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NOTE ON A THEOREM OF JACOBI'S, IN RELATION TO THE PROBLEM OF THREE BODIES.

THE following theorem of Jacobi's (*Comptes Rendus*, t. III., p. 61 (1836)) has not, I think, found its way in an explicit form into any treatise of physical astronomy. The theorem is as follows, viz. "Consider the movement of a point without mass round the Sun, disturbed by a planet the orbit of which is circular. Let xyz be the rectangular coordinates of the disturbed body, the orbit of the disturbing planet being taken as the plane of xy, and the Sun as the centre of coordinates; let a' be the distance of the disturbing planet, $n't$ its longitude, m' its mass, M the mass of the Sun: then we have, *rigorously*,

$$\frac{1}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] - n'\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)$$

$$= \frac{M}{(x^2+y^2+z^2)^{\frac{1}{2}}} + m'\left\{\frac{1}{(x^2+y^2+z^2-2a'(x\cos n't + y\sin n't)+a'^2)^{\frac{1}{2}}} - \frac{x\cos n't + y\sin n't}{a'^2}\right\} + C.$$

This is therefore a new integral equation, which, in the problem of three bodies, subsists, as regards the terms independent of the eccentricity of the disturbing planet, and which is rigorous as regards all the powers of the mass of such planet. In the *Lunar Theory* the Earth must be substituted in the place of the Sun, and the Sun taken as the disturbing planet."

To prove the theorem, as expressed in polar coordinates, I take the equations of motion in the form in which I have employed them in my "Memoir on the Theory of Disturbed Elliptic Motion" (*Memoirs*, vol. XXVII. p. 1 (1859)), [212], viz.

$$\frac{d}{dt}\frac{dr}{dt} - r\cos^2 y \left(\frac{dv}{dt}\right)^2 - r\left(\frac{dy}{dt}\right)^2 + \frac{n^2 a^3}{r^2} = \frac{d\Omega}{dr},$$

$$\frac{d}{dt}\left(r^2 \cos^2 y \frac{dv}{dt}\right) = \frac{d\Omega}{dv},$$

$$\frac{d}{dt}\left(r^2 \frac{dy}{dt}\right) + r^2 \cos y \sin y = \frac{d\Omega}{dy},$$

where

$$\Omega = m'\left\{\frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos H}} - \frac{r\cos H}{r'^2}\right\};$$

or, since $\cos H = \cos y \cos (v - v')$, and the Sun is considered as moving in a circular orbit (i.e. $r' = a'$, $v' = n't$), we have

$$\Omega = m'\left\{\frac{1}{\sqrt{r^2 + a'^2 - 2ra'\cos y \cos (v - n't)}} - \frac{r\cos y \cos(v - n't)}{a'^2}\right\};$$

so that Ω is a function of r, v, y and of t, which last quantity enters only in the combination $v - n't$. Hence the complete differential coefficient of Ω is

$$\frac{d(\Omega)}{dt} = \frac{d'\Omega}{dt} - n'\frac{d\Omega}{dv},$$

where $\dfrac{d'\Omega}{dt}$ denotes, as usual, the differential coefficient in regard to the time, in so far as it enters through the coordinates r, v, y of the disturbed body.

We have, as usual,

$$\frac{d}{dt}\frac{dr^2 + r^2(\cos^2 y\, dv^2 + dy^2)}{dt^2} = \frac{d'\Omega}{dt};$$

and, from the foregoing equation,

$$\frac{d'\Omega}{dt} = \frac{d(\Omega)}{dt} + n'\frac{d\Omega}{dv}$$

$$= \frac{d(\Omega)}{dt} + n'\frac{d}{dt}\left(r^2 \cos^2 y \frac{dv}{dt}\right);$$

hence, substituting this value, transposing, and integrating, we have

$$\left(\frac{dr}{dt}\right)^2 + r^2\left\{\cos^2 y \left(\frac{dv}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} - n'r^2 \cos^2 y \frac{dv}{dt} = \Omega + C,$$

which is Jacobi's equation expressed in terms of the coordinates r, v, y.

M. de Pontécoulant, in his *Lunar Theory* (1846), where the solar eccentricity is neglected, writes (p. 91),

$$\int d'R = R + m \int \frac{dR}{dv} \, dt,$$

($n' = mn = m$, since n is there put equal to unity); and combining this with the equation (p. 43),

$$\frac{r^2 dv}{(1 + s^2) \, dt} = h + \int \frac{dR}{dv} \, dt,$$

we have

$$\int d'R = R - mh + \frac{mr^2 \, dv}{(1 + s^2) \, dt};$$

and substituting this value of $\int d'R$ in the integral of *Vis Viva* (p. 41),

$$\left(\frac{dr}{dt}\right)^2 + \frac{r^2 dv^2}{(1 + s^2) \, dt^2} + \frac{r^2 ds^2}{(1 + s^2)^2 \, dt^2} - \frac{2}{r} + \frac{1}{a} = 2 \int d'R,$$

we have what is, in fact, Jacobi's equation.