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ON LAMBERT'S THEOREM FOR ELLIPTIC MOTION.

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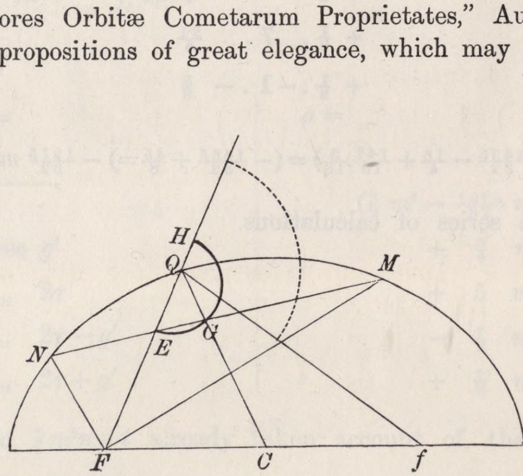
THE theorem referred to is that which gives the time of description of an elliptic arc in terms of the radius vectors and the chord. The demonstration given by the author in his "Insigniores Orbitæ Cometarum Proprietates," Augs. 1761, depends upon a series of geometrical propositions of great elegance, which may be thus stated.



Let FQ be a line given in magnitude and position, E a given point on this line, Qf a line given in magnitude only, the position thereof being determined by assigning a value to its variable inclination to the line FQ. With F, f, as foci describe an ellipse passing through the point Q (the axis major, $= FQ + Qf$, is of course a constant magnitude). Take C, the centre of the ellipse, and join CQ; through E draw a chord, MEN, conjugate to the diameter CQ and meeting it in G. Then treating the inclination as variable, 1°. The locus of G is a circle passing through E, and having its centre on the line FQ.

2°. The semichord GM or GN, and the sum $FM + FN$ of the radius vectors are respectively constant.

3°. The elliptic area NFM, divided by the square root of the latus rectum, is a constant.

It may also be mentioned, that taking 2θ to represent the external inclination (supplement of the angle FQf), and if, moreover, a is the semiaxis major, e the eccentricity, and u, u' the eccentric anomalies of the points M, N, then the square root of the latus rectum, or say $\sqrt{1-e^2}$, $\propto \sin\theta$, and moreover EM, EN, FM, FN, $e\cos u$, $e\cos u'$, $e\sin u$, $e\sin u'$, consists each of them of a constant part, *plus* a part which $\propto \cos\theta$; these expressions give as above $GM = GN = \frac{1}{2}(EM + EN) = \text{constant}$, $FM + FN = \text{constant}$; and they give moreover $e\cos u + e\cos u' = \text{constant}$; $e\sin u - e\sin u'$ $= \text{constant}$; $u - u' = \text{const.}$ The expression for the area is

$$\tfrac{1}{2}a^2\sqrt{1-e^2}\,\{u - u' - (e\sin u - e\sin u')\},$$

and consequently the area divided by $\sqrt{1-e^2}$ is a constant; that is, the area is, as stated above, proportional to the square root of the latus rectum.

Hence, assuming the dynamical theorem that for a given central force at F, varying inversely as the square of the distance, the time of describing the elliptic arc is proportional to the area divided by the square root of the latus rectum, the time of describing the elliptic arc is constant. But in the extreme case, where the point f lies in the line FQ produced in the direction from F to Q, the ellipse reduces itself to a finite right line, length $FQ + Qf$, which is considered to be described by a body falling from the extremity with an initial velocity zero; and the arc MN is a portion thereof given in magnitude, and having for its centre the point H (where EH is the diameter of the before-mentioned circle, the locus of G). Hence the time of describing the elliptic arc is equal to the time of describing, under the action of the same central force, a given right line, and as such it is at once obtainable in the form

$$\frac{a^{\frac{3}{2}}}{\sqrt{\mu}}\left(\phi - \phi' - (\sin\phi - \sin\phi')\right)$$

where ϕ, ϕ' are functions of the major axis $FQ + Qf$, and of FM, FN, or, what is the same thing, of $FQ + Qf$, and of the chord MN and sum of the radius vectors FM, FN. The preceding is the geometrical mode of getting out the result, without the assistance of any expression for the elliptic area, and latus rectum, and assuming only that we know the formula for rectilineal motion; but, if the expressions for the elliptic area and latus rectum are obtained, then the expression for the time is known, and the problem is solved, without the necessity of passing from the ellipse to the right line.

Writing $FQ=\rho$, $QF=\sigma$, and as before the exterior angle of inclination $=2\theta$, the actual expressions for the various lines of the figure are easily found to be

$$\tfrac{1}{2}(\rho+\sigma) \qquad\qquad , = a,$$

$$CF = Cf = \tfrac{1}{2}\sqrt{\rho^2+\sigma^2+2\rho\sigma\cos 2\theta}, = a,$$

$$CQ \qquad = \tfrac{1}{2}\sqrt{\rho^2+\sigma^2-2\rho\sigma\cos 2\theta}, = a',$$

$$CR \qquad = \sqrt{\rho\sigma} \qquad\qquad , = b',$$

where CR (not shown in the figure) denotes the semi-diameter conjugate to CQ.

$$1-e^2 = \frac{4\rho\sigma}{(\rho+\sigma)^2}\sin^2\theta,$$

$$\cos F = \frac{\rho+\sigma\cos 2\theta}{2ae}, \ \cos Q = \frac{\rho-\sigma\cos 2\theta}{2a'},$$

$$\sin F = \frac{\sigma\sin 2\theta}{2ae}, \ \sin Q = \frac{\sigma\sin 2\theta}{2a'}, \ \sin C = \frac{\rho\sigma\sin 2\theta}{4a'ae},$$

where F, C, Q, denote the angles of the triangle FCQ, respectively,

$$EG = \frac{2k\sigma}{\rho+\sigma}\cos\theta, \text{ and therefore } EH = \frac{2k\sigma}{\rho+\sigma},$$

$$QG = \frac{k}{\rho+\sigma}2a',$$

and, if for shortness $\Lambda = \sqrt{k\rho\sigma(\rho+\sigma-k)}$, then

$$(EM,\ EN) = \frac{2}{\rho+\sigma}(\Lambda \pm k\sigma\cos\theta),$$

$$(FM,\ FN) = \frac{1}{\rho+\sigma}\{\rho(\sigma+\rho)+k(\sigma-\rho)\pm 2\Lambda\cos\theta\},$$

so that

$$GM = GN = \tfrac{1}{2}(EM+EN) = \frac{2\Lambda}{\rho+\sigma},$$

$$\tfrac{1}{2}(FM+FN) = \frac{1}{\rho+\sigma}\{\rho(\sigma+\rho)+k(\sigma-\rho)\},$$

and moreover

$$(e\cos u,\ e\cos u') = \frac{1}{(\rho+\sigma)^2}\{(\sigma-\rho)(\rho+\sigma-2k)\mp 4\Lambda\cos\theta\},$$

$$(e\sin u,\ e\sin u') = \frac{1}{(\rho+\sigma)^2}\left\{\pm\frac{2(\sigma-\rho)\Lambda}{\sqrt{\rho\sigma}}+2(\rho+\sigma-2k)\sqrt{\rho\sigma}\cos\theta\right\},$$

so that

$$e\cos u + e\cos u' = \frac{2}{(\rho+\sigma)^2}(\rho+\sigma-2k),$$

$$e\sin u - e\sin u' = \frac{4}{(\rho+\sigma)^2}\,\frac{(\sigma-\rho)\Lambda}{\sqrt{\rho\sigma}},$$

$$u-u' = 2\tan^{-1}\frac{2\Lambda}{\sqrt{\rho\sigma}\,(\rho+\sigma-2k)} = \sin^{-1}\frac{4\Lambda(\rho+\sigma-2k)}{(\rho+\sigma)^2\sqrt{\rho\sigma}},$$

$$u-u'-(e\sin u - e\sin u') = \sin^{-1}\frac{4(\rho+\sigma-2k)\Lambda}{(\rho+\sigma)^2\sqrt{\rho\sigma}} - \frac{4(\sigma-\rho)\Lambda}{(\rho+\sigma)^2\sqrt{\rho\sigma}}$$

which is

$$= \phi-\phi'-(\sin\phi-\sin\phi'),$$

if

$$1-\cos\phi = \frac{1}{2a}(FM+FM+MN) = \frac{2}{(\rho+\sigma)^2}\{\rho(\sigma+\rho)+k(\sigma-\rho)+2\Lambda\},$$

$$1-\cos\phi' = \frac{1}{2a}(FM+FN-MN) = \frac{2}{(\rho+\sigma)^2}\{\rho(\sigma+\rho)+k(\sigma-\rho)-2\Lambda\}.$$

In fact we then have also

$$1+\cos\phi = \frac{2}{(\rho+\sigma)^2}\{\sigma(\sigma+\rho)-k(\sigma-\rho)-2\Lambda\},$$

$$1+\cos\phi' = \frac{2}{(\rho+\sigma)^2}\{\sigma(\sigma+\rho)-k(\sigma-\rho)+2\Lambda\},$$

and thence

$$\sin\tfrac{1}{2}\phi = \frac{1}{\rho+\sigma}\left(\sqrt{\rho(\rho+\sigma-k)}+\sqrt{k\sigma}\right),\quad \sin\tfrac{1}{2}\phi' = \left(\sqrt{\rho(\rho+\sigma-k)}-\sqrt{k\sigma}\right),$$

$$\cos\tfrac{1}{2}\phi = \frac{1}{\rho+\sigma}\left(\sqrt{\sigma(\rho+\sigma-k)}-\sqrt{k\rho}\right),\quad \cos\tfrac{1}{2}\phi' = \left(\sqrt{\sigma(\rho+\sigma-k)}+\sqrt{k\rho}\right),$$

whence

$$\sin\phi = \frac{2}{(\rho+\sigma)^2}\left\{\sqrt{\rho\sigma}(\rho+\sigma-2k)+\frac{(\sigma-\rho)\sqrt{\Lambda}}{\sqrt{\rho\sigma}}\right\},$$

$$\sin\phi' = \frac{2}{(\rho+\sigma)^2}\left\{\sqrt{\rho\sigma}(\rho+\sigma-2k)-\frac{(\sigma-\rho)\sqrt{\Lambda}}{\sqrt{\rho\sigma}}\right\},$$

and therefore
$$\sin\phi-\sin\phi' = \frac{4(\sigma-\rho)\sqrt{\Lambda}}{(\rho+\sigma)^2\sqrt{\rho\sigma}},$$

$$\sin\tfrac{1}{2}(\phi-\phi') = \frac{2k\Lambda}{(\rho+\sigma)\sqrt{\rho\sigma}},\quad \cos\tfrac{1}{2}(\phi-\phi') = \frac{\rho+\sigma-2k}{\rho+\sigma},$$

and therefore
$$\sin(\phi-\phi') = \frac{4(\rho+\sigma-2k)\Lambda}{(\rho+\sigma)^2\sqrt{\rho\sigma}},$$

which verifies the formula.