CHAPTER XXV.

LEJEUNE-DIRICHLET INTEGRALS, LIOUVILLE INTEGRALS, ETC.

958. We have seen that the formula (i_1 and i_2 both $+^{\text{ve}}$)

$$\int_0^1 x^{i_1-1}(1-x)^{i_2-1}\,dx=\frac{\Gamma(i_1)\,\Gamma(i_2)}{\Gamma(i_1+i_2)}$$

leads at once, by putting y for ax, to

$$\int_0^a x^{i_1-1}(a-x)^{i_2-1}\,dx=a^{i_1+i_2-1}\frac{\Gamma(i_1)\,\Gamma(i_2)}{\Gamma(i_1+i_2)}.$$

Now, consider the double integral

$$I=\iint x_1{}^{i_1-1}x_2{}^{i_2-1}\,dx_1\,dx_2$$

for all positive values of x_1 and x_2, which are such that their sum cannot be greater than unity.

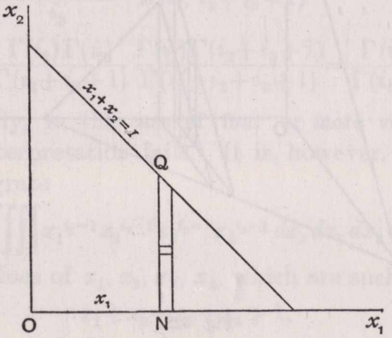

Fig. 321.

Then the limits for x_2 must be from 0 to $1-x_1$, x_1 remaining constant in the integration with regard to x_2, and the limits for x_1 will be from 0 to 1.

The geometrical interpretation is that we are adding up all such products as $x_1^{i_1-1}x_2^{i_2-1}\,\delta x_1\,\delta x_2$ as lie within the triangle formed by the axes Ox_1, Ox_2, and the straight line $x_1+x_2=1$. We use this notation rather than the ordinary x-y notation for Cartesians, because we propose to generalise the theorem for any number of variables. The limits must then be such as to add up all elements in a strip NQ parallel to the x_2-axis, *i.e.* x_2 increases from 0 to $1-x_1$, and in summing the strips, x_1 increases from $x_1=0$ to $x_1=1$.

Then $$I=\int_0^1 x_1^{i_1-1}\left[\frac{x_2^{i_2}}{i_2}\right]_0^{1-x_1}dx_1=\frac{1}{i_2}\int_0^1 x_1^{i_1-1}(1-x_1)^{i_2}\,dx_1$$

$$=\frac{1}{i_2}\,\frac{\Gamma(i_1)\,\Gamma(i_2+1)}{\Gamma(i_1+i_2+1)}=\frac{\Gamma(i_1)\,\Gamma(i_2)}{\Gamma(i_1+i_2+1)}.$$

959. Take next the **case of the triple integral**

$$I=\iiint x_1^{i_1-1}x_2^{i_2-1}x_3^{i_3-1}\,dx_1\,dx_2\,dx_3$$

for positive values of x_1, x_2, x_3, such that $x_1+x_2+x_3 \ngtr 1$.

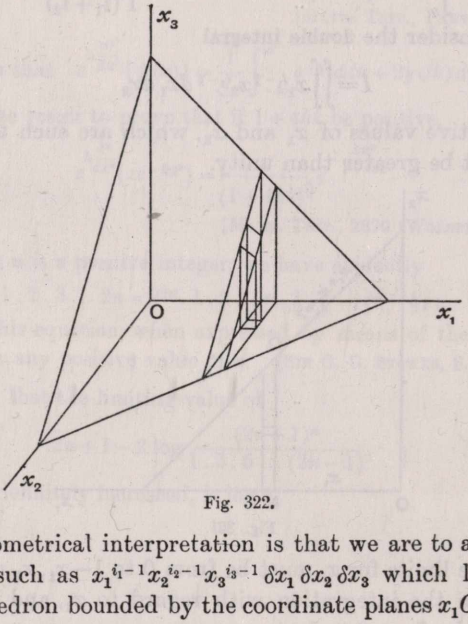

Fig. 322.

The geometrical interpretation is that we are to add up all elements such as $x_1^{i_1-1}x_2^{i_2-1}x_3^{i_3-1}\,\delta x_1\,\delta x_2\,\delta x_3$ which lie within the tetrahedron bounded by the coordinate planes x_1Ox_2, x_2Ox_3, x_3Ox_1 and the plane $x_1+x_2+x_3=1$.

Then dividing by planes parallel to the coordinate planes in the same way as explained in previous chapters, we have first to integrate with regard to x_3, keeping x_1 and x_2 constant, that is, for all values of x_3 which lie between $x_3=0$ and $x_3=1-x_1-x_2$, which, interpreted geometrically, means the addition of all elements which lie in an elementary prism parallel to the x_3-axis and whose ends lie respectively in the plane of $x_3=0$ and the plane $x_1+x_2+x_3=1$. Then, keeping x_1 constant, we have to integrate for all values of x_2 from $x_2=0$ to the value of x_2 which makes $1-x_1-x_2$ vanish; which means that we are to add up all the prisms which lie in a thin slice parallel to the plane of $x_1=0$. Finally, we are to integrate from $x_1=0$ to $x_1=1$, which means that we are to add up all the slices within the tetrahedron.

$$\text{Then}\quad I=\int_0^1\int_0^{1-x_1}\int_0^{1-x_1-x_2} x_1^{i_1-1}x_2^{i_2-1}x_3^{i_3-1}\,dx_1\,dx_2\,dx_3$$
$$=\int_0^1\int_0^{1-x_1} x_1^{i_1-1}x_2^{i_2-1}\frac{(1-x_1-x_2)^{i_3}}{i_3}\,dx_1\,dx_2$$
$$=\int_0^1 x_1^{i_1-1}\cdot\frac{B(i_2,\,i_3+1)}{i_3}(1-x_1)^{i_2+i_3}\,dx_1$$

[by applying the result $\int_0^k x^{i_1-1}(k-x)^{i_2-1}\,dx=k^{i_1+i_2-1}B(i_1,\,i_2)$].

$$\text{Hence}\quad I=\frac{B(i_2,\,i_3+1)}{i_3}\cdot B(i_1,\,i_2+i_3+1)$$
$$=\frac{\Gamma(i_2)\Gamma(i_3)}{\Gamma(i_2+i_3+1)}\,\frac{\Gamma(i_1)\Gamma(i_2+i_3+1)}{\Gamma(i_1+i_2+i_3+1)}=\frac{\Gamma(i_1)\Gamma(i_2)\Gamma(i_3)}{\Gamma(i_1+i_2+i_3+1)}.$$

960. Similarly, in the **case of four or more variables**; but geometrical interpretation fails. It is, however, clear that if we are to integrate

$$I=\iiiint x_1^{i_1-1}x_2^{i_2-1}x_3^{i_3-1}x_4^{i_4-1}\,dx_1\,dx_2\,dx_3\,dx_4$$

for positive values of x_1, x_2, x_3, x_4, which are such that

$$x_1+x_2+x_3+x_4 \ngtr 1,$$

(1) when x_1, x_2, x_3 are kept constant, x_4 will range from $x_4=0$ to such value of x_4 as will make

$$1-x_1-x_2-x_3-x_4$$

zero, *i.e.* from $x_4=0$ to $x_4=1-x_1-x_2-x_3$.

(2) Having integrated with regard to x_4, we now keep x_1, x_2 constant, and in integration with regard to x_3, x_3 must vary from $x_3=0$ to such value as will make $1-x_1-x_2-x_3$ vanish, *i.e.* x_3 must not exceed $1-x_1-x_2$, *i.e.* the limits are 0 and $1-x_1-x_2$.

(3) Integration with regard to x_4 and x_3 having now been completed, x_1 is to be kept constant whilst integration with regard to x_2 is effected, and x_2 must range from $x_2=0$ to such a value as will not make $1-x_1-x_2$ negative, *i.e.* x_2 must not exceed $1-x_1$. The limits are therefore 0 and $1-x_1$.

(4) Finally, the limits for x_1 are 0 to 1.

Hence

$$I=\int_0^1\int_0^{1-x_1}\int_0^{1-x_1-x_2}\int_0^{1-x_1-x_2-x_3} x_1^{i_1-1}x_2^{i_2-1}x_3^{i_3-1}x_4^{i_4-1}\,dx_1\,dx_2\,dx_3\,dx_4$$

$$=\int_0^1\int_0^{1-x_1}\int_0^{1-x_1-x_2} x_1^{i_1-1}x_2^{i_2-1}x_3^{i_3-1}\frac{(1-x_1-x_2-x_3)^{i_4}}{i_4}\,dx_1\,dx_2\,dx_3$$

$$=\int_0^1\int_0^{1-x_1} x_1^{i_1-1}x_2^{i_2-1}(1-x_1-x_2)^{i_3+i_4}\frac{B(i_3,\, i_4+1)}{i_4}\,dx_1\,dx_2$$

$$=\frac{B(i_3,\, i_4+1)}{i_4}\int_0^1 x_1^{i_1-1}(1-x_1)^{i_2+i_3+i_4}B(i_2,\, i_3+i_4+1)\,dx_1$$

$$=\frac{B(i_3,\, i_4+1)}{i_4}B(i_2,\, i_3+i_4+1)B(i_1,\, i_2+i_3+i_4+1)$$

$$=\frac{\Gamma(i_3)\Gamma(i_4)}{\Gamma(i_3+i_4+1)}\cdot\frac{\Gamma(i_2)\Gamma(i_3+i_4+1)}{\Gamma(i_2+i_3+i_4+1)}\cdot\frac{\Gamma(i_1)\Gamma(i_2+i_3+i_4+1)}{\Gamma(i_1+i_2+i_3+i_4+1)}$$

$$=\frac{\Gamma(i_1)\Gamma(i_2)\Gamma(i_3)\Gamma(i_4)}{\Gamma(i_1+i_2+i_3+i_4+1)},$$

and the rule indicated obviously holds for any number of integrations, viz.

$$\iiint\ldots\int x_1^{i_1-1}\,x_2^{i_2-1}\ldots x_n^{i_n-1}\,dx_1\,dx_2\ldots dx_n,$$

for positive values of the variables such that their sum does not exceed unity $=\dfrac{\Gamma(i_1)\Gamma(i_2)\ldots\Gamma(i_n)}{\Gamma(\sigma+1)}$, where $\sigma=i_1+i_2+\ldots+i_n$.

961. **An Extension.**

Similarly, if the limiting equation had been
$$x_1+x_2+\dots+x_n \not> c \quad \text{(instead of } \not> 1\text{)},$$
the limits would have been,

for x_n, from 0 to $c-x_1-x_2-\dots-x_{n-1}$;
for x_{n-1}, from 0 to $c-x_1-x_2-\dots-x_{n-2}$,
etc.;

but we may deduce the result from that already obtained by putting
$$x_1=cx_1', \quad x_2=cx_2', \text{ etc.},$$
so that
$$x_1'+x_2'+\dots \not> 1.$$

Thus we obtain
$$I=c^\sigma \iint\dots\int (x_1')^{i_1-1}(x_2')^{i_2-1}\dots(x_n')^{i_n-1}\,dx_1'\,dx_2'\dots dx_n',$$
$$=c^\sigma\frac{\Gamma(i_1)\,\Gamma(i_2)\dots\Gamma(i_n)}{\Gamma(\sigma+1)}, \text{ where } \sigma=i_1+i_2+\dots+i_n.$$

962. Dirichlet's Theorem.

We are now in a position to establish a remarkable theorem due to Gustav Peter Lejeune-Dirichlet,* who was successor to Gauss at Gottingen in 1855.†

The theorem is known as Dirichlet's Theorem, and is of great use in analysis.

The theorem is that when there are any number of variables $x_1, x_2, \dots x_n$, and integration is conducted for all positive values limited by the condition
$$\left(\frac{x_1}{a_1}\right)^{p_1}+\left(\frac{x_2}{a_2}\right)^{p_2}+\dots+\left(\frac{x_n}{a_n}\right)^{p_n} \not> 1,$$
then
$$I\equiv \iiint\dots\int x_1^{i_1-1}x_2^{i_2-1}x_3^{i_3-1}\dots x_n^{i_n-1}\,dx_1\,dx_2\,dx_3\dots dx_n$$
$$=\frac{a_1^{i_1}a_2^{i_2}\dots a_n^{i_n}}{p_1p_2\dots p_n}\cdot\frac{\Gamma\left(\frac{i_1}{p_1}\right)\left(\frac{i_2}{p_2}\right)\dots\Gamma\left(\frac{i_n}{p_n}\right)}{\Gamma\left(\frac{i_1}{p_1}+\frac{i_2}{p_2}+\dots+\frac{i_n}{p_n}+1\right)}=\frac{\prod_1^n\left\{\frac{a_r^{i_r}}{p_r}\Gamma\left(\frac{i_r}{p_r}\right)\right\}}{\Gamma\left(1+\sum_1^n\frac{i_r}{p_r}\right)},$$
the several quantities $i_1, i_2, i_3, \dots i_n$; $a_1, a_2, \dots a_n$; $p_1, p_2, \dots p_n$, being all positive, and Π denoting the product of the factors indicated.

* Liouville's *Journal*, vol. iv., p. 168.

† Cajori, *Hist. of Math.*, p. 367; Kummer, *Gedächnissrede auf G. P. Lejeune-Dirichlet*,

The limiting equation $\left(\frac{x_1}{a_1}\right)^{p_1}+\left(\frac{x_2}{a_2}\right)^{p_2}+\ldots \not> 1$ may be made linear by the change of variables $\xi_1=\left(\frac{x_1}{a_1}\right)^{p_1}$, $\xi_2=\left(\frac{x_2}{a_2}\right)^{p_2}$, etc., which give $\frac{1}{\xi_1}\frac{\partial \xi_1}{\partial x_1}=\frac{p_1}{x_1}$, $\frac{1}{\xi_2}\frac{\partial \xi_2}{\partial x_2}=\frac{p_2}{x_2}$, etc.,

and $$J'=p_1p_2\ldots p_n\frac{\xi_1}{x_1}\cdot\frac{\xi_2}{x_2}\cdot\frac{\xi_3}{x_3}\ldots\frac{\xi_n}{x_n}.$$

The transformed integral is then

$$I\equiv\frac{1}{p_1p_2\ldots p_n}\iint\ldots\int\frac{x_1^{i_1}x_2^{i_2}\ldots x_n^{i_n}}{\xi_1\xi_2\ldots\xi_n}d\xi_1 d\xi_2\ldots d\xi_n$$
$$=\frac{a_1^{i_1}a_2^{i_2}\ldots a_n^{i_n}}{p_1p_2\ldots p_n}\iint\ldots\int \xi_1^{\frac{i_1}{p_1}-1}\xi_2^{\frac{i_2}{p_2}-1}\ldots\xi_n^{\frac{i_n}{p_n}-1}d\xi_1 d\xi_2\ldots d\xi_n,$$

with the limiting equation $\xi_1+\xi_2+\ldots+\xi_n \not> 1$;

$$\therefore\ I=\frac{a_1^{i_1}a_2^{i_2}\ldots a_n^{i_n}}{p_1p_2\ldots p_n}\frac{\Gamma\left(\frac{i_1}{p_1}\right)\Gamma\left(\frac{i_2}{p_2}\right)\ldots\Gamma\left(\frac{i_n}{p_n}\right)}{\Gamma\left(\frac{i_1}{p_1}+\frac{i_2}{p_2}+\ldots+\frac{i_n}{p_n}+1\right)}=\frac{\prod_1^n\left\{\frac{a_r^{i_r}}{p_r}\Gamma\left(\frac{i_r}{p_r}\right)\right\}}{\Gamma\left(1+\sum_1^n\frac{i_r}{p_r}\right)}$$

as stated.

963. As before, if our limiting condition had been

$$\left(\frac{x_1}{a_1}\right)^{p_1}+\left(\frac{x_2}{a_2}\right)^{p_2}+\ldots+\left(\frac{x_n}{a_n}\right)^{p_n}\not> c \text{ (instead of } \not> 1\text{)},$$

we should have, after transformation as above,

$$\xi_1+\xi_2+\ldots+\xi_n\not> c,$$

and making the further transformation

$$\xi_1=c\xi_1',\quad \xi_2=c\xi_2',\ \ldots \text{ etc.},$$
$$\xi_1'+\xi_2'+\ldots+\xi_n'\not> 1,$$

and the result would be

$$I=\frac{a_1^{i_1}a_2^{i_2}\ldots a_n^{i_n}}{p_1p_2\ldots p_n}c^\sigma\frac{\Gamma\left(\frac{i_1}{p_1}\right)\Gamma\left(\frac{i_2}{p_2}\right)\ldots\Gamma\left(\frac{i_n}{p_n}\right)}{\Gamma(\sigma+1)},$$

where $$\sigma=\frac{i_1}{p_1}+\frac{i_2}{p_2}+\ldots+\frac{i_n}{p_n},$$

i.e. $$I=c^\sigma\prod_1^n\left\{\frac{a_r^{i_r}}{p_r}\Gamma\left(\frac{i_r}{p_r}\right)\right\}\Big/\Gamma\left(1+\sum_1^n\frac{i_r}{p_r}\right).$$

964. Ex. Find the centroid of an octant of the solid bounded by

$$\left(\frac{x}{a}\right)^{2k}+\left(\frac{y}{b}\right)^{2k}+\left(\frac{z}{c}\right)^{2k}=1,$$

the volume-density at any point being given by $\rho=\mu x^l y^m z^n$.

Here
$$\bar{x}=\frac{\iiint \rho x\,dx\,dy\,dz}{\iiint \rho\,dx\,dy\,dz}=\frac{\iiint x^{l+1}y^m z^n\,dx\,dy\,dz}{\iiint x^l y^m z^n\,dx\,dy\,dz}.$$

$$\text{The Numerator}=\frac{a^{l+2}b^{m+1}c^{n+1}}{2k\,.\,2k\,.\,2k}\,\frac{\Gamma\left(\frac{l+2}{2k}\right)\Gamma\left(\frac{m+1}{2k}\right)\Gamma\left(\frac{n+1}{2k}\right)}{\Gamma\left(\frac{l+2}{2k}+\frac{m+1}{2k}+\frac{n+1}{2k}+1\right)}.$$

$$\text{The Denominator}=\frac{a^{l+1}b^{m+1}c^{n+1}}{2k\,.\,2k\,.\,2k}\,\frac{\Gamma\left(\frac{l+1}{2k}\right)\Gamma\left(\frac{m+1}{2k}\right)\Gamma\left(\frac{n+1}{2k}\right)}{\Gamma\left(\frac{l+1}{2k}+\frac{m+1}{2k}+\frac{n+1}{2k}+1\right)}.$$

Hence
$$\bar{x}=a\frac{\Gamma\left(\frac{l+2}{2k}\right)}{\Gamma\left(\frac{l+1}{2k}\right)}\,\frac{\Gamma\left(\frac{l+m+n+3}{2k}+1\right)}{\Gamma\left(\frac{l+m+n+4}{2k}+1\right)}.$$

In the case of an octant of a uniform ellipsoid $l=m=n=0$, $k=1$,

$$\bar{x}=a\,\frac{1}{\Gamma(\frac{1}{2})}\,\frac{\Gamma(\frac{5}{2})}{\Gamma(3)}=a\,.\,\frac{\frac{3}{2}\,.\,\frac{1}{2}}{2}=\tfrac{3}{8}a.$$

Similarly for $\bar{y}$ and $\bar{z}$.

965. A Particular Case.

In the case when $\quad p_1=p_2=\ldots=p_n=1$

and $\quad a_1=a_2=\ldots=a_n=a$,

the theorem reduces back to

$$I=\iint\ldots\int x_1^{\,i_1-1}x_2^{\,i_2-1}\ldots x_n^{\,i_n-1}\,dx_1\,dx_2\ldots dx_n$$
$$=a^{i_1+i_2+\ldots+i_n}\frac{\Gamma(i_1)\Gamma(i_2)\ldots\Gamma(i_n)}{\Gamma(i_1+i_2+\ldots+i_n+1)},$$

and the limiting equation is

$$x_1+x_2+\ldots+x_n \not> a,$$

viz. the fundamental case of Art. 961 assumed.

966. Extension.

If the lower limits had not been zero in each case, but such that $x_1+x_2+\ldots+x_n$ is to be not less than b nor greater than a, *i.e.* $b < \Sigma x_r < a$; then plainly we must subtract from the result obtained, the integral found by making

$$x_1 + x_2 + \dots + x_n \ngtr b,$$

and the result will be

$$[a^{i_1+i_2+\dots+i_n} - b^{i_1+i_2+\dots+i_n}] \frac{\Gamma(i_1)\Gamma(i_2)\dots\Gamma(i_n)}{\Gamma(i_1+i_2+\dots+i_n+1)}.$$

967. If the difference between a and b be an infinitesimal difference δb, then to the first order

$$a^{i_1+\dots+i_n} - b^{i_1+\dots+i_n} = (b+\delta b)^{i_1+\dots+i_n} - b^{i_1+\dots+i_n}$$
$$= (i_1+i_2+\dots+i_n) b^{i_1+\dots+i_n-1}\delta b,$$

and the result will be

$$b^{i_1+i_2+\dots+i_n-1}\delta b \, \frac{\Gamma(i_1)\Gamma(i_2)\Gamma(i_3)\dots\Gamma(i_n)}{\Gamma(i_1+i_2+\dots+i_n)}.$$

For example, to verify this in a simple case, consider the volume of a triangular plate bounded by the coordinate planes, and the planes

$$x+y+z=b \quad \text{and} \quad x+y+z=b+\delta b.$$

Here $$i_1 = i_2 = i_3 = 1, \quad p_1 = p_2 = p_3 = 1,$$

$$V = b^2 \delta b \, . \frac{1\,.\,1\,.\,1}{2} = \tfrac{1}{2} b^2 \delta b = \delta\left(\frac{b}{3} \cdot \frac{b^2}{2}\right),$$

i.e. the change in the volume of the tetrahedron bounded by the coordinate planes, and the plane which makes intercepts b on the axes, when b increases to $b + \delta b$.

968. **Liouville's Extension.**

If we require to find the value of

$$I = \iint\dots\int x_1^{i_1-1} x_2^{i_2-1} \dots x_n^{i_n-1} f(x_1+x_2+\dots+x_n)\, dx_1\, dx_2 \dots dx_n,$$

subject to the conditions that $x_1, x_2, \dots x_n$ are all positive, but

$$x_1 + x_2 + \dots + x_n \ngtr a \text{ and } \nless b,$$

we may then take the case when

$$x_1 + x_2 + \dots + x_n$$

lies between v and $v + \delta v$, for which

$$x_1 + x_2 + \dots + x_n$$

differs from v by an infinitesimal ϵ.

Then for this limitation the integral takes the value

$$v^{i_1+i_2+\dots+i_n-1}\,\delta v\, f(v+\epsilon) \frac{\Gamma(i_1)\Gamma(i_2)\dots\Gamma(i_n)}{\Gamma(i_1+\dots+i_n)}$$

$$= v^{i_1+i_2+\dots+i_n-1}\,\delta v\, f(v) \frac{\Gamma(i_1)\dots\Gamma(i_n)}{\Gamma(i_1+\dots+i_n)}$$

to the first order of infinitesimals. And therefore, for the whole range of values from $v=b$ to $v=a$,

$$I=\frac{\Gamma(i_1)\Gamma(i_2)\dots\Gamma(i_n)}{\Gamma(i_1+i_2+\dots+i_n)}\int_b^a v^{i_1+i_2+\dots+i_n-1}f(v)\,dv.$$

969. Exactly in the same way, if we require

$$I=\iint\dots\int x_1^{i_1-1}\dots x_n^{i_n-1}f\left\{\left(\frac{x_1}{a_1}\right)^{p_1}+\dots+\left(\frac{x_n}{a_n}\right)^{p_n}\right\}dx_1\dots dx_n$$

for all positive values of the variables such that

$$\left(\frac{x_1}{a_1}\right)^{p_1}+\left(\frac{x_2}{a_2}\right)^{p_2}+\dots+\left(\frac{x_n}{a_n}\right)^{p_n}\not> h_1 \text{ and } \not< h_2.$$

Let $$\left(\frac{x_1}{a_1}\right)^{p_1}+\left(\frac{x_2}{a_2}\right)^{p_2}+\dots+\left(\frac{x_n}{a_n}\right)^{p_n}$$

lie between v and $v+\delta v$, $=v+\epsilon$, say, where ϵ is an infinitesimal.

Then for this limitation,

$$\Big[I\Big]_v^{v+\delta v}=\frac{a_1^{i_1}a_2^{i_2}\dots a_n^{i_n}}{p_1p_2\dots p_n}v^{k-1}\,\delta v\,f(v+\epsilon)\frac{\Gamma\left(\frac{i_1}{p_1}\right)\Gamma\left(\frac{i_2}{p_2}\right)\dots\Gamma\left(\frac{i_n}{p_n}\right)}{\Gamma(k)},$$

where $$k=\frac{i_1}{p_1}+\frac{i_2}{p_2}+\dots+\frac{i_n}{p_n},$$

and $\delta v f(v+\epsilon)$ differs from $f(v)\,\delta v$ by a second-order infinitesimal at most, supposing $f(v)$ and $f'(v)$ finite and continuous for the range. Hence in the limit, when we integrate with regard to v from $v=h_2$ to $v=h_1$,

$$I=\frac{a_1^{i_1}a_2^{i_2}\dots a_n^{i_n}}{p_1p_2\dots p_n}\,\frac{\Gamma\left(\frac{i_1}{p_1}\right)\Gamma\left(\frac{i_2}{p_2}\right)\dots\Gamma\left(\frac{i_n}{p_n}\right)}{\Gamma\left(\frac{i_1}{p_1}+\frac{i_2}{p_2}+\dots+\frac{i_n}{p_n}\right)}\int_{h_2}^{h_1}v^{k-1}f(v)\,dv,$$

where $$k=\frac{i_1}{p_1}+\frac{i_2}{p_2}+\dots+\frac{i_n}{p_n}.$$

This extension of Dirichlet's theorem is due to Liouville.*

970. **An Application.**

As an example of this theorem, consider

$$\iint\dots\int\frac{dx_1\,dx_2\dots dx_n}{\sqrt{a^2-x_1^2-x_2^2-\dots-x_n^2}}$$

for positive values of the variables with the condition

$$x_1^2+x_2^2+\dots+x_n^2=va^2\not> a^2.$$

* Liouville's *Journal*, vol. iv., p. 231.

Here $$p_1=p_2=\ldots=p_n=2,\quad i_1=i_2=\ldots=i_n=1,$$

$$a_1=a_2=\ldots=a_n=a\,;\quad h_1=1,\ h_2=0,\ k=\frac{n}{2}.$$

Then $$I=\frac{a^n}{2^n}\frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^n}{\Gamma\left(\frac{n}{2}\right)}\int_0^1\frac{v^{\frac{n}{2}-1}}{a\sqrt{1-v}}\,dv=\frac{a^{n-1}}{2^n}\frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^n}{\Gamma\left(\frac{n}{2}\right)}\int_0^1 v^{\frac{n}{2}-1}(1-v)^{\frac{1}{2}-1}dv$$

$$=\frac{a^{n-1}}{2^n}\frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^n}{\Gamma\left(\frac{n}{2}\right)}\cdot\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}=\frac{a^{n-1}}{2^n}\frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}.$$

Thus, for example, in the case $n=2$,

$$\iint\frac{dx_1\,dx_2}{\sqrt{a^2-x_1^2-x_2^2}}=\frac{a}{4}\frac{\pi^{\frac{3}{2}}}{\frac{1}{2}\pi^{\frac{1}{2}}}=\frac{\pi a}{2}.^*$$

Hence the area of the portion of a sphere $x^2+y^2+z^2=a^2$ which lies in the first octant, and which is

$$\iint\frac{a}{z}\,dx\,dy,\quad i.e.\ a\iint\frac{dx\,dy}{\sqrt{a^2-x^2-y^2}},\quad \text{is}\ =a\,.\,\frac{\pi a}{2},$$

and the area of the surface of the whole sphere $=4\pi a^2$.

Again ($n=3$), $$\iiint\frac{dx_1\,dx_2\,dx_3}{\sqrt{a^2-x_1^2-x_2^2-x_3^2}}=\frac{\pi^2a^2}{8}$$

(Gregory's *Examples*, p. 474).

and ($n=4$), $$\iiiint\frac{dx_1\,dx_2\,dx_3\,dx_4}{\sqrt{a^2-x_1^2-x_2^2-x_3^2-x_4^2}}=\frac{a^3}{16}\frac{\pi^{\frac{5}{2}}}{\Gamma(\frac{5}{2})}=\frac{\pi^2a^3}{12},$$

etc.

971. **Boole's Theorem.**

Consider $I=\iint\ldots\int F(a_1x_1+a_2x_2+\ldots+a_nx_n)\,dx_1dx_2\ldots dx_n$ for all real values of $x_1, x_2, \ldots x_n$ negative or positive, such that

$$x_1^2+x_2^2+\ldots \not> c^2.$$

Change the variables by the orthogonal transformation in the margin.

	u_1	u_2	u_3	...
x_1	l_1	l_2	l_3	...
x_2	m_1	m_2	m_3	...
x_3	n_1	n_2	n_3	...
...	...	...	...	...

Then $J=1$ and the relations of the transformation system are

$$\Sigma l^2=1,\ \text{etc.},$$
$$\Sigma lm=0,\ \text{etc.},$$

and $$\sum_1^n x_r^2=\sum_1^n u_r^2\,;$$

* Gregory's *Examples*, p. 474.

and suppose the transformation to have been so chosen that

$$a_1x_1+a_2x_2+\ldots+a_nx_n=ku_1, \quad \text{where } k^2=\sum_1^n a_r^2.$$

Then $$I=\underset{(n \text{ signs})}{\iint\ldots\int} F(ku_1)\,du_1\,du_2\ldots du_n.$$

Now for the first $n-1$ integrations, u_1 remains constant, and

$$\underset{(n-1 \text{ signs})}{\iint\ldots\int} du_2\,du_3\ldots du_n,$$

where $$u_2^2+u_3^2+\ldots+u_n^2 \ngtr c^2-u_1^2,$$

$$=2^{n-1}\frac{(c^2-u_1^2)^{\frac{n-1}{2}}}{2^{n-1}}\frac{(\Gamma\tfrac{1}{2})^{n-1}}{\Gamma\left(\dfrac{n+1}{2}\right)},$$

the first factor 2^{n-1} occurring because at each of the $n-1$ integrations the result is to be doubled to take into account the possible negative signs of the respective variables. Hence, dropping the suffix, we have

$$I=\frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\dfrac{n+1}{2}\right)}\int_{-c}^{c} F(ku)(c^2-u^2)^{\frac{n-1}{2}}du.$$

(See "Catalan's Theorem," Liouville's *Journal*, vol. vi., p. 81, and Boole's remarks upon it, *Cambridge Math. Journal*, vol. iii., p. 277.)

972. Consider next the integration

$$I=\underset{(n \text{ signs})}{\iint\ldots\int}\frac{F(a_1x_1+a_2x_2+\ldots+a_nx_n)}{\sqrt{c^2-x_1^2-x_2^2-\ldots-x_n^2}}\,dx_1\,dx_2\ldots dx_n,$$

where $$x_1^2+x_2^2+\ldots+x_n^2 \ngtr c^2,$$

for real values of $x_1, x_2, \ldots x_n$.

Changing the variables by the same orthogonal transformation as before,

$$I=\underset{(n \text{ signs})}{\iint\ldots\int}\frac{F(ku_1)}{\sqrt{c^2-u_1^2-u_2^2-u_3^2\ldots-u_n^2}}\,du_1\,du_2\ldots du_n.$$

Now for the first $n-1$ integrations, u_1 remains a constant, and

$$\underset{(n-1 \text{ signs})}{\iint\ldots\int}\frac{du_2\,du_3\ldots du_n}{(c^2-u_1^2-u_2^2-u_3^2\ldots-u_n^2)^{\frac{1}{2}}}=2^{n-1}\frac{(c^2-u_1^2)^{\frac{n}{2}-1}}{2^{n-1}}\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\dfrac{n}{2}\right)},$$

by Art. 970, the first factor 2^{n-1} being introduced because the several variables are not now restricted as to sign as was the case in Art. 970, so that at each of the $(n-1)$ integrations the result must be doubled. Also at the final integration the limits must be $-c$ to $+c$ for the same reason. Hence, dropping the suffix,

$$I=\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\int_{-c}^{c} F(ku)(c^2-u^2)^{\frac{n}{2}-1}du.*$$

973. **Further Generalisation.**

We next consider the still more general integral

$$I=\iint\ldots\int F\left(\frac{x_1^{\,2}}{a_1^{\,2}}+\ldots+\frac{x_n^{\,2}}{a_n^{\,2}}\right)f(A_1x_1+\ldots+A_nx_n)\,dx_1\ldots dx_n$$

for all real values of $x_1, x_2, \ldots x_n$, such that

$$\frac{x_1^{\,2}}{a_1^{\,2}}+\frac{x_2^{\,2}}{a_2^{\,2}}+\ldots+\frac{x_n^{\,2}}{a_n^{\,2}} \not> 1.$$

First we expand $F(v)$ in powers of $1-v$, say $\Sigma B_p(1-v)^p$ [or if it be possible to expand in positive *integral* powers of $1-v$, we may write $1-v=w$; then $F(v)=F(1-w)$, and by Maclaurin's theorem, we may put

$$F(v)=F(1)-wF'(1)+\frac{w^2}{2!}F''(1)-\ldots+(-1)^p\frac{w^p}{p!}F^{(p)}(1)+\ldots\Big].$$

Then we consider the integration of

$$\iint\ldots\int\left(1-\frac{x_1^{\,2}}{a_1^{\,2}}-\ldots-\frac{x_n^{\,2}}{a_n^{\,2}}\right)^p f(A_1x_1+\ldots+A_nx_n)\,dx_1\ldots dx_n.$$

If I_p be the result of this integration, the whole result will be

$$\Sigma B_pI_p$$

[or $\quad I_0F(1)-I_1F'(1)+\frac{I_2}{2!}F''(1)-\ldots+(-1)^p\frac{I_p}{p!}F^{(p)}(1)+\ldots,$ as the case may be].

To obtain I_p, first put

$$x_1=a_1\xi_1,\quad x_2=a_2\xi_2,\quad x_3=a_3\xi_3,\ \ldots\ x_n=a_n\xi_n.$$

Then $J=a_1a_2\ldots a_n$ and

$$\frac{I_p}{a_1a_2\ldots a_n}=\iint\ldots\int(1-\xi_1^{\,2}-\ldots-\xi_n^{\,2})^pf(A_1a_1\xi_1+\ldots+A_na_n\xi_n)d\xi_1\ldots d\xi_n.$$

* See Todhunter, *D.C.*, Art. 281; Gregory, *D. and I.C.*, p. 474.

Now make a further transformation to variables $u_1, u_2, \dots u_n$ by the orthogonal transformation formulae in the margin. The Jacobian of this system is unity, and

	u_1	u_2	...	u_n
ξ_1	l_1	l_2	...	l_n
ξ_2	m_1	m_2	...	m_n
...	...	...	...	...
ξ_n	...	...	...	...

$$\xi_1^2+\xi_2^2+\dots=u_1^2+u_2^2+\dots;$$

and further choose u_1 to be

$$(A_1 a_1 \xi_1 + A_2 a_2 \xi_2+\dots)/k,$$

where $k^2=A_1^2 a_1^2+\dots+A_n^2 a_n^2$.

Then $I_p=a_1 \dots a_n \iint \dots \int (1-u_1^2-\dots-u_n^2)^p f(ku_1)\, du_1 \dots du_n$.

In the integration with regard to $u_2, u_3, \dots u_n$, the remaining variable u_1 remains constant, and

$$\underbrace{\iint \dots \int}_{(n-1 \text{ signs})} (1-u_1^2-u_2^2-\dots-u_n^2)^p\, du_2\, du_3 \dots du_n,$$

$$=\frac{1}{2^{n-1}} \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^{n-1}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^{1-u_1^2} z^{\frac{n-1}{2}-1} (1-u_1^2-z)^p\, dz,$$

if restricted to positive values of u_2, u_3, etc.; and if the several variables may have full scope as to sign between the specified limits, each of these $n-1$ integrations must be doubled.

The result of the $n-1$ integrations is in that case

$$\frac{\Gamma\left(\frac{1}{2}\right)^{n-1}}{\Gamma\left(\frac{n-1}{2}\right)} \frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma(p+1)}{\Gamma\left(\frac{n+1}{2}+p\right)} (1-u_1^2)^{\frac{n-1}{2}+p}$$

$$=\frac{\pi^{\frac{n-1}{2}}\Gamma(p+1)}{\Gamma\left(\frac{n+1}{2}+p\right)} (1-u_1^2)^{\frac{n-1}{2}+p}.$$

Therefore, as the limits of the final integration with regard to u_1 are from -1 to $+1$,

$$I_p=a_1 a_2 \dots a_n \frac{\pi^{\frac{n-1}{2}}\Gamma(p+1)}{\Gamma\left(\frac{n+1}{2}+p\right)} \int_{-1}^{1} (1-u^2)^{\frac{n-1}{2}+p} f(ku)\, du,$$

it being now unnecessary to retain the suffix of the u. Hence

$$I=a_1a_2\ldots a_n\pi^{\frac{n-1}{2}}\Sigma B_p\frac{\Gamma(p+1)}{\Gamma\left(\frac{n+1}{2}+p\right)}\int_{-1}^{1}(1-u^2)^{\frac{n-1}{2}+p}f(ku)\,du,$$

where $$k^2=A_1^2a_1^2+A_2^2a_2^2+\ldots+A_n^2a_n^2.$$

This result, of course, includes former cases discussed.

974. **Extension.**

If the limits had been defined so that

$$x_1^2/a_1^2+x_2^2/a_2^2+\ldots+x_n^2/a_n^2 \ngtr a^2 \quad \text{(instead of } \ngtr 1\text{)},$$

we could deduce the new result from the former by writing

a_1a in place of a_1, a_2a in place of a_2, and so on,

and therefore ka in place of k;

and, finally, if the scope of the range of the variables is still further limited by

$$x_1^2/a_1^2+\ldots+x_n^2/a_n^2 \ngtr a^2 \quad \text{and} \quad \nless \beta^2,$$

we must subtract all cases for which $x_1^2/a_1^2+\ldots+x_n^2/a_n^2$ is $\ngtr \beta^2$, and we shall have

$$I/a_1a_2\ldots a_n\pi^{\frac{n-1}{2}}$$

$$=\Sigma B_p\frac{\Gamma(p+1)}{\Gamma\left(\frac{n+1}{2}+p\right)}\int_{-1}^{1}(1-u^2)^{\frac{n-1}{2}+p}\left[a^nf(kau)-\beta^nf(k\beta u)\right]du.$$

975. **Deductions.**

Compare with the foregoing results the series of integrals

$$\int x_1^{i_1-1}x_2^{i_2-1}\,dx_1, \qquad \text{where} \quad x_1+x_2=1,$$

$$\iint x_1^{i_1-1}x_2^{i_2-1}x_3^{i_3-1}\,dx_1\,dx_2, \qquad \text{where} \quad x_1+x_2+x_3=1,$$

etc.,

$$\iiint\ldots\int x_1^{i_1-1}\ldots x_n^{i_n-1}\,dx_1\ldots dx_{n-1}, \quad \text{where} \quad x_1+\ldots+x_{n-1}+x_n=1,$$

for positive values of the several variables.

Take for instance the second. Here $x_3=1-x_1-x_2$, and the integration

$$I\equiv\iint x_1^{i_1-1}x_2^{i_2-1}(1-x_1-x_2)^{i_3-1}\,dx_1\,dx_2$$

is to be conducted for all positive values of x_1, x_2, such that $x_1+x_2 \not> 1$,

Then
$$I=\frac{\Gamma(i_1)\Gamma(i_2)}{\Gamma(i_1+i_2)}\int_0^1 v^{i_1+i_2-1}(1-v)^{i_3-1}\,dv$$
$$=\frac{\Gamma(i_1)\Gamma(i_2)}{\Gamma(i_1+i_2)}\,\frac{\Gamma(i_1+i_2)\Gamma(i_3)}{\Gamma(i_1+i_2+i_3)}=\frac{\Gamma(i_1)\Gamma(i_2)\Gamma(i_3)}{\Gamma(i_1+i_2+i_3)}.$$

976. Similarly, in the general case,
$$I=\underset{(n-1\text{ signs})}{\iint\cdots\int} x_1^{i_1-1}x_2^{i_2-1}\dots x_{n-1}^{i_{n-1}-1}x_n^{i_n-1}\,dx_1\,dx_2\dots dx_{n-1}$$
for positive values of $x_1, x_2, \dots x_n$, such that $x_1+\dots+x_{n-1}+x_n=1$,
$$I=\underset{(n-1\text{ signs})}{\iint\cdots\int} x_1^{i_1-1}\dots x_{n-1}^{i_{n-1}-1}(1-x_1-\dots-x_{n-1})^{i_n-1}\,dx_1\dots dx_{n-1},$$
where $x_1+x_2+\dots+x_{n-1} \not> 1$
$$=\frac{\Gamma(i_1)\Gamma(i_2)\dots\Gamma(i_{n-1})}{\Gamma(i_1+i_2+\dots+i_{n-1})}\int_0^1 v^{i_1+i_2+\dots+i_{n-1}-1}(1-v)^{i_n-1}\,dv$$
$$=\frac{\Gamma(i_1)\Gamma(i_2)\dots\Gamma(i_{n-1})}{\Gamma(i_1+i_2+\dots+i_{n-1})}\,\frac{\Gamma(i_1+i_2+\dots+i_{n-1})\Gamma(i_n)}{\Gamma(i_1+i_2+\dots+i_{n-1}+i_n)}$$
$$=\frac{\Gamma(i_1)\Gamma(i_2)\dots\Gamma(i_n)}{\Gamma(i_1+i_2+\dots+i_n)}.$$

Thus, if $A\equiv\underset{(n\text{ signs})}{\iint\cdots\int} x_1^{i_1-1}\dots x_n^{i_n-1}\,dx_1\dots dx_n$, for $\sum_1^n x_r \not> 1$,

and
$$B\equiv\underset{(n-1\text{ signs})}{\iint\cdots\int} x_1^{i_1-1}\dots x_n^{i_n-1}\,dx_1\dots dx_{n-1},\ \text{for } \sum_1^n x_r=1,$$
we have
$$(i_1+i_2+\dots+i_n)A=B=\frac{\Gamma(i_1)\Gamma(i_2)\dots\Gamma(i_n)}{\Gamma(i_1+i_2+\dots+i_n)}.$$

977. In the same way, if we require the value of
$$I=\underset{(n-1\text{ signs})}{\iint\cdots\int} x_1^{i_1-1}x_2^{i_2-1}\dots x_{n-1}^{i_{n-1}-1}x_n^{i_n-p_n}\,dx_1\,dx_2\dots dx_{n-1},$$
for positive values of the variables, such that
$$\left(\frac{x_1}{a_1}\right)^{p_1}+\left(\frac{x_2}{a_2}\right)^{p_2}+\dots+\left(\frac{x_{n-1}}{a_{n-1}}\right)^{p_{n-1}}+\left(\frac{x_n}{a_n}\right)^{p_n}=1,$$

we have $$x_n = a_n\left\{1-\left(\frac{x_1}{a_1}\right)^{p_1}-\dots-\left(\frac{x_{n-1}}{a_{n-1}}\right)^{p_{n-1}}\right\}^{\frac{1}{p_n}},$$

and $$I=\underset{(n-1\text{ signs})}{\iint\dots\int} x_1^{i_1-1}x_2^{i_2-1}\dots x_{n-1}^{i_{n-1}-1}a_n^{i_n-p_n}$$

$$\times\left\{1-\left(\frac{x_1}{a_1}\right)^{p_1}-\dots-\left(\frac{x_{n-1}}{a_{n-1}}\right)^{p_{n-1}}\right\}^{\frac{i_n}{p_n}-1} dx_1\,dx_2\dots dx_{n-1},$$

where $$\left(\frac{x_1}{a_1}\right)^{p_1}+\dots+\left(\frac{x_{n-1}}{a_{n-1}}\right)^{p_{n-1}} \not> 1,$$

$$=\frac{a_1^{i_1}\dots a_{n-1}^{i_{n-1}}}{p_1\dots p_{n-1}}a_n^{i_n-p_n}\frac{\Gamma\left(\frac{i_1}{p_1}\right)\dots\Gamma\frac{(i_{n-1})}{(p_{n-1})}}{\Gamma\left(\frac{i_1}{p_1}+\dots+\frac{i_{n-1}}{p_{n-1}}\right)}\int_0^1 v^{\lambda-1}(1-v)^{\frac{i_n}{p_n}-1}\,dv,$$

where $$\lambda=\frac{i_1}{p_1}+\frac{i_2}{p_2}+\dots+\frac{i_{n-1}}{p_{n-1}};$$

$$\therefore I=\frac{p_n}{a_n^{p_n}}\frac{a_1^{i_1}a_2^{i_2}\dots a_n^{i_n}}{p_1p_2\dots p_n}\frac{\Gamma\left(\frac{i_1}{p_1}\right)\Gamma\left(\frac{i_2}{p_2}\right)\dots\Gamma\left(\frac{i_n}{p_n}\right)}{\Gamma\left(\frac{i_1}{p_1}+\frac{i_2}{p_2}+\dots+\frac{i_n}{p_n}\right)}.$$

978. Ex. Find the value of $\iint x^{\lambda-1}y^{\mu-1}z^{\nu-1}\,dx\,dy$ for all points of the ellipsoidal *surface* $x^2/a^2+y^2/b^2+z^2/c^2=1$ which lie in the positive octant.

Here $i_1=\lambda$, $i_2=\mu$, $i_3=\nu+1$, $p_1=p_2=p_3=2$, $a_1=a$, $a_2=b$, $a_3=c$,

$$I=\frac{2}{c^2}\frac{a^\lambda b^\mu c^{\nu+1}}{2\,.\,2\,.\,2}\frac{\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(\frac{\mu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\lambda+\mu+\nu+1}{2}\right)}.$$

Thus, for instance,

$$\iint z\,dx\,dy=\frac{2}{c^2}\frac{abc^3}{2\,.\,2\,.\,2}\frac{\pi\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{2}\right)}=\tfrac{1}{6}\pi abc=\tfrac{1}{8}\cdot\tfrac{4}{3}\pi abc.$$

979. **Relation of the Integral Forms discussed.**

We note then that the two integrals

$$A\equiv\underset{(n\text{ signs})}{\iint\dots\int} x_1^{i_1-1}x_2^{i_2-1}\dots x_n^{i_n-1}\,dx_1\,dx_2\dots dx_n,\ \text{for}\ \sum_1^n\left(\frac{x_r}{a_r}\right)^{p_r}\not>1,$$

$$B\equiv\underset{(n-1\text{ signs})}{\iint\dots\int} x_1^{i_1-1}x_2^{i_2-1}\dots x_{n-1}^{i_{n-1}-1}x_n^{i_n-p_n}\,dx_1\,dx_2\dots dx_{n-1},\ \text{for}\ \sum_1^n\left(\frac{x_r}{a_r}\right)^{p_r}=1,$$

for positive values of the variables in each case, are so related that

$$\sum_1^n \frac{i_r}{p_r} A = \frac{a_n^{p_n}}{p_n} B = \frac{a_1^{i_1} a_2^{i_2} \dots a_n^{i_n}}{p_1 p_2 \dots p_n} \frac{\Gamma\left(\frac{i_1}{p_1}\right)\Gamma\left(\frac{i_2}{p_2}\right)\dots\Gamma\left(\frac{i_n}{p_n}\right)}{\Gamma\left(\frac{i_1}{p_1}+\frac{i_2}{p_2}+\dots+\frac{i_n}{p_n}\right)}.$$

980. A LEMMA.

In order to abbreviate the work of the articles which follow, let us note that the Binomial expansion

$$(1-z)^{-n} = 1 + nz + \frac{n(n+1)}{2!} z^2 + \dots + \frac{n(n+1)\dots(n+r-1)}{r!} z^r + \dots$$

may be written as $\sum_0^\infty K_r^{(n)} z^r$, where $K_r^{(n)} = \frac{\Gamma(n+r)}{\Gamma(n)} \frac{1}{r!}$,

and that, writing $i_1 + i_2 = j_2$, $i_1 + i_2 + i_3 = j_3$, etc., we have

$$K_r^{(j_2)} \frac{\Gamma(i_1)\Gamma(i_2+r)}{\Gamma(i_1+i_2+r)} = \frac{\Gamma(j_2+r)}{\Gamma(j_2) r!} \cdot \frac{\Gamma(i_1)\Gamma(i_2+r)}{\Gamma(j_2+r)}$$

$$= \frac{\Gamma(i_1)\Gamma(i_2)}{\Gamma(j_2)} \cdot \frac{\Gamma(i_2+r)}{\Gamma(i_2) r!} = \frac{\Gamma(i_1)\Gamma(i_2)}{\Gamma(i_1+i_2)} K_r^{(i_2)},$$

$$K_r^{(j_3)} \frac{\Gamma(i_1)\Gamma(i_2)\Gamma(i_3+r)}{\Gamma(i_1+i_2+i_3+r)} = \frac{\Gamma(j_3+r)}{\Gamma(j_3) r!} \cdot \frac{\Gamma(i_1)\Gamma(i_2)\Gamma(i_3+r)}{\Gamma(j_3+r)}$$

$$= \frac{\Gamma(i_1)\Gamma(i_2)\Gamma(i_3)}{\Gamma(j_3)} \cdot \frac{\Gamma(i_3+r)}{\Gamma(i_3) r!} = \frac{\Gamma(i_1)\Gamma(i_2)\Gamma(i_3)}{\Gamma(i_1+i_2+i_3)} K_r^{(i_3)},$$

etc.,

and

$$K_\rho^{(j_3+r)} \frac{\Gamma(i_1)\Gamma(i_2+\rho)\Gamma(i_3+r)}{\Gamma(i_1+i_2+\rho+i_3+r)} = \frac{\Gamma(j_3+r+\rho)}{\Gamma(j_3+r)\rho!} \cdot \frac{\Gamma(i_1)\Gamma(i_2+\rho)\Gamma(i_3+r)}{\Gamma(j_3+\rho+r)}$$

$$= \frac{\Gamma(i_1)\Gamma(i_2)\Gamma(i_3+r)}{\Gamma(j_3+r)} \cdot \frac{\Gamma(i_2+\rho)}{\Gamma(i_2)\rho!} = \frac{\Gamma(i_1)\Gamma(i_2)\Gamma(i_3+r)}{\Gamma(i_1+i_2+i_3+r)} K_\rho^{(i_2)},$$

etc.

981. We propose now to consider integrals of the class

$$I_n = \iiint \dots \int \frac{x_1^{i_1-1} x_2^{i_2-1} \dots x_n^{i_n-1} f\left(\sum_1^n A_r x_r\right) dx_1 dx_2 \dots dx_n}{(\lambda + a_1 x_1 + a_2 x_2 + \dots + a_n x_n)^{i_1+i_2+\dots+i_n}}$$

for all positive values of the variables, such that

$$h_1 < A_1 x_1 + A_2 x_2 + \dots + A_n x_n < h_2,$$

all the letters involved representing positive quantities.

Putting

$$A_1x_1=\xi_1,\ A_2x_2=\xi_2,\ \text{etc., and}\ \frac{a_1}{A_1}=b_1,\ \frac{a_2}{A_2}=b_2,\ \text{etc.,}$$

$$I_n=\frac{1}{A_1^{i_1}A_2^{i_2}\dots A_n^{i_n}}\iint\dots\int\frac{\xi_1^{i_1-1}\dots\xi_n^{i_n-1}f(\xi_1+\dots+\xi_n)d\xi_1\dots d\xi_n}{(\lambda+b_1\xi_1+b_2\xi_2+\dots+b_n\xi_n)^{i_1+i_2+\dots+i_n}}.$$

Consider first the case of a double integral,

$$I_2=\frac{1}{A_1^{i_1}A_2^{i_2}}\iint\frac{\xi_1^{i_1-1}\xi_2^{i_2-1}f(\xi_1+\xi_2)}{(\lambda+b_1\xi_1+b_2\xi_2)^{i_1+i_2}}d\xi_1 d\xi_2,$$

a particular case of which is discussed by Todhunter (*Int. Calc.*, p. 263). Of the two quantities b_1, b_2, let b_1 be the one which is not less than the other. Then

$$\lambda+b_1\xi_1+b_2\xi_2\equiv\{\lambda+b_1(\xi_1+\xi_2)\}-(b_1-b_2)\xi_2,\quad =u-v,\ \text{say},$$

where $v=(b_1-b_2)\xi_2$. Then as $\lambda+b_1\xi_1+b_2\xi_2$ is a positive quantity, we have $v<u$, and

$$(\lambda+b_1\xi_1+b_2\xi_2)^{-(i_1+i_2)}=(u-v)^{-(i_1+i_2)}=u^{-(i_1+i_2)}\left(1-\frac{v}{u}\right)^{-(i_1+i_2)}$$

$$=u^{-(i_1+i_2)}\sum_0^\infty K_r^{(i_1+i_2)}(b_1-b_2)^r\left(\frac{\xi_2}{u}\right)^r,$$

a convergent binomial expansion. Hence the integral becomes

$$\frac{1}{A_1^{i_1}A_2^{i_2}}\iint\frac{\xi_1^{i_1-1}\xi_2^{i_2-1}f(\xi_1+\xi_2)}{u^{(i_1+i_2)}}\sum_0^\infty K_r^{(i_1+i_2)}(b_1-b_2)^r\left(\frac{\xi_2}{u}\right)^r d\xi_1 d\xi_2$$

$$=\frac{1}{A_1^{i_1}A_2^{i_2}}\sum_0^\infty K_r^{(i_1+i_2)}(b_1-b_2)^r\iint\frac{\xi_1^{i_1-1}\xi_2^{i_2+r-1}f(\xi_1+\xi_2)}{u^{i_1+i_2+r}}d\xi_1 d\xi_2,$$

and u being a function of $\xi_1+\xi_2$, we have, by Art. 968,

$$I_2=\frac{1}{A_1^{i_1}A_2^{i_2}}\sum_0^\infty K_r^{(i_1+i_2)}(b_1-b_2)^r\frac{\Gamma(i_1)\,\Gamma(i_2+r)}{\Gamma(i_1+i_2+r)}\int_{h_1}^{h_2}\frac{t^{i_1+i_2+r-1}f(t)}{(\lambda+b_1t)^{i_1+i_2+r}}dt$$

$$=\frac{1}{A_1^{i_1}A_2^{i_2}}\sum_0^\infty\frac{\Gamma(i_1)\Gamma(i_2)}{\Gamma(i_1+i_2)}K_r^{(i_2)}(b_1-b_2)^r\int_{h_1}^{h_2}\frac{t^{i_1+i_2+r-1}f(t)}{(\lambda+b_1t)^{i_1+i_2+r}}dt$$

$$=\frac{1}{A_1^{i_1}A_2^{i_2}}\frac{\Gamma(i_1)\Gamma(i_2)}{\Gamma(i_1+i_2)}\int_{h_1}^{h_2}\frac{t^{i_1+i_2-1}f(t)}{(\lambda+b_1t)^{i_1+i_2}}\sum_0^\infty K_r^{(i_2)}(b_1-b_2)^r\frac{t^r}{(\lambda+b_1t)^r}dt$$

$$=\frac{1}{A_1^{i_1}A_2^{i_2}}\frac{\Gamma(i_1)\Gamma(i_2)}{\Gamma(i_1+i_2)}\int_{h_1}^{h_2}\frac{t^{i_1+i_2-1}f(t)}{(\lambda+b_1t)^{i_1+i_2}}\left\{1-\frac{(b_1-b_2)t}{\lambda+b_1t}\right\}^{-i_2}dt$$

$$=\frac{1}{A_1^{i_1}A_2^{i_2}}\frac{\Gamma(i_1)\Gamma(i_2)}{\Gamma(i_1+i_2)}\int_{h_1}^{h_2}\frac{t^{i_1+i_2-1}f(t)}{(\lambda+b_1t)^{i_1}(\lambda+b_2t)^{i_2}}dt$$

$$=\frac{\Gamma(i_1)\Gamma(i_2)}{\Gamma(i_1+i_2)}\int_{h_1}^{h_2}\frac{t^{i_1+i_2-1}f(t)\,dt}{(A_1\lambda+a_1t)^{i_1}(A_2\lambda+a_2t)^{i_2}}.$$

982. Next take the case of the triple integral

$$I_3=\frac{1}{A_1^{i_1}A_2^{i_2}A_3^{i_3}}\iiint\frac{\xi_1^{i_1-1}\xi_2^{i_2-1}\xi_3^{i_3-1}f(\xi_1+\xi_1+\xi_3)\,d\xi_1\,d\xi_2\,d\xi_3}{(\lambda+b_1\xi_1+b_2\xi_2+b_3\xi_3)^{i_1+i_2+i_3}}.$$

Of these three quantities b_1, b_2, b_3, let b_1 be that which is not less than either of the other two. Then

$$\lambda+b_1\xi_1+b_2\xi_2+b_3\xi_3=\{\lambda+b_1(\xi_1+\xi_3)+b_2\xi_2\}-(b_1-b_3)\xi_3,\quad =u-v, \text{ say},$$

where $v=(b_1-b_3)\xi_3$, and is $<u$ and positive. Let $i_1+i_2+i_3=j_3$. Then

$$(\lambda+b_1\xi_1+b_2\xi_2+b_3\xi_3)^{-j_3}=u^{-j_3}\left(1-\frac{v}{u}\right)^{-j_3}=u^{-j_3}\sum_0^\infty K_r^{(j_3)}(b_1-b_3)^r\left(\frac{\xi_3}{u}\right)^r,$$

a convergent binomial expansion.

$$\therefore\ I_3=\sum_0^\infty\frac{(b_1-b_3)^r}{A_1^{i_1}A_2^{i_2}A_3^{i_3}}\iiint\frac{\xi_1^{i_1-1}\xi_2^{i_2-1}\xi_3^{i_3-1}f\left(\sum_1^3\xi_r\right)}{u^{j_3}}K_r^{(j_3)}\left(\frac{\xi_3}{u}\right)^r d\xi_1\,d\xi_2\,d\xi_3,$$

where u is, however, $\lambda+b_1(\xi_1+\xi_2)+b_2\xi_2$, and is not this time a function of the sum of the variables. Hence a further transformation is necessary.

We may write

$$u\equiv\lambda+b_1(\xi_1+\xi_2)+b_2\xi_2=[\lambda+b_1(\xi_1+\xi_2+\xi_3)]-(b_1-b_2)\xi_2$$
$$=U-V, \text{ say},$$

where $V\equiv(b_1-b_2)\xi_2$ is $<U$, and U is a function of $\xi_1+\xi_2+\xi_3$.

Also, writing $i_1+i_2+i_3+r=j_3'$ where necessary to shorten

$$u^{-j_3'}=U^{-j_3'}\left(1-\frac{V}{U}\right)^{-j_3'}=U^{-j_3'}\sum K_\rho^{(j_3')}(b_1-b_2)^\rho\left(\frac{\xi_2}{U}\right)^\rho,$$

a convergent binomial expansion.

Hence

$$\iiint\frac{\xi_1^{i_1-1}\xi_2^{i_2-1}\xi_3^{i_3+r-1}f(\Sigma\xi)}{u^{i_1+i_2+i_3+r}}d\xi_1\,d\xi_2\,d\xi_3$$

$$=\iiint\frac{\xi_1^{i_1-1}\xi_2^{i_2-1}\xi_3^{i_3+r-1}}{U^{j_3'}}f(\Sigma\xi)\sum_{\rho=0}^{\rho=\infty}K_\rho^{(j_3')}(b_1-b_2)^\rho\left(\frac{\xi_2}{U}\right)^\rho d\xi_1\,d\xi_2\,d\xi_3$$

$$=\iiint\sum_{\rho=0}^{\rho=\infty}K_\rho^{(j_3')}(b_1-b_2)^\rho\frac{\xi_1^{i_1-1}\xi_2^{i_2+\rho-1}\xi_3^{i_3+r-1}}{U^{j_3'+\rho}}f(\Sigma\xi)\,d\xi_1\,d\xi_2\,d\xi_3$$

$$=\int_{h_1}^{h_2}\sum_{\rho=0}^{\rho=\infty}K_\rho^{(j_3')}\frac{\Gamma(i_1)\,\Gamma(i_2+\rho)\,\Gamma(i_3+r)}{\Gamma(i_1+i_2+\rho+i_3+r)}(b_1-b_2)^\rho\frac{t^{j_3'+\rho-1}f(t)}{(\lambda+b_1t)^{j_3'+\rho}}dt$$

$$=\int_{h_1}^{h_2}\frac{\Gamma(i_1)\Gamma(i_2)\Gamma(i_3+r)}{\Gamma(i_1+i_2+i_3+r)}\frac{t^{j_3'-1}}{(\lambda+b_1t)^{j_3'}}\sum_{\rho=0}^{\rho=\infty}K_\rho^{(i_2)}(b_1-b_2)^\rho\frac{t^\rho}{(\lambda+b_1t)^\rho}f(t)\,dt$$

$$=\frac{\Gamma(i_1)\Gamma(i_2)\Gamma(i_3+r)}{\Gamma(i_1+i_2+i_3+r)}\int_{h_1}^{h_2}\frac{t^{j_3'-1}}{(\lambda+b_1t)^{j_3'}}\left\{1-\frac{(b_1-b_2)t}{\lambda+b_1t}\right\}^{-i_2}f(t)\,dt$$

$$=\frac{\Gamma(i_1)\Gamma(i_2)\Gamma(i_3+r)}{\Gamma(i_1+i_2+i_3+r)}\int_{h_1}^{h_2}\frac{t^{j_3'-1}f(t)\,dt}{(\lambda+b_1t)^{i_1+i_3+r}(\lambda+b_2t)^{i_2}};$$

$$\therefore\ I=\sum_{r=0}^{r=\infty}\frac{(b_1-b_3)^r}{A_1^{\ i_1}A_2^{\ i_2}A_3^{\ i_3}}K_r^{(j_3)}\frac{\Gamma(i_1)\Gamma(i_2)\Gamma(i_3+r)}{\Gamma(i_1+i_2+i_3+r)}\int_{h_1}^{h_2}\frac{t^{j_3'-1}f(t)\,dt}{(\lambda+b_1t)^{i_1+i_3+r}(\lambda+b_2t)^{i_2}}$$

$$=\frac{1}{A_1^{\ i_1}A_2^{\ i_2}A_3^{\ i_3}}\frac{\Gamma(i_1)\Gamma(i_2)\Gamma(i_3)}{\Gamma(i_1+i_2+i_3)}\int_{h_1}^{h_2}\frac{t^{j_3-1}f(t)}{(\lambda+b_1t)^{i_1+i_3}(\lambda+b_2t)^{i_2}}\sum_{r=0}^{r=\infty}K_r^{(i_3)}\frac{(b_1-b_3)^rt^r}{(\lambda+b_1t)^r}\,dt$$

$$=\frac{1}{A_1^{\ i_1}A_2^{\ i_2}A_3^{\ i_3}}\frac{\Gamma(i_1)\Gamma(i_2)\Gamma(i_3)}{\Gamma(i_1+i_2+i_3)}\int_{h_1}^{h_2}\frac{t^{j_3-1}f(t)}{(\lambda+b_1t)^{i_1+i_3}(\lambda+b_2t)^{i_2}}\left\{1-\frac{(b_1-b_3)t}{\lambda+b_1t}\right\}^{-i_3}dt$$

$$=\frac{1}{A_1^{\ i_1}A_2^{\ i_2}A_3^{\ i_3}}\frac{\Gamma(i_1)\Gamma(i_2)\Gamma(i_3)}{\Gamma(i_1+i_2+i_3)}\int_{h_1}^{h_2}\frac{t^{j_3-1}f(t)}{\prod_1^3(\lambda+b_st)^{i_s}}\,dt$$

$$=\frac{\Gamma(i_1)\Gamma(i_2)\Gamma(i_3)}{\Gamma(i_1+i_2+i_3)}\int_{h_1}^{h_2}\frac{t^{j_3-1}f(t)\,dt}{\prod_1^3(A_s\lambda+a_st)^{i_s}}.$$

983. Exactly the same process will hold for a multiple integral of higher order, so that in general we have

$$I_n=\frac{\Gamma(i_1)\Gamma(i_2)\ldots\Gamma(i_n)}{\Gamma(i_1+i_2+\ldots+i_n)}\int_{h_1}^{h_2}\frac{t^{i_1+i_2+\ldots+i_n-1}f(t)}{\prod_1^n(A_s\lambda+a_st)^{i_s}}\,dt.$$

984. **Extension.**

The result may obviously be extended to the integral

$$I_n=\iint\ldots\int\frac{x_1^{\ i_1-1}x_2^{\ i_2-1}\ldots x_n^{\ i_n-1}f\left(\sum_1^n A_rx_r^{\ a_r}\right)dx_1\,dx_2\ldots dx_n}{(\lambda+a_1x_1^{\ a_1}+a_2x_2^{\ a_2}+\ldots+a_nx_n^{\ a_n})^k},$$

where $$k=\frac{i_1}{a_1}+\frac{i_2}{a_2}+\ldots+\frac{i_n}{a_n},$$

all the letters involved being positive quantities and the conditions of the limits being

$$h_1<A_1x_1^{\ a_1}+A_2x_2^{\ a_2}+\ldots+A_nx_n^{\ a_n}<h_2.$$

For putting $A_1x_1^{a_1}=\xi_1$, $A_2x_2^{a_2}=\xi_2$, etc., $\dfrac{a_1}{A_1}=b_1$, $\dfrac{a_2}{A_2}=b_2$, etc., we have

$$I_n=\frac{1}{\prod\limits_1^n a_r A_r^{\frac{i_r}{a_r}}}\iint\ldots\int\frac{\xi_1^{\frac{i_1}{a_1}}\ldots\xi_n^{\frac{i_n}{a_n}}f(\xi_1+\ldots+\xi_n)}{(\lambda+b_1\xi_1+\ldots+b_n\xi_n)^k}\,\frac{d\xi_1 d\xi_2\ldots d\xi_n}{\xi_1\xi_2\ldots\xi_n}$$

$$=\frac{1}{a_1a_2\ldots a_n}\,\frac{\Gamma\left(\frac{i_1}{a_1}\right)\Gamma\left(\frac{i_2}{a_2}\right)\ldots\Gamma\left(\frac{i_n}{a_n}\right)}{\Gamma\left(\frac{i_1}{a_1}+\frac{i_2}{a_2}+\ldots+\frac{i_n}{a_n}\right)}\int_{h_1}^{h_2}\frac{t^{k-1}f(t)\,dt}{\prod\limits_1^n(A_s\lambda+a_st)^{\frac{i_s}{a_s}}}.$$

Thus in all such cases the multiple integral is reduced to a single integration.

985. **Differentiation with regard to a parameter contained in the integrand.**

In a multiple integral

$$u=\int_{a_1}^{b_1}\int_{a_2}^{b_2}\ldots\int_{a_n}^{b_n}\phi(x_1, x_2, \ldots x_n, c)\,dx_1dx_2\ldots dx_n,$$

which contains a constant c, differentiation with regard to c may be effected by the same rule as for a single integral, provided that the limits of the several integrals are all independent of c. That is

$$\frac{\partial u}{\partial c}=\int_{a_1}^{b_1}\int_{a_2}^{b_2}\ldots\int_{a_n}^{b_n}\frac{\partial\phi}{\partial c}\,dx_1dx_2\ldots dx_n.$$

The proof of this is the same as in the case of a single integral.

986. **Liouville's Integral.**

Consider the case

$$I=\int_0^\infty\int_0^\infty\ldots\int_0^\infty e^{-t}x_1^{\frac{1}{n}-1}x_2^{\frac{2}{n}-1}\ldots x_{n-1}^{\frac{n-1}{n}-1}\,dx_1dx_2\ldots dx_{n-1},\text{*}$$

where $t\equiv x_1+x_2+\ldots+x_{n-1}+\dfrac{a^n}{x_1x_2\ldots x_{n-1}}$,

an integral discussed by Liouville.

Differentiating with respect to a,

$$\frac{dI}{da}=-na^{n-1}\int_0^\infty\ldots\int_0^\infty e^{-t}x_1^{\frac{1}{n}-1}x_2^{\frac{2}{n}-1}\ldots x_{n-1}^{\frac{n-1}{n}-1}\frac{dx_1\ldots dx_{n-1}}{x_1x_2\ldots x_{n-1}}.$$

* Bertrand, *Calc. Intégral*, p. 476.

Now introduce another variable x_n defined by

$$x_1 x_2 \dots x_{n-1} x_n = a^n,$$

i.e. change to a system

$$x_1 = \frac{a^n}{x_2 x_3 \dots x_n}, \quad x_2 = x_2, \quad x_3 = x_3, \dots x_{n-1} = x_{n-1}.$$

Then $\quad J = \dfrac{\partial(x_1, x_2, \dots x_{n-1})}{\partial(x_2, x_3, \dots x_n)} = (-1)^{n-1} \dfrac{a^n}{x_2 x_3 \dots {x_n}^2}.$

Then $t \equiv x_1 + x_2 + \dots + x_{n-1} + \dfrac{a^n}{x_1 x_2 \dots x_{n-1}}$ is replaced by

$$x_2 + x_3 + \dots + x_n + \frac{a^n}{x_2 x_3 \dots x_n}, \ = t' \text{ say},$$

and $x_1^{\frac{1}{n}-1} x_2^{\frac{2}{n}-1} \dots x_{n-1}^{\frac{n-1}{n}-1} \dfrac{dx_1\, dx_2 \dots dx_{n-1}}{x_1 x_2 \dots x_{n-1}}$ is replaced by

$$J\left[\frac{a^n}{x_2 x_3 \dots x_n}\right]^{\frac{1}{n}-1} x_2^{\frac{2}{n}-1} x_3^{\frac{3}{n}-1} \dots x_{n-1}^{\frac{n-1}{n}-1} \frac{dx_2\, dx_3 \dots dx_n}{a^n/x_n},$$

i.e. $\quad (-1)^{n-1} a^{1-n} x_2^{\frac{1}{n}-1} x_3^{\frac{2}{n}-1} x_4^{\frac{3}{n}-1} \dots x_n^{\frac{n-1}{n}-1} dx_2 dx_3 \dots dx_n,$

and in the transformation of the multiple integral the sign is adjusted by a proper assignment of the limits.

Hence, as x_n is ∞ when x_1 is zero and *vice versa*, we have

$$\frac{dI}{da} = -na^{n-1} \int_0^\infty \dots \int_0^\infty a^{1-n} e^{-t'} x_2^{\frac{1}{n}-1} x_3^{\frac{2}{n}-1} \dots x_n^{\frac{n-1}{n}-1} dx_2 dx_3 \dots dx_n$$

$$= -nI \text{ (for if } a \text{ is increased } I \text{ is decreased).}$$

Hence $\dfrac{dI}{I} = -n\, da, \quad \log I = -na + \text{const.}, \quad I = Ce^{-na}.$

To find C, take the case $a = 0$.

Then I becomes

$$\int_0^\infty \int_0^\infty \dots \int_0^\infty e^{-(x_1 + x_2 + \dots + x_{n-1})} x_1^{\frac{1}{n}-1} x_2^{\frac{2}{n}-1} \dots x_{n-1}^{\frac{n-1}{n}-1} dx_1\, dx_2 \dots dx_{n-1},$$

and as the variables are independent and the limits constants, this may be written

$$\left[\int_0^\infty e^{-x_1} x_1^{\frac{1}{n}-1} dx_1\right] \times \left[\int_0^\infty e^{-x_2} x_2^{\frac{2}{n}-1} dx_2\right] \dots \times \left[\int_0^\infty e^{-x_{n-1}} x_{n-1}^{\frac{n-1}{2}-1} dx_{n-1}\right],$$

that is $\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\dots\Gamma\left(\frac{n-1}{n}\right)$ or $(2\pi)^{\frac{n-1}{2}}n^{-\frac{1}{2}}$.

Hence $$C=(2\pi)^{\frac{n-1}{2}}n^{-\frac{1}{2}}.$$

Hence the value of the integral is
$$I=(2\pi)^{\frac{n-1}{2}}n^{-\frac{1}{2}}e^{-na}.$$

987. **Liouville's Method of proving Gauss' Theorem.**

Consider the product
$$\Gamma(x)\Gamma\left(x+\frac{1}{n}\right)\Gamma\left(x+\frac{2}{n}\right)\dots\Gamma\left(x+\frac{n-1}{n}\right).$$

This may be written
$$\int_0^\infty e^{-x_1}x_1^{x-1}\,dx_1\times\int_0^\infty e^{-x_2}x_2^{x+\frac{1}{n}-1}\,dx_2\dots\times\int_0^\infty e^{-x_n}x_n^{x+\frac{n-1}{n}-1}\,dx_n$$
$$=\int_0^\infty\int_0^\infty\dots\int_0^\infty e^{-(x_1+x_2+\dots+x_n)}x_1^{x-1}x_2^{x+\frac{1}{n}-1}\dots x_n^{x+\frac{n-1}{n}-1}\,dx_1\,dx_2\dots dx_n.$$

Now change the variables according to the scheme
$$x_1=\frac{z^n}{x_2x_3\dots x_n},\quad x_2=x_2,\quad x_3=x_3\dots x_n=x_n.$$

Then $J=\dfrac{nz^{n-1}}{x_2x_3\dots x_n}$, and the integral may be written
$$\int_0^\infty\int_0^\infty\dots\int_0^\infty e^{-\left(x_2+x_3+\dots+x_n+\frac{z^n}{x_2x_3\dots x_n}\right)}\frac{nz^{n-1}}{x_2x_3\dots x_n}$$
$$\times\left(\frac{z^n}{x_2x_3\dots x_n}\right)^{x-1}x_2^{x+\frac{1}{n}-1}x_3^{x+\frac{2}{n}-1}\dots x_n^{x+\frac{n-1}{n}-1}\,dz\,dx_2\,dx_3\dots dx_n,$$

that is
$$n\int_0^\infty\int_0^\infty\dots\int_0^\infty e^{-t'}z^{nx-1}x_2^{\frac{1}{n}-1}x_3^{\frac{2}{n}-1}\dots x_n^{\frac{n-1}{n}-1}\,dz\,dx_2\,dx_3\dots dx_n$$
$$=n\int_0^\infty(2\pi)^{\frac{n-1}{2}}n^{-\frac{1}{2}}e^{-nz}z^{nx-1}\,dz,\quad\text{by the preceding article,}$$
$$=n^{\frac{1}{2}}(2\pi)^{\frac{n-1}{2}}\int_0^\infty e^{-nz}z^{nx-1}\,dz=n^{\frac{1}{2}-nx}(2\pi)^{\frac{n-1}{2}}\Gamma(nx),$$

viz.
$$n^{nx}\Gamma(x)\Gamma\left(x+\frac{1}{n}\right)\dots\Gamma\left(x+\frac{n-1}{n}\right)=n^{\frac{1}{2}}(2\pi)^{\frac{n-1}{2}}\Gamma(nx),$$

which is Gauss' result.

PROBLEMS.

1. Find the mass of the triangular lamina bounded by the axes of coordinates and the line $x+y=a$ for a law of surface density $\mu x^p y^q$.

2. Find the mass of the tetrahedron bounded by the coordinate planes and the plane $a^{-1}x+b^{-1}y+c^{-1}z=1$, the volume density being $\rho=\mu xyz$.

3. Find the centroid of the area in the first quadrant bounded by the lines $x+y=h_1$, $x+y=h_2$, for a law of surface density $\sigma=\mu x^p y^q$.

4. Find the centroid of the volume in the first octant bounded by the coordinate planes and the two planes

$$a^{-1}x+b^{-1}y+c^{-1}z=\delta_1, \quad a^{-1}x+b^{-1}y+c^{-1}z=\delta_2,$$

for the following laws of volume-density:

(i) $\rho=\mu(a^{-1}x+b^{-1}y+c^{-1}z)$, (ii) $\rho=\mu x^p y^q z^r$, (iii) $\rho=\mu(x^2+y^2+z^2)$.

5. Apply Dirichlet's theorem to find the mass of an octant of an ellipsoid in which the density at any point varies as the square of the product of the distances of the point from the principal sections of the ellipsoid.

6. Find the moment of inertia about the x-axis of the portion of the sphere $x^2+y^2+z^2=a^2$, which lies in the positive octant, supposing the law of volume density to be $\rho=\mu xyz$. Obtain the corresponding result for an octant of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$.

7. Find the mass of the positive octant of a sphere of radius R, whose centre is the origin, for a law of volume density

$$\rho=\mu(a,\ b,\ c,\ f,\ g,\ h)(x,\ y,\ z)^2.$$

8. Find the mass, centroid and moments of inertia about the axes, of the positive octant of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, for a law of volume density $\rho=\mu(x^2+y^2+z^2)$.

9. Show that the volume of the solid, the equation of whose surface is $a^{-4}x^4+b^{-4}y^4+c^{-4}z^4=1$, is $\dfrac{abc\sqrt{2}}{12\pi}\{\Gamma(\tfrac{1}{4})\}^4$.

10. A homogeneous solid is bounded by the surface

$$(x/a)^{\frac{2}{3}}+(y/b)^{\frac{2}{3}}+(z/c)^{\frac{2}{3}}=1.$$

Show that the centroid of the portion of it in the positive octant is the point

$$\left(\frac{21a}{128},\ \frac{21b}{128},\ \frac{21c}{128}\right).$$

[Oxf. II. Pub., 1901.]

11. Find the position of the centroid of the portion of the solid bounded by
$$(x/a)^{2l}+(y/b)^{2m}+(z/c)^{2n}=1,$$
which lies in the positive octant, the volume density being $\mu x^p y^q z^r$.

12. Show that $\iint x^{2l-1}y^{2m-1}\,dx\,dy$ for positive values of x and y, such that $x^2+y^2 \not> c^2$, is
$$\tfrac{1}{4}c^{2l+2m}\frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}.$$
[I. C. S., 1893.]

13. Obtain an expression for the value of
$$\iint x^{2l-1}y^{2m-1}f(ax^2+by^2)\,dx\,dy$$
for all positive values of x and y, such that $ax^2+by^2 \not> c^2$.

[I. C. S., 1893.]

14. Prove that the value of the volume integral
$$\iiint(\lambda x+\mu y+\nu z)^{2n}\,dx\,dy\,dz,$$
taken through the volume of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, λ, μ, ν being constants and n a positive integer, is
$$4\pi abc(\lambda^2a^2+\mu^2b^2+\nu^2c^2)^n/(2n+1)(2n+3).$$
[I. C. S., 1912.]

15. Find the value for positive values of x, y, z of
$$\iiint xyz\sin(x+y+z)\,dx\,dy\,dz$$
with condition $x+y+z \leqslant \frac{1}{2}\pi$. [I. C. S., 1899.]

16. Prove that
$$\int_0^\infty\int_0^\infty \phi(x+y)\,x^\alpha y^\beta\,dx\,dy$$
$$=\frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\int_0^\infty \phi(z)\,z^{\alpha+\beta+1}\,dz,$$
and extend the theorem to any number of variables. [Coll. γ, 1887.]

17. Prove that the area of the curve
$$(ax+by)^{2n}+(bx-ay)^{2n}=1 \quad\text{is}\quad \left[\Gamma\left(\frac{1}{2n}\right)\right]^2\Big/ n(a^2+b^2)\,\Gamma\left(\frac{1}{n}\right).$$
[Coll. γ, 1891.]

18. Find the volume enclosed by the surface
$$(x/a)^{2n}+(y/b)^{2n}+(z/c)^{2n}=1,$$
where n is an integer. [Math. Trip., Part II., 1919.

Show that the distance of the centroid of the portion for which x is positive from the plane $x=0$ is
$$\bar{x}=\frac{3a}{4}\,\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{3}{2n}\right)\Big/\,\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{1}{2n}\right).$$

19. Prove that $\iint\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{p-1} f(\alpha x+\beta y)\,dx\,dy$

$$=\sqrt{\pi}\,ab\,\frac{\Gamma(p)}{\Gamma(p+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{p-\frac{1}{2}}f(kt)\,dt,$$

where $k=(a^2\alpha^2+b^2\beta^2)^{\frac{1}{2}}$, the double integral being taken for all values of x and y, such that

$$x^2/a^2+y^2/b^2<1.$$ [γ, 1899.]

20. Show that, $xyzu$ being equal to a^4,

$$\int_0^\infty\int_0^\infty\int_0^\infty e^{-(x^4+y^4+z^4+u^4)}\,yz^2\,dx\,dy\,dz=\frac{\pi^{\frac{3}{2}}}{32\sqrt{2}\,e^{4a^4}}.$$

[St. John's, 1882.]

21. Show that

$$\iiint\frac{dx\,dy\,dz}{(\rho+\alpha x^2+\beta y^2+\gamma z^2)^{\frac{5}{2}}}=\frac{\pi}{6}\,\frac{abc}{\rho\sqrt{(\rho+a^2\alpha)(\rho+b^2\beta)(\rho+c^2\gamma)}},$$

where x, y, z have all positive values such that

$$x^2/a^2+y^2/b^2+z^2/c^2<1.$$ [Colleges γ, 1891.]

22. Prove that

$$\iint\frac{(1-x-y)^{k-1}x^{m-1}y^{n-1}}{(\rho+\alpha x+\beta y)^{k+m+n+1}}\,dx\,dy$$

$$=\frac{\Gamma(k)\,\Gamma(m)\,\Gamma(n)}{\Gamma(k+m+n+1)}\left\{\frac{k}{\rho}+\frac{m}{\rho+\alpha}+\frac{n}{\rho+\beta}\right\}\frac{1}{\rho^k(\rho+\alpha)^m(\rho+\beta)^n},$$

the integral extending to all positive values of x and y such that

$$x+y<1.$$ [Colleges γ, 1891.]

23. Show that

$$\iint\cdots\int\frac{x_1^{i_1-1}x_2^{i_2-1}\dots x_n^{i_n-1}f(x_1^{i_1}+x_2^{i_2}+\dots+x_n^{i_n})}{(\lambda+a_1x_1^{i_1}+a_2x_2^{i_2}+\dots+a_nx_n^{i_n})^n}\,dx_1\,dx_2\dots dx_n$$

$$=\frac{(-1)^{n-1}}{i_1i_2\dots i_n}\,\frac{1}{\Gamma(n)}\,\Sigma\,\frac{1}{(a_1-a_2)(a_1-a_3)\dots(a_1-a_n)}\int_0^1\frac{f(t)}{\lambda+a_1t}\,dt,$$

the summation referring to a cyclical change of letters from a_1 to a_n, and the integration being effected for all positive values of the variables for which $x_1^{i_1}+x_2^{i_2}+\dots \not> 1$.

24. Prove that, n, r being positive whole numbers,

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\frac{dx_1\,dx_2\dots dx_{2n}}{\left(a^2+\sum_1^{2n}x_r^2\right)^{\frac{2n+2r+1}{2}}}=\frac{\pi^n}{2a^{2r+1}}\,\frac{(n+r-1)!}{(2n+2r-1)!}\,\frac{(2r)!}{r!}.$$

[Math. Trip., 1870, Wolstenholme.]

25. Prove that

$$\int_0^{x_1}\frac{dx_2}{(x_1-x_2)^{\frac{n-1}{n}}}\int_0^{x_2}\frac{dx_3}{(x_2-x_3)^{\frac{n-1}{n}}}\int_0^{x_3}\frac{dx_4}{(x_3-x_4)^{\frac{n-1}{n}}}\cdots\int_0^{x_n}\frac{f'(\xi)\,d\xi}{(x_n-\xi)^{\frac{n-1}{n}}}$$
$$=\left\{\Gamma\left(\frac{1}{n}\right)\right\}^n\{f(x_1)-f(0)\}.$$

(See Ex. 30, Ch. XXIV.) [MATH. TRIPOS, 1875.]

26. Prove that

$$\int_0^\infty\int_0^\infty e^{-\left(x_1+x_2+\frac{a^3}{x_1x_2}\right)}x_1^{\frac{1}{3}}x_2^{\frac{2}{3}}\frac{dx_1}{x_1}\frac{dx_2}{x_2}=e^{-3a}\Gamma(\tfrac{1}{3})\Gamma(\tfrac{2}{3}).$$

[LIOUVILLE.]

27. If n be a positive integer, show that for an integration conducted over a triangle of area Δ in the x-y plane

$$\iint y^n\,dx\,dy=\Delta H_n,$$

where H_n is the arithmetic mean of the homogeneous products of the ordinates of the corners, and find the corresponding result for any plane polygon. [ROUTH, *Rigid Dyn.*, p. 425.]

28. Show that if the integration be conducted for all positive values of x_1, x_2, x_3, x_4 such that $x_1+x_2 \not> 1$ and $x_3+x_4 \not> 1$, then

$$\iiiint x_1^{i_1-1}x_2^{i_2-1}x_3^{i_3-1}x_4^{i_4-1}dx_1dx_2dx_3dx_4$$
$$=\Gamma(i_1)\Gamma(i_2)\Gamma(i_3)\Gamma(i_4)/\Gamma(i_1+i_2+1)\Gamma(i_3+i_4+1),$$

29. If $\quad t\equiv x_1^n+x_2^n+\dots+x_n^n\quad$ and $\quad x_1x_2\dots x_n=a^n,$

evaluate the integral

$$\int_0^\infty\int_0^\infty\dots\int_0^\infty e^{-t}x_1^1x_2^2x_3^3\dots x_{n-1}^{n-1}\frac{dx_1}{x_1}\frac{dx_2}{x_2}\dots\frac{dx_{n-1}}{x_{n-1}}.$$

30. If $t\equiv x_1^{\frac{n}{1}}+x_2^{\frac{n}{2}}+x_3^{\frac{n}{3}}+\dots x_n^{\frac{n}{n}}$ and $x_1^{\frac{1}{1}}x_2^{\frac{1}{2}}x_3^{\frac{1}{3}}\dots x_n^{\frac{1}{n}}=a$, show that

$$\int_0^\infty\int_0^\infty\dots\int_0^\infty e^{-t}dx_1dx_2\dots dx_{n-1}=\frac{n!}{n^{n+\frac{1}{2}}}\frac{(2\pi)^{\frac{n-1}{2}}}{e^{na}}.$$