CHAPTER XXIV.

EULERIAN INTEGRALS, GAUSS' Π FUNCTION, ETC.

853. **The Original Forms of the Eulerian Integrals.**

The properties of the two important integrals

$$I_1 \equiv \left(\frac{p}{q}\right) \equiv \int_0^1 \frac{x^{p-1}\,dx}{(1-x^n)^{\frac{n-q}{n}}} \quad \text{and} \quad I_2 \equiv \left[\frac{p}{q}\right] \equiv \int_0^1 \left(\log\frac{1}{x}\right)^{\frac{p}{q}-1} dx$$

were the subject of several remarkable memoirs by Euler. His investigations were published in the *Institutiones Calculi Integralis*, 1768-1770, and are of great importance in the general theory of Definite Integrals. The notation above, viz. $\left(\frac{p}{q}\right)$ and $\left[\frac{p}{q}\right]$, is that used by Euler, and the above forms are those in which the integrals were studied both by Euler and Lagrange. In each of these the value of the integral was supposed to change by the variation of p and q; the n which occurs in the first integral was supposed to be a constant.

Legendre, for the purpose of characterising these integrals and honouring their great discoverer, named them "Intégrales Eulériennes."* The second part of Legendre's *Exercices de Calcul Intégral* is devoted to a discussion of their properties. He adheres to the notation $\left(\frac{p}{q}\right)$ for the first integral, but suggests the notation $\Gamma\left(\frac{p}{q}\right)$ for the second, regarding $\Gamma(a)$ as a continuous function of a.

* *Exercices de Calcul Intégral*, par A. M. Legendre, 1811, p. 211.

854. **The More Convenient Modern Forms.**

The above forms of the integrals are not the most convenient in practice. Taking the first integral, write $x^n=y$, and put $p=nl$, $q=nm$.

Then

$$I_1=\int_0^1 \frac{x^{p-1}dx}{(1-x^n)^{\frac{n-q}{n}}}=\int_0^1 \frac{y^{\frac{p-1}{n}}\cdot\frac{1}{n}\frac{x}{y}dy}{(1-y)^{1-\frac{q}{n}}}=\frac{1}{n}\int_0^1 y^{l-1}(1-y)^{m-1}dy.$$

Taking the second integral and writing $\log\frac{1}{x}=y$, that is $x=e^{-y}$, and putting $\frac{p}{q}=n$,

$$I_2=\int_0^1\left(\log\frac{1}{x}\right)^{\frac{p}{q}-1}dx=\int_0^\infty e^{-y}y^{n-1}dy.$$

855. **Definition.**

We shall therefore define the FIRST AND SECOND EULERIAN INTEGRALS as

$$\mathrm{B}(l,m)\equiv\int_0^1 x^{l-1}(1-x)^{m-1}dx$$

and

$$\Gamma(n)\equiv\int_0^\infty e^{-x}x^{n-1}dx,$$

and refer to them respectively as the BETA and GAMMA Functions. This is now the commonly accepted notation and nomenclature.

856. In Gregory's *Examples* (p. 470), the digamma $\mathsf{F}(l, m)$ is used to denote what we have above defined as the Beta function. It will be observed that $\mathrm{B}(l, m)$ is n times the integral discussed by Euler, that is $n\left(\frac{p}{q}\right)$.

We shall assume in our subsequent work that all the quantities l, m, n are positive but not necessarily integral, and further that they are real unless the contrary be expressly stated.

857. **The Beta Function is symmetric in l and m, that is,**

$$\mathrm{B}(l,m)=\mathrm{B}(m,l).$$

If in the Beta function

$$\mathrm{B}(l,m)\equiv\int_0^1 x^{l-1}(1-x)^{m-1}dx$$

we write $1-y$ for x, we obtain

$$\mathrm{B}(l, m) = -\int_1^0 (1-y)^{l-1} y^{m-1} dy = \int_0^1 y^{m-1}(1-y)^{l-1} dy$$
$$= \int_0^1 x^{m-1}(1-x)^{l-1} dx = \mathrm{B}(m, l);$$

whence it appears that $\mathrm{B}(l, m)$ is a symmetric function of l and m, the l and m being interchangeable and

$$\mathrm{B}(l, m) \equiv \mathrm{B}(m, l).$$

This property might be exhibited by writing $\mathrm{B}(l, m)$ as

$$\mathrm{B}(l, m) = \frac{1}{2}\int_0^1 [x^{l-1}(1-x)^{m-1} + x^{m-1}(1-x)^{l-1}] dx.$$

858. **Case when l or m is a Positive Integer.**

When either of the two quantities l, m is a positive integer, the integration is expressible in finite terms.

Suppose m is a positive integer,

$$\mathrm{B}(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1} dx,$$

and by continued integration by parts

$$= \left[\frac{x^l}{l}(1-x)^{m-1} + \frac{x^{l+1}}{l(l+1)}(m-1)(1-x)^{m-2} \right.$$
$$+ \frac{x^{l+2}}{l(l+1)(l+2)}(m-1)(m-2)(1-x)^{m-3} + \dots$$
$$\left. + \frac{x^{l+m-1}}{l(l+1)\dots(l+m-1)}(m-1)(m-2)\dots 2 . 1\right]_0^1$$
$$= \frac{(m-1)!}{l(l+1)\dots(l+m-1)}.$$

Similarly, if l be a positive integer,

$$\mathrm{B}(l, m) = \frac{(l-1)!}{m(m+1)\dots(m+l-1)},$$

and if both be positive integers,

$$\mathrm{B}(l, m) = \frac{(l-1)!(m-1)!}{(l+m-1)!}.$$

859. **Various Forms of the Beta Function.**

The Beta function may be thrown into many other forms by a change of the variable, and therefore many other integrals are expressible in terms of the Beta function.

Thus: (1) Let $$y=\frac{x}{a}.$$

Then $$\mathrm{B}(l, m) \int_0^1 y^{l-1}(1-y)^{m-1}dy$$

$$=\int_0^a \left(\frac{x}{a}\right)^{l-1}\left(1-\frac{x}{a}\right)^{m-1}\frac{1}{a}\,dx$$

$$=\frac{1}{a^{l+m-1}}\int_0^a x^{l-1}(a-x)^{m-1}dx.$$

Hence $$\int_0^a x^{l-1}(a-x)^{m-1}dx=a^{l+m-1}\mathrm{B}(l, m).$$

(2) Let $$y=\frac{1}{1+x}.$$

Then $$\mathrm{B}(l, m)\equiv\int_0^1 y^{l-1}(1-y)^{m-1}dy$$

$$=\int_\infty^0 \frac{1}{(1+x)^{l-1}}\left(\frac{x}{1+x}\right)^{m-1}(-1)\frac{dx}{(1+x)^2}$$

$$=\int_0^\infty \frac{x^{m-1}}{(1+x)^{l+m}}\,dx,$$

and since l, m are interchangeable this must also

$$=\int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}\,dx,$$

which would have appeared immediately if we had made the substitution $y=\dfrac{x}{1+x}$ instead of $y=\dfrac{1}{1+x}$.

Note also that the symmetry in l, m may be exhibited as

$$\mathrm{B}(l, m)\equiv\frac{1}{2}\int_0^1 \frac{x^{l-1}+x^{m-1}}{(1+x)^{l+m}}\,dx;$$

whilst for all positive values of l and m we have

$$\int_0^1 \frac{x^{l-1}-x^{m-1}}{(1+x)^{l+m}}\,dx=0.$$

So that, for instance,

$$\int_0^1 \frac{x^5(1-x^6)}{(1+x)^{18}}\,dx=0; \quad \text{and} \quad \int_0^1 \frac{x^5(1+x^6)}{(1+x)^{18}}\,dx=2\mathrm{B}(6, 12).$$

(3) Putting $$\frac{y}{1+a}=\frac{x}{x+a}, \quad dy=a(a+1)\frac{dx}{(x+a)^2},$$

$$B(l, m)=\int_0^1 y^{l-1}(1-y)^{m-1}dy$$

$$=\int_0^1 (1+a)^{l-1}\left(\frac{x}{x+a}\right)^{l-1} a^{m-1}\left(\frac{1-x}{x+a}\right)^{m-1} a(a+1)\frac{dx}{(x+a)^2}$$

$$=a^m(1+a)^l\int_0^1 \frac{x^{l-1}(1-x)^{m-1}}{(a+x)^{l+m}}dx.$$

Hence $$\int_0^1 \frac{x^{l-1}(1-x)^{m-1}}{(a+x)^{l+m}}dx=\frac{B(l, m)}{a^m(1+a)^l}.$$

This is **Abel's transformation** (*Œuvres*, Vol. I., p. 93).

(4) Put $$y=\frac{x-b}{a-b}.$$

Then $$B(l, m)\equiv\int_0^1 y^{l-1}(1-y)^{m-1}dy$$

$$=\int_b^a \left(\frac{x-b}{a-b}\right)^{l-1}\left(\frac{a-x}{a-b}\right)^{m-1}\frac{dx}{a-b}$$

$$=\frac{1}{(a-b)^{l+m-1}}\int_b^a (x-b)^{l-1}(a-x)^{m-1}dx,$$

and $$\int_b^a (x-b)^{l-1}(a-x)^{m-1}dx=(a-b)^{l+m-1}B(l, m).$$

Here the limits have been changed to any arbitrary constants a and b.

(5) Transform by the formula $\dfrac{a}{x}-\dfrac{b}{y}=a-b.$

Here the limits remain unaltered, for if $y=1$ we have $x=1$, and if $y=0$, $x=0$.

$$B(l, m)=\int_0^1 y^{l-1}(1-y)^{m-1}dy$$

$$=\int_0^1\left\{\frac{bx}{a+(b-a)x}\right\}^{l-1}\left\{\frac{a(1-x)}{a+(b-a)x}\right\}^{m-1}\frac{ab\,dx}{\{a+(b-a)x\}^2}$$

$$=a^m b^l\int_0^1 \frac{x^{l-1}(1-x)^{m-1}}{\{a+(b-a)x\}^{l+m}}dx.$$

Hence $$\int_0^1 \frac{x^{l-1}(1-x)^{m-1}}{\{a+(b-a)x\}^{l+m}}dx=\frac{1}{a^m b^l}B(l, m);$$

also obviously $$\int_0^1 \frac{x^{l-1}(1-x)^{m-1}}{\{b+(a-b)x\}^{l+m}}dx=\frac{1}{a^l b^m}B(l, m);$$

and if we write $a-b=c$,

$$\int_0^1 \frac{x^{l-1}(1-x)^{m-1}}{(b+cx)^{l+m}}dx=\frac{1}{(b+c)^l b^m}B(l, m).$$

(6) In the last transformation, put $x = \sin^2\theta$.

Then $$\int_0^{\frac{\pi}{2}} \frac{\sin^{2l-2}\theta\cos^{2m-2}\theta}{(a\cos^2\theta + b\sin^2\theta)^{l+m}} 2\sin\theta\cos\theta\, d\theta = \frac{1}{a^m b^l} \mathrm{B}(l, m),$$

i.e. $$\int_0^{\frac{\pi}{2}} \frac{\sin^{2l-1}\theta\cos^{2m-1}\theta}{(a\cos^2\theta + b\sin^2\theta)^{l+m}} d\theta = \frac{1}{2a^m b^l} \mathrm{B}(l, m),$$

l, m, a and b being positive constants.

(7) $I = \int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\, d\theta$ is expressible in the same way in terms of a Beta function.

Let $\sin\theta = \sqrt{x}$, *i.e.* $\cos\theta\, d\theta = \frac{1}{2\sqrt{x}} dx$.

$$I = \int_0^1 x^{\frac{p}{2}} (1-x)^{\frac{q-1}{2}} \frac{1}{2\sqrt{x}} dx$$

$$= \frac{1}{2}\int_0^1 x^{\frac{p+1}{2}-1} (1-x)^{\frac{q+1}{2}-1} dx$$

$$= \frac{1}{2} \mathrm{B}\left(\frac{p+1}{2}, \frac{q+1}{2}\right).$$

This also follows from No. (6) by putting $a = b = 1$.

860. **Properties of the Gamma Function.**

Consider next the Gamma function, viz.

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx.$$

Integrating by parts

$$\Gamma(n) = \Big[-x^{n-1}e^{-x}\Big]_0^\infty + (n-1)\int_0^\infty e^{-x} x^{n-2}\, dx,$$

and whatever n may be, provided it be finite and >1, $-x^{n-1}e^{-x}$ vanishes at both limits.

Hence $\Gamma(n) = (n-1)\Gamma(n-1)$.

Similarly, $\Gamma(n-1) = (n-2)\Gamma(n-2)$,

and so on.

In the case then, where n is a positive integer,

$$\Gamma(n) = (n-1)(n-2)(n-3)\ldots 3.2.1\,\Gamma(1),$$

and $$\Gamma(1) = \int_0^\infty e^{-x}\, dx = \Big[-e^{-x}\Big]_0^\infty = 1,$$

whence $\Gamma(n) = (n-1)!$ in that case.

861. **Working Properties.**

We then have the properties

$$\Gamma(n+1)=n\,\Gamma(n), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\mathrm{I}.$$

$$\Gamma(1)=1\,; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\mathrm{II}.$$

and when n is a positive integer,

$$\Gamma(n+1)=n! \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\mathrm{III}.$$

The Gamma functions of the positive integers are then

$$\begin{aligned}
&\Gamma(1)=1,\\
&\Gamma(2)=1\,.\,1=1,\\
&\Gamma(3)=2\,\Gamma(2)=1\,.\,2,\\
&\Gamma(4)=3\,\Gamma(3)=1\,.\,2\,.\,3,\\
&\Gamma(5)=4\,\Gamma(4)=1\,.\,2\,.\,3\,.\,4,\\
&\qquad\text{etc.},
\end{aligned}$$

from which a rough idea of the march of $\Gamma(x)$ as a continuous function may be inferred, viz. a minimum existing somewhere between $x=1$ and $x=2$, and then after $x=2$ a quantity increasing more and more rapidly.

862. In any case the equation $\Gamma(n+1)=n\,\Gamma(n)$ furnishes a means of reduction of the Gamma function of any number greater than unity to a Gamma function of a number less than unity.

For instance

$$\Gamma(\tfrac{17}{3})=\tfrac{14}{3}\,\Gamma(\tfrac{14}{3})=\tfrac{14}{3}\,.\,\tfrac{11}{3}\,\Gamma(\tfrac{11}{3})=\tfrac{14}{3}\,.\,\tfrac{11}{3}\,.\,\tfrac{8}{3}\,\Gamma(\tfrac{8}{3})=\tfrac{14}{3}\,.\,\tfrac{11}{3}\,.\,\tfrac{8}{3}\,.\,\tfrac{5}{3}\,\Gamma(\tfrac{5}{3})$$
$$=\tfrac{14}{3}\cdot\tfrac{11}{3}\cdot\tfrac{8}{3}\cdot\tfrac{5}{3}\cdot\tfrac{2}{3}\,\Gamma(\tfrac{2}{3}).$$

That is, the Gamma function of any number greater than unity can be connected with the Gamma function of a number which is not greater than unity; so that it is already obvious that when we come to the calculation and tabulation of the numerical values of Gamma functions it will be unnecessary to tabulate $\Gamma(x)$ for any values of x except those which lie between 0 and 1.

863. **A Caution.**

The student should guard against the idea that the equations

$$\Gamma(x)=\int_0^\infty e^{-v}\,v^{x-1}\,dv \quad\text{and}\quad \Gamma(x+1)=x\,\Gamma(x)$$

are co-equivalent. They are not so. The latter is a consequence of the former, not the former of the latter. The latter is a functional or difference equation, viz.

$$\phi(x+1)=x\phi(x) \quad \text{or} \quad u_{x+1}=xu_x,$$

and such equations may have many solutions. What is proved is that $u_x=\int_0^\infty e^{-v}v^{x-1}\,dv$ is *a particular solution of* $u_{x+1}=xu_x$.

But so also are $A\int_0^\infty e^{-v}v^{x-1}\,dv$ when A is any constant, or such an expression as

$$\frac{A+B\cos^4 2\pi x}{C+D\sin^6 2\pi x}\int_0^\infty e^{-v}v^{x-1}\,dv$$

where A, B, C, D are constants, for these multipliers are not altered when x is increased by unity. Nor does it follow that $\int_0^\infty e^{-v}v^{x-1}\,dv$ occurs as a factor in all solutions of the difference equation.

The solution of $u_{x+1}=xu_x$ is obviously

$$Ax(x-1)(x-2)\ldots(r+1)ru_r$$

when A is either a constant or some arbitrary periodic function of x whose periodicity is unity, and which therefore does not alter when x is increased or decreased by any integer, and u_r any assumed initial value of u_x. We shall return to this matter later.

864. **Transformation of the Gamma Function.**

As in the case of the Beta function, transformations of the variable will give rise to other integrals.

(1) We have seen that $x=\log\frac{1}{y}$ or $y=e^{-x}$ produces

$$\Gamma(n)\equiv\int_0^\infty x^{n-1}e^{-x}\,dx=\int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy,$$

the form studied by Euler.

(2) If we write kx for x,

$$\Gamma(n)=\int_0^\infty e^{-kx}k^n x^{n-1}\,dx;$$

whence $$\int_0^\infty e^{-kx}x^{n-1}\,dx=\frac{\Gamma(n)}{k^n}$$

provided k be a real constant (see Arts. 1159 to 1162 and 1327).

(3) If we put $x^n=y$ where n is positive,

$$\Gamma(n)=\frac{1}{n}\int_0^\infty e^{-y^{\frac{1}{n}}}\,dy\,;$$

$$\therefore \int_0^\infty e^{-y^{\frac{1}{n}}}\,dy=n\,\Gamma(n)=\Gamma(n+1).$$

In this case, if we put $n=\frac{1}{2}$,

$$\int_0^\infty e^{-y^2}dy=\int_0^\infty e^{-x^2}dx=\tfrac{1}{2}\,\Gamma(\tfrac{1}{2}),$$

and this leads to an easy calculation of $\Gamma(\frac{1}{2})$.

For $$\{\tfrac{1}{2}\,\Gamma(\tfrac{1}{2})\}^2=\int_0^\infty e^{-x^2}dx\times\int_0^\infty e^{-y^2}dy,$$

and as x and y are independent variables and the limits constant, we may write this as

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy.$$

Now, regarding x, y as the Cartesian coordinates of a point we have to sum all such elements as $e^{-(x^2+y^2)}\,\delta x\,\delta y$ through an infinite square in the positive quadrant, two of whose sides are the coordinate axes.

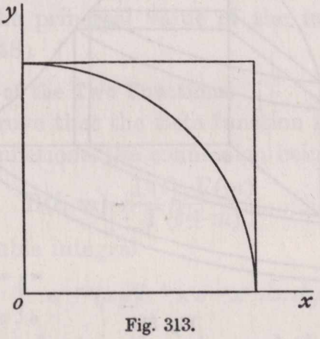

Fig. 313.

Transforming to polars, we have to sum

$$e^{-r^2}r\,\delta\theta\,\delta r$$

through the same square.

Let $x=a$, $y=a$, where $a=\infty$, be the other two sides of the square. Then for the portion of the square which lies inside the circle $x^2+y^2=a^2$ the limits for θ are 0 and $\frac{\pi}{2}$, and for r 0 and ∞.

Hence the portion within the circular quadrant contributes

$$\int_0^\infty \int_0^{\frac{\pi}{2}} e^{-r^2} r\,dr\,d\theta = \frac{\pi}{2}\int_0^\infty re^{-r^2}\,dr = \frac{\pi}{2}\Big[-\tfrac{1}{2}e^{-r^2}\Big]_0^\infty = \frac{\pi}{4}.$$

At points of the square outside the circle the elements are never greater than $e^{-a^2} r\,\delta\theta\,\delta r$, and when a is made sufficiently great this becomes an infinitesimal of higher degree than the second, and hence in the double integration disappears. Therefore the portion of the area between the circle and the square, exterior to the circle, contributes nothing.

Hence the value of $\Gamma(\frac{1}{2})$ is $\pm\sqrt{\pi}$, and as all the Gamma functions are from the definition essentially positive quantities,

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}.*$$

865. We may also regard the investigation of $\int_0^\infty e^{-u^2}\,du$ as the problem of finding the volume† bounded by the plane of x-y and the surface formed by the revolution about the z-axis of the curve $z = e^{-x^2}$, for this volume may be regarded as

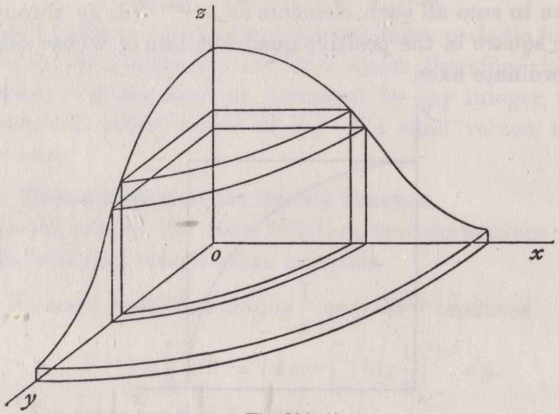

Fig. 314.

being built up of cylindrical shells whose axes coincide with the z-axis. The volume of this solid is then $\int_0^\infty 2\pi u\,du\,.\,z$, where u is the radius of a section parallel to the x-y plane,

$$= 2\pi\int_0^\infty ue^{-u^2}\,du = \pi.$$

* Euler, Tom. V., *des anciens Mémoires de Pétersbourg*, p. 44.

† Airy, *Errors of Observation*, p. 12.

But dividing it by planes parallel to the coordinate planes of $x=0$ and $y=0$, the volume is also expressed by

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dy\right]dx=\int_{-\infty}^{\infty} e^{-x^2}\,dx \times \int_{-\infty}^{\infty} e^{-y^2}dy$$

$$=4\left(\int_0^{\infty} e^{-x^2}\,dx\right)^2;$$

whence $$\int_0^{\infty} e^{-x^2}dx=\frac{\sqrt{\pi}}{2}.$$

This gives another geometrical interpretation to the work of the preceding article.

866. When n is diminished without limit $\int_0^{\infty} e^{-x}x^{n-1}\,dx$ becomes infinite. For the formula $\Gamma(n+1)=n\Gamma(n)$ holds for all positive values of n. Hence

$$Lt_{n=0}\Gamma(n)=Lt\frac{\Gamma(n+1)}{n}=Lt_{n=0}\frac{1}{n}=\infty,$$

$$i.e.\quad \Gamma(0)=\infty.$$

This is also obvious from the integral itself. For the integrand $\frac{e^{-x}}{x}$ (for the case $n=0$) takes an ∞ value at the lower limit, and the principal value of the integral becomes infinite (see Art. 348).

867. **Connection of the Two Functions.**

We shall next prove that the Beta function is expressible in terms of Gamma functions, the connection being

$$\mathrm{B}(l,\,m)=\frac{\Gamma(l)\,.\,\Gamma(m)}{\Gamma(l+m)}.$$

Consider the double integral

$$I=\int_0^{\infty}\int_0^{\infty} e^{-xy}(xy)^{l-1}\times e^{-x}x^m\,dx\,dy$$

[that is xy is written for x in the integrand of $\Gamma(l)$, and this is multiplied by the factors of the integrand of $\Gamma(m+1)$], *i.e.*

$$I=\int_0^{\infty}\int_0^{\infty} e^{-x(y+1)}x^{l+m-1}y^{l-1}\,dy\,dx.$$

Integrating first with regard to x, we have

$$I=\int_0^{\infty} y^{l-1}\frac{\Gamma(l+m)}{(1+y)^{l+m}}\,dy$$

$$=\Gamma(l+m)\,\mathrm{B}(l,\,m),\text{ by Art. 859 (2).}$$

But changing the order of integration, taking y first,

$$I=\int_0^\infty\int_0^\infty e^{-x}x^{l+m-1}y^{l-1}e^{-xy}\,dx\,dy$$

$$=\int_0^\infty e^{-x}x^{l+m-1}\frac{\Gamma(l)}{x^l}\,dx$$

$$=\Gamma(l)\int_0^\infty e^{-x}x^{m-1}\,dx$$

$$=\Gamma(l)\,\Gamma(m).$$

Hence $\quad \mathrm{B}(l,\,m)=\dfrac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)}.$

868. **Deductions.**

It further follows that

$$\mathrm{B}(l+m,\,n)=\frac{\Gamma(l+m)\,\Gamma(n)}{\Gamma(l+m+n)},$$

and therefore that

$$\mathrm{B}(l,\,m)\,\mathrm{B}(l+m,\,n)=\frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)},$$

which is a symmetric function of l, m, n. Hence we have

$$\mathrm{B}(l,\,m)\,\mathrm{B}(l+m,\,n)=\mathrm{B}(m,\,n)\,\mathrm{B}(m+n,\,l)=\mathrm{B}(n,\,l)\,\mathrm{B}(n+l,\,m).$$

Hence also

$$\mathrm{B}(l,\,m)\,\mathrm{B}(l+m,\,n)\,\mathrm{B}(l+m+n,\,p)=\frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)\,\Gamma(p)}{\Gamma(l+m+n+p)},\text{ etc.}$$

869. It now follows that the results of the transformations of the Beta function given in Art. 859 could be further expressed as Gamma functions.

Thus

$$\int_0^1\frac{x^{l-1}(1-x)^{m-1}dx}{(b+cx)^{l+m}}=\frac{1}{(b+c)^l b^m}\mathrm{B}(l,\,m)=\frac{1}{(b+c)^l b^m}\,\frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)},$$

$$\int_0^{\frac{\pi}{2}}\frac{\sin^{2l-1}\theta\cos^{2m-1}\theta}{(a\cos^2\theta+b\sin^2\theta)^{l+m}}d\theta=\frac{1}{2a^m b^l}\mathrm{B}(l,\,m)=\frac{1}{2a^m b^l}\,\frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)},$$

$$\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta\,d\theta=\frac{1}{2}\,\mathrm{B}\left(\frac{p+1}{2},\,\frac{q+1}{2}\right)=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q}{2}+1\right)},$$

etc.

The last of these integrals has already been used in earlier chapters, for convenience of calculation, with a temporary and limited definition of Γ.

870. We have also in Art. 859, Case 2, the integral

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{l+m}}dx = \mathrm{B}(l, m) = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)}.$$

Put $l+m=1$. Then, since $\Gamma(1)=1$, we have

$$\Gamma(m)\,\Gamma(1-m) = \int_0^\infty \frac{x^{m-1}}{1+x}dx,$$

where m is a positive proper fraction.

We have then to consider this integral next.

871. **The Integral** $I \equiv \int_0^\infty \frac{x^{n-1}}{1+x}dx$ where $0<n<1$.

The integration $\int_0^\infty$ may be separated into two parts, viz.

$$\int_0^1 + \int_1^\infty.$$

In the second part put $x=\frac{1}{y}$.

Then

$$\int_1^\infty \frac{x^{n-1}}{1+x}dx = \int_1^0 \frac{y^{1-n}}{1+\frac{1}{y}}\left(-\frac{1}{y^2}\right)dy = \int_0^1 \frac{y^{-n}}{1+y}dy = \int_0^1 \frac{x^{-n}}{1+x}dx.$$

Hence
$$I \equiv \int_0^1 \frac{x^{n-1}+x^{-n}}{1+x}dx,$$

and by division

$$\frac{1}{1+x} = 1-x+x^2-x^3+\ldots+(-1)^k x^k + (-1)^{k+1}\frac{x^{k+1}}{1+x}.$$

Hence

$$I \equiv \int_0^1 (x^{n-1}+x^{-n})(1-x+x^2-\ldots+(-1)^k x^k)$$
$$+(-1)^{k+1}\int_0^1 x^{k+1}\frac{x^{n-1}+x^{-n}}{1+x}dx$$

$$\left\{\frac{1}{n}-\frac{1}{1+n}+\frac{1}{2+n}-\frac{1}{3+n}+\ldots+(-1)^k\frac{1}{k+n}\right.$$
$$\left.+\frac{1}{1-n}-\frac{1}{2-n}+\frac{1}{3-n}-\ldots-(-1)^k\frac{1}{k-n}+(-1)^k\frac{1}{k-n+1}\right\}$$
$$+(-1)^{k+1}\int_0^1 x^{k+1}\frac{x^{n-1}+x^{-n}}{1+x}dx.$$

Now $\operatorname{cosec} z=$

$$\frac{1}{z}-\frac{1}{z+\pi}-\frac{1}{z-\pi}+\frac{1}{z+2\pi}+\frac{1}{z-2\pi}-\frac{1}{z+3\pi}-\frac{1}{z-3\pi}+\dots \text{ to } \infty.$$

(Hobson, *Trigonometry*, p. 335.)

Hence

$$\frac{1}{n}-\frac{1}{1+n}+\frac{1}{1-n}+\frac{1}{2+n}-\frac{1}{2-n}-\frac{1}{3+n}+\frac{1}{3-n}+\dots \text{ to } \infty$$

$$=\frac{\pi}{\sin n\pi};$$

and since in the limit when k is made indefinitely large the last term of the series for I, viz. $(-1)^k\dfrac{1}{k-n+1}$ becomes zero, the portion of I within the brackets becomes $\dfrac{\pi}{\sin n\pi}$.

Also as to the remainder, viz. $\displaystyle\int_0^1 x^{k+1}\frac{x^{n-1}+x^{-n}}{1+x}\,dx$, we may note that as x lies between 0 and 1 and is a positive proper fraction, x^{k+1} is diminished indefinitely by an infinite increase in k. If then this integration be expressed as a summation according to the definition of Art. 11, each term of the summation is diminished without limit, and may be regarded as an infinitesimal of the second or higher order when k is sufficiently increased.

Hence $$Lt_{k=\infty}\int_0^1 x^{k+1}\frac{x^{n-1}+x^{-n}}{1+x}\,dx=0,$$

and we are left with

$$\int_0^\infty \frac{x^{n-1}}{1+x}\,dx=\frac{\pi}{\sin n\pi} \quad \text{where} \quad 0<n<1.$$

872. An Important Result.

It now follows that

$$\Gamma(n)\,\Gamma(1-n)=\frac{\pi}{\sin n\pi} \quad (0<n<1).$$

As a particular case put $n=\frac{1}{2}$.

$$\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2=\frac{\pi}{\sin\dfrac{\pi}{2}}=\pi,$$

and $\Gamma(\frac{1}{2})=\sqrt{\pi}$, as has been seen before, Art. 864.

Again, put $n=\frac{1}{4}$.

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\frac{\pi}{\sin\frac{\pi}{4}}=\pi\sqrt{2}\,; \quad \therefore\ \Gamma\left(\frac{3}{4}\right)=\frac{\pi\sqrt{2}}{\Gamma\left(\frac{1}{4}\right)}.$$

Put $n=\frac{1}{6}$.

$$\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)=\frac{\pi}{\sin\frac{\pi}{6}}=2\pi\,; \quad \therefore\ \Gamma\left(\frac{5}{6}\right)=\frac{2\pi}{\Gamma\left(\frac{1}{6}\right)}, \text{ etc.}$$

Hence $\Gamma(\frac{3}{4})$, $\Gamma(\frac{5}{6})$, etc., are expressed in terms of Gamma functions of numbers which are $<\frac{1}{2}$; whence it will appear that if all Gamma functions were tabulated from $\Gamma(0)$ to $\Gamma(\frac{1}{2})$, all others could be found by this theorem, together with the theorem $\Gamma(n+1)=n\Gamma(n)$.

The result $\Gamma(n)\Gamma(1-n)=\dfrac{\pi}{\sin n\pi}$, was temporarily borrowed in an earlier chapter, Art. 592, in the calculation of a certain arc of a Lemniscate.

Since $\Gamma(1+n)=n\Gamma(n)$ and $\Gamma(n)\Gamma(1-n)=\dfrac{\pi}{\sin n\pi}$, this formula may be written

$$\Gamma(1+n)\,\Gamma(1-n)=\frac{n\pi}{\sin n\pi} \quad (0<n<1).$$

873. **To show that**

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\dots\Gamma\left(\frac{n-1}{n}\right)=\frac{(2\pi)^{\frac{n-1}{2}}}{n^{\frac{1}{2}}}.$$

We are now able to consider the continued product

$$P\equiv\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\dots\Gamma\left(\frac{n-1}{n}\right),$$

where *n for the present is any positive integer.*

By writing it down again in the reverse order, multiplying the results, and noting that

$$\Gamma\left(\frac{r}{n}\right)\Gamma\left(1-\frac{r}{n}\right)=\frac{\pi}{\sin\frac{r}{n}\pi} \quad (r<n),$$

we have $$P^2=\frac{\pi}{\sin\frac{\pi}{n}}\cdot\frac{\pi}{\sin\frac{2\pi}{n}}\cdot\frac{\pi}{\sin\frac{3\pi}{n}}\dots\frac{\pi}{\sin\frac{(n-1)\pi}{n}};$$

and since

$$\frac{\sin n\theta}{\sin\theta}=2^{n-1}\sin\left(\theta+\frac{\pi}{n}\right)\sin\left(\theta+\frac{2\pi}{n}\right)\sin\left(\theta+\frac{3\pi}{n}\right)\dots\sin\left(\theta+\frac{\overline{n-1}\,\pi}{n}\right)$$

(Hobson, *Trigonometry*, p. 117),

we have in the limit when $\theta=0$,

$$n=2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\dots\sin\frac{(n-1)\pi}{n}.$$

Hence $P^2=\frac{\pi^{n-1}}{n}2^{n-1}$, and P being positive, we have

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\dots\Gamma\left(\frac{n-1}{n}\right)=\frac{(2\pi)^{\frac{n-1}{2}}}{n^{\frac{1}{2}}}.$$

874. **Gauss' Π Function.**

Taking the original Eulerian form of the Gamma function, viz.

$$\Gamma(n)=\int_0^1\left(\log\frac{1}{x}\right)^{n-1}dx,$$

and remembering that $Lt_{\mu=\infty}\dfrac{1-x^{\frac{1}{\mu}}}{\frac{1}{\mu}}=\log\frac{1}{x}$ (*Diff. Calc.*, Art. 21)

we may write

$$\left(\log\frac{1}{x}\right)^{n-1}=\left\{\frac{\left(1-x^{\frac{1}{\mu}}\right)}{\frac{1}{\mu}}\right\}^{n-1}+\epsilon,$$

where ϵ is something which vanishes in the limit when μ becomes infinite.

Let us take μ as a positive integer.

Then $$\Gamma(n)=\int_0^1\mu^{n-1}\left(1-x^{\frac{1}{\mu}}\right)^{n-1}dx+\int_0^1\epsilon\,dx.$$

In the first integral put $x=y^\mu$.

Then $$\Gamma(n)=\mu^n\int_0^1 y^{\mu-1}(1-y)^{n-1}dy+\int_0^1\epsilon\,dx,$$

and as μ is a positive integer,

$$\int_0^1 y^{\mu-1}(1-y)^{n-1}dy=\frac{(\mu-1)!}{n(n+1)\dots(n+\mu-1)}\quad\text{(Art. 858).}$$

Hence $$\Gamma(n)=\mu^n\frac{(\mu-1)!}{n(n+1)\dots(n+\mu-1)}+\int_0^1\epsilon\,dx.$$

Hence, making μ increase without limit, the integral ultimately vanishes, and

$$\Gamma(n) = Lt_{\mu=\infty}\, \mu^n \frac{(\mu-1)!}{n(n+1)\dots(n+\mu-1)},$$

or, which is the same thing,

$$\Gamma(n) = Lt_{\mu=\infty}\, \mu^{n-1} \frac{\mu!}{n(n+1)\dots(n+\mu-1)};$$

and writing $n+1$ for n,

$$\Gamma(n+1) = Lt_{\mu=\infty}\, \mu^n \frac{1\,.\,2\,.\,3\dots\mu}{(n+1)\dots(n+\mu)}.$$

This limit is known as Gauss' Π function, and is written

$$\Pi(n) = Lt_{\mu=\infty}\, \mu^n \frac{1\,.\,2\,.\,3\dots\mu}{(n+1)\dots(n+\mu)},$$

or, which is the same thing,

$$Lt_{\mu=\infty} \frac{\mu^n}{\left(1+\frac{n}{1}\right)\left(1+\frac{n}{2}\right)\dots\left(1+\frac{n}{\mu}\right)}.$$

Here μ is integral, and n is essentially positive but not necessarily integral.

875. The limiting form at which we have arrived at the end of the last article plays an extremely important part in the development of the general theory of Gamma functions. It will be very desirable for the student to pay considerable attention to it, and we propose therefore, in due course, to consider at some length the general behaviour of the function $\dfrac{1\,.\,2\,.\,3\dots\mu}{(x+1)(x+2)(x+3)\dots(x+\mu)}\mu^x$ for different values of μ and for different values of x, and the only restriction we shall place upon it at present will be that μ is to be a positive integer, not necessarily large.

Two theorems, however, are required in dealing with such expressions as will arise, viz.

(1) **Wallis' Theorem**, which states that when n is a very large positive integer, $\dfrac{2\,.\,4\,.\,6\dots 2n}{1\,.\,3\,.\,5\dots(2n-1)}$ and $\sqrt{n\pi}$ become infinite in a ratio of equality, *i.e.*

$$Lt_{n=\infty} \frac{2\,.\,4\,.\,6\dots 2n}{1\,.\,3\,.\,5\dots(2n-1)} \frac{1}{\sqrt{n\pi}} = 1.$$

(2) **Stirling's Theorem**, which states that when n is a very large positive integer

$$1 . 2 . 3 \dots n \quad \text{and} \quad \sqrt{2n\pi} . n^n . e^{-n}$$

become infinite in a ratio of equality, that is

$$Lt_{n=\infty} \frac{n!\,e^n}{n^{n+\frac{1}{2}}} = \sqrt{2\pi}.$$

The first of these appears in most treatises on Trigonometry, for instance, Hobson's *Trigonometry*, p. 331, Ex. 1, but scarcely appears to receive the prominence in the text-books that it deserves. The second, Stirling's Theorem, is less available for the student; hence these theorems are reproduced here for present use.

876. **Digression on Wallis' and Stirling's Theorems.**

WALLIS. Expressing $\sin\theta$ as $\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\dots$ to ∞, and putting $\theta=\frac{\pi}{2}$, we have

$$\frac{2}{\pi}=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)\dots$$

$$=\frac{1.3}{2^2}\cdot\frac{3.5}{4^2}\cdot\frac{5.7}{6^2}\dots\frac{(2n-1)(2n+1)}{(2n)^2}\dots$$

$$=\frac{1^2.3^2.5^2\dots(2n-1)^2}{2^2.4^2.6^2\dots(2n)^2}(2n+1)\times(1-\epsilon),$$

where ϵ becomes indefinitely small when n becomes indefinitely large.

Hence, when n is large, we have

$$\frac{2.4.6\dots 2n}{1.3.5\dots(2n-1)}=\sqrt{\frac{\pi}{2}(2n+1)} \text{ ultimately;}$$

and since n is very great, we have

$$Lt\,\frac{2.4\dots 2n}{1.3\dots(2n-1)}\cdot\frac{1}{\sqrt{n\pi}}=1,$$

and $\frac{2.4\dots 2n}{1.3\dots(2n-1)}$ may be replaced by $\sqrt{n\pi}$, these expressions being ultimately equal. This is **Wallis' Theorem.**

877. STIRLING. Stirling's Theorem states that for very large values of n, $1.2.3 \ldots n$ and $\sqrt{2n\pi}\, e^{-n} n^n$ are ultimately equal.

Write $\qquad \phi(n)$ for $1.2.3 \ldots n$.

Then $\qquad \phi(2n) = 1.2.3 \ldots\ldots 2n$

and $\qquad 2^n \phi(n) = 2.4.6 \ldots 2n$.

Hence Wallis' Theorem, which may be written as

$$\frac{2^2.4^2.6^2 \ldots (2n)^2}{1.2.3.4 \ldots (2n-1).2n} = \sqrt{n\pi},$$

gives
$$\frac{2^{2n}[\phi(n)]^2}{\phi(2n)} = \sqrt{n\pi}.$$

Let $\qquad \dfrac{\phi(n)}{n^n\sqrt{2n\pi}}$ be called $\mathrm{F}(n)$.

Then $\qquad 2^{2n}[n^n\sqrt{2n\pi}\,\mathrm{F}(n)]^2 = \sqrt{n\pi}\,(2n)^{2n}\sqrt{4n\pi}\,\mathrm{F}(2n)$,

i.e. $\qquad \mathrm{F}(2n) = [\mathrm{F}(n)]^2$.

To solve this functional equation, write $2n$ for n.

Then $\qquad \mathrm{F}(2^2 n) = [\mathrm{F}(2n)]^2 = [\mathrm{F}(n)]^{2^2}$.

Similarly $\qquad \mathrm{F}(2^3 n) = [\mathrm{F}(n)]^{2^3}$, etc.,

and $\qquad \mathrm{F}(2^p n) = [\mathrm{F}(n)]^{2^p}$,

p being a positive integer.

Now, putting $\qquad 2^p n = x$,

$$\mathrm{F}(x) = \left\{[\mathrm{F}(n)]^{\frac{1}{n}}\right\}^x.$$

Let p increase indefinitely and n decrease indefinitely in such way as to keep the product $2^p n$ finite. Also let

$$Lt_{n=0}[\mathrm{F}(n)]^{\frac{1}{n}}$$

be called k.

Then $\mathrm{F}(x) = k^x$, which indicates the form of F to be exponential. We have to determine k.

Taking $\qquad 1.2.3 \ldots n \equiv \phi(n) = n^n\sqrt{2n\pi}\,k^n$,

change n to $n+1$.

$$1.2.3 \ldots n.(n+1) = (n+1)^{n+1}\sqrt{\overline{2n+1}\,\pi}\,k^{n+1}.$$

Hence, by division, $\quad n+1 = \dfrac{(n+1)^{n+1}}{n^n} \cdot \dfrac{\sqrt{n+1}}{\sqrt{n}} \cdot k$,

i.e.
$$k^{-1} = \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{n}\right)^{\frac{1}{2}}$$
$$= e$$

in the limit when n is indefinitely large. Hence $k=e^{-1}$, and therefore $1.2.3 \ldots n$ and $\sqrt{2n\pi}\, n^n e^{-n}$ become infinite with n, in a ratio of equality, or, what is the same thing,

$$Lt_{n=\infty} \frac{e^n n!}{n^{n+\frac{1}{2}}} = \sqrt{2\pi}.$$

This is Stirling's Theorem. The result will be considered further in a subsequent article (Art. 884).

This particular form of proof was given by Dr. E. J. Routh in lectures at Cambridge (see also Dr. Glaisher on Stirling's Theorem in the *Messenger of Mathematics*).

878. Illustrations of the Use of Stirling's Theorem.

Stirling's Theorem is useful in such cases as involve factorials of large numbers.

1. Thus the middle coefficient of the expansion of $(1+x)^{2n}$ where n is a positive integer, viz. $\frac{(2n)!}{(n!)^2}$, is ultimately when n is very large,

$$= \frac{\sqrt{4n\pi}\,(2n)^{2n} e^{-2n}}{2n\pi\, n^{2n} e^{-2n}} = \frac{2^{2n}}{\sqrt{n\pi}}.$$

This is the limiting form. It is of course infinite itself, but for large values of n a close approximation will be thus obtained. Thus, for instance, even taking a case when n is not *exceedingly* large, in calculating ${}^{40}C_{20} = \frac{40!}{(20!)^2}$ and $\frac{2^{40}}{\sqrt{20\pi}}$ from the logarithm tables the latter only exceeds the former by about 0·7 per cent.; and in calculating ${}^{100}C_{50} = \frac{100!}{(50!)^2}$ and $\frac{2^{100}}{\sqrt{50\pi}}$, the latter only exceeds the former by about 0·25 per cent.; and the error is diminishing as the magnitude of the numbers dealt with increases.

Ultimately, for exceedingly large values of n, the middle coefficients of the successive expansions $(1+x)^{2n}$, $(1+x)^{2n+2}$, etc., form what is nearly a G.P. with common ratio,

$$Lt \frac{2^{2n+2}}{\sqrt{(n+1)\pi}} \Big/ \frac{2^{2n}}{\sqrt{n\pi}}, \quad i.e.\ 4:1,$$

as is also directly obvious.

2. The n^{th} number of Bernoulli, viz. B_{2n-1} (see *Diff. Calc.*, p. 502), being given by

$$B_{2n-1} = \frac{2\,(2n)!}{(2\pi)^{2n}}\left(1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \ldots\right),$$

we have, when n is large,

$$B_{2n-1} = 2\frac{\sqrt{2n.2\pi}\,(2n)^{2n} e^{-2n}}{(2\pi)^{2n}}$$
$$= 4\pi^{-2n+\frac{1}{2}} e^{-2n} n^{2n+\frac{1}{2}}.$$

Similarly if $\frac{K_n}{n!}$ be the coefficient of x^n in the expansion of $\sec x + \tan x$, it is known that

$$K_n = \frac{2^{n+2} n!}{\pi^{n+1}} \{1 + (-\tfrac{1}{3})^{n+1} + (+\tfrac{1}{5})^{n+1} + (-\tfrac{1}{7})^{n+1} + \dots\},$$

which embraces the cases of Bernoullian numbers and Eulerian numbers together, viz.

$$K_{2n} \equiv \text{the } n^{\text{th}} \text{ Eulerian number},$$

$$K_{2n-1} \equiv \frac{2^{2n}(2^{2n}-1)}{2n} B_{2n-1}$$

(see *Diff. Calc.*, Art. 573, etc.),

and we have when n is very large,

$$K_n = \frac{2^{n+2}}{\pi^{n+1}} \sqrt{2n\pi}\, n^n e^{-n} = 2^{n+\frac{5}{2}} \left(\frac{n}{\pi}\right)^{n+\frac{1}{2}} e^{-n}.$$

In this expansion, viz.

$$\sec x + \tan x = 1 + K_1 \frac{x}{1!} + K_2 \frac{x^2}{2!} + K_3 \frac{x^3}{3!} + \dots,$$

the ratio of the $(n+1)^{\text{th}}$ term to the n^{th} is

$$\frac{K_n}{K_{n-1}} \frac{x}{n},$$

and when n is large this becomes

$$Lt \frac{2^{n+\frac{5}{2}} \left(\frac{n}{\pi}\right)^{n+\frac{1}{2}} e^{-n}}{2^{n+\frac{3}{2}} \left(\frac{n-1}{\pi}\right)^{n-\frac{1}{2}} e^{-n+1}} \cdot \frac{x}{n}$$

$$= Lt \frac{2}{\pi e} \cdot \frac{1}{\left(1 - \frac{1}{n}\right)^n} \cdot n^{\frac{1}{2}} (n-1)^{\frac{1}{2}} \cdot \frac{x}{n}$$

$$= Lt \frac{2}{\pi e} \cdot \frac{1}{e^{-1}} \cdot n \cdot \frac{x}{n} = \frac{2x}{\pi}.$$

It appears that, since $Lt \frac{K_n}{K_{n-1}} = \frac{2n}{\pi}$, the coefficients increase with great rapidity ultimately, and the series will be divergent for values of $x \not< \frac{\pi}{2}$.

3. In the series which gives rise to the Bernoullian numbers, viz.

$$\frac{x}{2} \coth \frac{x}{2} \equiv \frac{x}{e^x - 1} + \frac{x}{2} = 1 + B_1 \frac{x^2}{2!} - B_3 \frac{x^4}{4!} + B_5 \frac{x^6}{6!} - \dots + (-1)^{n-1} B_{2n-1} \frac{x^{2n}}{(2n)!} + \dots,$$

the ratio of the $(n+1)^{\text{th}}$ term to the n^{th} is

$$-\frac{B_{2n-1}}{B_{2n-3}} \frac{x^2}{(2n-1)(2n)},$$

and when n is large,

$$= - Lt \frac{4\pi^{-2n+\frac{1}{2}} e^{-2n} n^{2n+\frac{1}{2}}}{4\pi^{-2n+\frac{5}{2}} e^{-2n+2} (n-1)^{2n-\frac{3}{2}}} \cdot \frac{x^2}{(2n-1)2n}$$

$$= - Lt \frac{1}{\pi^2} \cdot \frac{1}{e^2} \frac{1}{\left(1-\frac{1}{n}\right)^{2n}} \cdot n^{\frac{1}{2}} (n-1)^{\frac{3}{2}} \cdot \frac{x^2}{(2n-1)2n}$$

$$= - Lt \frac{1}{\pi^2} \cdot \frac{1}{e^2} \cdot \frac{1}{e^{-2}} \cdot n^2 \cdot \frac{x^2}{4n^2}$$

$$= - \frac{x^2}{4\pi^2}.$$

The series is therefore divergent for values of $x^2 \not< (2\pi)^2$, and as

$$Lt \frac{B_{2n-1}}{B_{2n-3}} = Lt \frac{(2n-1)2n}{4\pi^2} = \frac{n^2}{\pi^2} \text{ ultimately,}$$

the Bernoullian numbers ultimately increase with great rapidity.

It will be noted that $\coth \frac{x}{2}$ becomes infinite if x have the unreal value $2\iota\pi$. When x is complex it is therefore necessary to limit expansion to the case for which the modulus of the complex is $< 2\pi$.*

879. A method of Calculation of the Numbers of Bernoulli and the Numbers of Euler is explained in the *Differential Calculus*, Art. 573. Both sets have been calculated for many coefficients of their respective series (see *Proceedings of the British Association* 1877), and probably far enough for all practical purposes for which they will ever be required. Several are quoted on pages 106 and 501 of the *Differential Calculus*. A few extra results are put upon record here for reference, for the convenience of the reader. Also, as we are about to deal with such sums as $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$ to $\infty \equiv S_p$, which for even values of p are to be found from

$$B_{2n-1} = \frac{2(2n)!}{(2\pi)^{2n}} S_{2n},$$

we tabulate a few of these results also.

$$B_1 = \tfrac{1}{6},\ B_3 = \tfrac{1}{30},\ B_5 = \tfrac{1}{42},\ B_7 = \tfrac{1}{30},\ B_9 = \tfrac{5}{66},\ B_{11} = \tfrac{691}{2730},\ B_{13} = \tfrac{7}{6},$$
$$B_{15} = \tfrac{3617}{510},\ B_{17} = \tfrac{43867}{798},\ B_{19} = \tfrac{1222277}{2310};$$

$$E_2 = 1,\ E_4 = 5,\ E_6 = 61,\ E_8 = 1385,\ E_{10} = 50521;$$

$$S_2 = \frac{\pi^2}{6},\ S_4 = \frac{\pi^4}{90},\ S_6 = \frac{\pi^6}{945},\ S_8 = \frac{\pi^8}{9450},\ S_{10} = \frac{\pi^{10}}{93555}.$$

The values of S_p up to S_{35} reduced to decimals will be found tabulated later for purposes of evaluation of integrals to be discussed (Art. 957).

880. For other methods of Calculation of Bernoulli's Numbers etc., see Boole, *Finite Differences*, Chapter VI.

881. We note that $B_1 > B_3 > B_5 < B_7 < B_9 <$ etc., and the coefficient B_5 is the smallest of Bernoulli's Numbers, after which they rapidly increase.

* See Bertrand, *Calc. Diff.*, Art. 412.

882. **The Value of $\Pi(\frac{1}{2})$.**

Consider next the case of Gauss' Π function for $n=\frac{1}{2}$.

$$\Pi\left(\frac{1}{2}\right)=Lt_{\mu=\infty}\frac{1.2.3\ldots\mu}{\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{2\mu+1}{2}}\mu^{\frac{1}{2}}$$

$$=Lt_{\mu=\infty}\frac{2^2.4^2.6^2\ldots(2\mu)^2}{1.2.3.4\ldots(2\mu)(2\mu+1)}\mu^{\frac{1}{2}}$$

$$=Lt_{\mu=\infty}\frac{2^{2\mu}(\mu\,!)^2}{(2\mu+1)\,!}\mu^{\frac{1}{2}}$$

$$=Lt_{\mu=\infty}\frac{2^{2\mu}2\mu\pi\,.\,\mu^{2\mu}\,.\,e^{-2\mu}}{\sqrt{(4\mu+2)\pi}\,(2\mu+1)^{2\mu+1}e^{-(2\mu+1)}}\mu^{\frac{1}{2}}$$

$$=Lt_{\mu=\infty}e\sqrt{\pi}\frac{1}{\left(1+\frac{1}{2\mu}\right)^{2\mu}}\frac{1}{\left(1+\frac{1}{2\mu}\right)}\frac{\mu^{\frac{1}{2}}}{2\sqrt{\mu+\frac{1}{2}}}$$

$$=e\sqrt{\pi}\,.\,\frac{1}{e}\,.\,1\,.\,\frac{1}{2}=\frac{\sqrt{\pi}}{2};$$

whence $$\Pi\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}.$$

It will be remembered that for positive values of n,

$$\Pi(n)=\Gamma(n+1);$$

therefore $\Gamma\left(\frac{3}{2}\right)=\Pi\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}$ and $\Gamma\left(\frac{3}{2}\right)=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$;

$$\therefore\ \Gamma(\tfrac{1}{2})=\sqrt{\pi},$$

which agrees with Art. 864.

883. **The Graph of $y=x^ne^{-x}$.**

We shall next study the nature of the family of curves

$$y=x^ne^{-x}$$

for various values of n.

The subject of integration in the Gamma Function $\Gamma(n+1)$ viz. x^ne^{-x}, has a maximum value when

$$nx^{n-1}e^{-x}-x^ne^{-x}=0,\quad i.e.\text{ when } x=n\quad (n>0),$$

and the maximum ordinate of the curve $y=x^ne^{-x}$ for positive values of x is n^ne^{-n}.

The graphs of the members of this family for $n=0$, $n=0{\cdot}5$ $n=1$, $n=2$ are shown in the accompanying figure for the first quadrant, which is all we require.

The case $n=0$, viz. $y=e^{-x}$, is a logarithmic curve, and cuts the y-axis at a point $y=1$. It has no maximum ordinate

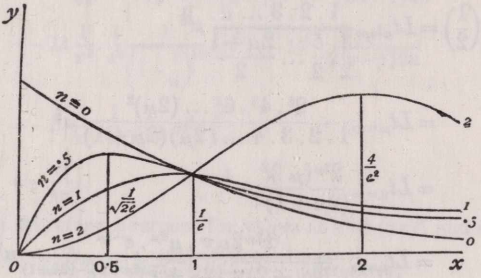

Fig. 315.

The case $n=0{\cdot}5$ has a maximum ordinate at $x=\frac{1}{2}$, viz. $\dfrac{1}{\sqrt{2e}}$, and then runs to the positive end of the x-axis asymptotically.

The case $n=1$ has a maximum at $x=1$, viz. $\dfrac{1}{e}$.

The case $n=2$ has a maximum at $x=2$, viz. $\dfrac{4}{e^2}$.

All the curves have the x-axis as an asymptote, and all go through the point $\left(1, \dfrac{1}{e}\right)$, where they cross.

For values of n between 0 and 1, the curves touch the y-axis at the origin.

The case $n=1$ touches the line $y=x$ at the origin.

The cases for $n > 1$ touch the x-axis at the origin.

The several maxima, viz. $n^n e^{-n}$, diminish for various values of n from $n=0$ to $n=1$, and then increase again, all the crests the curves lying upon $y=x^x e^{-x}$, *i.e.*

$$y=\left(\frac{x}{e}\right)^x$$

the least of the maximum ordinates being at $x=1$, and belonging to the curve $y=xe^{-x}$.

The area bounded by any of these curves $y=x^n e^{-x}$, the x-axis and the ordinate at $x=\infty$, is

$$\int_0^\infty e^{-x} x^n \, dx, \quad \textit{i.e.}\ \Gamma(n+1),$$

and increases without limit as n increases.

884. **Extension of Stirling's Theorem.**

We have shown (Stirling's Theorem) that when n *is a large positive integer*,

$$1.2.3\ldots n = \sqrt{2n\pi}\, n^n e^{-n},$$

the meaning of the equality sign being that these quantities become infinite in a ratio of equality.

We proceed to show that even when n is not integral, but still positive,
$$\Gamma(n+1) = \sqrt{2n\pi}\, n^n e^{-n},$$
when n is indefinitely increased.

We have
$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\, dx.$$

Let us transform this integral by putting

$$x^n e^{-x} = n^n e^{-n} e^{-\frac{n}{2}t^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

which is legitimate, as $n^n e^{-n}$ has been shown to be the maximum value of $x^n e^{-x}$.

Now, as t ranges from $-\infty$ through zero to $+\infty$,

x ranges from 0 through n to $+\infty$.

Thus
$$\frac{\Gamma(n+1)}{n^n e^{-n}} = \int_{-\infty}^{\infty} e^{-\frac{n}{2}t^2} \frac{dx}{dt}\, dt,$$

and we have to find $\frac{dx}{dt}$. Let $x = n(1+\tau)$.

Then
$$(n+n\tau)^n e^{-n} e^{-n\tau} = n^n e^{-n} e^{-\frac{n}{2}t^2};$$

$$\therefore (1+\tau)^n e^{-n\tau} = e^{-\frac{n}{2}t^2} \quad \text{and} \quad \log(1+\tau) - \tau = -\frac{t^2}{2}. \ \ldots(2)$$

Clearly τ vanishes with t, and as t can be expressed in terms of τ by expanding the logarithm, we can by the ordinary process of reversion of series expand τ in terms of t.

Let
$$\tau = A_1\frac{t}{1!} + A_2\frac{t^2}{2!} + A_3\frac{t^3}{3!} + \ldots.$$

Then, differentiating equation (2),

$$\tau\frac{d\tau}{dt} = t(1+\tau); \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

whence, by substituting the sèries for τ and equating coefficients, we can readily obtain the values of A_1, A_2, A_3, etc.

Now $$\frac{\Gamma(n+1)}{n^n e^{-n}}=\int_{-\infty}^{\infty} e^{-\frac{n}{2}t^2}\frac{dx}{dt}\,dt=n\int_{-\infty}^{\infty} e^{-\frac{n}{2}t^2}\frac{d\tau}{dt}\,dt$$

$$=n\int_{-\infty}^{\infty} e^{-\frac{n}{2}t^2}\left[A_1+A_2\frac{t}{1\,!}+A_3\frac{t^2}{2\,!}+A_4\frac{t^3}{3\,!}+\dots\right]dt$$

and $$\int_{-\infty}^{\infty} t^{2p}e^{-\kappa^2t^2}\,dt=\frac{1.3.5\dots(2p-1)}{2^p\kappa^{2p+1}}\sqrt{\pi},$$

by writing κt for x in the result of Art. 223, Ex. 4,

$$=\frac{\Gamma\left(\dfrac{2p+1}{2}\right)}{\kappa^{2p+1}},$$

and $$\int_{-\infty}^{\infty} t^{2p+1}e^{-\kappa^2t^2}\,dt=0,$$

as is obvious, for the negative elements of the summation cancel out the positive ones.

Hence

$$\frac{\Gamma(n+1)}{n^n e^{-n}}=n\left\{A_1\frac{\Gamma(\frac{1}{2})}{\left(\dfrac{n}{2}\right)^{\frac{1}{2}}}+\frac{A_3}{2\,!}\frac{\Gamma(\frac{3}{2})}{\left(\dfrac{n}{2}\right)^{\frac{3}{2}}}+\frac{A_5}{4\,!}\frac{\Gamma(\frac{5}{2})}{\left(\dfrac{n}{2}\right)^{\frac{5}{2}}}+\text{etc.}\right\}$$

$$=\sqrt{2n\pi}\left[A_1+\frac{1}{2}\cdot\frac{2}{n}\cdot\frac{A_3}{2\,!}+\frac{1}{2}\cdot\frac{3}{2}\left(\frac{2}{n}\right)^2\frac{A_5}{4\,!}+\dots\right],$$

and it remains to obtain the numerical values of the coefficients.

Substituting the series for τ in the differential equation (3),

$$\left(A_1\frac{t}{1\,!}+A_2\frac{t^2}{2\,!}+A_3\frac{t^3}{3\,!}+A_4\frac{t^4}{4\,!}+\dots\right)\times\left(A_1+A_2\frac{t}{1\,!}+A_3\frac{t^2}{2\,!}+\dots\right)$$
$$\equiv t\left(1+A_1\frac{t}{1\,!}+A_2\frac{t^2}{2\,!}+A_3\frac{t^3}{3\,!}+\dots\right);$$

whence $$\frac{A_1}{1\,!}A_1=1,$$

$$\frac{A_1}{1\,!}\frac{A_2}{1\,!}+\frac{A_2}{2\,!}A_1=\frac{A_1}{1\,!},$$

$$\frac{A_1}{1\,!}\frac{A_3}{2\,!}+\frac{A_2}{2\,!}\frac{A_2}{1\,!}+\frac{A_3}{3\,!}A_1=\frac{A_2}{2\,!},$$

and generally

$$\frac{A_1}{1\,!}\frac{A_n}{(n-1)\,!}+\frac{A_2}{2\,!}\frac{A_{n-1}}{(n-2)\,!}+\frac{A_3}{3\,!}\frac{A_{n-2}}{(n-3)\,!}+\dots+\frac{A_n}{n\,!}A_1=\frac{A_{n-1}}{(n-1)\,!},$$

i.e. $$nA_1A_n+\frac{n(n-1)}{1.2}A_2A_{n-1}+\frac{n(n-1)(n-2)}{1.2.3}A_3A_{n-2}+\ldots+A_nA_1=nA_{n-1},$$

i.e. $$(n+1)A_1A_n+\frac{(n+1)n}{1.2}A_2A_{n-1}+\frac{(n+1)n(n-1)}{1.2.3}A_3A_{n-2}+\ldots=nA_{n-1},$$

the series proceeding as far as the greatest binomial coefficient in $(1+z)^{n+1}$, and the last term of the series being halved if n be odd.

Thus
$$\begin{aligned}
&A_1=1,\\
&3A_1A_2=2A_1,\\
&4A_1A_3+3A_2{}^2=3A_2,\\
&5A_1A_4+10A_2A_3=4A_3,\\
&6A_1A_5+15A_2A_4+10A_3{}^2=5A_4,\\
&7A_1A_6+21A_2A_5+35A_3A_4=6A_5,\\
&8A_1A_7+28A_2A_6+56A_3A_5+35A_4{}^2=7A_6,\\
&\qquad\text{etc.,}
\end{aligned}$$

giving $A_1=1$, $A_2=\frac{2}{3}$, $A_3=\frac{1}{6}$, $A_4=-\frac{4}{45}$, $A_5=\frac{1}{36}$, $A_6=\frac{8}{189}$, $A_7=-\frac{139}{1080}$, $A_8=\frac{16}{81}$, $A_9=-\frac{571}{6480}$, etc.

Hence, finally,

$$\Gamma(n+1)=\sqrt{2n\pi}\,n^ne^{-n}\left[1+\frac{1}{12}\frac{1}{n}+\frac{1}{288}\frac{1}{n^2}+\ldots\right].$$

When n is indefinitely large, we therefore have

$$\Gamma(n+1)=\sqrt{2n\pi}\,n^ne^{-n},$$

which removes the limitation that n should be a positive integer, as supposed in Art. 877. Moreover, it will be noted that an expansion of $\dfrac{\Gamma(n+1)}{\sqrt{2n\pi}\,n^ne^{-n}}$ is effected in powers of $\dfrac{1}{n}$, viz.

$$\frac{\Gamma(n+1)}{\sqrt{2n\pi}\,n^ne^{-n}}=1+\frac{1}{12}\frac{1}{n}+\frac{1}{288}\frac{1}{n^2}-\frac{139}{51840}\frac{1}{n^3}-\ldots+\frac{A_{2p+1}}{2^pp!}\frac{1}{n^p}+\ldots,$$

the law of formation of A_{2p+1} being as above stated.

885. Ex. 1. In calculating $10\,!$ in this way,

$\log\sqrt{2\pi\,.\,10}\,.\,10^{10}e^{-10}=6{\cdot}3561451$ (Chambers' seven-figure logarithms);

$\therefore\ \sqrt{2\pi\,.\,10}\,.\,10^{10}e^{-10}=3598695$ (the last figure doubtful).

Carrying the series to four terms, viz.

$$1+\tfrac{1}{120}+\tfrac{1}{28800}-\tfrac{139}{51840000}\equiv 1{\cdot}00836537,$$

we get $\quad 10\,!=3598695\times 1{\cdot}00836537=3628799{\cdot}$ etc.

The true value is 3628800, so there is only an error in the last figure in the approximation.

Ex. 2. Calculate 100! Here

$$\log(100!) = \log\{\sqrt{2\pi\,.\,100}\,.\,100^{100}e^{-100}(1+\tfrac{1}{1200}+\tfrac{1}{2880000}-\dots)\}$$
$$=157\cdot9700036,$$

indicating a number of 158 figures, beginning with 933262, viz. $9\cdot33262\times10^{157}$.

[The logarithms from 1 to 100 add up to 157·9700038, which is in agreement with this result, except for the seventh figure of logarithms.]

886. **Properties of Gauss' Π Function.**

We may now proceed to discuss the nature and properties of Gauss' Π function.

Let us start again with a consideration of the expression

$$\Pi(x,\mu) \equiv \frac{1\,.\,2\,.\,3\dots\mu}{(x+1)(x+2)(x+3)\dots(x+\mu)}\mu^x,$$

where μ is a positive integer, not necessarily large, at present, and x is a fixed number, either real or unreal, positive or negative, integral or fractional, but finite. Call the expression $\Pi(x,\mu)$, and abbreviate it further into $\Pi(x)$ when in the limit μ is ∞, so that $\Pi(x)$ stands for $\Pi(x,\infty)$.

Consider the graphs of

$$y=\frac{1\,.\,2\,.\,3\dots\mu}{(x+1)(x+2)\dots(x+\mu)}\mu^x$$

for different values of μ.

There are μ asymptotes parallel to the y-axis.

y is positive from $x=\infty$ to $x=-1$,
negative from $x=-1$ to $x=-2$,
positive from $x=-2$ to $x=-3$,

and so on.

And if μ be >1, the x-axis is an asymptote at its negative extremity only;

also when $x=0$, $y=1$;

when $x=1$, $y=\dfrac{\mu}{\mu+1}$;

when $x=2$, $y=\dfrac{1\,.\,2\,\mu^2}{(\mu+1)(\mu+2)}$;

etc.;

and these ordinates approximate to 1, 1, 2!, 3!, ... as μ increases, whilst at the same time the number of asymptotes increases.

The cases of $\mu=1, 2, 3$ and 4 are shown in the accompanying figures, which are intended to exhibit graphically the general characteristics of the functions, but are not drawn to scale.

The case $\mu=1$ gives $y=\dfrac{1}{x+1}$, a rectangular hyperbola, with $y=0$, $x=-1$ for asymptotes.

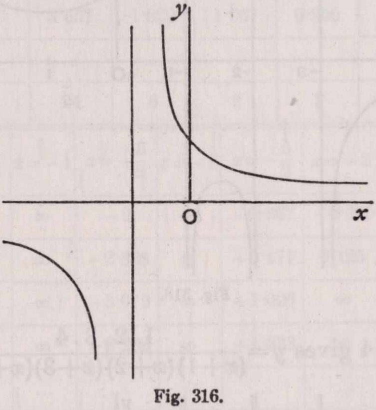

Fig. 316.

The case $\mu=2$ gives $y=\dfrac{1\,.\,2}{(x+1)(x+2)}\,2^x$.

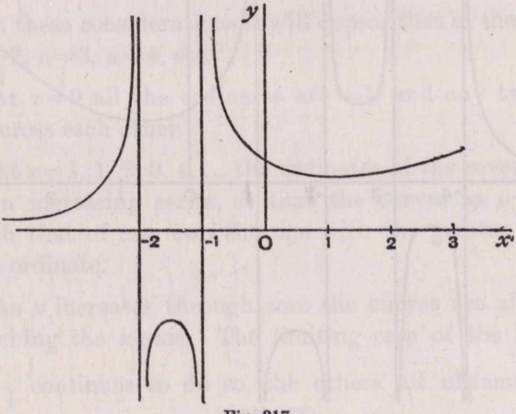

Fig. 317.

The case $\mu=3$ gives $y=\dfrac{1.2.3}{(x+1)(x+2)(x+3)}3^x$.

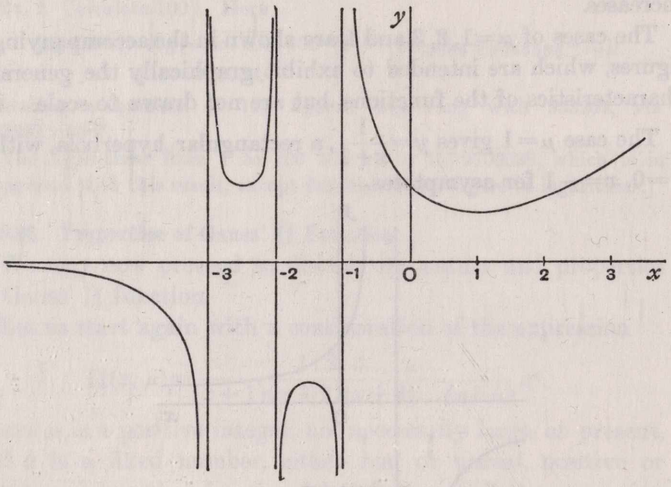

Fig. 318.

The case $\mu=4$ gives $y=\dfrac{1.2.3.4}{(x+1)(x+2)(x+3)(x+4)}4^x$.

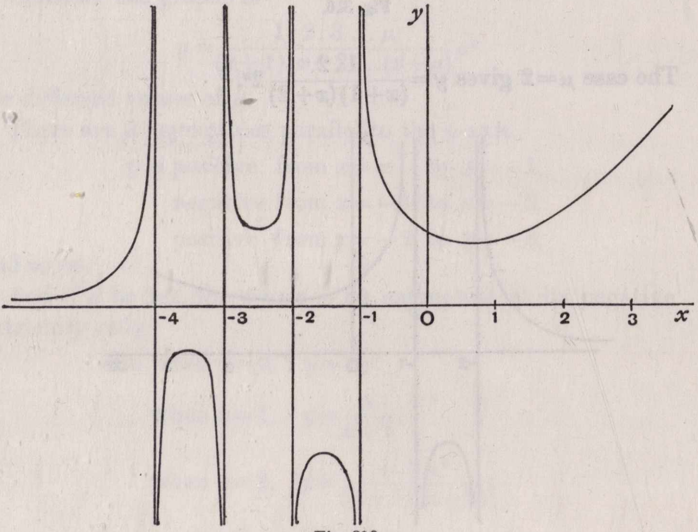

Fig. 319.

The lengths of the ordinates for various values of x and μ are shown in the table:

	$x=5$	$x=4$	$x=3$	$x=2$	$x=1$	$x=\frac{1}{2}$	$x=0$
$\mu=1$	0·167	0·200	0·250	0·333	0·500	0·667	1
$\mu=2$	1·524	1·067	0·800	0·667	0·667	0·754	1
$\mu=3$	4·339	2·314	1·350	0·900	0·750	0·792	1
$\mu=4$	8·127	3·657	1·829	1·067	0·800	0·813	1
...	...	...	...	...	...	...	...
$\mu=\infty$	120	24	6	2	1	0·886	1

	$x=-\frac{1}{2}$	$x=-1$	$x=-\frac{3}{2}$	$x=-2$	$x=-\frac{5}{2}$	$x=-3$	$x=-\frac{7}{2}$	$x=-4$
$\mu=1$	2	∞	-2	-1	$-0·667$	$-0·500$	$-0·400$	$-0·333$
$\mu=2$	1·886	∞	$-2·828$	∞	$+0·471$	0·125	0·047	0·021
$\mu=3$	1·847	∞	$-3·079$	∞	$+1·026$	∞	$-0·068$	$-0·012$
$\mu=4$	1·829	∞	$-3·200$	∞	$+1·333$	∞	$-0·200$	∞
...	...	...	...	...	...	...	...	...
$\mu=\infty$	1·772	∞	$-3·545$	∞	2·363	∞	$-0·945$	∞

887. **General Remarks.**

From these considerations it will appear that in these curves, viz. $\mu=2$, $\mu=3$, $\mu=4$, etc.,

(1) At $x=0$ all the ordinates are $=1$, and any two of the curves cross each other.

(2) At $x=\frac{1}{2}$, 1, 2, 3, 4, ... the ordinates of the several curves form an increasing series, so that the curves as μ increases are such that of any two the one with the greater μ has the greater ordinate.

(3) As x increases through zero the curves are all initially approaching the x-axis. The limiting case of the hyperbola $y=\dfrac{1}{x+1}$ continues to do so, the others all ultimately have

ordinates >1, and therefore have minimum ordinates in the first quadrant. Moreover it may be shown that

$\mu=2$ has a minimum ordinate between 1 and 2,
$\mu=3$ „ „ „ 0·9 and 1,
$\mu=4$ „ „ „ 0·7 and 0·8,
etc.

As μ increases, the minimum ordinate begins to approach the y-axis, but does not do so without limit. For in the case $\mu=\infty$ it lies somewhere between 0 and 1.

(4) On the negative side of the y-axis at $x=-\frac{1}{2}$ the successive ordinates of the curves $\mu=1$, $\mu=2$, $\mu=3$, etc., form a diminishing set.

(5) $\mu=1$ has one asymptote parallel to the y-axis,
$\mu=2$ has two asymptotes parallel to the y-axis,
$\mu=3$ has three asymptotes parallel to the y-axis,
etc.

$\mu=1$ is asymptotic to the x-axis at both ends.
$\mu=2$, $\mu=3$, $\mu=4$, etc., are only asymptotic to the x-axis at its negative end, and alternately from above and below the x-axis.

(6) Observe the behaviour between the several asymptotes.

Between $x=-1$ and $x=-2$ the several ordinates at $x=-\frac{3}{2}$ are all negative but numerically increasing, *i.e.* the more asymptotes there are the further do these branches recede from the x-axis. Similarly between the asymptotes $x=-2$ and $x=-3$, or any consecutive pair.

Note also that for each given value of μ the branch between two consecutive asymptotes has a numerically greater ordinate midway between those asymptotes than is the case for a branch between two consecutive asymptotes more remote from the y-axis.

(7) The limiting case

$$y=Lt_{\mu=\infty}\frac{1\,.\,2\,\ldots\,\mu}{(x+1)(x+2)\ldots(x+\mu)}\mu^x,\ \text{viz. } y=\Pi(x)$$

becomes, when x is positive, the curve $y=\Gamma(x+1)$, as has been shown.

The shape of this limiting form will be more carefully considered later in Art. 922.

But there is this difference between the functions

$$Lt_{\mu=\infty}\frac{1.2\ldots\mu}{(x+1)(x+2)\ldots(x+\mu)}\mu^x \quad\text{and}\quad \int_0^\infty e^{-v}v^x\,dv,$$

that though they coincide in value for all positive values of x, the former becomes infinite at the values $x=-1$, $x=-2$, $x=-3$, etc., but has finite values for other negative values of x, whilst the definite integral is permanently infinite for all negative values of $x+1$.

888. That the factor form has finite values, when μ becomes infinitely large, for negative values of x between the asymptotes may be made clear by taking a case. Take $x=-\frac{3}{2}$.

Then $$Lt_{\mu=\infty}\frac{1.2.3\ldots\mu}{\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\ldots\left(\frac{2\mu-3}{2}\right)}\mu^{-\frac{3}{2}}$$

$$=-Lt\frac{2.4.6\ldots 2\mu}{1.1.3.5\ldots(2\mu-3)}\frac{1}{\mu^{\frac{3}{2}}}$$

$$=-Lt\frac{2^2.4^2.6^2\ldots(2\mu)^2}{1.2.3.4\ldots(2\mu-3)(2\mu-2)(2\mu-1)(2\mu)}\frac{(2\mu-1)}{\mu^{\frac{3}{2}}}$$

$$=-Lt\,2^{2\mu}\frac{(\sqrt{2\mu\pi}\,\mu^\mu e^{-\mu})^2}{\sqrt{4\mu\pi}(2\mu)^{2\mu}e^{-2\mu}}\frac{(2\mu-1)}{\mu^{\frac{3}{2}}}$$

$$=-Lt\frac{2\pi\mu}{2\sqrt{\pi\mu}}\frac{2\mu-1}{\mu^{\frac{3}{2}}}=-\frac{2}{1}\sqrt{\pi}.$$

Similarly at $x=-\frac{5}{2}$ the corresponding limit is $\frac{2^2}{1.3}\sqrt{\pi}$,

at $x=-\frac{7}{2}$ the corresponding limit is $-\frac{2^3}{1.3.5}\sqrt{\pi}$,

and so on.

These mid-ordinates, half way between the successive asymptotes, thus form a regular descending series

$$-\frac{2}{1}\sqrt{\pi},\quad \frac{2^2}{1.3}\sqrt{\pi},\quad -\frac{2^3}{1.3.5}\sqrt{\pi},\quad \frac{2^4}{1.3.5.7}\sqrt{\pi},\ \text{etc.}$$

889. It is worth noticing that $\Pi(x,\mu)$ may be written as

$$\Pi(x,\mu)\equiv\frac{1.2.3\ldots\mu}{(x+1)(x+2)(x+3)\ldots(x+\mu)}\mu^x$$

$$\equiv\frac{\left(\frac{2}{1}\right)^x\left(\frac{3}{2}\right)^x\left(\frac{4}{3}\right)^x\ldots\left(\frac{\mu}{\mu-1}\right)^x\left(\frac{\mu+1}{\mu}\right)^x}{\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\left(1+\frac{x}{3}\right)\ldots\left(1+\frac{x}{\mu}\right)}\left(\frac{\mu}{\mu+1}\right)^x$$

$$=\frac{\left(1+\frac{1}{1}\right)^x\left(1+\frac{1}{2}\right)^x\left(1+\frac{1}{3}\right)^x\cdots\left(1+\frac{1}{\mu}\right)^x}{\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\left(1+\frac{x}{3}\right)\cdots\left(1+\frac{x}{\mu}\right)}\left(\frac{\mu}{\mu+1}\right)^x$$

$$=\left(\frac{\mu}{\mu+1}\right)^x \overset{r=\mu}{\underset{r=1}{P}} \frac{\left(1+\frac{1}{r}\right)^x}{\left(1+\frac{x}{r}\right)},$$

where $\overset{r=\mu}{\underset{r=1}{P}}$ indicates that the product of all such fractions as follow it is to be taken from $r=1$ to $r=\mu$.

And in the limit, when $\mu=\infty$,

$$\Pi(x)=\overset{r=\infty}{\underset{r=1}{P}} \frac{\left(1+\frac{1}{r}\right)^x}{1+\frac{x}{r}},$$

or, what is the same thing, when x is real and positive,

$$\Gamma(1+x)=\overset{r=\infty}{\underset{r=1}{P}} \frac{\left(1+\frac{1}{r}\right)^x}{\left(1+\frac{x}{r}\right)}.$$

890. **Reduction of $\Pi(x+1)$.**

Again,

$$\Pi(x+1,\,\mu)=\frac{1\,.\,2\,.\,3\ldots\mu}{(x+2)(x+3)(x+4)\ldots(x+\mu)(x+\mu+1)}\,\mu^{x+1}$$

$$=\mu\,\frac{x+1}{x+\mu+1}\,\Pi(x,\,\mu).$$

Hence

$$\Pi(x+1,\,\mu)=(x+1)\,\Pi(x,\,\mu)\times\frac{1}{1+\frac{x+1}{\mu}},$$

which is the law of connexion of the successive values of $\Pi(x,\,\mu)$ for unit differences in x.

In the case when μ is indefinitely increased, the factor

$$\left(1+\frac{x+1}{\mu}\right)^{-1}$$

becomes unity, and we are left with $\Pi(x+1)=(x+1)\,\Pi(x)$

and changing x to $x-1$, $\Pi(x)=x\Pi(x-1)$. This is true for all finite values of x, positive or negative.

In the case of values of $x>0$ we have $\Pi(x)=\Gamma(x+1)$, and therefore $\Gamma(x+1)=x\Gamma(x)$, the formula already established for the Gamma function.

891. **The Case when x is a Positive Integer.**

When x is a positive integer we may multiply the numerator and denominator of

$$\Pi(x, \mu)\equiv\frac{1 \cdot 2 \ldots \mu}{(x+1)(x+2)\ldots(x+\mu)}\mu^x \text{ by } x!$$

obtaining in that case $\quad \Pi(x, \mu)=\dfrac{x!\,\mu!}{(x+\mu)!}\mu^x,$

and then removing $\mu!$,

$$\Pi(x, \mu)=\frac{1 \cdot 2 \ldots x}{(\mu+1)(\mu+2)\ldots(\mu+x)}\mu^x$$

$$=\frac{1 \cdot 2 \ldots x}{\left(1+\frac{1}{\mu}\right)\left(1+\frac{2}{\mu}\right)\ldots\left(1+\frac{x}{\mu}\right)},$$

so that when μ is indefinitely increased, x remaining finite, $\Pi(x)$ becomes $x!$, which is in accordance with the result $\Gamma(x+1)=x!$ of Art. 860.

892. **Comparison of the Gamma Function with Gauss' Function.**

It will now be clear, from Art. 887, that the two functions $\Pi(x)$ and $\Gamma(x+1)$ are identical for all real values of x greater than -1; but that $\Pi(x)$ is a more general function, embracing real or unreal values of x quite unrestricted as to sign. That $\Pi(x)$ becomes infinite for all negative integral values of x, but has finite values for negative fractional values of x, whilst $\Gamma(x)$ defined as $\int_0^\infty e^{-v}v^{x-1}\,dv$ is infinite for all negative values of x. Graphically this means that the curves $y=\Pi(x-1)$ and $y=\Gamma(x)$ absolutely coincide for all positive values of x, but do not do so for negative values of x. If we had restricted the definition of Gauss' function, viz.

$$Lt_{\mu=\infty}\Pi(x, \mu)\equiv Lt_{\mu=\infty}\frac{1 \cdot 2 \cdot 3 \ldots \mu}{(x+1)(x+2)\ldots(x+\mu)}\mu^x,$$

to real values of x greater than -1, the identity of $\Pi(x)$ with Euler's Gamma function $\Gamma(x+1)$ would have been complete.

893. We have, from the definition,

$$\Pi(-x,\mu)\equiv\frac{1\,.\,2\,.\,3\ldots(\mu-1)\mu}{(1-x)(2-x)(3-x)\ldots(\mu-1-x)(\mu-x)}\mu^{-x}$$

and $\Pi(x-1,\mu)\equiv\dfrac{1\,.\,2\,.\,3\ldots(\mu-1)\mu}{x(x+1)(x+2)\ldots(x+\mu-1)}\mu^{x-1}.$

Hence multiplying them together, and assuming that x is not an integer,

$$\Pi(-x,\mu)\,\Pi(x-1,\mu)$$

$$=\frac{1}{x}\cdot\frac{1^2\,.\,2^2\,.\,3^2\ldots(\mu-1)^2}{(1^2-x^2)(2^2-x^2)(3^2-x^2)\ldots\{(\mu-1)^2-x^2\}}\,\frac{\mu}{\mu-x}$$

$$=\frac{1}{x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\ldots\left\{1-\frac{x^2}{(\mu-1)^2}\right\}}\,\frac{\mu}{\mu-x};$$

and when μ increases without limit, $Lt\,\dfrac{\mu}{\mu-x}=1$, x being finite, and we have

$$\Pi(-x)\,\Pi(x-1)=\frac{1}{x\left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{2^2}\right)\ldots\text{ to }\infty}=\frac{\pi}{\sin\pi x}.$$

It will be noticed that in proving this result no assumption has been made with regard to x except that it is not to be an integer, either positive or negative. For such values one or other of the Π functions would be infinite, as also of course would $\dfrac{\pi}{\sin x\pi}$.

Taking positive values of x less than unity, and remembering that in that case $\Pi(x)=\Gamma(x+1)$, we have

$$\Gamma(1-x)\,\Gamma(x)=\frac{\pi}{\sin x\pi},$$

as previously found.

894. If we were to base the discussion of the properties of $\Gamma(x)$ on this method of procedure, we could therefore infer the value of the definite integral $\displaystyle\int_0^1\frac{v^{x-1}}{1+v}\,dv$ of Art. 870 to be $\dfrac{\pi}{\sin x\pi}$, where $0<x<1$, instead of investigating the integral first and then deducing the result $\Gamma(1-x)\,\Gamma(x)=\dfrac{\pi}{\sin x\pi}$.

895. **An Unreal Value of x.**

We note also that if x be unreal and $=\iota y$,

$$\iota\Pi(-\iota y)\,\Pi(\iota y-1)=\frac{\pi}{\sinh \pi y};$$

but that Γ, as defined in the Eulerian manner, loses its meaning.

See, however, Art. 900 for an extension of the definition of Γ.

896. Both functions, viz. $\Pi(x)$ and $\Gamma(x+1)$, have been shown to satisfy the equation of differences

$$u_{x+1}=(x+1)u_x.$$

Let us see from this point of view what can be ascertained as to the nature of the function u_x.

It has already been stated that this equation necessitates one form of the result to be

$$u_x=Ax(x-1)(x-2)\ldots(r+1)ru_r,$$

where A is a constant or some arbitrary periodic function of x of unit periodicity, and u_r is some initial value of u_x to be chosen at pleasure.

Following Laplace's mode of procedure in such cases, assume as a trial solution,

$$u_x=\int t^x F(t)\,dt,\text{*}$$

where the form of $F(t)$ and the limits of integration are reserved for future choice.

Then, since $u_{x+1}=(x+1)u_x$,

$$\int t^{x+1}F(t)\,dt=(x+1)\int t^x F(t)\,dt$$

$$=\int F(t)\,(x+1)\,t^x\,dt$$

$$=[F(t)\,t^{x+1}]-\int t^{x+1}F'(t)\,dt,$$

the integration being by parts, and the square brackets denoting as usual that the term integrated is to be taken between the limits ultimately chosen.

Hence the choice must be such as to satisfy the equation

$$\int t^{x+1}[F(t)+F'(t)]\,dt=[F(t)\,t^{x+1}].$$

* See Boole, *Finite Differences*, p. 257.

Let us then take $F(t)$ so that $F'(t)+F(t)=0$, and the limits such that $[F(t)\,t^{x+1}]=0$.

Our choice is now complete, and there is no further latitude.

The first equation gives $\dfrac{F'(t)}{F(t)}=-1$, *i.e.* $F(t)=Ce^{-t}$, where C is an arbitrary constant as regards t.

This determines the form of the function F in our trial solution.

The limits must then be such as will satisfy the equation

$$[Ce^{-t}t^{x+1}]=0.$$

Supposing $x+1$ to be positive, this will be effected by taking $t=0$ and $t=\infty$, for in each case $Lt\,\dfrac{t^{x+1}}{e^t}=0$.

Hence a solution of the equation for positive values of $x+1$ is

$$u_x=C\int_0^\infty e^{-t}t^x\,dt$$
$$=C\Gamma(x+1).$$

So $u_x=C\Gamma(x+1)$ is a solution, provided $x+1$ be positive where C is any arbitrary constant *as regards* t.

To put the possible dependence upon x in evidence call C, v_x.

Then $$u_x=v_x\Gamma(x+1),$$

$$u_{x+1}=v_{x+1}\Gamma(x+2)=v_{x+1}(x+1)\Gamma(x+1),$$

but $$u_{x+1}=(x+1)\,u_x\,;$$

$$\therefore\ v_{x+1}=v_x,$$

whence it is clear that v_x is either an absolute constant or some arbitrary periodic function of x whose periodicity is unity, such as $\cos^n 2\pi x$ or $\dfrac{A+B\cos^p 2\pi x}{C+D\sin^p 2\pi x}$ where A, B, C, D are absolute constants, such functions returning to their original values when x is increased by unity.

Thus $u_x=f(x)\Gamma(x+1)$ satisfies the difference equation considered when $f(x)$ is such a periodic function as described.

It appears, therefore, that the equation $u_{x+1}=(x+1)u_x$ is not co-equivalent with $u_x=\Gamma(x+1)$, *i.e.* Euler's Gamma function, or with $u_x=\Pi(x)$, *i.e.* Gauss' Π function, but that these are particular forms of the solution, as has been previously pointed out.

897. **Euler's Constant.**

The limiting value when n is made infinitely great of

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\log n$$

is finite, positive and less than unity. This limit plays an important part in our subsequent work. It is called Euler's constant and denoted by γ. Its value has been computed to over 100 places of decimals (*Proc. Royal Society*, vol. xix. and vol. xx., p. 29).

The first twenty figures are*

$$\gamma \equiv 0{\cdot}577\ 215\ 664\ 901\ 532\ 860\ 60\dots.$$

We shall presently show how it is to be computed. For the present it is sufficient to show that it is a positive proper fraction, and this admits of elementary proof.

For

$$\frac{1}{r}+\log\frac{r}{r+1}=\frac{1}{r}-\log\left(1+\frac{1}{r}\right)$$

$$=\frac{1}{2r^2}-\frac{1}{3r^3}+\frac{1}{4r^4}-\frac{1}{5r^5}+\dots,\text{ a convergent series if } r\geqq 1,$$

$$=\frac{1}{r^2}\left(\frac{1}{2}-\frac{1}{3r}\right)+\frac{1}{r^4}\left(\frac{1}{4}-\frac{1}{5r}\right)+\dots$$

$$=\text{positive, since } r\geqq 1, \text{ for every bracket is positive;}$$

$$\therefore\ \left(\frac{1}{1}+\log\frac{1}{2}\right)+\left(\frac{1}{2}+\log\frac{2}{3}\right)+\dots+\left(\frac{1}{n}+\log\frac{n}{n+1}\right)\text{ is positive;}$$

$$\therefore\ \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}+\log\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\dots\frac{n}{n+1}\text{ is positive;}$$

$$\therefore\ \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\log(n+1)\text{ is positive;}$$

and as $$\log(n+1)>\log n,$$

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\log n\text{ is positive.}$$

*See Todhunter, *Integral Calculus*, p. 256; Serret, *Calc. Intégral*, p. 183; Legendre, *Exercices*, p. 295; De Morgan, *D. and I. Calculus*, p. 578.

Secondly,

$$\frac{1}{r}+\log\frac{r-1}{r}=\frac{1}{r}+\log\left(1-\frac{1}{r}\right)$$

$$=-\frac{1}{2r^2}-\frac{1}{3r^3}-\text{etc., a convergent series if } r>1\,;$$

$$\therefore\ \sum_2^n\left(\frac{1}{r}+\log\frac{r-1}{r}\right)=-\frac{1}{2}\sum_2^n\frac{1}{r^2}-\frac{1}{3}\sum_2^n\frac{1}{r^3}-\dots,\left\{\begin{array}{l}\text{which, when } n=\infty,\\ \text{are all convergent}\\ \text{series,}\end{array}\right.$$

$$=\text{a negative quantity.}$$

Therefore

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{n}+\log\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\dots\frac{n-1}{n}\text{ is a negative quantity,}$$

i.e. $$\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\frac{1}{n}-\log n\text{ is a negative quantity,}$$

and $\therefore$ $$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\log n\text{ is less than 1,}$$

and it has been shown to be positive.

Hence, making n increase indefinitely, γ is a positive proper fraction.

898. **Closer Limits for γ.**

Let $$u_n=\sum_1^n r^{-1}-\log(n+1),\quad v_n=\sum_1^n r^{-1}-\log n\quad (n>1).$$

Then $v_n-u_n=\log\left(1+\frac{1}{n}\right)=$ positive, if n be finite, and ultimately vanishing when $n=\infty$, *i.e.* $u_\infty=v_\infty=\gamma$.

Now $u_n-u_{n-1}=\frac{1}{n}+\log\frac{n}{n+1}=$ positive; $v_n-v_{n-1}=\frac{1}{n}+\log\frac{n-1}{n}=$ negative; therefore, as n increases, u_n increases and v_n decreases towards the common limit γ; and $u_n<\gamma<v_n$, whilst n remains finite.

Taking Bottomley's tables of Reciprocals and Napierian Logarithms, we readily find

$u_1=\cdot3069,\quad u_2=\cdot4014,\dots u_{10}=\cdot5311,\quad u_{20}=\cdot5532,\quad u_{30}=\cdot5610$, etc.

$v_1=1\cdot0000,\quad v_2=\cdot8069,\dots v_{10}=\cdot6264,\quad v_{20}=\cdot6020,\quad v_{30}=\cdot5938$, etc.

We thus have an approaching set of inferior and superior limits for γ, and note that it must lie between 0·56 and 0·60. It will be seen later that $\gamma=0\cdot5772\dots$ (Art. 917).

899. **Except for negative integral values of z, $\Pi(z)$ is Finite whatever z may be, Real or Complex.**

If $u_1, u_2, u_3, \dots u_n\dots$ be any series of real positive quantities, each of which is less than unity, the infinite products $\prod_{r=1}^{r=\infty}(1+u_r)$, $\prod_{r=1}^{r=\infty}(1-u_r)$ are convergent or divergent according as the infinite series Σu_r is convergent or divergent (see Smith's *Algebra*, p. 423,* and Hobson's *Trigonometry*, p. 319), and if the quantities $u_1, u_2, \ldots u_n \ldots$ be complex quantities, the modulus of each being less than unity, the product $\prod_{r=1}^{r=\infty}(1+u_r)$ converges if the series $\Sigma \bmod u_r$ converges. (See Hobson's *Trigonometry*, p. 320.)

It can be shown that though the infinite product

$$\overset{\infty}{\underset{1}{P}}\left(1+\frac{z}{n}\right), \quad \textit{i.e.}\ \left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\left(1+\frac{z}{3}\right)\left(1+\frac{z}{4}\right)\ldots \text{ to infinity,}$$

which occurs frequently in the present chapter, is obviously divergent, yet if we multiply the several factors by

$$e^{-\frac{z}{1}},\quad e^{-\frac{z}{2}},\quad e^{-\frac{z}{3}}, \text{ etc., respectively,†}$$

we arrive at a product

$$\overset{\infty}{\underset{1}{P}}\left[\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right],$$

which is absolutely convergent for all values of z positive or negative, real or complex.

For $$\log\left(1+\frac{z}{n}\right)=\frac{z}{n}-\frac{z^2}{2n^2}+\frac{z^3}{3n^3}-\ldots$$

is a series absolutely convergent if $\bmod z < n$ for some finite value of n; whence

$$e^{-\frac{z}{n}}=e^{-\log\left(1+\frac{z}{n}\right)}e^{-\frac{z^2}{2n^2}+\frac{z^3}{3n^3}-\ldots}$$

$$=\frac{1}{1+\frac{z}{n}}e^{-\frac{z^2}{2n^2}+\frac{z^3}{3n^3}-\ldots},$$

i.e. $$\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}=e^{-\frac{z^2}{2n^2}(1+\ldots)}$$

$$=1-\frac{z^2}{2n^2}(1+\epsilon_n), \text{ say,}$$

where ϵ_n is a series absolutely convergent which for finite values of z ultimately vanishes when n is infinitely large;

$$\therefore \overset{\infty}{\underset{1}{P}}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}=\overset{\infty}{\underset{1}{P}}\left[1-\frac{z^2}{2n^2}(1+\epsilon_n)\right].$$

* Also see Arndt, *Grunert*, xxi. 78.

† Weierstrass, *Abhandlungen Acad. of Berlin*, 1876. See also Hobson, *Trigonometry*, p. 327.

Suppose E the greatest of the moduli of $1+\epsilon_n$ for all values of z within a range for which the greatest modulus of z does not exceed a given finite quantity, then $\sum_1^\infty \bmod \frac{Ez^2}{2n^2}$ is an absolutely convergent series, and therefore also $\sum_1^\infty \frac{z^2}{2n^2}(1+\epsilon_n)$ is an absolutely convergent series, and since $\overset{\infty}{\underset{1}{P}}(1+u_n)$ is absolutely convergent when $\Sigma \bmod u_n$ is convergent,

$$\overset{\infty}{\underset{1}{P}}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$$

is an absolutely convergent product, as is also

$$\overset{\infty}{\underset{1}{P}}\left(1-\frac{z}{n}\right)e^{\frac{z}{n}}.$$

Now Gauss' Π function being defined as

$$\Pi(z)=Lt_{\mu=\infty}\frac{1\,.\,2\,.\,3\ldots\mu}{(z+1)(z+2)(z+3)\ldots(z+\mu)}\mu^z$$

can be written

$$=Lt_{\mu=\infty}\frac{\mu^z}{\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\left(1+\frac{z}{3}\right)\ldots\left(1+\frac{z}{\mu}\right)}$$

$$=Lt_{\mu=\infty}\frac{e^{z\left(\log\mu-\frac{1}{1}-\frac{1}{2}-\frac{1}{3}-\ldots-\frac{1}{\mu}\right)}}{\overset{\infty}{\underset{1}{P}}\left(1+\frac{z}{\mu}\right)e^{-\frac{z}{\mu}}}$$

$$=\frac{e^{-\gamma z}}{Lt_{\mu=\infty}\overset{\infty}{\underset{1}{P}}\left(1+\frac{z}{\mu}\right)e^{-\frac{z}{\mu}}},$$

where γ is Euler's constant, which shows that for all values of z, real or complex, positive or negative, excepting negative integral values,

$$\Pi(z)=\frac{e^{-\gamma z}}{\text{a finite function of } z},$$

and is therefore finite.

900. **Extension of Meaning of $\Gamma(z)$.**

So far it has been convenient to adhere to the Legendrian definition of the symbol $\Gamma(x)$, viz.

$$\Gamma(x)=\int_0^\infty e^{-v}v^{x-1}\,dv,$$

and to regard x in this Eulerian integral as representing a real variable. It has been shown to be identical with Gauss' Π function, $\Pi(x-1)$, for all real positive values of x. Having drawn attention to the difference of behaviour of the function defined as an integral and the factor-function of Gauss for negative values of x, it is scarcely worth while observing the distinction further, and we propose to extend the use of the symbol $\Gamma(z)$ to negative and unreal values of z, which means that, when z is negative or unreal, Γ is defined by

$$\Gamma(z+1) \equiv \Pi(z) = Lt_{\mu=\infty} \frac{1.2.3\ldots\mu}{(z+1)(z+2)\ldots(z+\mu)}\mu^z,$$

and that when z is positive it is defined either in this way or as $\int_0^\infty e^{-v}v^z\,dv$, and therefore we shall in general regard $\Pi(z)$ as identical with $\Gamma(z+1)$ or $z\Gamma(z)$ for all values of z.

901. Thus a meaning will be given to such an expression as $\Gamma(a+\sqrt{-1}\,b)$, viz.

$$Lt_{\mu=\infty} \frac{\mu^{a+\iota b}}{(a+\iota b)\left(1+\frac{a+\iota b}{1}\right)\left(1+\frac{a+\iota b}{2}\right)\ldots\left(1+\frac{a+\iota b}{\mu}\right)}$$

$$= \frac{e^{-\gamma(a+\iota b)}}{\text{a finite function of } (a+\iota b)} \quad \text{(Art. 899).}$$

902. Ex. 1. The modulus of $\Gamma(\frac{1}{2}+\iota a)$ is $\sqrt{\Gamma(\frac{1}{2}+\iota a)\Gamma(\frac{1}{2}-\iota a)}$

$$= \sqrt{\{\Gamma(\tfrac{1}{2}+\iota a)\Gamma(1-\overline{\tfrac{1}{2}+\iota a})\}} = \sqrt{\frac{\pi}{\sin(\frac{1}{2}+\iota a)\pi}} \quad \text{(Art. 895)}$$

$$= \sqrt{\frac{\pi}{\cosh a\pi}}.$$

Ex. 2. If $1, \alpha, \alpha^2, \ldots \alpha^{n-1}$ be the n^{th} roots of 1 (n odd), we have

$$(1+x)(1+\alpha x)(1+\alpha^2 x)\ldots(1+\alpha^{n-1}x) = 1+x^n,$$

and

$$1+\alpha+\alpha^2+\ldots+\alpha^{n-1} = 0.$$

Hence $\Pi(x)\Pi(\alpha x)\Pi(\alpha^2 x)\ldots\Pi(\alpha^{n-1}x) = \overset{r=n-1}{\underset{r=0}{P}}\Pi(\alpha^r x)$, say,

$$= Lt_{\mu=\infty} \overset{r=n-1}{\underset{r=0}{P}} \frac{\mu^{x\alpha^r}}{\left(1+\frac{x\alpha^r}{1}\right)\left(1+\frac{x\alpha^r}{2}\right)\ldots\left(1+\frac{x\alpha^r}{\mu}\right)}$$

$$= \frac{1}{\left(1+\frac{x^n}{1^n}\right)\left(1+\frac{x^n}{2^n}\right)\left(1+\frac{x^n}{3^n}\right)\ldots \text{ to } \infty}, \quad n>1;$$

$$\therefore\ \left(1+\frac{x^n}{1^n}\right)\left(1+\frac{x^n}{2^n}\right)\left(1+\frac{x^n}{3^n}\right)\dots \text{ to inf.}=\frac{1}{\Pi(x)\,\Pi(\alpha x)\,\Pi(\alpha^2 x)\dots\Pi(\alpha^{n-1}x)}$$

$$=\frac{1}{\overset{n-1}{\underset{0}{P}}\{\Pi(\alpha^r x)\}}=\frac{1}{\overset{n-1}{\underset{0}{P}}\,\Gamma(1+\alpha^r x)}=\frac{1}{x^n\overset{n-1}{\underset{0}{P}}\Gamma(\alpha^r x)};$$

thus $$x^n\left(1+\frac{x^n}{1^n}\right)\left(1+\frac{x^n}{2^n}\right)\left(1+\frac{x^n}{3^n}\right)\dots=\frac{1}{\Gamma(x)\,\Gamma(\alpha x)\,\Gamma(\alpha^2 x)\dots\Gamma(\alpha^{n-1}x)},$$

where $1, \alpha, \alpha^2, \dots$ are the n^{th} roots of unity.

903. Gauss' Theorem.

This theorem is a generalization of that of Art. 872, and includes it. It states that for any value of z

$$\frac{n^{nz}\Pi(z)\,\Pi\left(z-\frac{1}{n}\right)\Pi\left(z-\frac{2}{n}\right)\dots\Pi\left(z-\frac{n-1}{n}\right)}{\Pi(nz)}=(2\pi)^{\frac{n-1}{2}}n^{-\frac{1}{2}},$$

or, what is the same thing, as will be seen,

$$\frac{n^{nz}\Gamma(z)\,\Gamma\left(z+\frac{1}{n}\right)\Gamma\left(z+\frac{2}{n}\right)\dots\Gamma\left(z+\frac{n-1}{n}\right)}{\Gamma(nz)}=(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}}.$$

Let the left-hand member of the first equality be called $\phi(z)$. Then, first, we shall show that $\phi(z)$ is independent of z. By definition,

$$\Pi\left(z-\frac{r}{n}\right)=Lt_{\mu=\infty}\frac{\mu^{z-\frac{r}{n}}.1.2.3\dots\mu}{\left(1+z-\frac{r}{n}\right)\left(2+z-\frac{r}{n}\right)\dots\left(\mu+z-\frac{r}{n}\right)}$$

$$=Lt_{\mu=\infty}\frac{n^{\mu}\mu^{z-\frac{r}{n}}.1.2\dots\mu}{(n+nz-r)\,(2n+nz-r)\dots(\mu n+nz-r)};$$

$$\therefore\ n^{nz}\Pi(z)\,\Pi\left(z-\frac{1}{n}\right)\dots\Pi\left(z-\frac{n-1}{n}\right)=Lt\frac{n^{nz}\,n^{n\mu}\mu^{nz}\mu^{-\frac{n-1}{2}}(\mu\,!)^n}{D},$$

where D is the product of the factors

$$\begin{array}{llll}
n+nz, & 2n+nz, & 3n+nz, & \dots\dots\ \mu n+nz,\\
n+nz-1, & 2n+nz-1, & 3n+nz-1, & \dots\dots\ \mu n+nz-1,\\
n+nz-2, & 2n+nz-2, & 3n+nz-2, & \dots\dots\ \mu n+nz-2,\\
\vdots & \vdots & \vdots & \qquad\vdots\\
n+nz-(n-1), & 2n+nz-(n-1), & 3n+nz-(n-1), & \dots\ \mu n+nz-(n-1)
\end{array}$$

i.e.

$$[(nz+1)(nz+2)\dots(nz+n)][(nz+n+1)\dots(nz+2n)]\dots[\dots(nz+\mu n)]$$

$$=(nz+1)(nz+2)\dots(nz+\mu n).$$

Hence

$$n^{nz}\Pi(z)\Pi\left(z-\frac{1}{n}\right)\dots\Pi\left(z-\frac{n-1}{n}\right)=Lt\frac{n^{nz}n^{n\mu}\mu^{nz}\mu^{-\frac{n-1}{2}}(\mu!)^n}{(nz+1)(nz+2)\dots(nz+\mu n)}.$$

Again, writing $n\mu$ for μ in Gauss' expression for $\Pi(nz)$,

$$\Pi(nz)=Lt\frac{(n\mu)^{nz}(n\mu!)}{(nz+1)(nz+2)\dots(nz+n\mu)}.$$

Hence $\quad \phi(z)=Lt\dfrac{n^{nz}n^{n\mu}\mu^{nz}\mu^{-\frac{n-1}{2}}(\mu!)^n}{(n\mu)^{nz}(n\mu!)}$

$$=Lt_{\mu=\infty}\,n^{n\mu}\mu^{-\frac{n-1}{2}}\frac{(\mu!)^n}{(n\mu!)},$$

from which the z has disappeared.

Hence, $\phi(z)$ is independent of z. It remains to find its value. To do this we may either obtain the limit of the right-hand side directly, or avoid this by comparison with a known case, for a particular value of z, which will be a legitimate process, inasmuch as its value, not containing z at all, is an absolute numerical constant containing n.

Adopting the direct method and employing Stirling's result,

$$\phi(z)=Lt_{\mu=\infty}\,n^{n\mu}\mu^{-\frac{n-1}{2}}\frac{(\sqrt{2\mu\pi}\,\mu^{\mu}e^{-\mu})^n}{\sqrt{2n\mu\pi}\,(n\mu)^{n\mu}e^{-n\mu}}$$

$$=Lt\frac{n^{n\mu}\mu^{-\frac{n-1}{2}}(2\pi)^{\frac{n-1}{2}}\mu^{\frac{n}{2}}\mu^{n\mu}e^{-n\mu}}{\mu^{\frac{1}{2}}n^{\frac{1}{2}}(n\mu)^{n\mu}e^{-n\mu}}=\frac{(2\pi)^{\frac{n-1}{2}}}{n^{\frac{1}{2}}}.$$

Hence, finally,

$$\phi(z)\equiv\frac{n^{nz}\Pi(z)\Pi\left(z-\frac{1}{n}\right)\Pi\left(z-\frac{2}{n}\right)\dots\Pi\left(z-\frac{n-1}{n}\right)}{\Pi(nz)}=\frac{(2\pi)^{\frac{n-1}{2}}}{n^{\frac{1}{2}}}.$$

904. If we adopt the plan of comparison with a known case, take the case of a real value of z, viz. $z=0$.

Then, remembering that $\Pi(x)=\Gamma(1+x)$,

$$\phi(z)=\phi(0)\equiv\Gamma(1)\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\dots\Gamma\left(1-\frac{n-1}{n}\right)\Big/\Gamma(1);$$

or, reversing the order,

$$=\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\dots\Gamma\left(\frac{n-1}{n}\right)=\frac{(2\pi)^{\frac{n-1}{2}}}{n^{\frac{1}{2}}},\text{ by Art. 873.}$$

Writing $\Pi(z)=\Gamma(z+1)$, etc., we have

$$\frac{n^{nz}\Gamma(z+1)\Gamma\left(z+\frac{n-1}{n}\right)\Gamma\left(z+\frac{n-2}{n}\right)\dots\Gamma\left(z+\frac{1}{n}\right)}{\Gamma(nz+1)}=\frac{(2\pi)^{\frac{n-1}{2}}}{n^{\frac{1}{2}}},$$

i.e. reversing the order of the factors in the numerator, with the exception of $\Gamma(z+1)$, and writing $\Gamma(z+1)=z\Gamma(z)$ and $\Gamma(nz+1)=nz\Gamma(nz)$,

$$\frac{n^{nz}z\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\Gamma\left(z+\frac{2}{n}\right)\dots\Gamma\left(z+\frac{n-1}{n}\right)}{nz\Gamma(nz)}=\frac{(2\pi)^{\frac{n-1}{2}}}{n^{\frac{1}{2}}},$$

i.e. $$\frac{n^{nz}\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\Gamma\left(z+\frac{2}{n}\right)\dots\Gamma\left(z+\frac{n-1}{n}\right)}{\Gamma(nz)}=(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}},$$

which may be written as

$$\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\Gamma\left(z+\frac{2}{n}\right)\dots\Gamma\left(z+\frac{n-1}{n}\right)=\Gamma(nz)(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nz}.$$

905. **Cases of Gauss' Theorem.**

Putting $z=\frac{1}{n}$ we have the result of Art. 873, viz.

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\dots\Gamma\left(\frac{n-1}{n}\right)=(2\pi)^{\frac{n-1}{2}}n^{-\frac{1}{2}}.$$

Particular cases are

$$n=2,\quad \Gamma(x)\Gamma\left(x+\frac{1}{2}\right)=\Gamma(2x).(2\pi)^{\frac{1}{2}}2^{\frac{1}{2}-2x},$$

i.e. $$\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)=\frac{\pi^{\frac{1}{2}}}{2^{2x-1}}\Gamma(2x),$$

i.e. putting $\frac{p+1}{2}$ for x,

$$\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+2}{2}\right)=\frac{\pi^{\frac{1}{2}}}{2^p}\Gamma(p+1);$$

$n=3$ gives $\Gamma(x)\Gamma\left(x+\frac{1}{3}\right)\Gamma\left(x+\frac{2}{3}\right)=\frac{2\pi}{3^{3x-\frac{1}{2}}}\Gamma(3x)$, etc.

906. The case $n=2$ may be deduced directly from

$$\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta\,d\theta=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q}{2}+1\right)}.$$

For putting $q=p$, we have

$$\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^p\theta \, d\theta = \frac{\left\{\Gamma\left(\frac{p+1}{2}\right)\right\}^2}{2\Gamma(p+1)};$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^p 2\theta \, d\theta = 2^p \frac{\left\{\Gamma\left(\frac{p+1}{2}\right)\right\}^2}{2\Gamma(p+1)};$$

and writing $2\theta=\phi$,

$$\int_0^{\frac{\pi}{2}} \sin^p 2\theta \, d\theta = \tfrac{1}{2}\int_0^{\pi} \sin^p\phi \, d\phi = \int_0^{\frac{\pi}{2}} \sin^p\phi \, d\phi$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{p+2}{2}\right)};$$

$$\therefore 2^p \frac{\left\{\Gamma\left(\frac{p+1}{2}\right)\right\}^2}{2\Gamma(p+1)} = \frac{\Gamma\left(\frac{p+1}{2}\right)\sqrt{\pi}}{2\Gamma\left(\frac{p+2}{2}\right)},$$

$$\textit{i.e.}\quad 2^p\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+2}{2}\right) = \pi^{\frac{1}{2}}\Gamma(p+1).$$

907. An interesting proof of this result is due to M. Serret, (*Calc. Intég.*, p. 174).

Since $B(p, q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx$ we have

$$B(p, p)=\int_0^1 (x-x^2)^{p-1}dx = \int_0^1 \left[\tfrac{1}{4}-\left(\tfrac{1}{2}-x\right)^2\right]^{p-1}dx.$$

And since the integrand assumes equal values, whether we put $x=\frac{1}{2}+h$ or $\frac{1}{2}-h$, its values are symmetric about $x=\frac{1}{2}$.

Hence

$$B(p, p)=2\int_0^{\frac{1}{2}} \left[\tfrac{1}{4}-\left(\tfrac{1}{2}-x\right)^2\right]^{p-1}dx. \quad \text{Writing } \tfrac{1}{2}-x=\frac{\sqrt{z}}{2},$$

$$B(p, p)=2\int_1^0 \frac{1}{2^{2p-2}}(1-z)^{p-1}\left(-\frac{1}{4\sqrt{z}}\right)dz$$

$$=\frac{1}{2^{2p-1}}\int_0^1 z^{-\frac{1}{2}}(1-z)^{p-1}dz = \frac{1}{2^{2p-1}}B\left(\tfrac{1}{2}, p\right),$$

$$\textit{i.e.}\quad \frac{\Gamma(p)\Gamma(p)}{\Gamma(2p)} = \frac{1}{2^{2p-1}}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p)}{\Gamma\left(p+\frac{1}{2}\right)} \quad \text{or} \quad 2^{2p-1}\Gamma(p)\Gamma\left(p+\tfrac{1}{2}\right)=\sqrt{\pi}\,\Gamma(2p)$$

or writing $2p=q+1$,

$$2^q\Gamma\left(\frac{q+1}{2}\right)\Gamma\left(\frac{q+2}{2}\right)=\sqrt{\pi}\,\Gamma(q+1).$$

908. Another form of the general theorem is $\left(\text{writing } \frac{x}{n} \text{ for } z\right)$

$$\Gamma\left(\frac{x}{n}\right)\Gamma\left(\frac{x+1}{n}\right)\Gamma\left(\frac{x+2}{n}\right)\dots\Gamma\left(\frac{x+n-1}{n}\right)=\Gamma(x)(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-x},$$

i.e. $$\Gamma\left(\frac{x}{n}\right)\Gamma\left(\frac{x+1}{n}\right)\dots\Gamma\left(\frac{x+n}{n}\right)=\Gamma(x)\Gamma\left(1+\frac{x}{n}\right)(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-x}.$$

909. **To prove** $\int_x^{x+1}\log\Gamma(x)\,dx=x\log x-x+\frac{1}{2}\log 2\pi$.

Taking Gauss' Theorem for a real variable x,

$$\Gamma(x)\Gamma\left(x+\frac{1}{n}\right)\Gamma\left(x+\frac{2}{n}\right)\dots\Gamma\left(x+\frac{n-1}{n}\right)=\Gamma(nx)(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nx},$$

we have, upon taking logarithms,

$$\begin{aligned}\frac{1}{n}\log&\left\{\Gamma(nx)(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nx}\right\}\\ &=\frac{1}{n}\left\{\log\Gamma(x)+\log\Gamma\left(x+\frac{1}{n}\right)+\dots+\log\Gamma\left(x+\frac{n-1}{n}\right)\right\}\\ &=\sum\frac{1}{n}\log\Gamma\left(x+\frac{r}{n}\right),\text{ from } r=0 \text{ to } r=n-1,\\ &=\int_0^1\log\Gamma(x+y)\,dy,\text{ when } n \text{ is indefinitely increased,}\\ &=\int_x^{x+1}\log\Gamma(v)\,dv,\text{ if } v \text{ be put for } x+y.\end{aligned}$$

Thus, by Art. 884,

$$\begin{aligned}\int_x^{x+1}\log\Gamma(v)\,dv&=Lt_{n=\infty}\frac{1}{n}\log\left[\frac{\sqrt{2nx\pi}\,(nx)^{nx}e^{-nx}}{nx}(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nx}\right]\\ &=\tfrac{1}{2}\log 2\pi+x\log x-x=\log x^x e^{-x}(2\pi)^{\frac{1}{2}}.\end{aligned}$$

910. This expresses the area bounded by the x-axis, the curve $y=\log\Gamma(x)$, and two ordinates at unit distance.

Changing x to $x+1$, and adding to the former,

$$\int_x^{x+2}\log\Gamma(x)\,dx=\log\{x^x(x+1)^{x+1}e^{-x}e^{-(x+1)}(2\pi)^{\frac{2}{2}}\},$$

and so on, and more generally,

$$\int_x^{x+n}\log\Gamma(x)\,dx$$

$$=\log\left\{x^x(x+1)^{x+1}(x+2)^{x+2}\ldots(x+n-1)^{x+n-1}e^{-nx-\frac{(n-1)n}{2}}(2\pi)^{\frac{n}{2}}\right\},$$

where n is a positive integer.

911. **Expressions for the Differential Coefficients of the Function $\psi(x)$, $\log\Gamma(x+1)$, and Expansion of $\log\Gamma(x+1)$.**

Let us write $\psi(x)$ for $\dfrac{d}{dx}\log\Gamma(x)$, *i.e.* $\dfrac{\Gamma'(x)}{\Gamma(x)}$.

Then taking the logarithmic differential of Gauss' Theorem,

$$\Gamma(nx)=n^{nx}\,\Gamma(x)\,\Gamma\left(x+\frac{1}{n}\right)\ldots\Gamma\left(x+\frac{n-1}{n}\right)\Big/(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}},$$

$$n\psi(nx)=n\log n+\psi(x)+\psi\left(x+\frac{1}{n}\right)+\ldots+\psi\left(x+\frac{n-1}{n}\right),$$

and differentiating again,

$$n^2\psi'(nx)=\psi'(x)+\psi'\left(x+\frac{1}{n}\right)+\ldots+\psi'\left(x+\frac{n-1}{n}\right).$$

Hence

$$n\psi'(nx)=\sum\frac{1}{n}\,\psi'\left(x+\frac{r}{n}\right),\text{ from } r=0 \text{ to } r=n-1,$$

$$\textit{i.e. } Lt_{n=\infty}\,n\psi'(nx)=\int_0^1\psi'(x+y)\,dy=\Big[\,\psi(x+y)\Big]_{y=0}^{y=1}$$

$$=\psi(x+1)-\psi(x)=\frac{d}{dx}\log\Gamma(x+1)-\frac{d}{dx}\log\Gamma(x)$$

$$=\frac{d}{dx}\log\frac{\Gamma(x+1)}{\Gamma(x)}=\frac{d}{dx}\log x=\frac{1}{x},$$

i.e. $Lt_{n=\infty}(nx)\psi'(nx)=1$; or writing v for nx, $\psi'(v)=\dfrac{1}{v}$ in the limit when v is infinite, and therefore $\psi'(v)$ ultimately vanishes.

That is $\dfrac{d^2}{dx^2}\log\Gamma(x)$ vanishes when x is indefinitely increased.

Now $$\Gamma(x)=\frac{\Gamma(x+n+1)}{x(x+1)(x+2)\ldots(x+n)}.$$

Hence, taking the logarithmic differential,

$$\psi(x)=-\frac{1}{x}-\frac{1}{x+1}-\frac{1}{x+2}-\ldots-\frac{1}{x+n}+\psi(x+n+1),$$

and differentiating again,

$$\psi'(x)=\frac{1}{x^2}+\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\ldots+\frac{1}{(x+n)^2}+\psi'(x+n+1),$$

and it has just been proved that $\psi'(x+n+1)$ ultimately vanishes when n has been indefinitely increased.

$$\therefore \frac{d^2}{dx^2}\log\Gamma(x)\equiv\psi'(x)=\frac{1}{x^2}+\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\ldots \text{ to } \infty \ldots\text{.(1)}$$

The series (1) is obviously convergent for all values of $x>0$ becoming infinite at $x=0$.

Integrating this equation between limits 1 and x, we have

$$\psi(x)-\psi(1)=\left[-\frac{1}{x}\right]_1^x+\left[-\frac{1}{x+1}\right]_1^x+\left[-\frac{1}{x+2}\right]_1^x+\ldots$$

$$=\left(\frac{1}{1}-\frac{1}{x}\right)+\left(\frac{1}{2}-\frac{1}{x+1}\right)+\left(\frac{1}{3}-\frac{1}{x+2}\right)+\ldots\ldots\ldots\ldots\text{(2)}$$

which is a convergent series; for the test expression, viz.

$$Lt_{n=\infty}\, n\left(1-\frac{u_{n+1}}{u_n}\right)=Lt\,\frac{n(x+2n)}{(n+1)(x+n)}=2,$$

and is greater than unity. (See Smith's *Algebra*, Art. 342.)

Again, we have seen that

$$n\psi(nx)=n\log n+\psi(x)+\psi\left(x+\frac{1}{n}\right)+\ldots+\psi\left(x+\frac{n-1}{n}\right),$$

and putting $x=1$,

$$\psi(n)=\log n+\sum\frac{1}{n}\psi\left(1+\frac{r}{n}\right), \text{ from } r=0 \text{ to } r=n-1.$$

Hence when n increases indefinitely,

$$Lt_{n=\infty}[\psi(n)-\log n]=\int_0^1\psi(1+x)\,dx$$

$$=\Big[\log\Gamma(1+x)\Big]_0^1=\log\frac{\Gamma(2)}{\Gamma(1)}=\log 1=0.$$

That is, $\qquad Lt_{n=\infty}\left(\frac{\Gamma'(n)}{\Gamma(n)}-\log n\right)=0. \quad \ldots\ldots\ldots\ldots\ldots\text{(3)}$

Putting $x=\infty$ in equation (2),

$$\psi(\infty)-\psi(1)=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots \text{ to } \infty,$$

i.e. by equation (3),

$$-\psi(1)=Lt_{n=\infty}\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}-\log n\right)$$
$$=\text{Euler's Constant }\gamma,$$

i.e. $\psi(1)$, or $\left\{\dfrac{d}{dx}\log\Gamma(x+1)\right\}_{x=0}=-\gamma$. ....(4)

Hence, by equation (2),

$$\frac{d}{dx}\log\Gamma(x)\equiv\psi(x)=-\gamma+\left(\frac{1}{1}-\frac{1}{x}\right)+\left(\frac{1}{2}-\frac{1}{x+1}\right)+\ldots\text{ to }\infty$$

$$=-\gamma+\frac{1}{1}\frac{x-1}{x}+\frac{1}{2}\frac{x-1}{x+1}+\ldots+\frac{1}{n}\frac{x-1}{x+n-1}+\ldots\text{ to }\infty,\ \ldots\ldots(5)$$

which may also be written as

$$\frac{d}{dx}\log\Gamma(x+1)=Lt_{n=\infty}\left[\log n-\frac{1}{x+1}-\frac{1}{x+2}-\ldots-\frac{1}{x+n}\right].$$

Again, differentiating equation (1) $n-2$ times, we have

$$\frac{d^n}{dx^n}\log\Gamma(x)=(-1)^n(n-1)!\left[\frac{1}{x^n}+\frac{1}{(x+1)^n}+\frac{1}{(x+2)^n}+\ldots\text{ to }\infty\right],\tag{6}$$

i.e. $\psi^{(n-1)}(1)$, or $\left\{\dfrac{d^n}{dx^n}\log\Gamma(x)\right\}_{x=1}=(-1)^n(n-1)!\,S_n$,

where $S_n=\dfrac{1}{1^n}+\dfrac{1}{2^n}+\dfrac{1}{3^n}+\ldots$,

which is convergent if $n>1$; or, what is the same thing,

$$\left\{\frac{d^n}{dx^n}\log\Gamma(x+1)\right\}_{x=0}=(-1)^n(n-1)!\,S_n.\ \ldots\ldots\ldots\ldots(7)$$

Also $\left\{\log\Gamma(x+1)\right\}_{x=0}=\log\Gamma(1)=0$;

we thus have

$$\left\{\log\Gamma(x+1)\right\}_{x=0}=0;\quad\left\{\frac{d}{dx}\log\Gamma(x+1)\right\}_{x=0}=-\gamma;$$

and $\left\{\dfrac{d^n}{dx^n}\log\Gamma(x+1)\right\}_{x=0}=(-1)^n(n-1)!\,S_n$, where n is $\nless 2$.

Maclaurin's Theorem then gives

$$\log\Gamma(x+1)=-\gamma x+S_2\frac{x^2}{2}-S_3\frac{x^3}{3}+S_4\frac{x^4}{4}-\ldots+(-1)^nS_n\frac{x^n}{n}+\ldots,$$

a result otherwise established in a subsequent article, and which will be thrown into a more convergent form, by the addition of other known series, for working purposes. This series is convergent if x be numerically <1.

912. Collecting for convenience other useful results of the above article, we have

(a) $Lt_{x=\infty}\dfrac{d^2}{dx^2}\log\Gamma(x)=0$ and $Lt_{x=0}\dfrac{d^2}{dx^2}\log\Gamma(x)=\infty$, and in any case $\dfrac{d^2}{dx^2}\log\Gamma(x)=\dfrac{1}{x^2}+\dfrac{1}{(x+1)^2}+\dfrac{1}{(x+2)^2}+\dots$ to ∞, and is positive.

(b) $\dfrac{\Gamma'(n)}{\Gamma(n)}=\log n$ when n is infinitely large.

(c) $\left\{\dfrac{d}{dx}\log\Gamma(x+1)\right\}_{x=0}=-\gamma$, and $\therefore \left\{\dfrac{d}{dx}\log\Gamma(x)\right\}_{x=1}=-\gamma$.

(d) $\dfrac{d}{dx}\log\Gamma(x+1)=-\gamma+\left(\dfrac{1}{1}-\dfrac{1}{x+1}\right)+\left(\dfrac{1}{2}-\dfrac{1}{x+2}\right)+\dots$ to ∞.

(e) Since $\dfrac{d^2}{dx^2}\log\Gamma(x)$ is continuously positive for all positive values of x, $\dfrac{d}{dx}\log\Gamma(x)$ is an increasing function as x increases from 0 to ∞, starting from the value $-\infty$ at $x=0$; or, putting this geometrically, the tangent to the graph of $y=\log\Gamma(x)$ is continuously rotating in a counter-clockwise direction as x passes from zero to infinity; and the curve is always convex to the foot of the ordinate.

913. The student may note the following particular values of $\dfrac{d^2}{dx^2}\log\Gamma(x)$, *i.e.* $\psi'(x)$, viz. taking $\pi^2=9{\cdot}8696044011$,

$$\psi'(0) \qquad =\infty,$$

$$\psi'(\cdot 5)=\frac{1}{\left(\frac{1}{2}\right)^2}+\frac{1}{\left(\frac{3}{2}\right)^2}+\frac{1}{\left(\frac{5}{2}\right)^2}+\dots=4\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right)=4\,.\frac{\pi^2}{8}=\frac{\pi^2}{2}=4{\cdot}9348022,$$

$$\psi'(1)=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots=\frac{\pi^2}{6} \qquad =1{\cdot}6449341,$$

$$\psi'(1{\cdot}5)=4\left(\frac{1}{3^2}+\frac{1}{5^2}+\dots\right)=4\left(\frac{\pi^2}{8}-\frac{1}{1^2}\right)=\frac{\pi^2}{2}-4 \qquad = {\cdot}9348022,$$

$$\psi'(2)=\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots=\frac{\pi^2}{6}-1 \qquad = {\cdot}6449341,$$

$$\psi'(2{\cdot}5)=4\left(\frac{1}{5^2}+\frac{1}{7^2}+\dots\right)=4\left(\frac{\pi^2}{8}-\frac{1}{1^2}-\frac{1}{3^2}\right)=\frac{\pi^2}{2}-4{\cdot}4 \qquad = {\cdot}4903578,$$

$$\psi'(3)=\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\dots=\frac{\pi^2}{6}-\frac{1}{1^2}-\frac{1}{2^2}=\frac{\pi^2}{6}-1{\cdot}25 \qquad = {\cdot}3949341,$$

etc.

$$\psi'(\infty) \qquad =0,$$

which indicate how $\frac{d^2}{dx^2}\log\Gamma(x)$ is decreasing as x increases, but always remaining positive.

914. Since $\log\Gamma(x+1)=-\gamma x+S_2\frac{x^2}{2}-S_3\frac{x^3}{3}+S_4\frac{x^4}{4}-\ldots$, we may write $\Gamma(x+1)$ as

$$\Gamma(x+1)=e^{-\gamma x}e^{S_2\frac{x^2}{2}}e^{-S_3\frac{x^3}{3}}e^{S_4\frac{x^4}{4}}\ldots$$

$$=\left(1-\gamma x+\frac{\gamma^2x^2}{2!}-\frac{\gamma^3x^3}{3!}+\ldots\right)\left(1+S_2\frac{x^2}{2}\ldots\right)\left(1-S_3\frac{x^3}{3}\ldots\right)\ldots$$

$$=1-\gamma x+(\gamma_2{}^2+S_2)\frac{x^2}{2!}-(\gamma^3+3\gamma S_2+2S_3)\frac{x^3}{3!}+\ldots,$$

which expands $\Gamma(x+1)$ as far as cubes of x, and which might be useful for very small values of x, but the presence of powers of γ renders calculation troublesome, and less inconvenient methods of calculation will be given later.

915. It is noticeable, too, that

$$\frac{\log\Gamma(x+1)}{x}=-\gamma+S_2\frac{x}{2}-S_3\frac{x^2}{3}+S_4\frac{x^3}{4}-\ldots,$$

and that the several differential coefficients of this expression are therefore free from Euler's Constant γ, viz.

$$\frac{d^n}{dx^n}\frac{\log\Gamma(x+1)}{x}$$

$$=(-1)^{n-1}\left\{\frac{S_{n+1}}{n+1}n!-\frac{S_{n+2}}{n+2}\frac{(n+1)!}{1!}x+\frac{S_{n+3}}{n+3}\frac{(n+2)!}{2!}x^2-\ldots\right\}$$

$$=(-1)^{n-1}n!\left\{\frac{S_{n+1}}{n+1}-\frac{n+1}{1}\frac{S_{n+2}}{n+2}x+\frac{\overline{n+1}\,\overline{n+2}}{1.2}\frac{S_{n+3}}{n+3}x^2-\ldots\right\}.$$

And, similarly, if m be any positive integer,

$$\frac{d^n}{dx^n}x^m\log\Gamma(x+1)=\left(\frac{d}{dx}\right)^n\left[-\gamma x^{m+1}+\sum_2^\infty(-1)^r\frac{S_r}{r}x^{m+r}\right]$$

$$=-(m+1)_n\gamma x^{m+1-n}+\sum_2^\infty(-1)^r\frac{S_r}{r}(m+r)_nx^{m+r-n},$$

where $(m+1)_r$ denotes $(m+1)(m)(m-1)\ldots$ to r factors, if $n\leqslant m+1$, and is free from γ if $n>m+1$; also that

$$\left[\frac{d^{m+1}}{dx^{m+1}}(x^m\log\Gamma(x+1)\right]_{x=0}=-(m+1)!\,\gamma.$$

916. Expansion of $\log \Gamma(1+x)$ deduced from the Π Function.

The series

$$\log \Gamma(1+x) = -\gamma x + S_2 \frac{x^2}{2} - S_3 \frac{x^3}{3} + S_4 \frac{x^4}{4} - \dots$$

may be arrived at at once by taking the logarithm of the Gauss formula in the form

$$\Gamma(1+x) = Lt_{\mu=\infty} \frac{\mu^x}{\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\dots\left(1+\frac{x}{\mu}\right)},$$

viz.

$$\log \Gamma(1+x) = x \log \mu - \log\left(1+\frac{x}{1}\right) - \log\left(1+\frac{x}{2}\right) - \log\left(1+\frac{x}{3}\right) - \dots;$$

and expanding the logarithms, supposing $-1 < x < 1$,

$$\log \Gamma(1+x) = Lt\left[x(\log \mu - S_1) + S_2 \frac{x^2}{2} - S_3 \frac{x^3}{3} + \dots\right],$$

where $$S_r = \frac{1}{1^r} + \frac{1}{2^r} + \frac{1}{3^r} + \dots,$$

and $Lt(S_1 - \log \mu) =$ Euler's Constant γ, and the series S_r $(r > 1)$ are all convergent.

Hence,

$$\log \Gamma(1+x) = -\gamma x + S_2 \frac{x^2}{2} - S_3 \frac{x^3}{3} + S_4 \frac{x^4}{4} - \dots + (-1)^n S_n \frac{x^n}{n} + \dots; \quad (-1 < x < 1). \quad (1)$$

Now, the even terms may be removed by the addition of $\frac{1}{2} \log \frac{x\pi}{\sin x\pi}$.

For $$\frac{\sin x\pi}{x\pi} = \left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right)\left(1 - \frac{x^2}{3^2}\right)\dots \textit{ad inf.};$$

and taking logarithms and expanding,

$$0 = \tfrac{1}{2} \log \frac{x\pi}{\sin x\pi} - S_2 \frac{x^2}{2} - S_4 \frac{x^4}{4} - \dots \quad \dots\dots\dots\dots(2)$$

Adding to equation (1),

$$\log \Gamma(1+x) = \tfrac{1}{2} \log \frac{x\pi}{\sin x\pi} - \gamma x - S_3 \frac{x^3}{3} - S_5 \frac{x^5}{5} - \dots \quad \dots(3)$$

The coefficients S_3, S_5, ... all begin with a unit. This may be removed and the series reduced to a much more convergent form by the addition of the series for $\tanh^{-1} x$ to each side, viz.

$$\tanh^{-1} x = \tfrac{1}{2} \log \frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots.$$

And we then obtain

$$\log\Gamma(1+x)=\tfrac{1}{2}\log\frac{x\pi}{\sin x\pi}-\tanh^{-1}x+(1-\gamma)x$$
$$-(S_3-1)\frac{x^3}{3}-(S_5-1)\frac{x^5}{5}-\dots\dots\dots(4)$$

The values of γ, S_2, S_3, ... S_{35} are all calculated, and the tabulated results are given in Art. 957. Euler calculated S_2 to S_{15}. Legendre* gave the values S_2 to S_{35} to sixteen decimal places. The list in Art. 957 is taken from Legendre's list as given by De Morgan, *Diff. Calc.*, p. 554. The series (4) converges rapidly and is used for the calculation of the values of $\log\Gamma(x)$. Legendre gives a table of values of $L\Gamma(x)$, *i.e.* $10+\log\Gamma(x)$, from $L\Gamma(1{\cdot}000)$ to $L\Gamma(2{\cdot}000)$ to seven decimal places, in his *Exercices du Calcul Intégral*, pages 301 to 306. A table is also given by Bertrand, *Calc. Int.*, p. 285.

917. Calculation of Euler's Constant γ.

These series may be used for the calculation of Euler's Constant γ by taking a value of x, for which $\Gamma(x)$ is otherwise known, viz. $x=\frac{1}{2}$, for which $\Gamma(x)=\sqrt{\pi}$.

Equation (1) gives

$$\gamma=-\frac{1}{x}\log\Gamma(x+1)+S_2\frac{x}{2}-S_3\frac{x^2}{3}+S_4\frac{x^3}{4}-\dots;$$

and putting $x=\frac{1}{2}$,

$$\gamma=\log_e\frac{4}{\pi}+\frac{1}{2}S_2\cdot\frac{1}{2}-\frac{1}{3}S_3\frac{1}{2^2}+\frac{1}{4}S_4\frac{1}{2^3}-\dots\dots\dots(5)$$

Equation (3) gives, by changing the sign of x,

$$\log\Gamma(1-x)=\tfrac{1}{2}\log\frac{x\pi}{\sin x\pi}+\gamma x+S_3\frac{x^3}{3}+S_5\frac{x^5}{5}+\dots;$$

and putting $x=\frac{1}{2}$ in this,

$$\gamma=\log 2-\frac{1}{3}S_3\frac{1}{2^2}-\frac{1}{5}S_5\frac{1}{2^4}-\frac{1}{7}S_7\frac{1}{2^6}-\dots,\dots\dots\dots(6)$$

which is more rapidly convergent than the former.

Formula (4) gives

$$\log\frac{\sqrt{\pi}}{2}=\frac{1}{2}\log\frac{\pi}{2}-\frac{1}{2}\log 3+\frac{1-\gamma}{2}-\frac{S_3-1}{3}\frac{1}{2^3}-\frac{S_5-1}{5}\frac{1}{2^5}-\dots$$

i.e. $$\gamma=\log_e\frac{2e}{3}-\frac{S_3-1}{3}\frac{1}{2^2}-\frac{S_5-1}{5}\frac{1}{2^4}-\frac{S_7-1}{7}\frac{1}{2^6}-\dots\dots(7)$$

* *Traité des fonctions elliptiques*, Legendre.

This is the best of the three series to employ to find γ.

And with the aid of the tables of values of S_p the calculation to seven places, which is all that is likely to be wanted for ordinary purposes, may be readily performed.

The value of γ is

$$\gamma = \cdot 57721\ 56649 \cdot 01532\ 8606\ldots,$$

and $$1-\gamma = \cdot 42278\ 43350\ 98467\ 1394\ldots.$$

The value of $\log_e 10$ is of course required. It is

$$\log_e 10 = 2 \cdot 30258\ 50929\ 94045\ 68401\ 79914\ldots,$$

and the modulus $\log_{10} e = \cdot 43429\ 44819\ldots$.

918. The numerical calculation of values of $\log \Gamma(1+x)$, and therefore of $\Gamma(x)$ itself, will now present no difficulty. With the values of $\frac{S_3-1}{3}$, $\frac{S_5-1}{5}$, etc., inserted, the working formula stands* as

$$\begin{aligned}\log_e \Gamma(1+x) = \frac{1}{2}\log_e \frac{x\pi}{\sin x\pi} - \frac{1}{2}\log_e \frac{1+x}{1-x} &+ \cdot 4227843x \\ &- \cdot 06735230x^3 \\ &- \cdot 0073855x^5 \\ &- \cdot 0011927x^7 \\ &- \cdot 0002231x^9 \\ &- \text{etc.},\end{aligned}$$

and is rapidly convergent for the small values of x less than $x=\frac{1}{2}$, 2^{10} being 1024. Hence the last term $\cdot 0002231x^9$ in the case $x=\frac{1}{2}$ becomes $\cdot 0000004$, whilst for $x=\frac{1}{3}$, which is the largest value of x for which it will be necessary to use the series (see Art. 921), the error in omitting all the remaining terms of the series will not affect the seventh decimal place. Hence we have here all that is necessary for the construction of seven-figure tables for $\log \Gamma(x)$.

919. It is worth noting that the addition of $\log(1+x)$ and $\log(1-x)$ respectively to $\Gamma(1+x)$ and $\Gamma(1-x)$, viz.

$$\log \Gamma(1+x) = -\gamma x + S_2 \frac{x^2}{2} - S_3 \frac{x^3}{3} + S_4 \frac{x^4}{4} - \ldots$$

and $$\log \Gamma(1-x) = \gamma x + S_2 \frac{x^2}{2} + S_3 \frac{x^3}{3} + S_4 \frac{x^4}{4} + \ldots$$

* Bertrand, *Calc. Intégral*, p. 250.

give $\log\Gamma(1+x) = -\log(1+x)+(1-\gamma)x$

$$+(S_2-1)\frac{x^2}{2}-(S_3-1)\frac{x^3}{3}+(S_4-1)\frac{x^4}{4}-\dots$$

and $\log\Gamma(1-x) = -\log(1-x)-(1-\gamma)x$

$$+(S_2-1)\frac{x^2}{2}+(S_3-1)\frac{x^3}{3}+(S_4-1)\frac{x^4}{4}+\dots;$$

whence

$$\tfrac{1}{2}\log\frac{\Gamma(1+x)}{\Gamma(1-x)} = -\tanh^{-1}x+(1-\gamma)x-(S_3-1)\frac{x^3}{3}$$
$$-(S_5-1)\frac{x^5}{5}-\dots.$$

But $\tfrac{1}{2}\log\Gamma(1+x)\,\Gamma(1-x)=\tfrac{1}{2}\log\dfrac{\pi x}{\sin\pi x}$, *i.e.* adding,

$$\log\Gamma(1+x)=\tfrac{1}{2}\log\frac{\pi x}{\sin\pi x}-\tanh^{-1}x+(1-\gamma)x$$
$$-\sum_1^\infty (S_{2n+1}-1)\frac{x^{2n+1}}{2n+1},$$

the same series as before, which may be written

$$\log\Gamma(1+x)=\tfrac{1}{2}\log\left(\frac{\pi x}{\sin\pi x}\frac{1-x}{1+x}\right)+(1-\gamma)x-\sum_1^\infty(S_{2n+1}-1)\frac{x^{2n+1}}{2n+1};$$

and putting $x=1$, since $Lt_{x=1}\dfrac{1-x}{\sin\pi x}=\dfrac{-1}{\pi\cos\pi}=\dfrac{1}{\pi}$,

$$1-\gamma=\tfrac{1}{2}\log 2+\sum_1^\infty\frac{S_{2n+1}-1}{2n+1};$$

and putting $x=\tfrac{1}{2}$, since $\Gamma(\tfrac{3}{2})=\tfrac{1}{2}\sqrt{\pi}$,

$$1-\gamma=\log 1{\cdot}5+\sum_1^\infty\frac{S_{2n+1}-1}{(2n+1)2^{2n}} \quad \text{(cf. Art. 917).}$$

These series are given both by Serret and Bertrand for the calculation of $\Gamma(1+x)$ and γ.

The formulae

$$\log\Gamma(1+x)=\frac{1}{2}\log\frac{\pi x}{\sin\pi x}-\gamma x-\frac{1}{3}S_3x^3-\frac{1}{5}S_5x^5-\dots,$$

$$\log\Gamma(1-x)=\frac{1}{2}\log\frac{\pi x}{\sin\pi x}+\gamma x+\frac{1}{3}S_3x^3+\frac{1}{5}S_5x^5+\dots,$$

and $$\gamma=\log_e 2-\frac{1}{3}\frac{S_3}{2^2}-\frac{1}{5}\frac{S_5}{2^4}-\frac{1}{7}\frac{S_7}{2^6}-\dots,$$

were given by Legendre (*Exercices*, p. 299). But the addition of the series for $\tanh^{-1}x$ adds to the rapidity of the convergence.

920. Since $\Gamma(m)\Gamma(1-m)=\dfrac{\pi}{\sin m\pi}$, we have, on putting $\dfrac{1+x}{2}$ for m,

$$\Gamma\left(\frac{1+x}{2}\right)\Gamma\left(\frac{1-x}{2}\right)=\frac{\pi}{\sin\frac{1+x}{2}\pi}=\frac{\pi}{\cos\frac{x\pi}{2}}. \quad \ldots\ldots\ldots\text{(i)}$$

But $\quad \Gamma(x)=2^{1-2x}\sqrt{\pi}\,\dfrac{\Gamma(2x)}{\Gamma\left(\frac{1}{2}+x\right)}$ (Art. 905).

Hence, writing $\dfrac{x}{2}$ in place of x,

$$\Gamma\left(\frac{x}{2}\right)=2^{1-x}\sqrt{\pi}\,\frac{\Gamma(x)}{\Gamma\left(\frac{1}{2}+\frac{x}{2}\right)}. \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

From equations (i) and (ii), eliminating $\Gamma\left(\dfrac{1+x}{2}\right)$, we have

$$\Gamma(x)=\frac{\sqrt{\pi}}{2^{1-x}\cos\frac{x\pi}{2}}\,\frac{\Gamma\left(\frac{x}{2}\right)}{\Gamma\left(\frac{1-x}{2}\right)}. \ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

921. By means of the four formulae

$\Gamma(x)=(x-1)\Gamma(x-1)$, ...(1); $\Gamma(x)\Gamma(1-x)=\dfrac{\pi}{\sin x\pi}$, ...(2);

$\Gamma(x)=2^{1-2x}\sqrt{\pi}\,\dfrac{\Gamma(2x)}{\Gamma(\frac{1}{2}+x)}$, (3); $\Gamma(x)=\dfrac{\sqrt{\pi}}{2^{1-x}\cos\frac{x\pi}{2}}\,\dfrac{\Gamma\left(\frac{x}{2}\right)}{\Gamma\left(\frac{1-x}{2}\right)}$, (4);

it may be shown that $\Gamma(x)$ can be calculated for all values of x when those between $\Gamma(\frac{1}{6})$ and $\Gamma(\frac{1}{3})$ have been calculated.

(*a*) For $1<x<\infty$, reduce by continued application of formula (1) to a case $0<y<1$.

(*b*) For $\frac{2}{3}<x<1$, reduce by formula (2) to a case $0<y<\frac{1}{3}$.

(*c*) For $\frac{1}{3}<x<\frac{2}{3}$, reduce by formula (4) to a case $\frac{1}{6}<y<\frac{1}{3}$.

For if $x>\dfrac{1}{3}$, $\quad \dfrac{x}{2}>\dfrac{1}{6}$ and $\quad \dfrac{1-x}{2}<\dfrac{1}{3}$;

and if $\quad x<\dfrac{2}{3}$, $\quad \dfrac{x}{2}<\dfrac{1}{3}$ and $\quad \dfrac{1-x}{2}>\dfrac{1}{6}$.

(*d*) If $\frac{1}{6} < x < \frac{1}{3}$, the case needs no reduction.

(*e*) If $0 < x < \frac{1}{6}$, use formula (3). This involves $\Gamma(\frac{1}{2}+x)$, and $\frac{1}{2}+x$ lies between $\frac{1}{2}$ and $\frac{2}{3}$, and therefore falls under case (*c*), and an application of formula (4) reduces $\Gamma(x+\frac{1}{2})$ to cases in which the arguments lie as before, viz. $\frac{1}{6} < y < \frac{1}{3}$.

In $\Gamma(2x)$, which occurs in the numerator of formula (3), if $0 < x < \frac{1}{6}$, we have $0 < 2x < \frac{1}{3}$, and if $2x > \frac{1}{6}$, no further reduction is necessary.

But if $0 < x < \frac{1}{12}$, we have

$$0 < 2x < \tfrac{1}{6} \quad \text{and} \quad 0 < 4x < \tfrac{1}{3}.$$

We then use formula (3) with $2x$ written for x,

i.e.
$$\Gamma(2x) = \sqrt{\pi}\, 2^{1-4x} \frac{\Gamma(4x)}{\Gamma(\frac{1}{2}+2x)}.$$

Similarly if $0 < x < \frac{1}{24}$, use

$$\Gamma(4x) = \sqrt{\pi}\, 2^{1-8x} \frac{\Gamma(8x)}{\Gamma(\frac{1}{2}+4x)},$$

and so on.

Hence it follows that the use of series will be only necessary in the case of $\Gamma(x)$, where x lies from $\frac{1}{6}$ to $\frac{1}{3}$, and that when this group is calculated by the series, all others follow by the above rules.

922. **Graph of** $y = \Gamma(x) = \int_0^\infty e^{-z} z^{x-1}\, dz$.

Regarded as defined by the integral, it is plain that so long as x is real and positive $\Gamma(x)$ is a positive function, and that it becomes infinite if $x = 0$, as may also be seen from the fact that $\Gamma(x) = \dfrac{1}{x}\Gamma(x+1)$, and therefore $\Gamma(0) = \dfrac{\Gamma(1)}{0} = \infty$.

We have seen that

$$\frac{d^2}{dx^2}\log\Gamma(x) = \frac{1}{x^2} + \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \dots,$$

and therefore is infinite when $x = 0$, but for all values of x from 0 to ∞ it remains positive and finite. Hence

$$\frac{d}{dx}\log\Gamma(x), \quad i.e. \ \frac{\Gamma'(x)}{\Gamma(x)},$$

is an increasing function of x, and its value at $x = 0$ is obviously $-\infty$, for

$$\frac{d}{dx}\log\Gamma(x) = -\gamma + \left(\frac{1}{1} - \frac{1}{x}\right) + \left(\frac{1}{2} - \frac{1}{x+1}\right) + \dots \quad \text{(Art. 911).}$$

Also, when x is $+\infty$,

$$\frac{d \log \Gamma(x)}{dx} = -\gamma + \tfrac{1}{1} + \tfrac{1}{2} + \dots \text{ to } \infty = +\infty.$$

Hence $\dfrac{\Gamma'(x)}{\Gamma(x)}$ increases from $-\infty$ through zero to $+\infty$ as x increases from 0 to ∞ and as $\Gamma(x)$ remains positive throughout, $\Gamma'(x)$ changes from negative to positive once, and once only, as x increases from 0 to ∞.

Therefore $\Gamma(x)$ has one, and only one, stationary value, and that is a minimum, and $\Gamma(x)$ decreases from ∞ when $x=0$ to $\Gamma(1)=1$ when $x=1$, and since $\Gamma(2)=1$ and $\Gamma(1)=1$, the ordinates at $x=1$ and $x=2$ are equal, and the minimum lies somewhere between $x=1$ and $x=2$, and is numerically less than unity. From $x=2$ to $x=\infty$ the value of $\Gamma(x)$ is continually increasing.

The curve then

(*a*) lies entirely on the upper side of the x-axis;
(*b*) it is asymptotic to the y-axis;
(*c*) it has a minimum between $x=1$ and $x=2$;
(*d*) it recedes from the x-axis from $x=2$ to $x=\infty$.

The equation to find the exact position of the minimum ordinate is $\dfrac{d\,\Gamma(x)}{dx}=0$, or writing $x=1+t$, $\dfrac{d}{dt}\Gamma(1+t)=0$.

Also
$$\frac{d \log \Gamma(1+t)}{dt} = \frac{\Gamma'(1+t)}{\Gamma(1+t)}.$$

Hence
$$\frac{d}{dt}\Gamma(1+t) = \Gamma(1+t)\left[-\frac{1}{1+t} + (1-\gamma) + (S_2-1)t - (S_3-1)t^2 + \dots\right],$$

and t is to be found by trial from

$$\frac{1}{1+t} = 0{\cdot}422784\dots + (S_2-1)t - (S_3-1)t^2 + \dots;$$

and substituting for S_2 and S_3 their values in decimals to a few places, an approximate value for t may be obtained, and by the usual approximation methods the result may be found as nearly as desired. Serret gives the result to seven places, viz.

$$t = 0{\cdot}4616321\dots$$

i.e. the abscissa of the minimum ordinate is

$$x = 1 + t = 1{\cdot}4616321\ldots,$$

and the value of the corresponding ordinate is found to be

$$y = 0{\cdot}8856032\ldots.^*$$

In the tables for $L\,\Gamma(x)$, *i.e.* $10 + \log\Gamma(x)$, we find in the vicinity of the minimum

x	$L\,\Gamma(x)$	x	$L\,\Gamma(x)$
1·45	9·9472677	1·463	9·9472396
1·46	9·9472397	1·47	9·9472539
1·461	9·9472393	1·48	9·9473079
1·462	9·9472392		

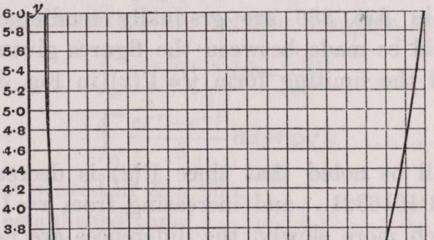

Fig. 320.

So we see from the tables that the minimum ordinate is in the vicinity of 1·462, and the value of the corresponding

* Bertrand gives 0·8556032, page 283, and again page 284, line 3, and the result is given elsewhere. This is evidently an error. The result is given correctly in Serret, *Calc. Intég.*, p. 186.

logarithm, $\bar{1}\cdot 9472392$, indicates an ordinate $0\cdot 885603$ approximately. The minimum ordinate is reached, therefore, a little earlier in the march of x from 1 to 2 than the half-way $1\cdot 5$, which might have been expected from the very rapid fall of value in $\Gamma(x)$ between $\Gamma(0)=\infty$ and $\Gamma(1)=1$ and the much slower rise on passing $x=2$, $\Gamma(2)=1$, $\Gamma(3)=2$, $\Gamma(4)=6$, $\Gamma(5)=24$, etc.

For large values of x, $\dfrac{\Gamma(x+1)}{x}$ approximates to $\dfrac{\sqrt{2x\pi}\,x^x e^{-x}}{x}$, and the graph of $y=\Gamma(x)$ to the curve $y=\sqrt{\dfrac{2\pi}{x}}\left(\dfrac{x}{e}\right)^x$.

We have now seen to what shape the several curves in the graphs in Art. 886 are gradually tending, and comparison should be made between the figures given there and the graph of the limiting form $y=\Gamma(x)$ in Fig. 320 of this article.

923. It will be noted that since $\Gamma(x)$ is decreasing from $x=0$ to $x=1\cdot 4616321\ldots$ and increasing from $x=1\cdot 4616321\ldots$ to $x=\infty$ much more slowly, the differences are negative for the first part of the march of $\Gamma(x)$ and positive for the second. Similarly for the differences in the tables which give $\log\Gamma(x)$ or $L\,\Gamma(x)$. The tabulation is only effected from $x=1$ to $x=2$, for by virtue of the reduction formula $\Gamma(x+1)=x\Gamma(x)$ this is all that is necessary. In using the tables care should be observed with regard to the change of sign of the differences, and those who wish to make close calculations should observe the remarks made by Bertrand, *Calc. Intég.*, p. 284, with regard to the behaviour of the differences both of the first and second orders.

924. The rule of interpolation commonly used is

$$u_x=u_0+x\,\Delta u_0+\frac{x(x-1)}{1\,.\,2}\,\Delta^2 u_0+\ldots$$

(Boole, *Finite Differences*, Art. 2),

rather than the ordinary rule of proportional parts, which stops at the second term.

925. **Expressions for**

$$\frac{d}{dx}\log\Gamma(x),\quad \frac{d^2}{dx^2}\log\Gamma(x),\quad \frac{d^n}{dx^n}\log\Gamma(x),\quad \text{etc.,}$$

as definite integrals.

The expressions for $\frac{d}{dx}\log\Gamma(x)$, $\frac{d^2}{dx^2}\log\Gamma(x)$, etc., viz.

$$\frac{d}{dx}\log\Gamma(x)=Lt_{n=\infty}\left\{\log n-\left(\frac{1}{x}+\frac{1}{x+1}+\frac{1}{x+2}+\ldots+\frac{1}{x+n-1}\right)\right\}, \quad (1)$$

$$\frac{d^2}{dx^2}\log\Gamma(x)=\frac{1}{x^2}+\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\ldots \text{ to } \infty, \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\frac{d^n}{dx^n}\log\Gamma(x)=(-1)^n\Gamma(n)\left[\frac{1}{x^n}+\frac{1}{(x+1)^n}+\frac{1}{(x+2)^n}+\ldots \text{ to } \infty\right], \quad (3)$$

can readily be converted into definite integrals by aid of the results

$$\int_0^\infty e^{-\beta x}\beta^{n-1}\,d\beta=\frac{\Gamma(n)}{x^n} \quad \ldots\ldots\ldots\ldots\ldots\ldots (a)$$

and

$$\int_0^\infty \frac{e^{-z}-e^{-kz}}{z}\,dz=\log k. \quad \ldots\ldots\ldots\ldots\ldots (b)$$

(a) has been proved in Art. 864.

(b) can be established thus:

$$\int_0^\infty e^{-kz}\,dz=\left[-\frac{e^{-kz}}{k}\right]_0^\infty=\frac{1}{k}.$$

Integrating with regard to k between limits 1 and k,

$$\log k=\int_0^\infty\left[-\frac{e^{-kz}}{z}\right]_1^k dz=\int_0^\infty\frac{e^{-z}-e^{-kz}}{z}\,dz.$$

To convert

$$\frac{d}{dx}\log\Gamma(x)=Lt_{n=\infty}\left\{\log n-\frac{1}{x}-\frac{1}{x+1}-\frac{1}{x+2}-\ldots-\frac{1}{x+n-1}\right\},$$

the right side may be written, by aid of (a) and (b),

$$=Lt_{n=\infty}\left[\int_0^\infty\left(\frac{e^{-\beta}-e^{-n\beta}}{\beta}-e^{-\beta x}-e^{-\beta(x+1)}-\ldots-e^{-\beta(x+n-1)}\right)d\beta\right]$$

$$=Lt_{n=\infty}\left[\int_0^\infty\left(\frac{e^{-\beta}-e^{-n\beta}}{\beta}-e^{-\beta x}\frac{1-e^{-n\beta}}{1-e^{-\beta}}\right)d\beta\right]$$

$$=Lt_{n=\infty}\left[\int_0^\infty\left(\frac{e^{-\beta}}{\beta}-\frac{e^{-x\beta}}{1-e^{-\beta}}\right)d\beta-\int_0^\infty e^{-n\beta}\left(\frac{1}{\beta}-\frac{e^{-x\beta}}{1-e^{-\beta}}\right)d\beta\right]$$

$$=\int_0^\infty\left(\frac{e^{-\beta}}{\beta}-\frac{e^{-x\beta}}{1-e^{-\beta}}\right)d\beta, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A)$$

for the second integral disappears when n is made infinite.

926. With regard to $I_0^\infty \equiv \int_0^\infty e^{-n\beta}\left(\frac{1}{\beta}-\frac{e^{-x\beta}}{1-e^{-\beta}}\right)d\beta$, it may be desirable to make a closer investigation, for though for all values of β between ϵ and infinity where ϵ is a given small finite quantity the factor $e^{-n\beta}$ destroys the integrand when n is made infinite, there may be some doubt as to the behaviour of the expression in the immediate proximity of the lower limit.

We note that

$$\frac{1}{\beta}-\frac{e^{-x\beta}}{1-e^{-\beta}}=x-\frac{1}{2}-\left\{\frac{x(x-1)}{2}+\frac{1}{12}\right\}\beta+\dots,$$

and is finite for all given positive values of x, however small β may be, tending to the finite limit $x-\frac{1}{2}$ when β is indefinitely diminished.

Let K be its greatest numerical value between

$$\beta=0 \quad \text{and} \quad \beta=\epsilon.$$

Then the portion of the integral I between 0 and ϵ does not exceed $K\int_0^\epsilon e^{-n\beta}\,d\beta$, *i.e.* $K\frac{1-e^{-n\epsilon}}{n}$, and therefore vanishes in the limit when n is indefinitely increased.

Hence $\int_0^\infty e^{-n\beta}\left\{\frac{1}{\beta}-\frac{e^{-x\beta}}{1-e^{-\beta}}\right\}d\beta$ vanishes when n is made infinite, for all positive finite values of x.

927. To convert

$$\frac{d^n}{dx^n}\log\Gamma(x)\equiv(-1)^n\Gamma(n)\left[\frac{1}{x^n}+\frac{1}{(x+1)^n}+\frac{1}{(x+2)^n}+\dots\ ad\ inf.\right],$$

the right-hand side may be written by theorem (a),

$$=(-1)^n\int_0^\infty\left[e^{-x\beta}\beta^{n-1}+e^{-(x+1)\beta}\beta^{n-1}+e^{-(x+2)\beta}\beta^{n-1}+\dots\right]d\beta;$$

$$\therefore\ \frac{d^n}{dx^n}\log\Gamma(x)=(-1)^n\int_0^\infty\frac{\beta^{n-1}e^{-x\beta}}{1-e^{-\beta}}\,d\beta \quad (n \nless 2), \quad \dots\text{(B)}$$

and this includes the case

$$\frac{d^2}{dx^2}\log\Gamma(x)=\int_0^\infty\frac{\beta e^{-x\beta}}{1-e^{-\beta}}\,d\beta. \quad \dots\dots\dots\dots\text{(C)}$$

928. The same method of treatment will apply in many other cases.

Thus the sum

$$S_p=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\dots \quad (p>1)$$

$$=\frac{1}{\Gamma(p)}\int_0^\infty \beta^{p-1}(e^{-\beta}+e^{-2\beta}+e^{-3\beta}+\dots)\,d\beta$$

$$=\frac{1}{\Gamma(p)}\int_0^\infty \frac{\beta^{p-1}e^{-\beta}}{1-e^{-\beta}}\,d\beta$$

$$=\frac{1}{\Gamma(p)}\int_0^\infty \frac{\beta^{p-1}e^{-\frac{\beta}{2}}}{e^{\frac{\beta}{2}}-e^{-\frac{\beta}{2}}}=\frac{1}{2\Gamma(p)}\int_0^\infty \frac{\beta^{p-1}e^{-\frac{\beta}{2}}}{\sinh\frac{\beta}{2}}\,d\beta. \text{ ....(D)}$$

929. Again,

$$s_p=\frac{1}{1^p}+\frac{1}{3^p}+\frac{1}{5^p}+\frac{1}{7^p}+\dots \quad (p>1)$$

$$=\frac{1}{\Gamma(p)}\int_0^\infty \beta^{p-1}(e^{-\beta}+e^{-3\beta}+e^{-5\beta}+\dots)\,d\beta$$

$$=\frac{1}{\Gamma(p)}\int_0^\infty \frac{\beta^{p-1}e^{-\beta}}{1-e^{-2\beta}}\,d\beta=\frac{1}{2\Gamma(p)}\int_0^\infty \frac{\beta^{p-1}}{\sinh\beta}\,d\beta, \text{ ....(E)}$$

Similarly

$$s_p'=\frac{1}{1^p}-\frac{1}{3^p}+\frac{1}{5^p}-\frac{1}{7^p}+\dots$$

$$=\frac{1}{\Gamma(p)}\int_0^\infty \frac{\beta^{p-1}e^{-\beta}}{1+e^{-2\beta}}\,d\beta=\frac{1}{2\Gamma(p)}\int_0^\infty \frac{\beta^{p-1}}{\cosh\beta}\,d\beta. \text{ ....(F)}$$

And whenever such series occur the conversion to a definite integral form follows at once. For instance, in the expansion (*Diff. Calc.*, Art. 574)

$$\sec x+\tan x=1+A_1\frac{x}{1!}+A_2\frac{x^2}{2!}+A_3\frac{x^3}{3!}+\dots,$$

$$A_n=\frac{2^{n+2}n!}{\pi^{n+1}}\left\{1+\left(-\frac{1}{3}\right)^{n+1}+\left(\frac{1}{5}\right)^{n+1}+\left(-\frac{1}{7}\right)^{n+1}+\dots\right\};$$

$$\therefore A_n=2\left(\frac{2}{\pi}\right)^{n+1}\int_0^\infty \beta^n[e^{-\beta}+e^{-3\beta}+e^{-5\beta}+e^{-7\beta}+\dots]\,d\beta, \quad n \text{ odd},$$

and $$=2\left(\frac{2}{\pi}\right)^{n+1}\int_0^\infty \beta^n[e^{-\beta}-e^{-3\beta}+e^{-5\beta}-e^{-7\beta}+\dots]\,d\beta, \quad n \text{ even};$$

$$A_n=2\left(\frac{2}{\pi}\right)^{n+1}\int_0^\infty \frac{\beta^n e^{-\beta}}{1+(-1)^n e^{-2\beta}}\,d\beta. \text{ ....(G)}$$

Thus the n^{th} Bernoullian number

$$B_{2n-1}=\frac{2n}{2^{2n}(2^{2n}-1)}A_{2n-1}=\frac{2n}{(2^{2n}-1)\pi^{2n}}\int_0^\infty \frac{\beta^{2n-1}}{\sinh\beta}d\beta;\ \ldots\text{(H)}$$

and the n^{th} Eulerian number

$$E_{2n}=A_{2n}=\left(\frac{2}{\pi}\right)^{2n+1}\int_0^\infty \frac{\beta^{2n}}{\cosh\beta}d\beta.\ \ldots\ldots\ldots\ldots\text{(I)}$$

If we write B_{2n-1} as

$$B_{2n-1}=\frac{2(2n)!}{(2\pi)^{2n}}\left[1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\ldots\right]=2(2n)!\sum_1^\infty\frac{1}{(2r\pi)^{2n}}$$

$$=4n\int_0^\infty \beta^{2n-1}(e^{-2\pi\beta}+e^{-4\pi\beta}+e^{-6\pi\beta}+\ldots)\,d\beta,$$

we have

$$B_{2n-1}=4n\int_0^\infty\frac{\beta^{2n-1}e^{-2\pi\beta}}{1-e^{-2\pi\beta}}d\beta=2n\int_0^\infty\frac{\beta^{2n-1}e^{-\pi\beta}}{\sinh\pi\beta}d\beta,\ \ldots\ldots\ldots\text{(J)}$$

a result due to Plana. (*Mem. de l'Acad. de Turin*, 1820.)*

930. **Another Method of obtaining Expressions for $\log\Gamma(x)$, $\frac{d}{dx}\log\Gamma(x)$, $\frac{d^2}{dx^2}\log\Gamma(x)$, … $\frac{d^n}{dx^n}\log\Gamma(x)$ as Definite Integrals** is as follows:

Differentiating the equation $\Gamma(x)=\int_0^\infty e^{-\alpha}\alpha^{x-1}\,d\alpha$, we have

$$\frac{d\Gamma(x)}{dx}=\int_0^\infty e^{-\alpha}\alpha^{x-1}\log\alpha\,d\alpha.\ \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

But $\quad \int_0^\infty e^{-\alpha z}dz=\left[-\frac{e^{-\alpha z}}{\alpha}\right]_0^\infty=\frac{1}{\alpha},$

and integrating this between limits 1 and α with regard to α,

$$\log\alpha=\int_0^\infty\frac{e^{-z}-e^{-\alpha z}}{z}dz.\ \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\therefore\ \frac{d\Gamma(x)}{dx}=\int_0^\infty e^{-\alpha}\alpha^{x-1}\left\{\int_0^\infty\frac{e^{-z}-e^{-\alpha z}}{z}dz\right\}d\alpha$$

$$=\int_0^\infty\int_0^\infty \alpha^{x-1}\frac{e^{-\alpha-z}-e^{-\alpha(1+z)}}{z}\,d\alpha\,dz;$$

* See Boole, *Fin. Diff.*, p. 110.

and changing the order of integration,

$$=\int_0^\infty\int_0^\infty \alpha^{x-1}\frac{e^{-\alpha-z}-e^{-\alpha(1+z)}}{z}\,dz\,d\alpha=\Gamma(x)\int_0^\infty \frac{1}{z}\left\{e^{-z}-\frac{1}{(1+z)^x}\right\}dz;$$

$$\therefore\ \frac{d\log\Gamma(x)}{dx}\equiv\frac{1}{\Gamma(x)}\frac{d\Gamma(x)}{dx}=\int_0^\infty \frac{1}{z}\left\{e^{-z}-\frac{1}{(1+z)^x}\right\}dz. \quad\text{.....(3)}$$

Integrating this with regard to x between limits $x=1$ and $x=x$,

$$\log\Gamma(x)=\int_0^\infty \frac{1}{z}\left\{(x-1)e^{-z}-\frac{(1+z)^{-1}-(1+z)^{-x}}{\log(1+z)}\right\}dz. \quad\text{....(4)}$$

Putting $x=2$,

$$0=\int_0^\infty \frac{1}{z}\left\{e^{-z}-\frac{z(1+z)^{-2}}{\log(1+z)}\right\}dz.$$

Multiply this by $x-1$ and subtract from equation (4);

$$\log\Gamma(x)=\int_0^\infty \left\{(x-1)(1+z)^{-2}-\frac{(1+z)^{-1}-(1+z)^{-x}}{z}\right\}\frac{dz}{\log(1+z)}. \quad (5)$$

Now put $1+z=e^\beta$,

$$\log\Gamma(x)=\int_0^\infty \left\{(x-1)e^{-\beta}-\frac{e^{-\beta}-e^{-x\beta}}{1-e^{-\beta}}\right\}\frac{d\beta}{\beta}. \quad\text{.........(6)}$$

Differentiating this with regard to x,

$$\frac{d}{dx}\log\Gamma(x)=\int_0^\infty \left(\frac{e^{-\beta}}{\beta}-\frac{e^{-x\beta}}{1-e^{-\beta}}\right)d\beta; \quad\text{..............(7)}$$

and a further differentiation with regard to x gives

$$\frac{d^2}{dx^2}\log\Gamma(x)=\int_0^\infty \frac{\beta e^{-x\beta}}{1-e^{-\beta}}\,d\beta. \quad\text{..................(8)}$$

Differentiating (8) $n-2$ times with regard to x, we get

$$\frac{d^n}{dx^n}\log\Gamma(x)=(-1)^n\int_0^\infty \frac{\beta^{n-1}e^{-x\beta}}{1-e^{-\beta}}\,d\beta \quad (n \nless 2) \quad\text{..........(9)}$$

Results (6), (7), (8), (9) give $\log\Gamma(x)$, and its differential coefficients expressed as definite integrals.

From (9), expanding $(1-e^{-\beta})^{-1}$, we have

$$\frac{d^n}{dx^n}\log\Gamma(x)=(-1)^n\int_0^\infty \beta^{n-1}(e^{-x\beta}+e^{-(x+1)\beta}+e^{-(x+2)\beta}+\dots)\,d\beta$$

$$=(-1)^n\,\Gamma(n)\left[\frac{1}{x^n}+\frac{1}{(x+1)^n}+\frac{1}{(x+2)^n}+\dots\text{ to }\infty\right],$$

the formula of Art. 911 (6).

And so far as formulae (7), (8) and (9) are concerned, these definite integral forms are the same as those obtained in Arts. 925 to 927 from the result of Art. 911 (6).

931. **Approximate Summation. Maclaurin's Formula.**

As we are dealing with many series of the form

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\dots \quad (p>1),$$

and other forms in which in some cases an exact summation has not been effected, it is desirable to explain the method usually adopted for approximate evaluation of such summations.

Defining the symbols E, Δ as in *Differential Calculus*, Art. 550, viz. such that

$$Eu_x=u_{x+1} \quad \text{and} \quad \Delta u_x=u_{x+1}-u_x=Eu_x-u_x \quad \text{or} \quad (E-1)u_x;$$

and also remembering the symbolical form of Taylor's theorem,

$$e^{hD}u_x=u_{x+h}, \text{ where } D\equiv\frac{d}{dx},$$

we have the following identity of operators:

$$E\equiv e^D\equiv\Delta+1,$$

and it was pointed out in the *Differential Calculus* that these operative symbols obey the same elementary rules of algebra as quantities, viz. the three fundamental rules:

(*a*) the associative law,

(*b*) the commutative law,

(*c*) the index law for positive integral exponents,

with the exception that they are not commutative with regard to variables. Hence, bearing this exception in mind, there is an algebra of operators bearing formal analogy with the ordinary algebra of quantities, and such theorems as the binomial, multinomial or exponential expansions hold.

Let us define another symbol, Σ, to be such that

$$\Sigma u_x=u_{x-1}+u_{x-2}+u_{x-3}+\dots+u_a,$$

where u_a is some fixed term of the series.

Then $$\Sigma u_{x+1}-\Sigma u_x=u_x,$$

i.e. $$\Sigma\,\Delta u_x=u_x,$$

and therefore Σ represents the inverse of the operation Δ, which may be written as $\frac{1}{\Delta}$ or Δ^{-1}; and since $\Delta\{f(x)+C\}$, where C is a constant and $f(x)$ is any function of x, is equal to

$$[f(x+1)+C]-[f(x)+C]=f(x+1)-f(x),$$

so that the constant disappears, so in reversing the process, if such reversal be possible, we must restore the constant, so that we shall regard Σu_x as $\Delta^{-1}u_x+C$ where C is an arbitrary constant to be determined in each special case.

In this respect the symbol of finite summation, or integration, Σ behaves exactly as the sign $\int dx$ of the integral calculus.

Thus $$\Sigma u_x \equiv C+\frac{1}{E-1}u_x \equiv C+\frac{1}{e^D-1}u_x.$$

Now it has been shown that

$$\frac{t}{e^t-1}=1-\frac{t}{2}+\frac{B_1}{2!}t^2-\frac{B_3}{4!}t^4+\frac{B_5}{6!}t^6-\dots \quad (\textit{Diff. Calc.}, \text{Art. } 148);$$

whence dividing out by t and writing D in place of t, we have the following equivalence of operators, viz.

$$\frac{1}{e^D-1}\equiv\frac{1}{D}-\frac{1}{2}+\frac{B_1}{2!}D-\frac{B_3}{4!}D^3+\frac{B_5}{6!}D^5-\dots,$$

in which all the operations on the right side represent direct differentiations except the first, which represents an integration.

Applying this to any function of x, viz. u_x,

$$\Sigma u_x=C+\int u_x\,dx-\frac{1}{2}u_x+\frac{B_1}{2!}\frac{du_x}{dx}-\frac{B_3}{4!}\frac{d^3u_x}{dx^3}+\frac{B_5}{6!}\frac{d^5u_x}{dx^5}-\dots.$$

For this and many other formulae derived from the same principles, the student may consult Boole, *Finite Differences*, p. 89, etc.

932. Apply this theorem to the case of the series

$$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{x}.$$

Here $$u_x=\frac{1}{x},\quad \Sigma u_x=1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{x-1}.$$

Hence

$$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{x}=\frac{1}{x}+C+\int\frac{dx}{x}-\frac{1}{2x}+\frac{B_1}{2!}\frac{d}{dx}\left(\frac{1}{x}\right)-\frac{B_3}{4!}\frac{d^3}{dx^3}\left(\frac{1}{x}\right)+\dots$$

$$=C+\log_e x+\frac{1}{2x}-\frac{B_1}{2}\cdot\frac{1}{x^2}+\frac{B_3}{4}\frac{1}{x^4}-\frac{B_5}{6}\cdot\frac{1}{x^6}+\dots$$

The constant C must be determined in such examples, either by reference to some known case of the summation, or by absolute calculation of the result for a particular value of x, and when once found, the formula can be used with the determined constant for summation for other values of x.

In the present case, putting $x=\infty$,

$$C=Lt_{x=\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{x}-\log x\right)=\text{Euler's constant}=\gamma.$$

If this be available (see Art. 897) the series can be used for the calculation of the harmonic series to any degree of approximation required. If C be not available take the case $x=10$, and insert the values of Bernoulli's coefficients, viz.

$B_1=\frac{1}{6}$, $B_3=\frac{1}{30}$, $B_5=\frac{1}{42}$, $B_7=\frac{1}{30}$, $B_9=\frac{5}{66}$, etc. (see Art. 879).

Now

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}+\tfrac{1}{9}+\tfrac{1}{10}=2{\cdot}928\ 968\ 254\dots$$

Also $\qquad \log_e 10=2{\cdot}302\ 585\ 09;$

$$\therefore\ 2{\cdot}928\ 968\ 25\dots-2{\cdot}302\ 585\ 09\dots$$

$$=C+\frac{1}{20}-\frac{1}{12}\cdot\frac{1}{10^2}+\frac{1}{120}\cdot\frac{1}{10^4}-\frac{1}{252}\cdot\frac{1}{10^6}+\frac{1}{240}\cdot\frac{1}{10^8}-\dots$$

$$\cdot 626\ 383\ 16\dots=C+\cdot 049\ 167\ 496;$$

$$\therefore\ C=\cdot 577\ 215\ 66\dots \quad \text{(Euler's constant)},$$

which is correct to eight places of decimals.

Hence to the same degree of approximation we may now proceed to sum the series to any other number of terms by the result

$$1+\frac{1}{2}+\dots+\frac{1}{x}=\cdot 57721566\dots+\log_e x+\frac{1}{2x}-\frac{B_1}{2}\frac{1}{x^2}+\frac{B_3}{4}\frac{1}{x^4}-\text{etc.}\dots.$$

It will be noted that to obtain eight decimal places of Euler's constant only three of the terms on the right-hand side affected the result.

933. Take the case

$$\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\dots+\frac{1}{x^n}\quad (n>1).$$

Here $\qquad u_n=\dfrac{1}{x^n},$

$$\Sigma u_n+\frac{1}{x^n}=\frac{1}{x^n}+C+\int\frac{dx}{x^n}-\frac{1}{2}\frac{1}{x^n}+\frac{B_1}{2\,!}\frac{d}{dx}\frac{1}{x^n}-\frac{B_3}{4\,!}\frac{d^3}{dx^3}\left(\frac{1}{x^n}\right)+\frac{B_5}{6\,!}\frac{d^5}{dx^5}\left(\frac{1}{x^n}\right)-\dots$$

$$=\frac{1}{x^n}+C-\frac{1}{n-1}\frac{1}{x^{n-1}}-\frac{1}{2x^n}-\frac{n}{12}\frac{1}{x^{n+1}}+\frac{n(n+1)(n+2)}{720}\frac{1}{x^{n+3}}-\dots$$

except in the case $n=1$, when $\log x$ replaces $-\frac{1}{n-1}\frac{1}{x^{n-1}}$.

Hence $\quad \frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\dots+\frac{1}{x^n} \quad (n>1)$

$$=C-\frac{1}{n-1}\frac{1}{x^{n-1}}+\frac{1}{2}\frac{1}{x^n}-\frac{n}{12}\frac{1}{x^{n+1}}+\frac{n(n+1)(n+2)}{720}\frac{1}{x^{n+3}}-\text{etc.},$$

and this series can be calculated to any degree of approximation when C has been found.

In the case when n is even, the exact sums for an infinite number of terms are known for the earlier values of n. The values for $n=2, 4, 6, 8, 10$ are given in Art. 879.

When this is the case the exact value of C is known, *e.g.* if $n=2$, $C=\frac{\pi^2}{6}$ (Euler), and

$$\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{x^2}=\frac{\pi^2}{6}-\frac{1}{x}+\frac{1}{2x^2}-\frac{1}{6}\frac{1}{x^3}+\frac{1}{30}\frac{1}{x^5}-\frac{1}{42}\frac{1}{x^7}+\text{etc.}$$

If $n=4$, $C=\frac{\pi^4}{90}$ (Euler), and for even values of n higher than 10, C can be found from $C=\frac{(2\pi)^{2n}}{2(2n)\,!}B_{2n-1}$. (See Art. 879.)

934. For odd indices we proceed as in Art. 932, and the value of the constant is to be calculated, as it is not available otherwise.

Thus, if $n=3$,

$$\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\dots+\frac{1}{x^3}=C-\frac{1}{2x^2}+\frac{1}{2x^3}-\frac{1}{4}\frac{1}{x^4}+\frac{1}{12}\frac{1}{x^6}-\frac{1}{12}\frac{1}{x^8}+\dots.$$

Take the case $x=10$. It will be found to give $C=1{\cdot}202056903\dots$ to the first nine places of decimals, and to that approximation with this value of C the formula can be used for finding the sum of any other number of terms.

The value of C is the sum to infinity, in all these examples, viz. $\sum_{r=1}^{r=\infty}\frac{1}{r^n}$, except when $n=1$, a case which has been considered.

935. Consider finally the case

$$\log 1+\log 2+\log 3+\dots+\log x.$$

Here $\qquad u_x=\log x;$

$$\therefore\ \log(x\,!)=C+\log x+\int\log x\,dx-\frac{1}{2}\log x+\frac{1}{6}\frac{1}{2\,!}\frac{d}{dx}\log x$$

$$-\frac{1}{30}\frac{1}{4\,!}\frac{d^3}{dx^3}\log x+\frac{1}{42}\frac{1}{6\,!}\frac{d^5}{dx^5}(\log x)-\dots$$

$$=C+\log x+x(\log x-1)-\frac{1}{2}\log x+\frac{1}{12}\frac{1}{x}-\frac{1}{360}\frac{1}{x^3}+\frac{1}{1260}\frac{1}{x^5}-\dots$$

$$=C-x+x\log x+\frac{1}{2}\log x+\frac{1}{12}\frac{1}{x}-\frac{1}{360}\frac{1}{x^3}+\frac{1}{1260}\frac{1}{x^5}-\dots,$$

and when x is made very large

$$\log(\sqrt{2x\pi}\,x^x e^{-x})=C+x\log x+\frac{1}{2}\log x-x\,;$$

$$\therefore\ C=\log\sqrt{2\pi}\,;$$

$$\therefore\ \log(1\,.\,2\,.\,3\dots x)=\frac{1}{2}\log 2\pi-x+\left(x+\frac{1}{2}\right)\log x+\frac{1}{12}\frac{1}{x}-\frac{1}{360}\frac{1}{x^3}+\frac{1}{1260}\frac{1}{x^5}-\dots,$$

i.e. $\quad 1\,.\,2\,.\,3\dots x=\sqrt{2\pi x}\,x^x e^{-x}e^{\frac{1}{12x}}e^{-\frac{1}{360x^3}}e^{\frac{1}{1260x^5}}\dots,$*

i.e. $\quad 1\,.\,2\,.\,3\,.\dots x=\sqrt{2\pi x}\,x^x e^{-x}\left[1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51840x^3}\dots\right]$

as a close approximation. (Cf. Arts. 877, 884.)

936. It will be seen that the formula

$$\Sigma u_x=C+\int u_x\,dx-\frac{1}{2}u_x+\frac{B_1}{2!}\frac{du_x}{dx}-\text{etc.}$$

will be of the greatest service when methods of exact summation fail. The student should, however, test the formula for himself in cases with known results, such as

$$1^3+2^3+\dots+x^3=\frac{x^2(x+1)^2}{4},$$

to gain familiarity with it.

Enough has been said to show that the summations we require in the present chapter, such as

$$S_r=\frac{1}{1^r}+\frac{1}{2^r}+\frac{1}{3^r}+\dots+\frac{1}{x^r}\quad(r>1),$$

can be readily calculated, when wanted, to any degree of approximation which may be required, without the labour of calculating out each term separately, except for a few terms to determine the value of the constant. We have, for finding C, chosen 10 terms for the obvious reason that the arithmetical calculations of the right-hand member of the equality are thereby much simplified.

* See De Morgan, *Differential Calculus*, p. 312.

937. **A Theorem due to Cauchy.**

It is a well-known theorem in trigonometry that

$$\cot z = \frac{1}{z} - \sum_1^m \frac{2z}{r^2\pi^2 - z^2} + R_m,$$

where R_m is a quantity which may be made as small as we please by taking m large enough (see Hobson, *Trigonometry*, Art. 293). This is so whether z is real or complex. Also, when m is indefinitely increased the series is absolutely convergent for all values of z, with the exception of such as are expressed by $z = \pm r\pi$ for integral values of r.

Writing $\frac{iz}{2}$ in place of z, we have

$$\frac{1}{2}\coth\frac{z}{2} = \frac{1}{z} + \sum_1^m \frac{2z}{4r^2\pi^2 + z^2} + R'_m,$$

where R'_m, like R_m, can be made indefinitely small by increasing m without limit, and

$$\frac{1}{2}\coth\frac{z}{2} = \frac{1}{2}\left(\frac{e^z+1}{e^z-1}\right),$$

and can be written either as

$$\frac{1}{e^z-1} + \frac{1}{2} \quad \text{or as} \quad \frac{e^z}{e^z-1} - \frac{1}{2}, \quad i.e. \quad \frac{1}{1-e^{-z}} - \frac{1}{2}.$$

$$\left.\begin{array}{ll}\text{Hence} & \dfrac{1}{e^z-1} + \dfrac{1}{2} - \dfrac{1}{z} \\ \text{or} & \dfrac{1}{1-e^{-z}} - \dfrac{1}{2} - \dfrac{1}{z}\end{array}\right\} = \sum_1^m \frac{2z}{4r^2\pi^2 + z^2} + R'_m.$$

Now, by division,

$$\frac{1}{a^2+z^2} = \frac{1}{a^2} - \frac{z^2}{a^4} + \frac{z^4}{a^6} - \ldots + (-1)^{n-1}\frac{z^{2n-2}}{a^{2n}} + (-1)^n \frac{z^{2n}}{a^{2n+2}}\epsilon,$$

where $\epsilon = \frac{a^2}{a^2+z^2}$ and is a positive proper fraction for all real values of z, and the series would be convergent, and could be continued to infinity, provided $z < a$ if real, or mod. $z < a$ if z be complex.

Write in this identity $a = 2\pi, 4\pi, 6\pi \ldots 2m\pi$ successively, and indicate by suffixes 1, 2, 3, ..., the corresponding values of ϵ, and let $S_r{}^m$ denote

$$\frac{1}{1^r} + \frac{1}{2^r} + \frac{1}{3^r} + \ldots + \frac{1}{m^r}.$$

Then we arrive at m equations of the type

$$\frac{1}{(2r\pi)^2+z^2}=\frac{1}{(2r\pi)^2}-\frac{z^2}{(2r\pi)^4}+\ldots+(-1)^{n-1}\frac{z^{2n-2}}{(2r\pi)^{2n}}+(-1)^n\frac{z^{2n}}{(2r\pi)^{2n+2}}\epsilon_r,$$

and, adding these equations together,

$$\sum_1^m\frac{1}{4r^2\pi^2+z^2}=\frac{S_2{}^m}{(2\pi)^2}-\frac{S_4{}^m z^2}{(2\pi)^4}+\ldots+(-1)^{n-1}\frac{S_{2n}^m z^{2n-2}}{(2\pi)^{2n}}+\frac{(-1)^n S_{2n+2}^m z^{2n}}{(2\pi)^{2n+2}}\epsilon',$$

where $$\epsilon' S_{2n+2}^m=\sum_1^m\frac{\epsilon_r}{r^{2n+2}},$$

and if η be the greatest of the quantities $\epsilon_1, \epsilon_2, \ldots,$

$$\epsilon' S_{2n+2}^m<\eta\sum_1^m\frac{1}{r^{2n+2}},\quad i.e.\ \epsilon'<\eta,$$

and therefore ϵ' is also, like $\epsilon_1, \epsilon_2, \epsilon_3$, etc., a positive proper fraction.

We thus have, taking e^z to have its principal value,

$$\left(\frac{1}{e^z-1}+\frac{1}{2}-\frac{1}{z}\right)=\frac{2S_2{}^m}{(2\pi)^2}z-\frac{2S_4{}^m}{(2\pi)^4}z^3+\frac{2S_6{}^m}{(2\pi)^6}z^5-\ldots$$
$$+(-1)^{n-1}\frac{2S_{2n}^m}{(2\pi)^{2n}}z^{2n-1}+(-1)^n\frac{2S_{2n+2}^m}{(2\pi)^{2n+2}}z^{2n+1}\epsilon'+R'_m,$$

and if we increase m without limit, the series $S_2{}^m, S_4{}^m, S_6{}^m$, being all convergent,

$$Lt_{m=\infty}S_r{}^m=\frac{1}{1^r}+\frac{1}{2^r}+\ldots \text{ to } \infty=S_r, \quad\text{and}\quad Lt\,R'_m=0.$$

Hence

$$\left(\frac{1}{e^z-1}+\frac{1}{2}-\frac{1}{z}\right)=\frac{2S_2}{(2\pi)^2}z-\frac{2S_4}{(2\pi)^4}z^3+\frac{2S_6}{(2\pi)^6}z^5-\ldots$$
$$+(-1)^{n-1}\frac{2S_{2n}}{(2\pi)^{2n}}z^{2n-1}+(-1)^n\frac{2S_{2n+2}}{(2\pi)^{2n+2}}z^{2n+1}\Theta.$$

where Θ is a positive proper fraction; or, what is the same thing, $\left(\frac{1}{1-e^{-z}}-\frac{1}{2}-\frac{1}{z}\right)=$ the same expression.

And if we write $\frac{B_{2n-1}}{(2n)!}$ for $\frac{2S_{2n}}{(2\pi)^{2n}}$, we have

$$\left.\begin{array}{l}\frac{1}{z}\left(\frac{1}{e^z-1}+\frac{1}{2}-\frac{1}{z}\right)\\ \text{or}\\ \frac{1}{z}\left(\frac{1}{1-e^{-z}}-\frac{1}{2}-\frac{1}{z}\right)\end{array}\right\}=\frac{B_1}{2!}-\frac{B_3}{4!}z^2+\frac{B_5}{6!}z^4-\ldots+(-1)^{n-1}\frac{B_{2n-1}}{(2n)!}z^{2n-2}$$
$$+(-1)^n\frac{B_{2n+1}}{(2n+2)!}z^{2n}\Theta,$$

where $0<\Theta<1$ for all real values of z.

938. Now Cauchy has shown that Maclaurin's Theorem for the expansion of a continuous function of x, viz. $F(x)$, for the case of a real variable, still holds for a complex variable which is such that its modulus has a value lower than that for which $F(x)$ ceases to be finite or continuous (see Art. 1299).

The function $\dfrac{1}{e^z-1}+\dfrac{1}{2}-\dfrac{1}{z}$ only becomes infinite for values of z which are given by $z=2\lambda\iota\pi$, where λ is a positive or negative integer other than zero. This function is therefore capable of expansion by Maclaurin's Theorem in a convergent series within the circle of convergence of radius 2π for any real or complex value of z, whose modulus is $<2\pi$, and the form of that expansion has been given in *Diff. Calc.*, Art. 148, as

$$\frac{1}{z}\left(\frac{1}{e^z-1}+\frac{1}{2}-\frac{1}{z}\right)=\frac{B_1}{2!}-\frac{B_3}{4!}z^2+\frac{B_5}{6!}z^4-\dots \text{ to infinity}$$

or
$$\frac{z}{e^z-1}=1-\frac{z}{2}+\frac{B_1}{2!}z^2-\frac{B_3}{4!}z^4+\frac{B_5}{6!}z^6-\dots,$$

and the various coefficients were defined as Bernoulli's numbers.

This series then is convergent when z is a real variable which lies between -2π and $+2\pi$, exclusive. It is also true and convergent when z is a complex variable and z lies within a circle of convergence of radius 2π.

And when the infinite series is not convergent, *i.e.* when z does not lie between the limits specified, the series may be stopped at any term $(-1)^{n-1}\dfrac{B_{2n-1}}{(2n)!}z^{2n-2}$, and the error is then numerically less than the next term, $(-1)^n\dfrac{B_{2n+1}}{(2n+2)!}z^{2n}$.

This theorem is due to Cauchy.

939. Lemma. As a preliminary to what follows we may remark that such an integral as $\displaystyle\int_a^x \frac{\theta}{x^p}dx$, where $0<\theta<1$, lies intermediate between $\theta_1\displaystyle\int_a^x \frac{1}{x^p}dx$ and $\theta_2\displaystyle\int_a^x \frac{1}{x^p}dx$, where θ_1 and θ_2 are the greatest and least values of θ between $x=a$ and $x=x$. Therefore $\displaystyle\int_a^x \frac{\theta}{x^p}dx=\Theta\int_a^x \frac{dx}{x^p}$ for some value of Θ between θ_1 and θ_2, and therefore, if θ_1 and θ_2 are positive proper fractions, so also must Θ be a positive proper fraction.

940. Now we have established the equation

$$\psi'(x) \equiv \frac{d^2}{dx^2} \log \Gamma(x) = \int_0^\infty \frac{\beta e^{-x\beta}}{1-e^{-\beta}} d\beta \quad \text{(Art. 930, 8)};$$

or, what is the same thing,

$$\psi'(x+1) \equiv \frac{d^2}{dx^2} \log \Gamma(x+1) = \int_0^\infty \frac{\beta e^{-(x+1)\beta}}{1-e^{-\beta}} d\beta = \int_0^\infty e^{-x\beta} \frac{\beta}{e^\beta - 1} d\beta,$$

Hence, substituting for $\frac{\beta}{e^\beta-1}$, the finite series established by Cauchy (Art. 937),

$$\psi'(x+1) \equiv \frac{d^2}{dx^2} \log \Gamma(x+1) = \int_0^\infty e^{-x\beta} \Big[1 - \frac{\beta}{2} + \frac{B_1}{2!}\beta^2 - \frac{B_3}{4!}\beta^4 + \dots$$

$$+ (-1)^{n-1} \frac{B_{2n-1}}{(2n)!} \beta^{2n} + (-1)^n \frac{B_{2n+1}}{(2n+2)!} \beta^{2n+2} \Theta \Big] d\beta,$$

$$(0 < \Theta < 1),$$

$$= \frac{1}{x} - \frac{1}{2} \cdot \frac{\Gamma(2)}{x^2} + \frac{B_1}{2!} \frac{\Gamma(3)}{x^3} - \frac{B_3}{4!} \frac{\Gamma(5)}{x^5} + \dots + (-1)^{n-1} \frac{B_{2n-1}}{(2n)!} \frac{\Gamma(2n+1)}{x^{2n+1}}$$

$$+ (-1)^n \frac{B_{2n+1}}{(2n+2)!} \frac{\Gamma(2n+3)}{x^{2n+3}} \Theta, \quad (0 < \Theta < 1),$$

i.e.

$$\psi'(x+1) \equiv \frac{d^2}{dx^2} \log \Gamma(x+1) = \frac{1}{x} - \frac{1}{2x^2} + \frac{B_1}{x^3} - \frac{B_3}{x^5} + \dots$$

$$+ (-1)^{n-1} \frac{B_{2n-1}}{x^{2n+1}} + (-1)^n \frac{B_{2n+1}}{x^{2n+3}} \Theta, \quad (0 < \Theta < 1).$$

Integrating this result,

$$\psi(x+1) \equiv \frac{d}{dx} \log \Gamma(x+1) = A + \log x + \frac{1}{2x} - \frac{B_1}{2x^2} + \frac{B_3}{4x^4} - \dots$$

$$- (-1)^{n-1} \frac{B_{2n-1}}{2n\, x^{2n}} - (-1)^n \frac{B_{2n+1}}{(2n+2) x^{2n+2}} \Theta_1,$$

where $0 < \Theta_1 < 1$, by the lemma of the last article, A being a constant to be determined.

Let x become infinite. Then

$$A = Lt_{x=\infty} \left[\frac{\Gamma'(x+1)}{\Gamma(x+1)} - \log x \right] = Lt_{x=\infty} \left[\frac{\Gamma'(x+1)}{\Gamma(x+1)} - \log(x+1) \right]$$

$$+ Lt_{x=\infty} \log\left(1 + \frac{1}{x}\right) = 0, \quad \text{by Art. 911 (3).}$$

Hence

$$\psi(x+1)\equiv\frac{d}{dx}\log\Gamma(x+1)=\log x+\frac{1}{2x}-\frac{B_1}{2x^2}+\frac{B_3}{4x^4}-\dots$$
$$-(-1)^{n-1}\frac{B_{2n-1}}{2n\,x^{2n}}-(-1)^n\frac{B_{2n+1}}{(2n+2)x^{2n+2}}\Theta_1,$$
$$(0<\Theta_1<1).$$

Again integrating,

$$\log\Gamma(x+1)=A'+x(\log x-1)+\tfrac{1}{2}\log x+\frac{B_1}{1.2}\frac{1}{x}-\frac{B_3}{3.4}\cdot\frac{1}{x^3}+\dots$$
$$+(-1)^{n-1}\frac{B_{2n-1}}{(2n-1)2n}\frac{1}{x^{2n-1}}+(-1)^n\frac{B_{2n+1}}{(2n+1)(2n+2)}\frac{1}{x^{2n+1}}\Theta_2,$$

$(0<\Theta_2<1)$, by the lemma, where A' is a constant to be determined.

Let x become an infinite integer,

$$A'=Lt_{x=\infty}[\log\Gamma(x+1)-x(\log x-1)-\tfrac{1}{2}\log x]$$
$$=Lt_{x=\infty}[\log(\sqrt{2x\pi}\,x^xe^{-x})-(x+\tfrac{1}{2})\log x+x]$$
$$=\log\sqrt{2\pi}.$$

Hence

$$\log\Gamma(x+1)=\tfrac{1}{2}\log 2\pi+(x+\tfrac{1}{2})\log x-x+\frac{B_1}{1.2}\frac{1}{x}-\frac{B_3}{3.4}\frac{1}{x^3}+\dots$$
$$+(-1)^{n-1}\frac{B_{2n-1}}{(2n-1)2n}\frac{1}{x^{2n-1}}+(-1)^n\frac{B_{2n+1}}{(2n+1)(2n+2)}\frac{1}{x^{2n+1}}\Theta_2,$$
$$(0<\Theta_2<1).$$

This result is also due to Cauchy.

941. The series, if carried to infinity, is known as Stirling's Series. It is divergent, however great x may be. For the general term

$$\frac{B_{2n-1}}{(2n-1)2n}\frac{1}{x^{2n-1}}=\frac{1}{(2n-1)2n}\cdot\frac{1}{x^{2n-1}}\frac{2(2n)!}{(2\pi)^{2n}}S_{2n},$$

and the ratio of this term to the preceding term is

$$\frac{(2n-3)(2n-2)}{(2\pi x)^2}\times\frac{S_{2n}}{S_{2n-2}},$$

i.e. ultimately $\frac{n^2}{\pi^2x^2}$, and however great x may be, will ultimately be >1 when n is large enough. The formula can, nevertheless, be made useful for approximative purposes for calculating $\Gamma(x+1)$. For, as in the series of Art. 938, the error in stopping at the term involving $\frac{1}{x^{2n-1}}$ has been shown to be $\Theta\frac{B_{2n+1}}{(2n+1)(2n+2)}\frac{1}{x^{2n+1}}$ $(0<\Theta<1)$, *i.e.* the error is less than the succeeding term. And as the ratio of two consecutive terms, viz. $\frac{(2n-3)(2n-2)}{(2\pi x)^2}\frac{S_{2n}}{S_{2n-2}}$, is less than unity until $(2n-3)(2n-2)\frac{S_{2n}}{S_{2n-2}}$ exceeds $4\pi^2x^2$, the absolute values of the several terms go on diminishing until this happens, and then increase again. Hence the closest approximation will be obtained by continuing the series until that term is reached which precedes the smallest term.

942. We have as successive approximations

$$\log\Gamma(x+1)>\tfrac{1}{2}\log 2\pi+(x+\tfrac{1}{2})\log x-x,$$

$$\log\Gamma(x+1)<\tfrac{1}{2}\log 2\pi+(x+\tfrac{1}{2})\log x-x+\frac{B_1}{1\,.\,2}\frac{1}{x},$$

$$\log\Gamma(x+1)>\tfrac{1}{2}\log 2\pi+(x+\tfrac{1}{2})\log x-x+\frac{B_1}{1\,.\,2}\frac{1}{x}-\frac{B_3}{3\,.\,4}\frac{1}{x^3},$$

$$\log\Gamma(x+1)<\tfrac{1}{2}\log 2\pi+(x+\tfrac{1}{2})\log x-x$$
$$+\frac{B_1}{1\,.\,2}\frac{1}{x}-\frac{B_3}{3\,.\,4}\frac{1}{x^3}+\frac{B_5}{5\,.\,6}\frac{1}{x^5}, \text{ etc.}$$

And since $B_1=\frac{1}{6}$, $B_3=\frac{1}{30}$, $B_5=\frac{1}{42}$, etc.,

$$\Gamma(x+1)$$
$$>\sqrt{2\pi x}\,x^xe^{-x},$$
$$<\sqrt{2\pi x}\,x^xe^{-x}e^{\frac{1}{12x}},$$
$$>\sqrt{2\pi x}\,x^xe^{-x}e^{\frac{1}{12x}-\frac{1}{360x^3}}, \text{ etc.,}$$

i.e.

$$\Gamma(x+1)$$
$$>\sqrt{2\pi x}\,x^xe^{-x},$$
$$<\sqrt{2\pi x}\,x^xe^{-x}\left(1+\frac{1}{12x}+\frac{1}{288x^2}\cdots\right),$$
$$>\sqrt{2\pi x}\,x^xe^{-x}\left(1+\frac{1}{12x}+\frac{1}{2(12x)^2}-\frac{139}{30(12x)^3}-\frac{571}{120(12x)^4}\cdots\right),$$

etc.

943. In order to facilitate calculation from the series

$$\log\Gamma(x+1)=\frac{1}{2}\log 2\pi+\left(x+\frac{1}{2}\right)\log x-x$$
$$+\frac{B_1}{1\,.\,2}\cdot\frac{1}{x}-\frac{B_3}{3\,.\,4}\frac{1}{x^3}+\frac{B_5}{5\,.\,6}\frac{1}{x^5}-\dots,$$

it is desirable to arrange so that x shall not be small.

For this purpose Legendre puts $x=4+a$; whence

$$\log\Gamma(x+1)=\log x+\log\Gamma(x)=\log x$$
$$+\log\Gamma(a)+\log a(a+1)(a+2)(a+3)$$

and

$$\log_{10}\Gamma(a)=\frac{1}{2}\log_{10}2\pi+\left(x-\frac{1}{2}\right)\log_{10}x-\mu x+\frac{\mu B_1}{1\,.\,2}\frac{1}{x}-\frac{\mu B_3}{3\,.\,4}\cdot\frac{1}{x^3}$$
$$+\frac{\mu B_5}{5\,.\,6}\cdot\frac{1}{x^5}-\dots-\log_{10}a(a+1)(a+2)(a+3),$$

where μ is the modulus of the logarithm tables, viz.

$$\mu=\log_{10}e=\cdot4342944819\dots.$$

Thus, if $\log_{10}\Gamma(1\cdot25)$ be required, $x=5\cdot25$, and

$$\log_{10}\Gamma(1\cdot25)=\frac{1}{2}\log_{10}2\pi+4\cdot75\log_{10}5\cdot25-\mu5\cdot25+\frac{\mu}{12}\frac{1}{5\cdot25}-\text{etc.}$$
$$-\log_{10}[(1\cdot25)(2\cdot25)(3\cdot25)(4\cdot25)],$$

and by this artifice it is possible to avoid the calculation of all but the earlier terms of the series. We could make $x=5+a$, $6+a, \dots$, equally well, and the choice is in the hands of the calculator.

Legendre remarks as to his calculations of the seven-figure tables of $\log\Gamma(x)$ with regard to the above: "de cette manière on n'a jamais eu besoin de calculer plus de deux ou trois termes de la série $\dfrac{mA'}{1\,.\,2k}-\dfrac{mB'}{3\,.\,4k^2}+\dfrac{mC'}{5\,.\,6k^5}-$etc., pour avoir $\log\Gamma(a)$ approché jusqu'à sept décimales, dans tout l'intervalle depuis $a=1$ jusqu'à $a=2$" (*Exercices*, p. 300).

Legendre's m, k, A', B', C' are what we have called μ, x, B_1, B_3, B_5 respectively.

944. **The Case when x is a Commensurable Number.**

We have established the result

$$\frac{d}{dx}\log\Gamma(x)=\int_0^\infty\left(\frac{e^{-\beta}}{\beta}-\frac{e^{-\beta x}}{1-e^{-\beta}}\right)d\beta. \quad \text{(Art. 930 (7).)}$$

And we have seen that Euler's constant γ is the value of

$$-\frac{d}{dx}\log\Gamma(x) \quad \text{when} \quad x=1 \quad \text{(Art. 911 (4).)}$$

that is $$\gamma=-\int_0^\infty\left(\frac{e^{-\beta}}{\beta}-\frac{e^{-\beta}}{1-e^{-\beta}}\right)d\beta.$$

Hence, adding

$$\frac{d}{dx}\log\Gamma(x)+\gamma=\int_0^\infty\frac{e^{-\beta}-e^{-\beta x}}{1-e^{-\beta}}d\beta.$$

In the case when x is a commensurable number * this integral can be reduced to the integration of a rational integral algebraic expression, and the integration effected in finite terms in terms of the ordinary algebraic, logarithmic and inverse circular fnnctions.

Let $x=\frac{p}{q}$, where p and q are positive integers, and let $e^{-\beta}=t^q$.

Then $$\frac{d}{dx}\log\Gamma(x)+\gamma=q\int_0^1\frac{t^q-t^p}{t(1-t^q)}dt,$$

and the integrand is a rational integral algebraic function of t.

If $q=1$, *i.e.* if x be an integer, the value of $\frac{d}{dx}\log\Gamma(x)$ is given by

$$\frac{d}{dx}\log\Gamma(x)+\gamma=\int_0^1\frac{1-t^{x-1}}{1-t}dt$$

$$=\int_0^1(1+t+t^2+\ldots+t^{x-2})\,dt$$

$$=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{x-1},$$

as might be expected from Art. 911 (2).

945. **Expansion of $\Gamma(x+1)$ derived from the Integral Definition (De Morgan).**

The expansion of $\log\Gamma(1+x)$ in powers of x may be obtained directly from the definition of $\Gamma(1+x)$ as $\int_0^\infty e^{-v}v^x\,dv$.

For we have $$Lt_{a=0}\left(\frac{1-e^{-av}}{a}\right)^x=v^x.$$

Hence $$\Gamma(1+x)=Lt_{a=0}\int_0^\infty\frac{e^{-v}(1-e^{-av})^x}{a^x}dv.$$

* See Serret, *Calc. Intégral*, p. 184.

Let $e^{-av}=y$. Then $a\,dv=-\dfrac{dy}{y}$, and

$$\Gamma(1+x)=Lt\,\frac{1}{a^{x+1}}\int_0^1 y^{\frac{1}{a}-1}(1-y)^x dy$$

$$=Lt\,\frac{1}{a^{x+1}}B\left(\frac{1}{a},\, x+1\right)\quad \left(\text{Let } \frac{1}{a}=b, \text{ a positive integer.}\right)$$

$$=Lt_{b=\infty}\, b^{x+1}\frac{\Gamma(b)\Gamma(x+1)}{\Gamma(b+x+1)},$$

i.e.
$$Lt_{b=\infty}\, b^{x+1}\frac{\Gamma(b)}{\Gamma(x+b+1)}=1;$$

$$\therefore\ Lt_{b=\infty}\frac{(x+b)(x+b-1)\ldots(x+1)\,\Gamma(x+1)}{(b-1)(b-2)\ldots 1\,.\,b^{x+1}}=1,$$

i.e.

$$\log\Gamma(1+x)=Lt\left[x\log b-\log\left(1+\frac{x}{1}\right)-\log\left(1+\frac{x}{2}\right)-\ldots \textit{ad inf.}\right],$$

or, expanding the logarithms, assuming $x<1$,

$$\log\Gamma(1+x)=Lt\left[-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{b}-\log b\right)x\right.$$
$$\left.+\frac{1}{2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{b^2}\right)x^2-\frac{1}{3}\left(\frac{1}{1^3}+\frac{1}{2^3}+\ldots+\frac{1}{b^3}\right)x^3+\ldots\right],$$

and when b is indefinitely increased

$$\log\Gamma(1+x)=-\gamma x+S_2\frac{x^2}{2}-S_3\frac{x^3}{3}+S_4\frac{x^4}{4}-\ldots$$

for values of x, $\qquad 0<x<1.$

This investigation is due to De Morgan.*

It was felt desirable to deduce this series directly from the integral, rather than to base it upon results deduced from the property $\Gamma(x+1)=x\Gamma(x)$, *i.e.* the difference equation $u_{x+1}=xu_x$, inasmuch as Legendre's tables of the values of the Gamma function are derived from this series and others obtained from it. And in default of direct derivation of the series from the integral itself, some doubt might be felt as to whether Legendre's tabulated results were the values of the integral itself or the values of the integral multiplied by some periodic function of x whose period is unity, which, as explained in Art. 863, would equally be a solution of the difference equation.

* De Morgan, *Diff. Calc.*, p. 584.

946. From De Morgan's investigation given above, the formal identification of $\Gamma(x+1)$ with $\Pi(x)$ for all positive values of x, may proceed as follows:

$$\Pi(x)=Lt_{\mu=\infty}\,\mu^x\bigg/\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\dots\left(1+\frac{x}{\mu}\right);$$

$$\therefore\ \log\Pi(x)=Lt_{\mu=\infty}\left[x\log\mu-\log\left(1+\frac{x}{1}\right)-\log\left(1+\frac{x}{2}\right)-\dots\right.$$
$$\left.-\log\left(1+\frac{x}{\mu}\right)\right],$$

and if $x<1$, $\quad =-\gamma x+\frac{S_2}{2}x^2-\frac{S_3}{3}x^3+\dots;$

$$\therefore\ \Pi(x)=\Gamma(x+1)\text{ if }x<1\text{ and positive.}$$

If x lies between 1 and 2, say $x=1+\xi$, then, since

$$\left.\begin{array}{l}\Pi(1+\xi)=(1+\xi)\,\Pi(\xi)\\ \text{and }\Gamma(2+\xi)=(1+\xi)\,\Gamma(1+\xi)\end{array}\right\}\text{and }\Pi(\xi)=\Gamma(1+\xi)\ (0<\xi<1),$$

it follows that $\Pi(1+\xi)=\Gamma(2+\xi)$,

i.e. $\qquad \Pi(x)=\Gamma(1+x)$ when x lies between 1 and 2.

Similarly if x lies between 2 and 3, etc.

Hence, for all positive values of x, $\Pi(x)$ and $\Gamma(1+x)$ are identical.

947. **The Integration of** $\int_0^a e^{-v}v^n dv$, (a *not* infinite, $n>-1$).

In considering the integration of $e^{-v}v^n dv$ between limits 0 and a, where a is not infinite, we must have recourse to either

(1) an expression in series

or (2) a continued fraction.

$$(1)\qquad I_n\equiv\int_0^a e^{-v}v^n dv=\left[e^{-v}\frac{v^{n+1}}{n+1}\right]_0^a+\frac{1}{n+1}\int_0^a e^{-v}v^{n+1}dv$$
$$=\frac{e^{-a}a^{n+1}}{n+1}+\frac{1}{n+1}I_{n+1},$$

and by the continued use of this rule,

$$I_n=\frac{e^{-a}a^{n+1}}{n+1}\left[1+\frac{a}{n+2}+\frac{a^2}{(n+2)(n+3)}+\frac{a^3}{(n+2)(n+3)(n+4)}\right.$$
$$\left.+\dots\ ad\ inf.\right],$$

a series which is always convergent for any finite value of a, but only slowly so if a be >1. A little consideration will show that the integral remainder is ultimately infinitely small. Or we may proceed thus:

$$\text{Let}\quad J_n \equiv \int_a^\infty e^{-v} v^n dv = \Big[-e^{-v} v^n\Big]_a^\infty + nJ_{n-1}$$
$$= e^{-a} a^n + nJ_{n-1};$$

whence

$$J_n = e^{-a} a^n \left[1 + \frac{n}{a} + \frac{n(n-1)}{a^2} + \dots + \frac{n(n-1)\dots(n-r+1)}{a^r}\right] + n(n-1)\dots(n-r)J_{n-r-1}.$$

If n be a positive integer, the integration can be effected in finite terms. But if n be negative or fractional, the series on the right-hand side is divergent if continued to infinity whatever a may be. The terms however ultimately take alternate signs, and when such is the case, and when there is convergence for a certain number of terms, and then ultimate divergence, we can apply the principle adopted in Arts. 938, 941, the convergent part making a continual approximation to the arithmetical value of the function under consideration, and the error being less than the first term omitted.*

If then J_n be thus approximated to,

$$I_n = \int_0^a e^{-v} v^n dv = \left(\int_0^\infty - \int_a^\infty\right) e^{-v} v^n dv,$$

and

$$I_n = \Gamma(n+1) - J_n.$$

948. (2) De Morgan has shown how such an integral as $\int_v^\infty e^{-v} v^n dv$ can be converted into a continued fraction.

When this is done $\int_0^v e^{-v} v^n dv = \Gamma(n+1) - \int_v^\infty e^{-v} v^n dv$, as before.

Let $\int_v^\infty e^{-v} v^n dv = e^{-v} v^n V$, where V is some function of v.

Then differentiating with regard to v,

$$-e^{-v} v^n = e^{-v} v^n V' + ne^{-v} v^{n-1} V - e^{-v} v^n V;$$
$$\therefore\ vV' + nV - vV = -v,$$

or

$$vV' = (v-n)V - v.$$

Consider the equation

$$vV' = (v - a_1)V - v + b_1 V^2. \quad \text{.........................(1)}$$

*De Morgan, *Differential Calculus*, p. 226 and p. 590.

Putting $V=\dfrac{1}{1+k_1\dfrac{V_1}{v}}$, we derive an equation

$$vV_1'=(v-a_2)V_1-v+b_2V_1^2, \qquad \text{.......................(2)}$$

where $\qquad b_1-a_1=k_1, \quad b_2=k_1=b_1-a_1, \quad a_2=-(a_1+1).$

Putting $V_1=\dfrac{1}{1+k_2\dfrac{V_2}{v}}$ in equation (2), we derive an equation

$$vV_2'=(v-a_3)V_2-v+b_3V_2^2, \qquad \text{.......................(3)}$$

where $\qquad b_2-a_2=k_2, \quad b_3=k_2, \quad a_3=-(a_2+1),$

and so on.

Then
$$V=\frac{1}{1+}\,\frac{k_1v^{-1}}{1+}\,\frac{k_2v^{-1}}{1+}\,\frac{k_3v^{-1}}{1+}\text{ etc.}$$

In our case

$$\begin{array}{lll} a_1=n, & b_1=0, & k_1=-n=b_2\,; \\ a_2=-(1+n), & b_2=-n, & k_2=1=b_3\,; \\ a_3=n, & b_3=1, & k_3=-(n-1)=b_4\,; \\ a_4=-(n+1), & b_4=-(n-1), & k_4=2=b_5\,; \\ a_5=n, & b_5=2, & k_5=2-n=b_6\,; \\ & \text{etc.}\,; & \end{array}$$

whence

$$\int_v^\infty e^{-v}v^n\,dv=e^{-v}v^n\left[\frac{1}{1-}\,\frac{nv^{-1}}{1+}\,\frac{v^{-1}}{1-}\,\frac{(n-1)v^{-1}}{1+}\,\frac{2v^{-1}}{1-}\,\frac{(n-2)v^{-1}}{1+}\text{ etc.}\right].$$

The expression converges rapidly for large values of v.

The process above employed by De Morgan is similar to that employed by Boole, *Differential Equations*, p. 92, in the solution of Riccati's equation

$$x\frac{dy}{dx}-ay+by^2=cx^n.$$

The equation we have just solved is a very similar equation, viz.

$$x\frac{dy}{dx}+a_1y-b_1y^2=-x+xy.$$

949. More generally, consider the differential equation

$$P+Qy+Ry^2+S\frac{dy}{dx}=0,$$

where P, Q, R, S are functions of x alone.

Let $\qquad X_1=Ax^\alpha, \quad X_2=Bx^\beta, \quad X_3=Cx^\gamma,$ etc.

Take $y_1, y_2, y_3, \dots$ successive new dependent variables, such that

$$y=\frac{X_1}{1+y_1}, \quad y_1=\frac{X_2}{1+y_2}, \quad y_2=\frac{X_3}{1+y_3}, \text{ etc.}$$

Then when $A, B, C, \dots \alpha, \beta, \gamma, \dots$ have been properly determined, we have

$$y=\frac{Ax^\alpha}{1+}\,\frac{Bx^\beta}{1+}\,\frac{Cx^\gamma}{1+\dots},$$

viz. a solution in the form of a continued fraction. [LACROIX, t. II., p. 288.]

To begin with, using accents for differentiations,

$$y'=\frac{X'_1(1+y_1)-X_1y'_1}{(1+y_1)^2};$$

$$\therefore\ P+Q\frac{X_1}{1+y_1}+R\frac{X_1^2}{(1+y_1)^2}+S\frac{X'_1(1+y_1)-X_1y'_1}{(1+y_1)^2}=0,$$

i.e. $(P+QX_1+RX_1^2+SX'_1)+(2P+QX_1+SX'_1)y_1+Py_1^2-SX_1y'_1=0,$

or $$P_1+Q_1y_1+R_1y_1^2+S_1y'_1=0,$$

where $$\left.\begin{aligned}P_1&\equiv P+QX_1+RX_1^2+SX'_1,\\ Q_1&\equiv 2P+QX_1+SX'_1,\\ R_1&\equiv P,\\ S_1&\equiv -SX_1.\end{aligned}\right\}$$

At the second substitution, viz. $y_1=\frac{X_2}{1+y_2}$, the differential equation becomes

$$P_2+Q_2y_2+R_2y_2^2+S_2y'_2=0,$$

where P_2, Q_2, R_2, S_2 are formed from P_1, Q_1, R_1, S_1 in the same way as the latter were formed from P, Q, R, S, and so on.

Again assuming the expansion of y in powers of x to be of the form $Ax^\alpha+A_1x^{\alpha+1}+\ldots$ and the expansion of y_1 to be $Bx^\beta+B_1x^{\beta+1}+\ldots$, and so on, we can by substitution in the several differential equations they satisfy obtain the values of A and α, B and β, etc., by an examination of the lowest order terms occurring, and thus express y in the form of a continued fraction.

950. Development of $\psi(a+x)\equiv\frac{d}{dx}\log\Gamma(a+x)$ in a Factorial Series.

Since

$$\Delta\psi(a+x)=\psi(a+x+1)-\psi(a+x)=\frac{d}{dx}[\log\Gamma(a+x+1)-\log\Gamma(a+x)]$$
$$=\frac{d}{dx}\log(a+x)=\frac{1}{a+x},$$

we have

$$\Delta^2\psi(a+x)=\Delta\frac{1}{a+x}=\frac{1}{a+x+1}-\frac{1}{a+x}=\frac{(-1)}{(a+x)(a+x+1)},$$

$$\Delta^3\psi(a+x)=\Delta^2\frac{1}{a+x}=\frac{(-1)(-2)}{(a+x)(a+x+1)(a+x+2)},$$

and generally

$$\Delta^n\psi(a+x)=\Delta^{n-1}\frac{1}{a+x}=\frac{(-1)^{n-1}(n-1)!}{(a+x)(a+x+1)\ldots(a+x+n-1)}.$$

Let

$$\psi(a+x)=A_0+A_1\frac{x^{(1)}}{1!}+A_2\frac{x^{(2)}}{2!}+A_3\frac{x^{(3)}}{3!}+\ldots+A_n\frac{x^{(n)}}{n!}+\ldots,$$

where $x^{(n)}\equiv x(x-1)\ldots(x-n+1)$.

Then $$\Delta\psi(a+x)=A_1+A_2\frac{x^{(1)}}{1!}+A_3\frac{x^{(2)}}{2!}+\dots,$$

$$\Delta^2\psi(a+x)=A_2+A_3\frac{x^{(1)}}{1!}+A_4\frac{x^{(2)}}{2!}+\dots,$$

etc.

Hence

$$A_0=\psi(a+0),\quad A_1=\Delta\psi(a+0),\quad A_2=\Delta^2\psi(a+0),\dots \text{ etc.},$$

where $\Delta^n\psi(a+0)$ means the value of $\Delta^n\psi(a+x)$ when x is put $=0$.

Hence

$$\psi(a+x)\equiv\frac{d}{dx}\log\Gamma(a+x)=\psi(a)+\frac{x}{a}-\frac{1}{2}\,\frac{x(x-1)}{a(a+1)}+\frac{1}{3}\,\frac{x(x-1)(x-2)}{a(a+1)(a+2)}$$
$$-\frac{1}{4}\,\frac{x(x-1)(x-2)(x-3)}{a(a+1)(a+2)(a+3)}+\dots,$$

a series which will terminate in the case when x is a positive integer and is in any case convergent for real and positive values of x and a.

The value of $\psi(a)$, *i.e.* $\frac{d}{da}\log_e\Gamma(a)$, can be found for any particular value of a by means of the series

$$\frac{d}{dx}\log_e\Gamma(x+1)=\log_e x+\frac{1}{2x}-\frac{B_1}{2x^2}+\frac{B_3}{4x^4}-\text{ etc.}$$

of Art. 940.

951. In the case when $a=1$, we have

$$\psi(1+x)=\psi(1)+\frac{x}{1!}-\frac{1}{2}\,\frac{x(x-1)}{2!}+\frac{1}{3}\,\frac{x(x-1)(x-2)}{3!}$$
$$-\frac{1}{4}\,\frac{x(x-1)(x-2)(x-3)}{4!}+\dots$$

and $$-\psi(1)=\gamma \text{ (Euler's constant).}$$

Since $\Delta x^{(n)}=nx^{(n-1)}$, this may be written symbolically as

$$\psi(1+x)=-\gamma+\Delta\left(\frac{1}{\Delta}-\frac{1}{2\Delta^2}+\frac{1}{3\Delta^3}-\dots\right)x=-\gamma+\Delta\log\left(1+\frac{1}{\Delta}\right)x,$$

i.e. $$\frac{d}{dx}\log\Gamma(1+x)=-\gamma+\Delta\log\left(\frac{E}{\Delta}\right)x.$$

952. Other properties of the ψ function are:

Since $\Gamma(x+1)=x\Gamma(x)$, we have by logarithmic differentiation

$$\psi(x+1)-\psi(x)=\frac{1}{x}. \qquad \dots\dots(a)$$

Since $\Gamma(x)\,\Gamma(1-x)=\frac{\pi}{\sin x\pi}$, we have similarly

$$\psi(x)-\psi(1-x)=-\pi\cot x\pi. \qquad \dots\dots(b)$$

Since $2^{2x}\,\Gamma(x)\,\Gamma(\frac{1}{2}+x)=2\sqrt{\pi}\,\Gamma(2x)$, we have similarly

$$\psi(x)+\psi(\tfrac{1}{2}+x)=2\psi(2x)-2\log 2. \qquad \dots\dots(c)$$

Since $2\,\Gamma(x)\,\Gamma\left(\frac{1-x}{2}\right)=\frac{2^x\sqrt{\pi}}{\cos\frac{x\pi}{2}}\Gamma\left(\frac{x}{2}\right)$, we have similarly

$$\psi(x)-\tfrac{1}{2}\psi\left(\frac{1-x}{2}\right)=\tfrac{1}{2}\psi\left(\frac{x}{2}\right)+\log 2+\frac{\pi}{2}\tan\frac{x\pi}{2}.\quad\ldots\ldots\ldots\ldots(d)$$

Since $\Gamma(x)\,\Gamma\left(x+\frac{1}{n}\right)\Gamma\left(x+\frac{2}{n}\right)\ldots\Gamma\left(x+\frac{n-1}{n}\right)=n^{-nx+\frac{1}{2}}(2\pi)^{\frac{n-1}{2}}\Gamma(nx)$,

we have similarly

$$\psi(x)+\psi\left(x+\frac{1}{n}\right)+\psi\left(x+\frac{2}{n}\right)+\ldots+\psi\left(x+\frac{n-1}{n}\right)=n\psi(nx)-n\log n.\quad(e)$$

953. The equation $\Delta\psi(a+x)=\dfrac{1}{a+x}$ is of considerable service in summation of series.

1. A sum of the form

$$\frac{1}{a+b}+\frac{1}{a+2b}+\frac{1}{a+3b}+\ldots \text{ to } n \text{ terms, viz.}$$

$$S=\sum_{r=1}^{r=n}\frac{1}{a+rb} \text{ can be written}$$

$$=\frac{1}{b}\sum_1^n\frac{1}{\frac{a}{b}+r}=\frac{1}{b}\Sigma\Delta\psi\left(\frac{a}{b}+r\right)$$

$$=\frac{1}{b}\left[\psi\left(\frac{a}{b}+r\right)\right]_1^{n+1}=\frac{1}{b}\left[\psi\left(\frac{a}{b}+n+1\right)-\psi\left(\frac{a}{b}+1\right)\right].$$

2. A sum of the form

$$S=\frac{1}{a+b}-\frac{1}{a+2b}+\frac{1}{a+3b}-\frac{1}{a+4b}+\ldots \text{ to } 2n \text{ terms}$$

$$=\frac{1}{2b}\sum_1^n\frac{1}{\frac{a-b}{2b}+r}-\frac{1}{2b}\sum_1^n\frac{1}{\frac{a}{2b}+r}$$

$$=\frac{1}{2b}\sum_1^n\Delta\psi\left(\frac{a-b}{2b}+r\right)-\frac{1}{2b}\sum_1^n\Delta\psi\left(\frac{a}{2b}+r\right)$$

$$=\frac{1}{2b}\left[\psi\left(\frac{a-b}{2b}+r\right)\right]_1^{n+1}-\frac{1}{2b}\left[\psi\left(\frac{a}{2b}+r\right)\right]_1^{n+1}.$$

E.g. (*a*) $\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots+\dfrac{1}{2n-1}$

$$=\tfrac{1}{2}\sum_0^{n-1}\frac{1}{\frac{1}{2}+r}=\tfrac{1}{2}\sum_0^{n-1}\Delta\psi(\tfrac{1}{2}+r)=\tfrac{1}{2}\left[\psi(\tfrac{1}{2}+r)\right]_0^n=\tfrac{1}{2}[\psi(\tfrac{1}{2}+n)-\psi(\tfrac{1}{2})].$$

(*b*) $\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+$ *ad inf.*

$$=\tfrac{1}{4}\sum_0^\infty\frac{1}{\frac{1}{4}+r}-\tfrac{1}{4}\sum_0^\infty\frac{1}{\frac{3}{4}+r}$$

$$=\tfrac{1}{4}\Sigma\Delta\psi(\tfrac{1}{4}+r)-\tfrac{1}{4}\Sigma\Delta\psi(\tfrac{3}{4}+r)$$

$$=\tfrac{1}{4}\left[\psi(\tfrac{1}{4}+r)\right]_0^\infty-\tfrac{1}{4}\left[\psi(\tfrac{3}{4}+r)\right]_0^\infty$$

$$=\tfrac{1}{4}[\psi(\tfrac{3}{4})-\psi(\tfrac{1}{4})].$$

But by (b) ($x=\frac{3}{4}$), $\quad \psi(\frac{3}{4})-\psi(\frac{1}{4})=\pi$;

$$\therefore \text{ the series is} = \frac{\pi}{4},$$

which is well known otherwise, being Gregory's series for $\tan^{-1}1$.

3. Sum the series

$$S=\tfrac{1}{1}-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\textit{ ad inf.}$$

Here

$$\begin{aligned} S &= \tfrac{1}{2}\sum_0^\infty \frac{1}{\frac{1}{2}+r}-\tfrac{1}{2}\sum_0^\infty \frac{1}{1+r} \\ &= \tfrac{1}{2}\Sigma\Delta\psi(\tfrac{1}{2}+r)-\tfrac{1}{2}\Sigma\Delta\psi(1+r) \\ &= \tfrac{1}{2}\Big[\psi(\tfrac{1}{2}+r)\Big]_0^\infty-\tfrac{1}{2}\Big[\psi(1+r)\Big]_0^\infty \\ &= \tfrac{1}{2}[\psi(1)-\psi(\tfrac{1}{2})]. \end{aligned}$$

Now by (c) ($x=\frac{1}{2}$), $\quad \psi(1)+\psi(\frac{1}{2})=2\psi(1)-2\log 2$;

$$\therefore \psi(1)-\psi(\tfrac{1}{2})=2\log 2;$$

$\therefore S=\log 2$, which is well known otherwise.

We may note that it follows that

$$\begin{aligned} \psi(\tfrac{1}{2}) = \psi(1)-2\log 2 &= -\gamma-2\log 2 \\ &= -0{\cdot}5772157-1{\cdot}3862944 \\ &= -1{\cdot}9635101\ldots. \end{aligned}$$

By (c), $\psi(\frac{1}{4})+\psi(\frac{3}{4})=2\psi(\frac{1}{2})-2\log 2=2\{\psi(1)-2\log 2\}-2\log 2$

$$= -2\gamma-6\log 2$$

and $\quad \psi(\frac{3}{4})-\psi(\frac{1}{4})=\pi$.

Hence

$$\left.\begin{aligned} \psi(\tfrac{3}{4}) &= \frac{\pi}{2}-\gamma-3\log 2, \\ \psi(\tfrac{1}{4}) &= -\frac{\pi}{2}-\gamma-3\log 2 \\ \text{and} \quad \psi(\tfrac{1}{2}) &= -\gamma-2\log 2. \end{aligned}\right\}$$

954. Gauss has established a remarkable result, giving for the function $\psi(x)$ the value of $\psi(1-x)+\psi(x)$ in a series of trigonometric terms in the case when x is any commensurable proper fraction. This result taken with

$$\psi(1-x)-\psi(x)=\pi\cot x\pi$$

will enable us to calculate the value of $\psi(x)$ in all such cases.

The theorem is given by Bertrand in Art. 307 of his *Calcul Intégral*. For shortness we shall denote

$$\log x \text{ by } lx, \quad \psi\left(\frac{r}{q}\right) \text{ by } \psi_r, \quad \cos r\theta \text{ by } c_r, \quad \log 4\sin^2\frac{r\theta}{2} \text{ by } L_r.$$

Then when $\quad \theta=\dfrac{2\pi}{q}$ or $\dfrac{4\pi}{q}$ or $\dfrac{6\pi}{q}\ldots$ or $\dfrac{2(q-1)\pi}{q}$,

$$c_q=c_{2q}=c_{3q}=\ldots=1; \quad c_{q+r}=c_{2q+r}=\ldots=c_r; \quad c_1+c_2+\ldots+c_q\equiv\sum_1^q c_r=0.$$

Writing the fundamental equation

$$\psi(x)=Lt_{n=\infty}\left[\log n-\frac{1}{x}-\frac{1}{x+1}-\frac{1}{x+2}-\dots-\frac{1}{x+n-1}\right] \text{ as}$$

$$\psi(x)=\left(l1-\frac{1}{x}\right)+\left(l\frac{2}{1}-\frac{1}{x+1}\right)+\dots+\left(l\frac{n}{n-1}-\frac{1}{x+n-1}\right)+\left(l\frac{n+1}{n}-\frac{1}{x+n}\right)+\dots,$$

and putting $x=\frac{r}{q}$, where $r \not> q$, and both are positive integers, we have

$$\psi_r=\left(l1-\frac{q}{r}\right)+\left(l\frac{2}{1}-\frac{q}{q+r}\right)+\dots+\left(l\frac{n}{n-1}-\frac{q}{(n-1)q+r}\right)+\left(l\frac{n+1}{n}-\frac{q}{nq+r}\right)+\dots. \quad (A$$

Taking $r=1, 2, 3, \dots q$ in this equation, multiplying by $\cos\theta$, $\cos 2\theta$, $\cos 3\theta$, ... $\cos q\theta$ respectively, and adding, we get

$$\sum_1^q c_r\psi_r=\left(\sum_1^q c_r l1-\sum_1^q\frac{q}{r}c_r\right)+\left(\sum_1^q c_r l\frac{2}{1}-\sum_1^q\frac{q}{q+r}c_r\right)+\dots+\left(\sum_1^q c_r l\frac{n+1}{n}-\sum_1\frac{q}{nq+r}c_r\right)+\dots$$

Now the coefficients of log 1, log $\frac{2}{1}$, log $\frac{3}{2}$, etc., all vanish and since $c_r=c_{q+r}$, etc., the remaining terms form a continuous series to infinity, viz.

$$-q\left[\sum_1^q\frac{c_r}{r}+\sum_1^q\frac{c_{q+r}}{q+r}+\sum_1^q\frac{c_{2q+r}}{2q+r}+\dots\right]=-q\sum_1^\infty\frac{c_r}{r}=\frac{q}{2}\log 4\sin^2\frac{\theta}{2}=\frac{q}{2}L_1;$$

$$\therefore \sum_1^q c_r\psi_r=\frac{q}{2}L_1,$$

viz. an equation connecting $\psi_1, \psi_2, \psi_3, \dots \psi_{q-1}, \psi_q$, the last of which terms is $\psi\left(\frac{q}{q}\right)=\psi(1)=-\gamma$, where γ is Euler's constant. That is

$$c_1\psi_1+c_2\psi_2+c_3\psi_3+\dots+c_{q-1}\psi_{q-1}=\frac{q}{2}L_1+\gamma.$$

So far θ has stood for any of the quantities $\frac{2\pi}{q}$, $\frac{4\pi}{q}$, ... or $2\frac{q-1}{q}\pi$. Say the first. Then similar results will hold for the rest, *i.e.* if we take 2θ, 3θ, ... $(q-1)\theta$ in place of θ. We thus get $q-1$ linear equations from which we can find $\psi\left(\frac{1}{q}\right)$, $\psi\left(\frac{2}{q}\right)$, ... $\psi\left(\frac{q-1}{q}\right)$, viz.

$$\left.\begin{array}{l}
c_1\psi_1+c_2\psi_2+\dots+\quad c_p\psi_p+\dots+\quad c_{q-p}\psi_{q-p}+\dots+\quad c_{q-1}\psi_{q-1}=\frac{q}{2}L_1+\gamma,\\
c_2\psi_1+c_4\psi_2+\dots+\quad c_{2p}\psi_p+\dots+\quad c_{2(q-p)}\psi_{q-p}+\dots+\quad c_{2(q-1)}\psi_{q-1}=\frac{q}{2}L_2+\gamma,\\
c_3\psi_1+c_6\psi_2+\dots+\quad c_{3p}\psi_p+\dots+\quad c_{3(q-p)}\psi_{q-p}+\dots+\quad c_{3(q-1)}\psi_{q-1}=\frac{q}{2}L_3+\gamma,\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
c_{q-1}\psi_1+\quad\dots+c_{(q-1)p}\psi_p+\dots+c_{(q-1)(q-p)}\psi_{q-p}+\dots+c_{(q-1)(q-1)}\psi_{q-1}=\frac{q}{2}L_{q-1}+\gamma;
\end{array}\right\}$$

and in addition we have

$$c_q\psi_1+c_{2q}\psi_2+\dots+\quad c_{qp}\psi_p+\dots+c_{q(q-p)}\psi_{q-p}+\dots+c_{q(q-1)}\psi_{q-1}=-(q-1)\gamma-q\log q,$$

which is merely a case of the identity (*e*) of Art. 952, for the coefficients $\cos q\theta$, $\cos 2q\theta$, etc., each $=1$.

To solve these equations we multiply them, and the identity, respectively by $c_p, c_{2p}, c_{3p}, \dots c_{qp}$.

Now note that $c_\lambda c_\mu + c_{2\lambda} c_{2\mu} + c_{3\lambda} c_{3\mu} + \ldots + c_{q\lambda} c_{q\mu}$ for any integral values of λ, μ (the last term being unity, since $q\theta =$ a multiple of 2π)

$$= \tfrac{1}{2} \sum_1^q c_{(\lambda+\mu)r} + \tfrac{1}{2} \sum_1^q c_{(\lambda-\mu)r};$$

and that each of these sums is zero, except in the two cases $\lambda \pm \mu =$ a multiple of q, and that in the cases we have to consider λ and μ each range in value from 0 to $q-1$. Hence the only cases of this kind are when $\lambda = \mu$ or $\lambda = q - \mu$, and both would happen if $\lambda = \mu = q - \mu$, *i.e.* if q be even, and $\lambda = \mu = \frac{q}{2}$.

If $\lambda = \mu$, $\frac{1}{2} \sum_1^q c_{(\lambda-\mu)r} = \frac{1}{2} \sum_1^q 1 = \frac{q}{2}$; if $\lambda = q - \mu$, $\frac{1}{2} \sum_1^q c_{(\lambda+\mu)r} = \frac{1}{2} \sum_1^r 1 = \frac{q}{2}$,

and when q is even and $\lambda = \mu = \frac{q}{2}$, $\frac{1}{2} \sum_1^q c_{(\lambda+\mu)r} + \frac{1}{2} \sum_1^q c_{(\lambda-\mu)r} = q$.

The latter case will occur when, q being even and therefore $q-1$ odd, there is a middle term in the system of unknowns, viz. $\psi_p = \psi_{q-p} = \psi(\frac{1}{2})$, and the case need not be distinguished from the others. Thus, after multiplication by $c_p, c_{2p}, \ldots c_{qp}$ and addition, the coefficients of all the unknowns vanish except those of ψ_p and ψ_{q-p}, and the coefficients of these terms are each $\frac{q}{2}$; and if $q-1$ be odd and $p = \frac{q}{2}$, all vanish except that of $\psi(\frac{1}{2})$, which is the middle unknown of the series, and the coefficient of this term will be q.

And on the right-hand side we have

$$\frac{q}{2}(c_p L_1 + c_{2p} L_2 + \ldots + c_{(q-1)p} L_{q-1}) + \gamma(c_p + c_{2p} + \ldots + c_{qp}) - q\gamma c_{qp} - q \log q \,.\, c_{qp}$$
$$= \frac{q}{2}(c_p L_1 + c_{2p} L_2 + \ldots + c_{(q-1)p} L_{q-1}) - q\gamma - q \log q.$$

In the bracket, terms equidistant from the ends pair, but if q be even there will be an unpaired term left in the middle of the series. This term is $\frac{q}{2} \cos \frac{q}{2} p\theta \log 4 \sin^2 \frac{q\theta}{4}$ which reduces, since $q\theta = 2\pi$, to $q(-1)^p \log 2$.

Hence the right-hand side becomes

$$q\left(c_p L_1 + c_{2p} L_2 + \ldots + c_{\frac{q-1}{2}p} L_{\frac{q-1}{2}}\right) - q\gamma - q \log q \quad (q \text{ odd}),$$

or $$q\left(c_p L_1 + c_{2p} L_2 + \ldots + c_{\frac{q-2}{2}p} L_{\frac{q-2}{2}}\right) - q\gamma - q \log q + q(-1)^p \log 2 \quad (q \text{ even}).$$

We thus have

$$\psi\left(1 - \frac{p}{q}\right) + \psi\left(\frac{p}{q}\right) = 2\left\{\sum_1^{(q-1)/2} c_{rp} L_r - \gamma - \log q\right\} \quad (q \text{ odd}),$$

or $$= 2\left\{\sum_1^{(q-2)/2} c_{rp} L_r - \gamma - \log q + (-1)^p \log 2\right\} \quad (q \text{ even}),$$

and this, as pointed out above with

$$\psi\left(1 - \frac{p}{q}\right) - \psi\left(\frac{p}{q}\right) = \pi \cot \frac{p}{q}\pi,$$

will enable us by addition and subtraction to obtain both

$$\psi\left(1-\frac{p}{q}\right) \text{ and } \psi\left(\frac{p}{q}\right)$$

for any integral values of p and q ($p<q$).

It will be observed that these theorems give the tangents of the slopes of the curve $y=\log\Gamma(x)$ at equal distances on opposite sides of the ordinate at $x=0\cdot5$.

Ex. If $p=1$, $q=3$,

$$\psi(\tfrac{2}{3})-\psi(\tfrac{1}{3})=\pi\cot\frac{\pi}{3}=\frac{\pi}{\sqrt{3}},$$

$$\begin{aligned}\psi(\tfrac{2}{3})+\psi(\tfrac{1}{3})&=2\left[-\gamma-\log 3+\cos\frac{2\pi}{3}\log 4\sin^2\frac{\pi}{3}\right]\\&=2[-\gamma-\log 3-\tfrac{1}{2}\log 3]\\&=-2\gamma-3\log 3;\end{aligned}$$

$$\therefore\ \left.\begin{aligned}\psi(\tfrac{2}{3})&=-\gamma-\tfrac{3}{2}\log 3+\frac{\pi}{2\sqrt{3}},\\ \psi(\tfrac{1}{3})&=-\gamma-\tfrac{3}{2}\log 3-\frac{\pi}{2\sqrt{3}}.\end{aligned}\right\}$$

955. List of Results.

As the results obtained in the present chapter are very numerous and necessarily scattered over many pages in the gradual development of the theory of Eulerian integrals, it may be convenient to the reader to have the principal facts arrived at collected together for ready reference. A synopsis is therefore added in two groups, the second group referring more particularly to the ψ function, which entails some repetition.

Group I.

1. $B(l, m)=B(m, l)=\int_0^1 x^{l-1}(1-x)^{m-1}\,dx.$ (Art. 857.)

2. If l, m be positive integers, $B(l, m)=\dfrac{(l-1)!\,(m-1)!}{(l+m-1)!}$.

If l only be a positive integer,

$$B(l, m)=\frac{(l-1)!}{m(m+1)\dots(m+l-1)}. \quad \text{(Art. 858.)}$$

3. $B(l, m)=\int_0^\infty \dfrac{x^{m-1}}{(1+x)^{l+m}}dx=\int_0^\infty \dfrac{x^{l-1}}{(1+x)^{l+m}}\,dx.$ (Art. 859 (2).)

4. $\int_b^a (x-b)^{l-1}(a-x)^{m-1}\,dx=(a-b)^{l+m-1}B(l, m).$

(Art. 859 (4).)

5. $\int_0^{\frac{\pi}{2}} \sin^p\theta\cos^q\theta\,d\theta = \dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q}{2}+1\right)}.$ (Arts. 859, 869.)

6. $\int_0^{\frac{\pi}{2}} \dfrac{\sin^{2l-1}\theta\cos^{2m-1}\theta}{(a\cos^2\theta+b\sin^2\theta)^{l+m}}\,d\theta = \dfrac{1}{2a^m b^l}B(l, m).$ (Arts. 859, 869.)

7. $\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}\,dx,\quad \dfrac{\Gamma(n)}{k^n}=\int_0^\infty x^{n-1}e^{-kx}\,dx,$

$$\Gamma(1+x)=\overset{\infty}{\underset{1}{P}}\frac{\left(1+\frac{1}{r}\right)^x}{\left(1+\frac{x}{r}\right)},\quad \Pi(x)=Lt_{\mu=\infty}\frac{1\,.\,2\,\ldots\,\mu}{(n+1)(n+2)\ldots(n+\mu)}\mu^x,$$

(Arts. 854, 864, 874, 889.)

8. $\Gamma(n+1)=n\Gamma(n)=\Pi(n).$
$\Pi(n+1)=(n+1)\Pi(n).$ (Arts. 860, 890.)

9. $\Gamma(\tfrac{1}{2})=\sqrt{\pi}=\Pi(-\tfrac{1}{2}).$ (Arts. 864, 882.)

10. $\Gamma(x)\Gamma(1-x)=\pi\operatorname{cosec}x\pi=\Pi(-x)\Pi(x-1).$
$\Gamma(1+x)\Gamma(1-x)=x\pi\operatorname{cosec}x\pi.$ (Arts. 872, 893.)

11. $\int_0^\infty \dfrac{x^{l-1}}{1+x}\,dx=\dfrac{\pi}{\sin l\pi}\quad (0<l<1).$ (Art. 871.)

12. $\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right)=\dfrac{(2\pi)^{\frac{n-1}{2}}}{n^{\frac{1}{2}}}.$ (Art. 873.)

13. $n^{nx}\Gamma(x)\Gamma\left(x+\frac{1}{n}\right)\Gamma\left(x+\frac{2}{n}\right)\ldots\Gamma\left(x+\frac{n-1}{n}\right)=\Gamma(nx)(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}},$

$$\Gamma(x)\Gamma(x+\tfrac{1}{2})=\frac{\pi^{\frac{1}{2}}}{2^{2x-1}}\Gamma(2x),\quad \Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+2}{2}\right)=\frac{\pi^{\frac{1}{2}}}{2^p}\Gamma(p+1).$$

(Arts. 903, 905.)

14. $Lt_{n=\infty}\dfrac{1\,.\,2\,.\,3\,\ldots\,n}{\sqrt{2n\pi}\,n^n e^{-n}}=1.$ (Art. 877).

15. $\dfrac{\Gamma(n+1)}{\sqrt{2n\pi}\,n^n e^{-n}}=\sum_0^\infty\dfrac{A_{2p+1}}{2^p p\,!}\,\dfrac{1}{n^p}.$ (Art. 884.)

16. $\gamma=0{\cdot}57721566\ldots=Lt_{n=\infty}\left(\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n}-\log n\right).$

(Arts. 897, 917.)

17. $\int_x^{x+n}\log\Gamma(x)\,dx=\log\left[\dfrac{x^x(x+1)^{x+1}\ldots(x+n-1)^{x+n-1}(2\pi)^{\frac{n}{2}}}{e^{nx+\frac{(n-1)n}{2}}}\right].$

(Art. 910.)

18. $\dfrac{d}{dx}\log\Gamma(x) = Lt_{n=\infty}\left[\log n - \dfrac{1}{x} - \dfrac{1}{x+1} - \dfrac{1}{x+2} - \ldots - \dfrac{1}{x+n-1}\right]$

$= -\gamma + \left(\dfrac{1}{1} - \dfrac{1}{x}\right) + \left(\dfrac{1}{2} - \dfrac{1}{x+1}\right) + \ldots \textit{ ad inf.}$ (Art. 911 (5).)

19. $\dfrac{d^2}{dx^2}\log\Gamma(x) = \dfrac{1}{x^2} + \dfrac{1}{(x+1)^2} + \dfrac{1}{(x+2)^2} + \ldots \textit{ ad inf.}$ (Art. 911 (1).)

20. $Lt_{n=\infty}\left(\dfrac{\Gamma'(n)}{\Gamma(n)} - \log n\right) = 0.$ (Art. 911 (3).)

21. $\log\Gamma(1+x) = -\gamma x + S_2\dfrac{x^2}{2} - S_3\dfrac{x^3}{3} + S_4\dfrac{x^4}{4} - \ldots.$ (Arts. 911, 916.)

22. $\log\Gamma(1+x) = \tfrac{1}{2}\log\dfrac{x\pi}{\sin x\pi} - \tanh^{-1}x + (1-\gamma)x$

$-(S_3-1)\dfrac{x^3}{3} - (S_5-1)\dfrac{x^5}{5} - \ldots.$ (Art. 919.)

23. Min. ordinate of $y = \Gamma(x)$ is at $x = 1{\cdot}4616\ldots.$ (Art. 922).

24. $\log\Gamma(x) = \displaystyle\int_0^\infty\left[(x-1)e^{-\beta} - \frac{e^{-\beta} - e^{-x\beta}}{1-e^{-\beta}}\right]\frac{d\beta}{\beta}.$ (Art. 930 (6).)

25. $\dfrac{d}{dx}\log\Gamma(x) = \displaystyle\int_0^\infty\left(\frac{e^{-\beta}}{\beta} - \frac{e^{-x\beta}}{1-e^{-\beta}}\right)d\beta;$ (Art. 925.)

also $= \displaystyle\int_0^\infty\left\{e^{-\beta} - \frac{1}{(1+\beta)^x}\right\}\frac{d\beta}{\beta}.$ (Art. 930 (3).)

26. $\dfrac{d^n}{dx^n}\log\Gamma(x) = (-1)^n\displaystyle\int_0^\infty\frac{\beta^{n-1}e^{-x\beta}}{1-e^{-\beta}}\,d\beta \;(n \nless 2).$ (Art. 930 (9).)

27. $S_p = \dfrac{1}{1^p} + \dfrac{1}{2^p} + \dfrac{1}{3^p} + \ldots = \dfrac{1}{2\Gamma(p)}\displaystyle\int_0^\infty\frac{\beta^{p-1}e^{-\frac{\beta}{2}}}{\sinh\frac{\beta}{2}}\,d\beta,$

$s_p = \dfrac{1}{1^p} + \dfrac{1}{3^p} + \dfrac{1}{5^p} + \ldots = \dfrac{1}{2\Gamma(p)}\displaystyle\int_0^\infty\frac{\beta^{p-1}}{\sinh\beta}\,d\beta.$

$s_p' = \dfrac{1}{1^p} - \dfrac{1}{3^p} + \dfrac{1}{5^p} - \ldots = \dfrac{1}{2\Gamma(p)}\displaystyle\int_0^\infty\frac{\beta^{p-1}}{\cosh\beta}\,d\beta.$ (Arts. 928, 929.)

28. $B_{2n-1} = \dfrac{2n}{(2^{2n}-1)\pi^{2n}}\displaystyle\int_0^\infty\frac{\beta^{2n-1}}{\sinh\beta}\,d\beta = 2n\int_0^\infty\frac{\beta^{2n-1}e^{-\pi\beta}}{\sinh\pi\beta}\,d\beta,$

$E_{2n} = \left(\dfrac{2}{\pi}\right)^{2n+1}\displaystyle\int_0^\infty\frac{\beta^{2n}}{\cosh\beta}\,d\beta.$ (Art. 929.)

29. $\Sigma u_x = C + \int u_x\,dx - \frac{1}{2}u_x + \frac{B_1}{2\,!}\frac{du_x}{dx} - \frac{B_3}{4\,!}\frac{d^3u_x}{dx^3} + \frac{B_5}{6\,!}\frac{d^5u_x}{dx^5} - \dots$.

(Art. 931.)

30. $$\frac{d}{dx}\log\Gamma(x+1) = \log x + \frac{1}{2x} - \frac{B_1}{2x^2} + \frac{B_3}{4x^4} - \dots$$
$$-(-1)^{n-1}\frac{B_{2n-1}}{2n}\frac{1}{x^{2n}} - (-1)^n\frac{B_{2n+1}}{(2n+2)}\frac{1}{x^{2n+2}}\Theta \quad (0<\Theta<1).$$

(Art. 940.)

31. $$\log\Gamma(x+1) = \tfrac{1}{2}\log 2\pi + (x+\tfrac{1}{2})\log x - x + \frac{B_1}{1\,.\,2}\frac{1}{x} - \frac{B_3}{3\,.\,4}\frac{1}{x^3} + \dots$$
$$+(-1)^{n-1}\frac{B_{2n-1}}{(2n-1)2n}\frac{1}{x^{2n-1}} + (-1)^n\frac{B_{2n+1}}{(2n+1)(2n+2)}\frac{1}{x^{2n+1}}\Theta$$

$(0<\Theta<1)$. (Art. 940.)

32. $$\frac{\Gamma(x+1)}{\sqrt{2\pi x}\,x^xe^{-x}} = 1 + \frac{1}{12x} + \frac{1}{2(12x)^2} - \frac{139}{30(12x)^3} - \frac{571}{120(12x)^4} + \dots.$$

See also No. 15. (Art. 942.)

956. II. Group of ψ Formulae.

Since the ψ-function, viz. $\psi(x) = \frac{d}{dx}\log\Gamma(x)$, is a very interesting function, and very useful in itself, we gather together the principal results which refer to this function in particular.

1. $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = Lt_{n=\infty}\left[\log n - \frac{1}{x} - \frac{1}{x+1} - \dots - \frac{1}{x+n-1}\right]$.

(Art. 911.)

2. $\psi(0) = -\infty$, $\psi(1) = -\gamma$, $\psi(1{\cdot}4616\dots) = 0$, $\psi(\infty) = \infty$.

(Arts. 911 (3), 922, 923.)

3. $\psi(x) - \psi(1) = \left(\frac{1}{1} - \frac{1}{x}\right) + \left(\frac{1}{2} - \frac{1}{x+1}\right) + \left(\frac{1}{3} - \frac{1}{x+2}\right) + \dots$.

(Art. 911.)

4. $\psi'(x) = \frac{1}{x^2} + \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \dots$. (Art. 911.)

5. $\psi(x) = \int_0^\infty \left(\frac{e^{-\beta}}{\beta} - \frac{e^{-x\beta}}{1-e^{-\beta}}\right)d\beta = \int_0^\infty \left\{e^{-\beta} - \frac{1}{(1+\beta)^x}\right\}\frac{d\beta}{\beta}$.

(Arts. 925, 930 (3) and (7).)

6. $\psi'(x)=\int_0^\infty \frac{\beta e^{-x\beta}}{1-e^{-\beta}}\,d\beta.$ (Art. 930 (8).)

7. $\psi(x+1)=\log x+\frac{1}{2x}-\frac{B_1}{2x^2}+\frac{B_3}{4x^4}-\ldots.$ (Art. 940.)

8. $\psi'(x+1)=\frac{1}{x}-\frac{1}{2x^2}+\frac{B_1}{x^3}-\frac{B_3}{x^5}+\ldots.$ (Art. 940.)

9. $\psi(x)+\gamma=\int_0^\infty \frac{e^{-\beta}-e^{-x\beta}}{1-e^{-\beta}}\,d\beta.$ (Art. 944.)

10. $\psi(x)+\gamma=\int_0^1 \frac{1-t^{x-1}}{1-t}\,dt$ (x integral). (Art. 944.)

11. $\psi(1+a)-\psi(1+b)=\int_0^1 \frac{t^b-t^a}{1-t}\,dt.$ (From 10.)

12. $\Delta\psi(a+x)=\frac{1}{a+x}.$ (Art. 950.)

13. $\psi(x+1)-\psi(x)=\frac{1}{x}.$ (Art. 952.)

14. $\psi(1-x)-\psi(x)=\pi\cot x\pi.$ (Art. 952.)

15. $\psi(\frac{1}{2}+x)-\psi(\frac{1}{2}-x)=\pi\tan x\pi.$ (From 14.)

16. $\psi(x)+\psi(\frac{1}{2}+x)=2\psi(2x)-2\log 2.$ (Art. 952.)

17. $\psi(x)-\frac{1}{2}\psi\left(\frac{1-x}{2}\right)=\frac{1}{2}\psi\left(\frac{x}{2}\right)+\log 2+\frac{\pi}{2}\tan\frac{x\pi}{2}.$ (Art. 952.)

18. $\psi(x)+\psi\left(x+\frac{1}{n}\right)+\psi\left(x+\frac{2}{n}\right)+\ldots+\psi\left(x+\frac{n-1}{n}\right)$

$=n\psi(nx)-n\log n.$ (Art. 952.)

19. $\psi(a+x)=\psi(a)+\frac{x}{a}-\frac{1}{2}\frac{x(x-1)}{a(a+1)}+\frac{1}{3}\frac{x(x-1)(x-2)}{a(a+1)(a+2)}-\text{etc.}$ (Art. 950.)

20. $\psi\left(1-\frac{p}{q}\right)+\psi\left(\frac{p}{q}\right)$

$=2\left[\psi(1)-\log q+\sum_1^{\frac{q-1}{2}}\cos\frac{2rp\pi}{q}\log 4\sin^2\frac{r\pi}{q}\right]$ (q odd) (Art. 953.)

$=2\left[\psi(1)-\log q+\sum_1^{\frac{q-2}{2}}\cos\frac{2rp\pi}{q}\log 4\sin^2\frac{r\pi}{q}\right]+(-1)^p 2\log 2$ (q even).

957. Table of Values of $S_p=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots$ *ad inf.* up to $p=35$, which is the last in which the tenth decimal place is affected; all remaining ones to this approximation may be regarded as $=1$. (De Morgan, *D.C.*, p. 554.)

p	S_p to sixteen places of decimals.
1	0·57721 56649 01532 9... + log ∞ (Euler's Const. + ∞)
2	1·64493 40668 48226 4
3	1·20205 69031 59594 3
4	1·08232 32337 11138 2
5	1·03692 77551 43370 0
6	1·01734 30619 84449 1
7	1·00834 92773 81922 7
8	1·00407 73561 97944 3
9	1·00200 83928 26082 2
10	1·00099 45751 27818 0
11	1·00049 41886 04119 4
12	1·00024 60865 53308 0
13	1·00012 27133 47578 5
14	1·00006 12481 35058 7
15	1·00003 05882 36307 0
16	1·00001 52822 59408 6
17	1·00000 76371 97637 9
18	1·00000 38172 93265 0
19	1·00000 19082 12716 6
20	1·00000 09539 62033 9
21	1·00000 04769 32986 8
22	1·00000 02384 50502 7
23	1·00000 01192 19926 0
24	1·00000 00596 08189 1
25	1·00000 00298 03503 5
26	1·00000 00149 01554 8
27	1·00000 00074 50711 8
28	1·00000 00037 25334 0
29	1·00000 00018 62659 7
30	1·00000 00009 31327 4
31	1·00000 00004 65662 9
32	1·00000 00002 32831 2
33	1·00000 00001 16415 5
34	1·00000 00000 58207 7
35	1·00000 00000 29103 8

The sixteenth decimal place is not always the sixteenth occurring, but the nearest in consideration of terms to follow, *e.g.* γ has for its 16th, 17th, etc., ... figures 8606 ....

$\log_e 10 = 2{\cdot}30258\ 50929\ 94045\ 6840\ \ldots$

Euler's Const. $=\gamma=0{\cdot}57721\ 56649\ 01532\ 8606\ \ldots$

$\mu=0{\cdot}43429\ 44819\ \ldots$

PROBLEMS.

1. Show that (i) $\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})=\frac{2\pi}{\sqrt{3}}$; (ii) $\Gamma(\frac{1}{3})\Gamma(\frac{5}{6})=\pi^{\frac{1}{2}}2^{\frac{1}{3}}\Gamma(\frac{2}{3})$.

2. Show that $3^{\frac{1}{2}}\{\Gamma(\frac{1}{3})\}^2=\pi^{\frac{1}{2}}2^{\frac{1}{3}}\Gamma(\frac{1}{6})$.

3. Show that $\Gamma(\cdot 1)\Gamma(\cdot 2)\Gamma(\cdot 3)\ldots\Gamma(\cdot 9)=\frac{(2\pi)^{\frac{9}{2}}}{\sqrt{10}}$.

4. Show that $2^n \Gamma(n+\frac{1}{2}) = 1 \,.\, 3 \,.\, 5 \ldots (2n-1)\sqrt{\pi}$, where n is a positive integer. [Oxford II. P., 1888.]

5. Show that $\Gamma(\frac{3}{2}-x)\,\Gamma(\frac{3}{2}+x) = (\frac{1}{4}-x^2)\,\pi \sec \pi x$, provided
$$-1 < 2x < 1.$$

6. Show by means of the transformation $xy = u$, $y = u + v$, that
$$\int_0^1 \int_0^1 \frac{(1-x)^{m-1} y^m (1-y)^{n-1}}{(1-xy)^{m+n-1}}\, dx\, dy = B(m, n).$$
[Coll. γ, 1901.]

7. By means of the integral $\int_0^1 x^{m-1}(1-x^a)^n dx$, prove that
$$\frac{1}{(m)\, n!} - \frac{1}{(m+a)(n-1)!\, 1!} + \frac{1}{(m+2a)(n-2)!\, 2!} - \ldots + \frac{(-1)^n}{(m+na)\, n!}$$
$$= \frac{a^n}{m(m+a)(m+2a)\ldots(m+na)}.$$
[St. John's, 1884.]

Show that this integral may be expressed as $\dfrac{n!\,\Gamma\left(\dfrac{m}{a}\right)}{a\Gamma\left(\dfrac{m}{a}+n+1\right)}$.

8. Show that the product of the series
$$1 + \frac{1}{2}\cdot\frac{1}{17} + \frac{1\,.\,3}{2\,.\,4}\cdot\frac{1}{33} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{1}{49} + \text{etc.}$$
and $$\frac{1}{9} + \frac{1}{2}\cdot\frac{1}{25} + \frac{1\,.\,3}{2\,.\,4}\cdot\frac{1}{41} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{1}{57} + \text{etc.} \quad \text{is } \frac{\pi}{16}.$$
[Colleges α, 1883.]

9. Prove by the substitution $x^2 = \xi$ that
$$\int_0^\infty e^{-x^2} x^{2n} dx = \sqrt{\pi}\,\frac{1\,.\,3\,.\,5\ldots(2n-1)}{2\,.\,4\,.\,6\ldots 2n}\int_0^\infty e^{-x^2} x^{2n+1} dx,$$
where n is a positive integer.

[See also Art. 223 (5).] [Colleges α, 1890.]

10. Show that if K be any positive constant,
$$\int_0^K \int_0^{K-x} e^{-x-y} x^{l-1} y^{m-1}\, dx\, dy = \int_0^1 (1-v)^{l-1} v^{m-1} dv \,.\, \int_0^K e^{-u} u^{l+m-1} du,$$
and by proceeding to a limit express $B(l, m)$ in terms of Gamma functions. [Oxf. II. P., 1902.]

11. Show that the sum of the series
$$\frac{1}{n+1} + m\frac{1}{n+2} + \frac{m(m+1)}{2!}\,\frac{1}{n+3} + \frac{m(m+1)(m+2)}{3!}\,\frac{1}{n+4} + \ldots$$
is $$\Gamma(n+1)\,\Gamma(1-m)/\Gamma(n-m+2),$$
where $n > -1$, and $m < 1$. [Coll. γ, 1899.]

12. From the value in Gamma functions of $\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta\, d\theta$, show that

$$2^p\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+2}{2}\right)=\sqrt{\pi}\,\Gamma(p+1)$$

for all real values of p. [TRINITY, 1886.]

13. Prove that $\int_5^\infty e^{-x^2}dx = e^{-25}\times 0{\cdot}09811$ nearly. [TRINITY, 1896.]

14. Prove that

$$\Gamma(n)=\frac{1}{n}\frac{\left(1+\frac{1}{1}\right)^n}{\left(1+\frac{n}{1}\right)}\cdot\frac{\left(1+\frac{1}{2}\right)^n}{\left(1+\frac{n}{2}\right)}\cdot\frac{\left(1+\frac{1}{3}\right)^n}{\left(1+\frac{n}{3}\right)}\ldots \text{to } \infty$$

[OXFORD II. P., 1888.]

and $$\Gamma(n+1)=\prod_{r=1}^{r=\infty}\frac{\left(1+\frac{1}{r}\right)^n}{\left(1+\frac{n}{r}\right)}.$$

[OXFORD II. P., 1903.]

15. Show that, when x is positive,

$$2^{2x-1}B(x,x)=\sqrt{\pi}\frac{\Gamma(x)}{\Gamma(x+\frac{1}{2})}=\sum_{n=0}^{\infty}\frac{2n\,!}{2^{2n}\,n\,!\,n\,!}\,\frac{1}{x+n}.$$

[MATH. TRIP., 1897.]

16. Prove that, if x be positive,

$$x\left(\frac{1+x}{2}\right)^{\frac{1}{2}}\left(\frac{2+x}{3}\right)^{\frac{1.3}{2.4}}\left(\frac{3+x}{4}\right)^{\frac{1.3.5}{2.4.6}}\ldots \text{to } \infty = e^{\sqrt{\pi}\int_1^x \frac{\Gamma(x)}{\Gamma(x+\frac{1}{2})}dx}.$$

[MATH. TRIPOS, 1897.]

17. Show that, when x is a real positive quantity not greater than unity,

$$c\,\Gamma(x)=f(x)+\sum_{n=0}^{\infty}\frac{1}{x(x+1)(x+2)\ldots(x+n)}$$

where $f(x)$ is a function of x not greater than unity.

[MATH. TRIPOS, 1897.]

18. If n lie between zero and unity, prove that

$$\int_0^{\frac{\pi}{2}}(\tan x)^n\,dx=\frac{1}{2}\frac{\pi}{\sin\frac{n+1}{2}\pi}.$$

[COLL. α, 1890.]

19. Show that the perimeter of a loop of the curve $r^n=a^n\cos n\theta$ is

$$\frac{a}{n}2^{\frac{1}{n}-1}\left(\Gamma\frac{1}{2n}\right)^2\Big/\left(\Gamma\frac{1}{n}\right).$$

20. Show that if x, y be a point on the ellipse $x^2/a^2+y^2/b^2=1$, and $2r$ be the conjugate diameter, and the integral be taken round the whole perimeter, then

$$\int \frac{x^l y^l}{r^{2l+3}}\,ds = \frac{2\left\{\Gamma\left(\frac{l+1}{2}\right)\right\}^2}{\Gamma(l+1)}\cdot\frac{1}{ab}.$$ [Colleges, 1892.]

21. Express in Gamma functions

$$\int_0^1 (1-x^n)^{\frac{1}{n}}\,dx.$$ [Trinity, 1896.]

22. Express in Gamma functions the area of the curve $yc^x=ax^c$ ($c>0$) for positive values of x (0 to ∞), also the volume generated by its revolution round the axis of x. [St. John's, 1883.]

23. If $2\sin n\pi\,\Gamma(n)\,\phi(n)=(2\pi)^n\phi(1-n)\{(-\iota)^{n-1}+\iota^{n-1}\}$ where $\iota=\sqrt{-1}$ and $\phi(n)$ is some function of n, prove that

$$\Gamma\left(\frac{n}{2}\right)\pi^{-\frac{n}{2}}\phi(n)$$

remains unaltered when $1-n$ is written for n. [Colleges α, 1881.]

24. Prove that

$$\int_a^\infty e^{-t^2}dt=\frac{e^{-a^2}}{2a}\left[\frac{1}{1+}\,\frac{q}{1+}\,\frac{2q}{1+}\,\frac{3q}{1+}\,\frac{4q}{1+}\text{ etc.}\right],\quad\text{where}\quad q=\frac{1}{2a^2}.$$

[De Morgan, *Diff. Cal.*, p. 591.]

25. Prove that

$$\int_v^\infty e^{-v}\log v\,dv=e^{-v}\left[\log v+\frac{v^{-1}}{1+}\,\frac{v^{-1}}{1+}\,\frac{v^{-1}}{1+}\,\frac{2v^{-1}}{1+}\,\frac{2v^{-1}}{1+}\,\frac{3v^{-1}}{1+}\,\frac{3v^{-1}}{1+}\text{ etc.}\right].$$

[De Morgan, p. 591.]

26. Prove that

$$\frac{d}{dx}\log\Gamma(1+x)=-\gamma+x-\frac{1}{2}\,\frac{x(x-1)}{1\,.\,2}+\frac{1}{3}\,\frac{x(x-1)(x-2)}{1\,.\,2\,.\,3}-\ldots.$$

[De Morgan, p. 593.]

27. If $\phi(x)=\dfrac{d}{dx}\log\Gamma(1+x)$ and x be a positive integer, show that

$$\phi(x)=\phi(0)+1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{x}.$$

Prove further that

$$\phi(0)=\int_0^\infty e^{-x}\log x\,dx,$$

and has a finite value. [I. C. S., 1898.]

28. If $(1+x)^n = 1 + A_1 x + A_2 x^2 + \dots$, where n is any positive quantity, prove that

$$1 + A_1{}^2 + A_2{}^2 + \dots = \frac{2^{2n}}{\sqrt{\pi}} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}.$$

[MATH. TRIPOS, 1895.]

29. Prove that if

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^n(\theta x),$$

$$\int_0^\infty \frac{f^n(\theta x)}{x^r} dx = \frac{\Gamma(n+1)\,\Gamma(r)}{\Gamma(n+r)} \int_0^\infty \frac{f^n(x)}{x^r} dx,$$

r being any positive quantity.

[If $r > 1$ both integrals generally $= \infty$.]

[WOLSTENHOLME, *Educ. Times.*]

30. Prove by changing the order of integration or otherwise that

$$\int_0^x \frac{dy}{\sqrt{x-y}} \int_0^y \frac{f'(\xi)\, d\xi}{\sqrt{y-\xi}} = \pi \{f(x) - f(0)\}.$$

[MATH. TRIPOS, 1875.]

31. Show that

$$\int \frac{dx}{1+x^n} = \frac{x}{1+} \; \frac{\frac{x^n}{n+1}}{1+} \; \frac{\frac{n^2 x^n}{(n+1)(2n+1)}}{1+} \; \frac{\frac{(n+1)^2 x^n}{(2n+1)(3n+1)}}{1+} \; \frac{\frac{(2n)^2 x^n}{(3n+1)(4n+1)}}{1+} \; \frac{\frac{(2n+1)^2 x^n}{(4n+1)(5n+1)}}{1+} \text{ etc.}$$

[LACROIX, *Calc. Diff.*, vol. ii., p. 292.]

Deduce expressions for $\log \overline{1+x}$ and $\tan^{-1} x$ as continued fractions.

32. Prove that

$$\overset{\infty}{\underset{1}{P}} \left(1 + \frac{x^3}{n^3}\right) = x^{-3} / \Gamma(x)\, \Gamma(x\omega)\, \Gamma(x\omega^2), \text{ where } \omega = e^{\frac{2\iota\pi}{3}}.$$

[ST. JOHN'S, 1891.]

33. Evaluate the modulus of $\Gamma(\frac{1}{2} + \sqrt{-1}\,a)$. [SMITH'S PRIZE, 1875.]

34. Show that for very large integral values of n, $\Gamma(n+\frac{1}{2})$ is very nearly the geometric mean between $\Gamma(n)$ and $\Gamma(n+1)$.

[OXFORD, 1892.]

35. If b be a large whole number, show that, provided $x > -1$,

$$(x+1)(x+2)\dots(x+b) = b^x \frac{\Gamma(b+1)}{\Gamma(x+1)}, \text{ very nearly.}$$

[DE MORGAN, *Diff. Calc.*, p. 585.]

36. Writing $\phi(x) \equiv e^x . x!/\sqrt{2\pi}\, x^{x+\frac{1}{2}}$, prove by the aid of Wallis' theorem that $\phi(2x) = [\phi(x)]^2$ when x is large.

Then show that for any value of x,

(a) $\dfrac{\phi(x)}{\phi(x+1)} = e^{-1+(x+\frac{1}{2})\log\left(1+\frac{1}{x}\right)}$.

(b) $\log \dfrac{\phi(x)}{\phi(x+1)} = \dfrac{1}{12x^2} - \dfrac{1}{12x^3} + \dfrac{3}{40x^4} - \ldots + \dfrac{(n-1)}{2n(n+1)} \dfrac{(-1)^n}{x^n} + \ldots$.

(c) $\dfrac{\phi(x)}{\phi(x+1)} < e^{\frac{1}{12x^2}}$. (d) $\log \dfrac{\phi(x)}{\phi(x+1)} < \dfrac{1}{12x(x+1)}$.

(e) $\dfrac{\phi(x)}{\phi(2x)} = \dfrac{\theta_0}{x^2} + \dfrac{\theta_1}{(x+1)^2} + \dfrac{\theta_2}{(x+2)^2} + \ldots + \dfrac{\theta_{x-1}}{(2x-1)^2}$,

where $\theta_0, \theta_1, \theta_2, \ldots$ are numbers between 0 and $\frac{1}{12}$.

(f) $\dfrac{\phi(x)}{\phi(2x)} = e^{\frac{\theta}{x}} \quad (0 < \theta < \frac{1}{12})$,

and finally deduce Stirling's theorem,

$$1 . 2 . 3 \ldots x = \sqrt{2\pi}\, e^{-x} x^{x+\frac{1}{2}} (1 + \epsilon_x),$$

where ϵ_x denotes a positive quantity which vanishes when $x = \infty$.

[Serret, *Calc. Intég.*, p. 207.]

37. Show that, if x be a whole number,

$$\log \Gamma(x+1) = \tfrac{1}{2}\log 2\pi - x + (x + \tfrac{1}{2})\log x + \sum_{m=0}^{m=\infty} \left[\left(x + m + \frac{1}{2}\right) \log\left(1 + \frac{1}{x+m}\right) - 1 \right].$$

[Gudermann.]

38. Show that

$$1 . 2 . 3 \ldots x > \sqrt{2\pi x}\, x^x e^{-x} \quad \text{and} \quad < \sqrt{2\pi x}\, x^x e^{-x+\frac{1}{12x}}$$

when x is large. [Serret, *Calc. Intég.*, p. 213.]

39. Writing

$$\phi(x) = Lt_{m=\infty} \frac{(m!)^n n^{mn+1}}{(mn)!\, m^{\frac{n-1}{2}}}, \quad \psi(m) = \frac{m!}{m^{m+\frac{1}{2}}}, \quad \text{and} \quad u_n = \sqrt{n}\, Lt \frac{[\psi(m)]^n}{\psi(mn)},$$

prove that

$$u_n = u_2^{\,n-1}, \quad u_2 = \frac{\phi(2)}{\sqrt{2}} = \sqrt{2\pi}, \quad \phi(n) = n^{\frac{1}{2}} (2\pi)^{\frac{n-1}{2}}.$$

Hence deduce Gauss' theorem,

$$n^{nx}\Gamma(x)\Gamma\left(x + \frac{1}{n}\right) \ldots \Gamma\left(x + \frac{n-1}{n}\right) = (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2}} \Gamma(nx).$$

[Serret, *Calc. Intégral*, p. 190.]

40. Prove that

$$\sum_1^\infty \left\{ \frac{1}{(x+r)^n} - \frac{1}{r^n} \right\} = \frac{(-1)^n}{\Gamma(n)} \int_0^1 \frac{1-v^x}{1-v} (\log v)^{n-1}\, dv.$$

[Cf. DE MORGAN, *Diff. C.*, p. 594.]

41. Prove that

$$\frac{d}{dx} \log \Gamma(x) = \log x + \int_0^\infty \left\{ e^{-xt} - (1+t)^{-x} \right\} \frac{dt}{t},$$

and that $$\frac{1}{\Gamma(x+1)} = e^{Cx} \prod_{n=1}^{\infty} \left[\left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}} \right],$$

where C is a certain constant. [MATH. TRIPOS, Pt. II., 1915.]

42. If the binomial expansion for a positive index be written

$$(a+b)^n = \Sigma \binom{n}{r} a^r b^{n-r},$$

show that $$\Sigma \binom{n}{r} B(n-r+1,\ r+1) = 1.$$

Prove also that

$$\frac{2\pi}{3\sqrt{3}} = 1 + \frac{(1!)^2}{3!} + \frac{(2!)^2}{5!} + \frac{(3!)^2}{7!} + \frac{(4!)^2}{9!} + \dots.$$

43. Show that (1000)! lies between

$$4{\cdot}02387 \times 10^{2567} \quad \text{and} \quad 4{\cdot}02388 \times 10^{2567},$$

and is a number with 2568 figures in the ordinary system of numeration, its logarithm being 2567·6046442….

[COURNOT, *Théorie des Fonctions*, vol. ii., p. 472.]

44. Show that if

$$\log \Gamma(x+1) = \log \sqrt{2\pi} + (x + \tfrac{1}{2}) \log x - x + \frac{B_1}{1 \,.\, 2x} - \frac{B_3}{3 \,.\, 4x^3} + \dots$$
$$+ (-1)^{n-1} \frac{B_{2n-1}}{(2n-1)(2n)x^{2n-1}} + (-1)^n \frac{R}{(2n+2)!},$$

then $$R = \int_0^\infty e^{-\alpha x} \alpha^{2n} f^{2n+2}(\theta\alpha)\, d\alpha,$$

where $f(\alpha) \equiv \dfrac{\alpha}{e^\alpha - 1}$ and θ is a positive proper fraction.

[LIOUVILLE, *Journal de Mathématiques*, Tom. iv., p. 317.]

If λ_{2n+2} be the maximum numerical value of $f^{2n+2}(\alpha)$ between the limits $\alpha = 0$, $\alpha = \infty$, show that

$$\frac{R}{(2n+2)!} < \frac{\lambda_{2n+2}}{(2n+1)(2n+2)} \frac{1}{x^{2n+1}},$$

and examine the nature of the approximation attained by the omission of all the terms which contain Bernoulli's coefficients.

[LIOUVILLE, *J. de M.*; also COURNOT, *Théorie des Fonctions*, p. 474.]

45. Starting with

$$\log\Gamma(x)=\int_0^\infty\left[(x-1)e^{-\beta}-\frac{e^{-\beta}-e^{-x\beta}}{1-e^{-\beta}}\right]\frac{d\beta}{\beta}$$
$$=\int_0^\infty (P+Qe^{-x\beta})\,d\beta,\text{ say,}$$

and putting R for the two terms with negative indices in the development of Q in ascending powers of β, namely $\dfrac{1}{\beta^2}+\dfrac{1}{2\beta}$, let

$$F(x)=\int_0^\infty (P+Re^{-x\beta})\,d\beta \quad\text{and}\quad \varpi(x)=\int_0^\infty (Q-R)e^{-x\beta}d\beta.$$

Then show that

(1) $\varpi(\tfrac12)=\tfrac12\log\dfrac{e}{2}$. (2) $F(\tfrac12)=\tfrac12\log\dfrac{2\pi}{e}$.

(3) $F(x)-F(\tfrac12)=\tfrac12-x+(x-\tfrac12)\log x$. (4) $\Gamma(x)=e^{-x}x^{x-\frac12}\sqrt{2\pi}\,e^{\varpi(x)}$.

(5) That when x is large $e^{\varpi(x)}$ differs but little from unity.

(6) $$\log\Gamma(x+1)=\tfrac12\log 2\pi+(x+\tfrac12)\log x-x$$
$$+\int_0^\infty\left(\frac{1}{1-e^{-\beta}}-\frac{1}{\beta}-\frac{1}{2}\right)e^{-\beta x}\frac{d\beta}{\beta},\quad\text{and}$$

(7) Deduce the equation,

$$\log\Gamma(x+1)=\tfrac12\log(2\pi)+(x+\tfrac12)\log x-x+\frac{B_1}{1\,.\,2}\frac{1}{x}-\frac{B_3}{3\,.\,4}\frac{1}{x^3}+\dots$$
$$+(-1)^{n-1}\frac{B_{2n-1}}{(2n-1)\,2n}\frac{1}{x^{2n-1}}+(-1)^n\frac{B_{2n+1}}{(2n+1)(2n+2)}\frac{1}{x^{2n+1}}\Theta,$$
$$0<\Theta<1.$$

[Bertrand, *Calc. Intégral*, p. 265.]

46. Show that

(1) $$\gamma=\int_0^\infty\left(\frac{1}{1-e^{-\beta}}-\frac{1}{\beta}\right)e^{-\beta}d\beta.$$

(2) $$\log\Gamma(x+1)=\int_0^\infty\frac{e^{-\beta}}{\beta}\left\{x-\frac{1-e^{-\beta x}}{1-e^{-\beta}}\right\}d\beta.$$

[Todhunter, *Int. Calc.*, p. 392.]

47. If A_r be the acute angle whose tangent is the n^{th} power of the reciprocal of the r^{th} of the prime numbers 2, 3, 5, ..., show that

$$\cos 2A_1\cos 2A_2\cos 2A_3\cos 2A_4\dots\text{ to }\infty=2\frac{B_{2n}}{B_n^2}\frac{\{(2n)!\}^2}{(4n)!},$$

where B_n is the n^{th} number of Bernoulli. [Math. Tripos, 1897.]

48. If $I=\displaystyle\int_0^1\frac{dx}{\sqrt{1-x^3}}$, show that

$$\Gamma(\tfrac16)=\pi^{\frac16}2^{\frac59}3^{\frac58}I^{\frac23},\quad \Gamma(\tfrac13)=\pi^{\frac13}2^{\frac49}3^{\frac18}I^{\frac13},$$
$$\Gamma(\tfrac23)=\pi^{\frac23}2^{\frac59}3^{-\frac18}I^{-\frac13},\quad \Gamma(\tfrac56)=\pi^{\frac56}2^{\frac49}3^{-\frac58}I^{-\frac23}.$$

49. If $I=\int_0^1 \frac{dx}{\sqrt{1-x^5}}$ and $J=\int_0^1 \frac{x\,dx}{\sqrt{1-x^5}}$, show that

$$\Gamma(\tfrac{1}{10})=\pi^{\frac{1}{10}}2^{\frac{11}{25}}5^{\frac{4}{5}}S_1^{-\frac{3}{5}}S_4^{\frac{2}{5}}I^{\frac{1}{5}}J^{\frac{3}{5}}, \qquad \Gamma(\tfrac{2}{10})=\pi^{\frac{1}{5}}2^{\frac{2}{25}}5^{\frac{3}{5}}S_1^{-\frac{1}{5}}S_4^{-\frac{1}{5}}I^{\frac{2}{5}}J^{\frac{1}{5}},$$

$$\Gamma(\tfrac{3}{10})=\pi^{\frac{3}{10}}2^{-\frac{2}{25}}5^{\frac{2}{5}}S_1^{\frac{1}{5}}S_3^{-1}S_4^{\frac{1}{5}}I^{\frac{3}{5}}J^{-\frac{1}{5}}, \quad \Gamma(\tfrac{4}{10})=\pi^{\frac{2}{5}}2^{-\frac{11}{25}}5^{\frac{1}{5}}S_1^{-\frac{2}{5}}S_4^{-\frac{2}{5}}I^{-\frac{1}{5}}J^{\frac{2}{5}},$$

where $S_1=\sin\frac{\pi}{10}$, $S_2=\sin\frac{2\pi}{10}$, $S_3=\sin\frac{3\pi}{10}$, $S_4=\sin\frac{4\pi}{10}$,

and write down the values of $\Gamma(\frac{6}{10})$, $\Gamma(\frac{7}{10})$, $\Gamma(\frac{8}{10})$, $\Gamma(\frac{9}{10})$, in similar form.

50. Show that $\displaystyle\int_0^\infty x^2[\log(1+e^x)-x]\,dx=\frac{7\pi^4}{360}$.

[OXFORD I. P., 1914.]

51. Prove that the volume in the positive octant bounded by the planes $x=0$, $y=0$, $z=h$ and the surface $z/c=x^m/a^m+y^m/b^m$ is equal to

$$abh\left(\frac{h}{c}\right)^{\frac{2}{m}}\frac{\left\{\Gamma\left(\frac{1}{m}\right)\right\}^2}{2(m+2)\,\Gamma\left(\frac{2}{m}\right)}.$$

[MATH. TRIP., PART II., 1913.]

52. Prove that $\displaystyle e^{h\frac{d^2}{dx^2}}\{\phi(x)\}=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-y^2}\phi(x+2y\sqrt{h})\,dy$,

and apply the result to prove that if $1+4hk$ be positive,

$$e^{h\frac{d^2}{dx^2}}\{xe^{-kx^2}\}=\frac{x}{(1+4hk)^{\frac{3}{2}}}e^{-\frac{kx^2}{1+4hk}}.$$

[MATH. TRIP., 1870 (WOLSTENHOLME).]

53. When n is a positive integer, we have evidently

$$1\,.\,2\,.\,3\,\ldots\,2n=2^{2n}.\,1\,.\,2\,\ldots\,n\,.\,\tfrac{1}{2}\,.\,\tfrac{3}{2}\,\ldots\,(n-\tfrac{1}{2});$$

prove that this equation, when expressed by means of the function Γ, is true for any positive value of n. [SIR G. G. STOKES, S. P., 1870.]

54. Prove that the limiting value of

$$2n+1-2\log\frac{(2n+1)^n}{1\,.\,3\,.\,5\,\ldots\,(2n-1)},$$

when n is indefinitely increased, is $\log 2$. [R. P.]

But there is this difference between the functions

$$Lt_{\mu=\infty}\frac{1.2\ldots\mu}{(x+1)(x+2)\ldots(x+\mu)}\mu^x \quad\text{and}\quad \int_0^\infty e^{-v}v^x\,dv,$$

that though they coincide in value for all positive values of x, the former becomes infinite at the values $x=-1$, $x=-2$, $x=-3$, etc., but has finite values for other negative values of x, whilst the definite integral is permanently infinite for all negative values of $x+1$.

888. That the factor form has finite values, when μ becomes infinitely large, for negative values of x between the asymptotes may be made clear by taking a case. Take $x=-\frac{3}{2}$.

Then $$Lt_{\mu=\infty}\frac{1.2.3\ldots\mu}{\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\ldots\left(\frac{2\mu-3}{2}\right)}\mu^{-\frac{3}{2}}$$

$$=-Lt\frac{2.4.6\ldots2\mu}{1.1.3.5\ldots(2\mu-3)}\frac{1}{\mu^{\frac{3}{2}}}$$

$$=-Lt\frac{2^2.4^2.6^2\ldots(2\mu)^2}{1.2.3.4\ldots(2\mu-3)(2\mu-2)(2\mu-1)(2\mu)}\frac{(2\mu-1)}{\mu^{\frac{3}{2}}}$$

$$=-Lt\,2^{2\mu}\frac{(\sqrt{2\mu\pi}\,\mu^\mu e^{-\mu})^2}{\sqrt{4\mu\pi}(2\mu)^{2\mu}e^{-2\mu}}\frac{(2\mu-1)}{\mu^{\frac{3}{2}}}$$

$$=-Lt\frac{2\pi\mu}{2\sqrt{\pi\mu}}\frac{2\mu-1}{\mu^{\frac{3}{2}}}=-\frac{2}{1}\sqrt{\pi}.$$

Similarly at $x=-\frac{5}{2}$ the corresponding limit is $\frac{2^2}{1.3}\sqrt{\pi}$,

at $x=-\frac{7}{2}$ the corresponding limit is $-\frac{2^3}{1.3.5}\sqrt{\pi}$,

and so on.

These mid-ordinates, half way between the successive asymptotes, thus form a regular descending series

$$-\frac{2}{1}\sqrt{\pi},\quad \frac{2^2}{1.3}\sqrt{\pi},\quad -\frac{2^3}{1.3.5}\sqrt{\pi},\quad \frac{2^4}{1.3.5.7}\sqrt{\pi},\ \text{etc.}$$

889. It is worth noticing that $\Pi(x,\mu)$ may be written as

$$\Pi(x,\mu)\equiv\frac{1.2.3\ldots\mu}{(x+1)(x+2)(x+3)\ldots(x+\mu)}\mu^x$$

$$\equiv\frac{\left(\frac{2}{1}\right)^x\left(\frac{3}{2}\right)^x\left(\frac{4}{3}\right)^x\cdots\left(\frac{\mu}{\mu-1}\right)^x\left(\frac{\mu+1}{\mu}\right)^x}{\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\left(1+\frac{x}{3}\right)\cdots\left(1+\frac{x}{\mu}\right)}\left(\frac{\mu}{\mu+1}\right)^x$$