}\right) dt \\
&+ \cos (T-\theta) \frac{d\Omega}{dF} dt \\
&- \sin F \sin (T-\theta) \frac{d\Omega}{dS} dt,
\end{aligned}$$

and the right-hand side is

$$\begin{aligned}
&-\cot F \sin (T-\theta) \frac{d\Omega}{dT} dt \\
&+ \cos (T-\theta) \frac{d\Omega}{dF} dt \\
&- \operatorname{cosec} F \sin (T-\theta) \frac{d\Omega}{dS} dt,
\end{aligned}$$

or we have finally

$$\begin{aligned}
k \sin\phi\, d\theta = &-\sin (T-\theta) \left(\cot F \frac{d\Omega}{dT} + \operatorname{cosec} F \frac{d\Omega}{dS}\right) dt \\
&+ \cos (T-\theta) \frac{d\Omega}{dF} dt.
\end{aligned}$$

The expression for $d\sigma$ can be obtained from either of those for $d\theta$, by the equation $d\sigma = \cos\phi\, d\theta$, and we have thus the values of $d\sigma$, $d\theta$, $d\phi$.

X.

It remains to find the expressions for dg and $d\varpi$. We have

$$g = \quad n \int \frac{-C dr}{(B-A)\, pq},$$

$$T_2 = \varpi + \int \frac{k (h - Cr^2)}{k^2 - C^2 r^2} \, \frac{-C dr}{(B-A)\, pq},$$

where p, q, are considered as functions of r, h, k, and where, besides, n and the inferior limit r_0 of the integrals are functions of h and k. We may write

$$dg = \frac{-nC}{(B-A)\,pq}\,dr + \mathfrak{A}\,dh + \mathfrak{B}\,dk,$$

$$dT_2 = d\varpi + \frac{k(h-Cr^2)}{k^2-C^2r^2}\,\frac{-C\,dr}{(B-A)\,pq}\,dr + \mathfrak{C}\,dh + \mathfrak{D}\,dk,$$

where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, are functions which contain integrals with respect to r, and there is not any algebraical relation between them, except the equation $\mathfrak{B} = -2n\mathfrak{C}$, which will be obtained presently. I retain, therefore, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, in the formulæ. The first equation gives the value of dg: from the second equation we have

$$d\varpi = dT_2 + \frac{Ck(h-Cr^2)}{(k^2-C^2r^2)(B-A)\,pq}\,dr - \mathfrak{C}\,dh - \mathfrak{D}\,dk,$$

or substituting for dT_2 its value $\cos F_2 dS_2$, the expression for which has been obtained above [p. 489], we have

$$d\varpi = \frac{-C^2(Ch-k^2)\,r^2dr}{k(k^2-C^2r^2)(B-A)\,pq} - \frac{\tfrac{1}{2}Cr\,dh}{k(B-A)\,pq} + \frac{C(h-Cr^2)\,r\,dk}{(k^2-C^2r^2)(B-A)\,pq}$$
$$+ \frac{Ck^2(h-Cr^2)\,dr}{k(k^2-C^2r^2)(B-A)\,pq} - \mathfrak{C}\,dh - \mathfrak{D}\,dk,$$

or reducing, this is

$$d\varpi = \frac{Ch\,dr}{k(B-A)\,pq} + \left[\frac{-\tfrac{1}{2}Cr}{k(B-A)\,pq} - \mathfrak{C}\right]dh + \left[\frac{C(h-Cr^2)\,r}{(k^2-C^2r^2)(B-A)\,pq} - \mathfrak{D}\right]dk,$$

which might be retained in this form. I obtain, however, a different form as follows, viz., we have

$$d\varpi = \cos F_2\,dS_2 + \frac{Ck(h-Cr^2)\,dr}{(k^2-C^2r^2)(B-A)\,pq} - \mathfrak{C}\,dh - \mathfrak{D}\,dk;$$

or using the other form of dS_2 [p. 490], and substituting also for dr its value from the equation $C\,dr = -\dfrac{d\Omega}{dS}\,dt$, we have

$$d\varpi = -\frac{1}{k}\cot F_2\left\{\sin(S_2-S)\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)dt + \cos(S_2-S)\frac{d\Omega}{dF}\,dt\right\}$$
$$-\frac{k(h-Cr^2)}{(k^2-C^2r^2)(B-A)\,pq}\,\frac{d\Omega}{dS}\,dt - \mathfrak{C}\,dh - \mathfrak{D}\,dk.$$

We have now to prove the before mentioned equation $BC = -2n\mathfrak{C}$. We have

$$\mathfrak{B} = \frac{d}{dk}\,n\int\frac{-C\,dr}{(B-A)\,pq} = n\int dr\,\frac{d}{dk}\,\frac{-C}{(B-A)\,pq} + \frac{dn}{dk}\int\frac{-C\,dr}{(B-A)\,pq} + \frac{nC}{(B-A)\,p_0q_0}\,\frac{dr_0}{dk},$$

$$\mathfrak{C} = \frac{d}{dh}\int\frac{k(h-Cr^2)}{k^2-C^2r^2}\,\frac{-C\,dr}{(B-A)\,pq} = \int dr\,\frac{d}{dh}\,\frac{-Ck(h-Cr^2)}{(k^2-C^2r^2)(B-A)\,pq} + \frac{Ck(h-Cr_0^2)}{(k^2-C^2r_0^2)(B-A)\,p_0q_0}\,\frac{dr_0}{dh},$$

where p_0, q_0, are the values of p and q corresponding to $r = r_0$. If we assume that n is a function of h only, the term multiplied by $\frac{dn}{dk}$ will disappear, and by properly determining r_0 as a function of h and k, we can, as regards the terms which contain r_0, satisfy the equation $\mathfrak{B} = -2n\mathfrak{C}$; the condition for this is

$$\frac{dr_0}{dk} = \frac{-2k(h - Cr_0^2)}{k^2 - C^2r_0^2}\frac{dr_0}{dh},$$

which will be satisfied if r_0 is determined as a function of h and k by an equation of the form

$$r_0 = f\left(\frac{h - Cr_0^2}{k^2 - C^2r_0^2}\right),$$

where f denotes an arbitrary function.

It remains to show that, as regards the terms involving integrals, we have the same relation $\mathfrak{B} = -2n\mathfrak{C}$, and this will be the case if

$$\frac{d}{dk}\frac{-C}{(B-A)pq} = -2\frac{d}{dh}\frac{-Ck(h - Cr^2)}{(k^2 - C^2r^2)(B-A)pq},$$

or, what is the same thing, if

$$\frac{d}{dk}\frac{1}{pq} = -2\frac{d}{dh}\frac{k(h - Cr^2)}{(k^2 - C^2r^2)pq} = \frac{-2k}{(k^2 - C^2r^2)}\frac{d}{dh}\frac{h - Cr^2}{pq},$$

in which equation p, q, are considered as functions of h, k, given by the equations

$$Ap^2 + Bq^2 + Cr^2 = h,$$

$$A^2p^2 + B^2q^2 + C^2r^2 = k^2.$$

We find without difficulty

$$\frac{d\,.\,pq}{dh} = \frac{-\tfrac{1}{2}(A^2p^2 - B^2q^2)}{AB(B-A)pq}, \quad \frac{d\,.\,pq}{dk} = \frac{k(Ap^2 - Bq^2)}{AB(B-A)pq},$$

and thence

$$\frac{d}{dh}\frac{h - Cr^2}{pq} = \frac{1}{pq} + \frac{\tfrac{1}{2}(h - Cr^2)(A^2p^2 - B^2q^2)}{AB(B-A)p^3q^3}$$

$$= \frac{1}{AB(B-A)p^3q^3}\left[AB(B-A)p^2q^2 + \tfrac{1}{2}(Ap^2 + Bq^2)(A^2p^2 - B^2q^2)\right]$$

$$= \frac{1}{AB(B-A)p^3q^3}\tfrac{1}{2}(Ap^2 - Bq^2)(A^2p^2) + (B^2q^2)$$

$$= \frac{\tfrac{1}{2}(k^2 - C^2r^2)(Ap^2 - Bq^2)}{AB(B-A)p^3q^3},$$

and the right-hand side of the equation in question is therefore

$$= \frac{-k(Ap^2 - Bq^2)}{AB(B-A)p^3q^3},$$

which is obviously also the value of the left-hand side. Hence, under the assumed relations $\left(n \text{ a function of } h \text{ only, and } r_0 = f\left(\frac{h - Cr_0^2}{k^2 - C^2r_0^2}\right)\right)$, we have the above-mentioned equation $\mathfrak{B} = -2n\mathfrak{C}$.

XI.

It will be convenient to recapitulate the various formulæ for the variations, as follows; we have

$$\tfrac{1}{2}dh = (-p\sin S + q\cos S)\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)dt$$

$$+ (\ p\cos S + q\sin S)\frac{d\Omega}{dF}dt - r\frac{d\Omega}{dS}dt,$$

and

$$\tfrac{1}{2}dh = \frac{k(h - Cr^2)}{k^2 - C^2r^2}\left\{\cos(S_2 - S)\sin F_2\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)dt - \sin(S_2 - S)\sin F_2\frac{d\Omega}{dF}dt - \cos F_2\frac{d\Omega}{dS}dt\right.$$

$$+ \frac{k(B-A)pq}{k^2 - C^2r^2}\left\{\sin(S_2 - S)\sin F_2\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)dt + \cos(S_2 - S)\sin F_2\frac{d\Omega}{dF}dt\right\}$$

$$+ \frac{r(Ch - k^2)}{k^2 - C^2r^2}\frac{d\Omega}{dS}dt;$$

$$kdk = (-Ap\sin S + Bq\cos S)\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)dt$$

$$+ (\ Ap\cos S + Bq\sin S)\frac{d\Omega}{dF}dt - Cr\frac{d\Omega}{dS}dt,$$

and

$$dk = \cos(S_2 - S)\sin F_2\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)dt - \sin(S_2 - S)\sin F_2\frac{d\Omega}{dF}dt - \cos S_2\frac{d\Omega}{dS}dt;$$

$$kd\phi = -\cos\phi\cos(T - \theta)\left(\cot F\frac{d\Omega}{dT} + \operatorname{cosec} F\frac{d\Omega}{dS}\right)dt$$

$$- \sin\phi\frac{d\Omega}{dT}dt - \cos\phi\sin(T - \theta)\frac{d\Omega}{dF}dt,$$

and

$$kd\phi = -\left\{\sin(T_2-\sigma)\sin(S_2-S)+\cos(T_2-\sigma)\cos(S_2-S)\cos F_2\right\}\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt$$

$$-\left\{\sin(T_2-\sigma)\cos(S_2-S)-\cos(T_2-\sigma)\sin(S_2-S)\cos F_2\right\}\frac{d\Omega}{dF}dt$$

$$-\left\{\qquad -\cos(T_2-\sigma)\sin F_2\qquad\right\}\frac{d\Omega}{dS}dt;$$

$$k\sin\phi d\theta = -\sin(T-\theta)\left(\cot F\frac{d\Omega}{dT}+\operatorname{cosec} F\frac{d\Omega}{dS}\right)dt$$

$$+\cos(T-\theta)\frac{d\Omega}{dF}dt,$$

and

$$k\sin\phi d\theta = \left\{\cos(T_2-\sigma)\sin(S_2-S)-\sin(T_2-\sigma)\cos(S_2-S)\cos F_2\right\}\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt$$

$$+\left\{\cos(T_2-\sigma)\cos(S_2-S)+\sin(T_2-\sigma)\sin(S_2-S)\cos F_2\right\}\frac{d\Omega}{dF}dt$$

$$+\left\{\qquad -\sin(T_2-\sigma)\sin F_2\qquad\right\}\frac{d\Omega}{dS}dt;$$

$$d\sigma = \cos\phi d\theta;$$

$$dg = \frac{-nC}{(B-A)pq}dr \qquad + \mathfrak{A}dh + \mathfrak{B}dk,$$

and

$$dg = \frac{n}{(B-A)pq}\frac{d\Omega}{dS}dt + \mathfrak{A}dh + \mathfrak{B}dk;$$

$$d\varpi = \frac{Chdr}{k(B-A)pq} + \left\{\frac{-\frac{1}{2}Cr}{k(B-A)pq}-\mathfrak{C}\right\}dh + \left\{\frac{C(h-Cr^2)r}{(k^2-C^2r^2)(B-A)pq}-\mathfrak{D}\right\}dk,$$

and

$$d\varpi = -\frac{1}{k}\cot F_2\left\{\sin(S_2-S)\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)dt+\cos(S_2-S)\frac{d\Omega}{dF}dt\right\}$$

$$+\frac{B\sin^2 S_2 + A\cos^2 S_2}{k(B-A)\sin^2 F_2\sin S_2\cos S_2}\frac{d\Omega}{dS}dt - \mathfrak{C}dh - \mathfrak{D}dk;$$

where it will be remembered that $\mathfrak{B} = -2n\mathfrak{C}$. The different forms for the variations of the same element are, or may be, each of them useful. The first expressions for dg and $d\varpi$ respectively are to be considered as giving these variations in terms of dr, dh, dk; and the second expressions are those obtained by substituting for these quantities their values, but in the terms multiplied by the integral expressions $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, which, on account of these multipliers, do not unite with the other terms, dh, dk are retained as standing for their values. The following equations may be added,

$$dT_2 = d\varpi - \frac{Ck(h - Cr^2)}{(k^2 - C^2r^2)(B - A)pq} dr + \mathfrak{C}dh + \mathfrak{D}dk,$$

$$dF_2 = \frac{-Ck \sin F_2}{k^2 - C^2r^2} dr + \frac{k \sin F_2 \cos F_2}{k^2 - C^2r^2} dk,$$

$$dS_2 = \frac{-C(Ch - k^2)r}{(k^2 - C^2r^2)(B - A)pq} dr - \frac{\frac{1}{2}dh}{(B - A)pq} + \frac{(h - Cr^2)kdk}{(k^2 - C^2r^2)(B - A)pq},$$

which give dF_2, dS_2, in terms of dr, dh, dk; and I call to mind, also, the equations

$$Ap = -k \sin S_2 \sin F_2, \qquad Bq = k \cos S_2 \sin F_2, \qquad Cr = k \cos F_2.$$

XII.

To find the differential coefficients of Ω with respect to the elements, I proceed as follows; considering the function first under the form $\Omega = \Omega(T, S, F)$, the total differential is

$$\frac{d\Omega}{dT} dT + \frac{d\Omega}{dS} dS + \frac{d\Omega}{dF} dF,$$

which must be equal to the total differential of Ω considered under the form $\Omega = \Omega\ (h, k, g, \varpi, \sigma, \theta, \phi)$; that is, it must be

$$= \frac{d\Omega}{dh} dh + \frac{d\Omega}{dk} dk + \frac{d\Omega}{dg}(ndt + dg) + \frac{d\Omega}{d\varpi} d\varpi + \frac{d\Omega}{d\theta} d\theta + \frac{d\Omega}{d\sigma} d\sigma + \frac{d\Omega}{d\phi} d\phi;$$

where, as elsewhere dg denotes only the part of the variation of g, which depends on the variation of the constants; so that the total variation of g is represented by $ndt + dg$. The value of $d\Omega$,

$$= \frac{d\Omega}{dT} dT + \frac{d\Omega}{dS} dS + \frac{d\Omega}{dF} dF,$$

is to be obtained in a form comparable with the last-mentioned expression, by means of the formulæ *suprà*, Art. VIII., which, for shortness, I represent as follows:

$$\begin{aligned} -\sin S \sin F\, dT + \cos S\, dF &= dP, \\ \cos S \sin F\, dT + \sin S\, dF &= dQ, \\ -dS + \cos F\, dT \qquad\qquad &= dR; \end{aligned}$$

these equations give

$$\begin{aligned} dF &= \qquad \cos S\, dP + \sin S\, dQ, \\ dT &= \operatorname{cosec} F(-\sin S\, dP + \cos S\, dQ), \\ dS &= \quad \cot F(-\sin S\, dP + \cos S\, dQ) - dR; \end{aligned}$$

and we then have

$$d\Omega = (\ \cos S\,dP + \sin S\,dQ)\frac{d\Omega}{dS}$$

$$+(-\sin S\,dP + \cos S\,dQ)\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)$$

$$-dR\frac{d\Omega}{dS};$$

and substituting for dP, dQ, dR, their values, the resulting expression may, for shortness, be represented by $d\Omega = d_1\Omega + d_2\Omega + d_3\Omega$, where

$$d_1\Omega = (\ p\cos S + q\sin S)\frac{d\Omega}{dF}\,dt$$

$$+(-p\sin S + q\cos S)\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)dt$$

$$-r\frac{d\Omega}{dS}\,dt.$$

$$d_2\Omega = \left\{\cos(S_2 - S)\sin F_2\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right) - \sin(S_2 - S)\sin F_2\frac{d\Omega}{dF} - \cos F_2\frac{d\Omega}{dS}\right\}dT_2$$

$$+\left\{\sin(S_2 - S)\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right) + \cos(S_2 - S)\frac{d\Omega}{dF}\right\}dF_2$$

$$+\frac{d\Omega}{dS}\,dS_2,$$

which, for shortness, I represent by $d_2\Omega = X\,dT_2 + Y\,dF_2 + Z\,dS_2$; and

$$d_3\Omega = \left\{\begin{array}{l}(\ \alpha_2\cos S + \beta_2\sin S)(-\sin\sigma\sin\phi\,d\theta + \cos\sigma\,d\phi)\\ +(\ \alpha_2'\cos S + \beta_2'\sin S)(\ \cos\sigma\sin\phi\,d\theta + \sin\sigma\,d\phi)\\ +(\ \alpha_2''\cos S + \beta_2''\sin S)(\ -d\sigma + \cos\phi\,d\theta\ )\end{array}\right\}\frac{d\Omega}{dF}$$

$$+\left\{\begin{array}{l}(-\alpha_2\sin S + \beta_2\cos S)(-\sin\sigma\sin\phi\,d\theta + \cos\sigma\,d\phi)\\ +(-\alpha_2'\sin S + \beta_2'\cos S)(\ \cos\sigma\sin\phi\,d\theta + \sin\sigma\,d\phi)\\ (-\alpha_2''\sin S + \beta_2''\cos S)(\ -d\sigma + \cos\phi\,d\theta\ )\end{array}\right\}\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)$$

$$-\left\{\begin{array}{l}\gamma_2(-\sin\sigma\sin\phi\,d\theta + \cos\sigma\,d\phi)\\ +\gamma_2'(\ \cos\sigma\sin\phi\,d\theta + \sin\sigma\,d\phi)\\ +\gamma_2''(\ -d\sigma + \cos\phi\,d\theta\ )\end{array}\right\}\frac{d\Omega}{dS},$$

or substituting for α_2, &c., their values, and reducing, $d_3\Omega =$

$$\left\{\begin{array}{l}([\cos(S_2-S)\sin(T_2-\sigma)-\sin(S_2-S)\cos(T_2-\sigma)\cos F_2]\sin\phi-\sin(S_2-S)\sin F_2\cos\phi)\,d\theta\\ +(\cos(S_2-S)\cos(T_2-\sigma)+\sin(S_2-S)\sin(T_2-\sigma)\cos F_2)\,d\phi+\sin(S_2-S)\sin F_2 d\sigma\end{array}\right\}\frac{d\Omega}{dF}$$

$$+\left\{\begin{array}{l}([\sin(S_2-S)\sin(T_2-\sigma)+\cos(S_2-S)\cos(T_2-\sigma)\cos F_2]\sin\phi+\cos(S_2-S)\sin F_2\cos\phi)\,d\theta\\ +(\sin(S_2-S)\cos(T_2-\sigma)-\cos(S_2-S)\sin(T_2-\sigma)\cos F_2)\,d\phi-\cos(S_2-S)\cos F_2 d\sigma\end{array}\right\}\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)$$

$$-\left\{(-\cos(T_2-\sigma)\sin F_2\sin\phi+\cos F_2\cos\phi)\,d\theta+\sin(T_2-\sigma)\sin F_2 d\phi-\cos F_2 d\sigma\right\}\frac{d\Omega}{dS}.$$

Hence, comparing $d_1\Omega+d_2\Omega+d_3\Omega$ with the other expression for $d\Omega$, and observing that dT_2, dS_2, dF_2, do not involve $d\theta$, $d\sigma$, $d\phi$, we have

$$d_1\Omega=\frac{d\Omega}{dg}\,ndt,$$

$$d_2\Omega=\frac{d\Omega}{dh}\,dh+\frac{d\Omega}{dk}\,dk+\frac{d\Omega}{dg}\,dg+\frac{d\Omega}{d\varpi}\,d\varpi,$$

$$d_3\Omega=\frac{d\Omega}{d\theta}\,d\theta+\frac{d\Omega}{d\sigma}\,d\sigma+\frac{d\Omega}{d\phi}\,d\phi.$$

XIII.

The first equation gives

$$n\frac{d\Omega}{dg}=(-p\sin S+q\cos S)\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)$$

$$+(p\cos S+q\sin S)\frac{d\Omega}{dF}-r\frac{d\Omega}{dS},$$

and, comparing this with the first form of $\tfrac{1}{2}dh$, we have

$$dh=2n\frac{d\Omega}{dg}\,dt.$$

The third equation gives

$$\frac{d\Omega}{d\theta}=\left\{[\sin(S_2-S)\sin(T_2-\sigma)+\cos(S_2-S)\cos(T_2-S)\cos F_2]\sin\phi+\cos(S_2-S)\sin F_2\cos\phi\right\}\left(\operatorname{cosec} F\frac{d\Omega}{dT}+\cot F\frac{d\Omega}{dS}\right)$$

$$+\left\{[\cos(S_2-S)\sin(T_2-\sigma)-\sin(S_2-S)\cos(T_2-\sigma)\cos F_2]\sin\phi-\sin(S_2-S)\sin F_2\cos\phi\right\}\frac{d\Omega}{dF}$$

$$+\left\{\cos(T_2-\sigma)\sin F_2\sin\phi-\cos F_2\cos\phi\right\}\frac{d\Omega}{dS},$$

$$\frac{d\Omega}{d\sigma} = -\cos(S_2 - S)\cos F_2\left(\operatorname{cosec} F\frac{d\Omega}{dT} - \cot F\frac{d\Omega}{dS}\right)$$
$$+ \sin(S_2 - S)\sin F_2\frac{d\Omega}{dF} + \cos F_2\frac{d\Omega}{dS},$$
$$\frac{d\Omega}{d\phi} = \left\{\sin(S_2 - S)\cos(T_2 - \sigma) - \cos(S_2 - S)\sin(T_2 - \sigma)\cos F_2\right\}\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)$$
$$+ \left\{\cos(S_2 - S)\cos(T_2 - \sigma) + \sin(S_2 - S)\sin(T_2 - \sigma)\cos F_2\right\}\frac{d\Omega}{dF}$$
$$- \sin(T_2 - \sigma)\sin F_2\frac{d\Omega}{dS},$$

and thence

$$\cot\phi\frac{d\Omega}{d\sigma} + \operatorname{cosec}\phi\frac{d\Omega}{d\theta} =$$
$$\left\{\sin(S_2 - S)\sin(T_2 - \sigma) + \cos(S_2 - S)\cos(T_2 - \sigma)\cos F_2\right\}\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)$$
$$+ \left\{\cos(S_2 - S)\sin(T_2 - \sigma) - \sin(S_2 - S)\cos(T_2 - \sigma)\cos F_2\right\}\frac{d\Omega}{dF}$$
$$+ \cos(T_2 - \sigma)\sin F_2\frac{d\Omega}{dS};$$

and we have therefore

$$d\phi = -\frac{\cot\phi}{k}\frac{d\Omega}{d\sigma}dt - \frac{\operatorname{cosec}\phi}{k}\frac{d\Omega}{d\theta}dt,$$
$$d\sigma = \frac{\cot\phi}{k}\frac{d\Omega}{d\phi}dt,$$
$$d\theta = \frac{\operatorname{cosec}\phi}{k}\frac{d\Omega}{d\phi}dt.$$

The second equation, viz.

$$\frac{d\Omega}{dh}dh + \frac{d\Omega}{dk}dk + \frac{d\Omega}{dg}dg + \frac{d\Omega}{d\varpi}d\varpi = d_2\Omega = XdT_2 + YdF_2 + ZdS_2,$$

if we substitute for dg, dT_2, dF_2, dS_2 their values, becomes

$$\frac{d\Omega}{dh}dh + \frac{d\Omega}{dk}dk + \frac{d\Omega}{dg}\left(-\frac{nC}{(B-A)pq}dr + \mathfrak{A}dh + \mathfrak{B}dk\right) + \frac{d\Omega}{d\varpi}d\varpi$$
$$= X\left\{d\varpi - \frac{Ck(h - Cr^2)}{(k^2 - C^2r^2)(B-A)pq}dr + \mathfrak{C}dh + \mathfrak{D}dk\right\}$$
$$+ Y\left\{-\frac{Ck\sin F_2}{k^2 - C^2r^2}dr + \frac{k\sin F_2\cos F_2}{k^2 - C^2r^2}dk\right\}$$
$$+ Z\left\{-\frac{C(Ch - k^2)r}{(k^2 - C^2r^2)(B-A)pq}dr - \frac{\tfrac{1}{2}dh}{(B-A)pq} + \frac{(n - Cr^2)kdk}{(k^2 - C^2r^2)(B-A)pq}\right\}.$$

The comparison of the terms involving $d\varpi$ gives at once $\frac{d\Omega}{d\varpi} = X$, or, substituting for X its value,

$$\frac{d\Omega}{d\varpi} = \cos(S_2 - S)\sin F_2\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)$$

$$- \sin(S_2 - S)\sin F_2\frac{d\Omega}{dF} - \cos F_2\frac{d\Omega}{dS};$$

and comparing with the expression for dk, we find

$$dk = \frac{d\Omega}{d\varpi}\,dt.$$

The terms involving dr give

$$n\frac{d\Omega}{dg} = \frac{k(h - Cr^2)}{k^2 - C^2r^2}X + \frac{k\sin F_2(B-A)pq}{k^2 - C^2r^2}Y + \frac{(Ch - k^2)r}{k^2 - C^2r^2}Z,$$

or, substituting for X, Y, Z, their values,

$$n\frac{d\Omega}{dg} = \frac{k(h - Cr^2)}{k^2 - C^2r^2}\left\{\cos(S_2 - S)\sin F_2\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right) - \sin(S_2 - S)\sin F_2\frac{d\Omega}{dF} - \cos F_2\frac{d\Omega}{dS}\right\}$$

$$+ \frac{k(B-A)pq}{k^2 - C^2r^2}\left\{\sin(S_2 - S)\sin F_2\left(\operatorname{cosec} F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right) + \cos(S_2 - S)\sin F_2\frac{d\Omega}{dF}\right\}$$

$$+ \frac{r(Ch - k^2)}{k^2 - C^2r^2}\frac{d\Omega}{dS},$$

which agrees with the second form for $\frac{1}{2}dh$, and gives as before

$$dh = 2n\frac{d\Omega}{dg}\,dt;$$

the terms involving dh give

$$\frac{d\Omega}{dh} + \mathfrak{A}\frac{d\Omega}{dg} = \mathfrak{C}X - \frac{\frac{1}{2}}{(B-A)pq}Z,$$

and thence

$$-2n\frac{d\Omega}{dh} = 2n\mathfrak{A}\frac{d\Omega}{dg} - 2n\mathfrak{C}X + \frac{n}{(B-A)pq}Z;$$

or, substituting for $-2n\mathfrak{C}$ its value $\mathfrak{B}$, for X the value $\frac{d\Omega}{d\varpi}$, and for Z its value $\frac{d\Omega}{dS}$, we have

$$-2n\frac{d\Omega}{dh} = 2n\mathfrak{A}\frac{d\Omega}{dg} + \mathfrak{B}\frac{d\Omega}{d\varpi} + \frac{n}{(B-A)pq}\frac{d\Omega}{dS},$$

where, on the right hand, $\frac{d\Omega}{dg}$ and $\frac{d\Omega}{d\varpi}$ may be considered as standing for given functions of $\frac{d\Omega}{dT}$, $\frac{d\Omega}{dS}$, $\frac{d\Omega}{dF}$; for the present purpose, however, multiplying by dt, and substituting for $2n\frac{d\Omega}{dg}$ and $\frac{d\Omega}{d\varpi}$ the values dh and dk, we have

$$-2n\frac{d\Omega}{dh}dt = \mathfrak{A}dh + \mathfrak{B}dk + \frac{n}{(B-A)pq}\frac{d\Omega}{dS}dt,$$

which agrees with the foregoing value of dg, or we have

$$dg = -2n\frac{d\Omega}{dh}dt.$$

The comparison of the terms involving dk gives

$$\frac{d\Omega}{dk} + \mathfrak{B}\frac{d\Omega}{dg} = \mathfrak{D}X + \frac{k\sin F_2\cos F_2}{k^2 - C^2r^2}Y + \frac{(h-Cr^2)k}{(k^2-C^2r^2)(B-A)pq}Z;$$

or, substituting and reducing,

$$-\frac{d\Omega}{dk} = -2n\mathfrak{C}\frac{d\Omega}{dg} - \mathfrak{D}\frac{d\Omega}{d\varpi} - \frac{1}{k}\cot F_2\left\{\cos(S_2-S)\frac{d\Omega}{dF} + \sin(S_2-S)\left(\operatorname{cosec}F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)\right\}$$
$$-\frac{(h-Cr^2)k}{(k^2-C^2r^2)(B-A)pq}\frac{d\Omega}{dS},$$

where, on the right-hand side, $\frac{d\Omega}{dg}$ and $\frac{d\Omega}{d\varpi}$ may be considered as standing for given functions of $\frac{d\Omega}{dT}$, $\frac{d\Omega}{dS}$, $\frac{d\Omega}{dF}$; for the present purpose, however, multiplying by dt and putting for $2n\frac{d\Omega}{dg}dt$ and $\frac{d\Omega}{d\varpi}dt$ the values dh and dk, we have

$$-\frac{d\Omega}{dk}dt = -\mathfrak{C}dh - \mathfrak{D}dk - \frac{1}{k}\cot F_2\left\{\cos(S_2-S)\frac{d\Omega}{dF}dt + \sin(S_2-S)\left(\operatorname{cosec}F\frac{d\Omega}{dT} + \cot F\frac{d\Omega}{dS}\right)dt\right\}$$
$$-\frac{(h-Cr^2)k}{(k^2-C^2r^2)(B-A)pq}\frac{d\Omega}{dS}dt,$$

which agrees with the foregoing value of $d\varpi$, or we have

$$d\varpi = -\frac{d\Omega}{dk}dt,$$

and we have thus the system of formulæ for the variation of the elements in the problem of rotation, given in Art. I.