CHAPTER XXVII.

DEFINITE INTEGRALS (II.).

LOGARITHMIC AND EXPONENTIAL FUNCTIONS INVOLVED.

1071. In the class of definite integrals we are about to discuss, it will be convenient to remember the result

$$\int_0^1 x^p (\log x)^n \, dx = (-1)^n \frac{n!}{(p+1)^{n+1}}.$$

This is the result of integration by parts,

$$\int_0^1 x^p (\log x)^n \, dx = \left[\frac{x^{p+1}}{p+1} (\log x)^n\right]_0^1 - \frac{n}{p+1}\int_0^1 x^p (\log x)^{n-1} \, dx$$

$$= -\frac{n}{p+1}\int_0^1 x^p (\log x)^{n-1} \, dx$$

$$= (-1)^2 \frac{n(n-1)}{(p+1)^2}\int_0^1 x^p (\log x)^{n-2} \, dx = \text{etc.}$$

$$= (-1)^n \frac{n!}{(p+1)^{n+1}}.$$

Or we might obtain the same result by the transformation $x=e^{-y}$, viz.

$$\int_0^1 x^p (\log x)^n \, dx = \int_\infty^0 e^{-py}(-1)^n y^n (-e^{-y}) \, dy = (-1)^n \int_0^\infty y^n e^{-(p+1)y} \, dy$$

$$= (-1)^n \frac{\Gamma(n+1)}{(p+1)^{n+1}} = (-1)^n \frac{n!}{(p+1)^{n+1}},$$

including the case $\int_0^1 (\log x)^n \, dx = (-1)^n \Gamma(n+1) = (-1)^n n!$.

1072. Again, let $F(x) \equiv A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \dots$ be supposed a convergent series for all values of x between $x=0$ and $x=1$, and such that

$$Lt_{x=1} F(x) \left(\log \frac{1}{x}\right)^p \text{ is zero or finite when } x=1,$$

so that even when the series for $F(x)$ ceases to be convergent when $x=1$, the final element of the summation indicated by the integration $\int_0^1 F(x)\left(\log\frac{1}{x}\right)^p dx$ will have no effect. Then we shall have, by putting $x=e^{-y}$,

$$I\equiv\int_0^1\left(\log\frac{1}{x}\right)^p F(x)\,dx=\int_0^\infty y^p e^{-y}F(e^{-y})\,dy$$

$$=\Gamma(p+1)\left(\frac{A_0}{1^{p+1}}+\frac{A_1}{2^{p+1}}+\frac{A_2}{3^{p+1}}+\dots\right),$$

and therefore I can be expressed in finite terms whenever $F(x)$ is such that this series is capable of summation.

An extensive class of definite integrals arises from this fact.

1073. It will be well to recount several previous results obtained. We have now used the symbol S_p to denote the complete series

$$S_p\equiv\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\frac{1}{5^p}+\dots\ ad\ inf.\quad (p>1),$$

and the numerical values of S_p up to S_{35} are tabulated in Art. 957.

Also, if $\sec x+\tan x=1+K_1\frac{x}{1!}+K_2\frac{x^2}{2!}+K_3\frac{x^3}{3!}+\dots$, then

$$K_n\cdot\frac{\pi^{n+1}}{2^{n+2}n!}=1+(-\tfrac{1}{3})^{n+1}+(\tfrac{1}{5})^{n+1}+(-\tfrac{1}{7})^{n+1}+(\tfrac{1}{9})^{n+1}+\dots\ ad\ inf.,$$

and rules were given (*Diff. Calc.*, Art. 573) for the calculation of K_n, the results being

$$K_1=1,\quad K_2=1,\quad K_3=2,\quad K_4=5,\quad K_5=16,$$
$$K_6=61,\quad K_7=272,\quad K_8=1385,\quad K_9=7936,\quad \text{etc.},$$

K_{2n} being the n^{th} "Eulerian" number $\equiv E_{2n}$; whilst K_{2n-1} is the n^{th} "Prepared Bernoullian" number $\equiv\frac{2^{2n}(2^{2n}-1)}{2n}B_{2n-1}$, B_{2n-1} being the n^{th} Bernoullian number itself.

Also we have seen that

$$S_{2n}\equiv\frac{\pi^{2n}}{2(2n-1)!\,(2^{2n}-1)}K_{2n-1}\equiv\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\dots=\frac{(2\pi)^{2n}}{2(2n)!}B_{2n-1},$$

$$\frac{\pi^{2n+1}}{2^{2n+2}(2n)!}K_{2n}\equiv\frac{1}{1^{2n+1}}-\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}-\frac{1}{7^{2n+1}}+\dots=\frac{\pi^{2n+1}}{2^{2n+2}(2n)!}E_{2n},$$

and we have the particular results

$\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\dots=\frac{\pi}{4}$ (Euler) $\qquad \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots=\frac{\pi^4}{90}$ (Euler)

$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\dots=\frac{\pi^3}{32}$ (Tchebechef) $\qquad \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots=\frac{\pi^2}{8}$ (Euler)

$\frac{1}{1^5}-\frac{1}{3^5}+\frac{1}{5^5}-\dots=\frac{5\pi^5}{1536}$ (Tchebechef) $\qquad \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots=\frac{\pi^4}{96}$ (Euler)

$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots=\frac{\pi^2}{6}$ (Euler) $\qquad \frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots=\log 2$

$$\left.\begin{aligned} s_p &\equiv \frac{1}{1^p}+\frac{1}{3^p}+\frac{1}{5^p}+\frac{1}{7^p}+\dots=\left(1-\frac{1}{2^p}\right)S_p \\ \sigma_p &\equiv \frac{1}{1^p}-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\dots=\left(1-\frac{2}{2^p}\right)S_p \end{aligned}\right\} (p>1).$$

1074. One class of series of this nature will not be obtainable from the tabulated results of Art. 957, viz.

$$\frac{1}{1^{2n}}-\frac{1}{3^{2n}}+\frac{1}{5^{2n}}-\frac{1}{7^{2n}}+\frac{1}{9^{2n}}-\dots\equiv s_{2n}', \text{ say};$$

and so far as the author is aware the values of this series for various values of n have not been tabulated, and it would appear that there is no method of obtaining the values except from the series itself or from some transformation of it to render it more rapidly convergent. The most troublesome case for direct calculation is the case when $n=1$, on account of the slow rate of convergence. But in this isolated case, viz.

$$s_2'\equiv\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\dots,$$

the value has been shown by Mr. J. W. L. Glaisher to be

0·91596 55941 77219 01505 ....

(*Proceedings of the London Math. Soc.*, 1876-7.)

Mr. Glaisher arrived at this result by means of the identity

$$\frac{t}{\sin t\cos t}=\sec^2 t-\tfrac{1}{3}\tan^2 t\sec^2 t+\tfrac{1}{5}\tan^4 t\sec^2 t-\dots,$$

a form of Gregory's series, which upon integration yields

$$\tan x-\frac{1}{3^2}\tan^3 x+\frac{1}{5^2}\tan^5 x-\dots=\int_0^x\frac{2t}{\sin 2t}\,dt=\frac{1}{2}\int_0^{2x}\frac{T}{\sin T}\,dT$$

$$=\frac{1}{2}\int_0^{2x}\left[1-\frac{2T^2}{T^2-\pi^2}+\frac{2T^2}{T^2-2^2\pi^2}-\dots\right]dT,$$

and expanding the fractions in powers of T and integrating,

$$=x+\frac{1}{3}\frac{\sigma_2}{\pi^2}(2x)^3+\frac{1}{5}\frac{\sigma_4}{\pi^4}(2x)^5+\dots,$$

where $$\sigma_{2n}=\frac{1}{1^{2n}}-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\frac{1}{4^{2n}}+\dots=\left(1-\frac{2}{2^{2n}}\right)S_{2n};$$

whence, putting $\frac{\pi x}{2}$ for x, Mr. Glaisher obtained the remarkable series

$$\tan\frac{\pi x}{2}-\frac{1}{3^2}\tan^3\frac{\pi x}{2}+\frac{1}{5^2}\tan^5\frac{\pi x}{2}-\dots=\frac{\pi}{2}\left[x+\frac{2}{3}\sigma_2x^3+\frac{2}{5}\sigma_4x^5+\dots\right],$$

and putting $x=\frac{1}{2}$, $s_2'\equiv\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\dots=\frac{\pi}{2}\left[\frac{1}{2}+\frac{1}{3}\frac{\sigma_2}{2^2}+\frac{1}{5}\frac{\sigma_4}{2^4}+\dots\right]$,

whence the value above given may be derived. The details of the calculation are given in Mr. Glaisher's paper (*loc. cit.*).

1075. It is to be remarked that in approximating to a case of the general series $\frac{1}{1^n}-\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}-\dots$, if we retain any specified number of terms, the error in rejecting the remainder of the series is less than the first of the rejected terms. *E.g.* if

$$s_2'=\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\epsilon,\text{ say,}$$

then $$\epsilon=\frac{1}{9^2}-\left(\frac{1}{11^2}-\frac{1}{13^2}\right)-\text{etc., and }\therefore\text{ is }<\frac{1}{9^2};$$

and since $$\epsilon=\left(\frac{1}{9^2}-\frac{1}{11^2}\right)+\left(\frac{1}{13^2}-\frac{1}{15^2}\right)+\dots,\text{ it is }>0,$$

and the error in taking 4 terms lies between 0 and $\frac{1}{81}$. Similarly, and more generally, if we retain r terms the error is less than the $(r+1)^{\text{th}}$ term.

The series for s_4', s_6', etc., are much more rapidly convergent than that for s_2', and therefore the calculations direct from the series are much less laborious.

For immediate convenience we may note that to six figures

$$s_2'=\cdot 915,966,\quad s_4'=\cdot 988,944,$$
$$s_6'=\cdot 998,685,\quad s_8'=\cdot 999,850.$$

1076. The integrals which follow are arranged in groups according to their forms. Where it is thought necessary the working is fully given. In some cases two or three of the steps are given, and in other cases merely the result is stated. It is intended that these should be WORKED BY THE STUDENT FOR HIS OWN PRACTICE. In some cases it will be seen that by treatment of the same integral by different methods various identities may be established.

1077. GROUP A. EXAMPLES OF **Integrals of form**

$$\int_0^1 \frac{x^q\left(\log\frac{1}{x}\right)^p}{1\pm x}\,dx.$$

1. $I=\int_0^1 \frac{\log\frac{1}{x}}{1-x}\,dx.$ Putting $x=e^{-y}$, we have

$$I=\int_0^\infty y\left(e^{-y}+e^{-2y}+e^{-3y}+\ldots\right)dy=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}.$$

2. Show that $\int_0^1 \frac{\log\frac{1}{x}}{1+x}\,dx=\left(1-\frac{2}{2^2}\right)\frac{\pi^2}{6}=\frac{\pi^2}{12}.$

3. Show that

$$\int_0^1 \frac{\left(\log\frac{1}{x}\right)^2}{1-x}\,dx=2\,!\left(\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\ldots\right)=2S_3=2{\cdot}40411\ldots.$$

4. Show that $\int_0^1 \frac{\left(\log\frac{1}{x}\right)^2}{1+x}\,dx=\frac{3}{2}S_3.$

5. Show that

$$\int_0^1 \frac{\left(\log\frac{1}{x}\right)^3}{1-x}\,dx=\frac{\pi^4}{15},\quad \int_0^1 \frac{\left(\log\frac{1}{x}\right)^3}{1+x}\,dx=\frac{7\pi^4}{120}.$$

6. Show that

$$\int_0^1 \frac{\left(\log\frac{1}{x}\right)^{2n-1}}{1-x}\,dx=\frac{(2\pi)^{2n}}{4n}B_{2n-1},\quad \int_0^1 \frac{\left(\log\frac{1}{x}\right)^{2n-1}}{1+x}\,dx=\frac{2^{2n-1}-1}{2n}\pi^{2n}B_{2n-1}.$$

7. Show that

$$\int_0^1 \frac{\left(\log\frac{1}{x}\right)^{2n}}{1-x}\,dx=(2n)\,!\,S_{2n+1},\quad \int_0^1 \frac{\left(\log\frac{1}{x}\right)^{2n}}{1+x}\,dx=(2n)!\,\frac{2^{2n}-1}{2^{2n}}S_{2n+1}.$$

It is to be noted that integrals with integrands of the same character as the above multiplied by rational integral algebraic polynomials present no difficulty, thus :

8. $\int_0^1 x\frac{\log\frac{1}{x}}{1-x}\,dx=\int_0^\infty y\left(e^{-2y}+e^{-3y}+e^{-4y}+\ldots\right)dy=\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}-\frac{1}{1^2}.$

9. Show that $\int_0^1 x^2\frac{\log\frac{1}{x}}{1-x}\,dx=\frac{\pi^2}{6}-\frac{1}{1^2}-\frac{1}{2^2}.$

10. Show that

$$\int_0^1 (ax^2+bx+c)\frac{\log\frac{1}{x}}{1-x}\,dx=(a+b+c)\frac{\pi^2}{6}-\frac{a+b}{1^2}-\frac{a}{2^2}.$$

1078. In some of the simpler cases, viz. when the power of the logarithmic factor is the first, we way write $1-y$ for x, and expand the logarithm.

Thus

$$\int_0^1 \frac{\log x}{1-x}\,dx = \int_1^0 \frac{\log(1-y)}{y}(-1)\,dy = \int_0^1 \frac{\log(1-y)}{y}\,dy$$

$$= -\int_0^1 \left(1+\frac{y}{2}+\frac{y^2}{3}+\dots\right)dy = -\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right) = -\frac{\pi^2}{6}.$$

Examples.

1. Prove that $\int_0^1 \tanh^{-1}x\,\frac{dx}{x} = \frac{\pi^2}{8} = \int_0^a \tanh^{-1}\frac{x}{a}\,\frac{dx}{x}.$

2. Deduce from (2), Art. 1077, by putting $x=\tan^2\theta$,

$$\int_0^{\frac{\pi}{4}} \tan\theta \log\cot\theta\,d\theta = \frac{\pi^2}{48}.$$

3. Deduce from (6), Art. 1077, by putting $x=\sin^2\theta$,

$$\int_0^{\frac{\pi}{2}} \tan\theta\,(\log\operatorname{cosec}\theta)^{2n-1}\,d\theta = \frac{\pi^{2n}}{4n}B_{2n-1}.$$

4. Prove that

$$\int_0^{\frac{\pi}{4}} \tan\theta\,(\log\cot\theta)^{2n-1}\,d\theta = \frac{2^{2n-1}-1}{2^{2n+1}}\,\frac{\pi^{2n}}{n}B_{2n-1}.$$

1079. Group B. Examples of **Integrals of form**

$$\int_0^1 \frac{x^q\left(\log\frac{1}{x}\right)^p}{1\pm x^2}\,dx.$$

Prove that

1. $\int_0^1 \frac{\log\frac{1}{x}}{1-x^2}\,dx = \frac{\pi^2}{8}$, $\int_0^1 \frac{\log\frac{1}{x}}{1+x^2}\,dx = s_2' = \cdot 915966\dots$ approximately.

2. $\int_0^1 \frac{\left(\log\frac{1}{x}\right)^2}{1-x^2}\,dx = 2\left(1-\frac{1}{2^3}\right)S_3 = 2\cdot 103599\dots$, $\int_0^1 \frac{\left(\log\frac{1}{x}\right)^2}{1+x^2}\,dx = \frac{\pi^3}{16}$.

3. $\int_0^1 \frac{\left(\log\frac{1}{x}\right)^3}{1-x^2}\,dx = \frac{\pi^4}{16}$, $\int_0^1 \frac{\left(\log\frac{1}{x}\right)^3}{1+x^2}\,dx = 6s_4' = 5\cdot 9336\dots$

4. $\int_0^1 \frac{\left(\log\frac{1}{x}\right)^4}{1-x^2}\,dx = 4!\left(1-\frac{1}{2^5}\right)S_5 = 24\cdot 10857\dots$, $\int_0^1 \frac{\left(\log\frac{1}{x}\right)^4}{1+x^2}\,dx = \frac{5\pi^5}{64}$.

5. $\int_0^1 \frac{\left(\log\frac{1}{x}\right)^5}{1-x^2}dx=\frac{\pi^6}{8}$, $\int_0^1 \frac{\left(\log\frac{1}{x}\right)^5}{1+x^2}dx=5!\,s_6'=119\cdot842\ldots$.

6. $\int_0^1 \frac{\left(\log\frac{1}{x}\right)^6}{1-x^2}dx=6!\left(1-\frac{1}{2^7}\right)S_7$, $\int_0^1 \frac{\left(\log\frac{1}{x}\right)^6}{1+x^2}dx=\frac{61\pi^7}{2^8}$.

7. $\int_0^1 \frac{\left(\log\frac{1}{x}\right)^{2n-1}}{1-x^2}dx=\frac{\pi^{2n}(2^{2n}-1)}{4n}B_{2n-1}$,

$\int_0^1 \frac{\left(\log\frac{1}{x}\right)^{2n}}{1+x^2}dx=\frac{\pi^{2n+1}}{2^{2n+2}}E_{2n}$;

and in the same way as 8, 9, 10 of Group A, prove that

8. $\int_0^1 x\frac{\log\frac{1}{x}}{1-x^2}dx=\frac{\pi^2}{24}$. [Euler, *Nov. Com. Pet.*, vol. xix.]

9. $\int^1 x^2\frac{\log\frac{1}{x}}{1-x^2}dx=\frac{\pi^2}{8}-1$.

10. $\int_0^1 x^3\frac{\log\frac{1}{x}}{1-x^2}dx=\frac{\pi^2}{24}-\frac{1}{4}$, and so on for similar cases.

11. Putting $x=\sin\theta$ in No. 7 (1st part) and $x=\tan\theta$ in No. 7 (2nd part), show that, if n be a positive integer,

(i) $\int_0^{\frac{\pi}{2}} \sec\theta\,(\log\operatorname{cosec}\theta)^{2n-1}\,d\theta=\int_0^{\frac{\pi}{2}}\operatorname{cosec}\theta\,(\log\sec\theta)^{2n-1}\,d\theta$

$=\pi^{2n}\frac{2^{2n}-1}{4n}B_{2n-1}$;

(ii) $\int_0^{\frac{\pi}{4}}(\log\cot\theta)^{2n}\,d\theta=\frac{\pi^{2n+1}}{2^{2n+2}}E_{2n}$.

1080. Group C. Examples of **Integrals of type**

$$\int_0^1 \frac{x^m\left(\log\frac{1}{x}\right)^p}{(1\pm x)^q}dx,$$

p and q being positive integers ($p \nless q$).

1. Putting $x=e^{-y}$, we have

$$\int_0^1 \frac{\left(\log\frac{1}{x}\right)^p}{(1-x)^2}dx=\int_0^\infty y^p(e^{-y}+2e^{-2y}+3e^{-3y}+\ldots)\,dy$$

$$=p!\left(\frac{1}{1^{p+1}}+\frac{2}{2^{p+1}}+\frac{3}{3^{p+1}}+\ldots\right)=p!\,S_p \quad (p>1).$$

Prove that

2. $\int_0^1 \frac{\left(\log \frac{1}{x}\right)^p}{(1+x)^2}\,dx = p!\left(1-\frac{2}{2^p}\right)S_p \quad (p>1).$ 3. $\int_0^1 \frac{\log \frac{1}{x}}{(1+x)^2}\,dx = \log_e 2.$

4. $\int_0^1 \frac{\left(\log \frac{1}{x}\right)^p}{(1-x)^3}\,dx = \frac{p!}{2!}(S_{p-1}+S_p).$

5. $\int_0^1 \frac{\left(\log \frac{1}{x}\right)^p}{(1-x)^4}\,dx = \frac{p!}{3!}(S_{p-2}+3S_{p-1}+2S_p).$

6. $\int_0^1 \frac{\left(\log \frac{1}{x}\right)^p}{(1-x)^q}\,dx = \frac{p!}{(q-1)!}(S_{p-q+2}+P_1 S_{p-q+3}+\dots+P_{q-2}S_p),$

where P_r is the sum of the products r at a time of 1, 2, 3, ... $(q-2)$.

7. $\int_0^1 \frac{\left(\log \frac{1}{x}\right)^p}{(1+x)^q}\,dx = \frac{p!}{(q-1)!}(\sigma_{p-q+2}+P_1\sigma_{p-q+3}+\dots+P_{q-2}\sigma_p),$

where $\sigma_r = \frac{1}{1^r}-\frac{1}{2^r}+\frac{1}{3^r}-\dots = \left(1-\frac{2}{2^r}\right)S_r$.

8. $\int_0^1 x^2 \frac{\left(\log \frac{1}{x}\right)^3}{(1-x^2)^2}\,dx = 3\left(\tfrac{7}{8}S_3 - \frac{\pi^4}{96}\right), \quad \int_0^1 x^2 \frac{\left(\log \frac{1}{x}\right)^3}{(1+x^2)^2}\,dx = 3\left(s_4' - \frac{\pi^3}{32}\right).$

9. $\int_0^{\frac{\pi}{4}} \frac{\log\cot\theta}{(\sin\theta+\cos\theta)^2}\,d\theta = \log 2.$ (Put $x=\tan\theta$ in 3.)

10. $\int_0^{\frac{\pi}{4}} \sin 2\theta \log\cot\theta\, d\theta = \tfrac{1}{2}\log 2.$ (Put $x=\tan^2\theta$ in 3.)

1081. GROUP D. Various Forms containing Radicals.

1. $$I = \int_0^1 \frac{\log\frac{1}{x}}{\sqrt{1-x}}\,dx = \int_0^\infty y\left(e^{-y}+\frac{1}{2}e^{-2y}+\frac{1.3}{2.4}e^{-3y}+\dots\right)dy$$

$$= \frac{1}{1^2}+\frac{1}{2}\cdot\frac{1}{2^2}+\frac{1.3}{2.4}\frac{1}{3^2}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{4^2}+\dots.$$

Again putting $x=\sin^2\theta$,

$$I = -\int_0^{\frac{\pi}{2}} \log\sin^2\theta . 2\sin\theta\, d\theta = -4\int_0^{\frac{\pi}{2}} \sin\theta\log\sin\theta\, d\theta$$

$$= -4\left[-\cos\theta\log\sin\theta + \log\tan\frac{\theta}{2}+\cos\theta\right]_0^{\frac{\pi}{2}}$$

$$= -4\left[\cos\theta(1-\log 2)+2\sin^2\frac{\theta}{2}\log\sin\frac{\theta}{2} - 2\cos^2\frac{\theta}{2}\log\cos\frac{\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= -4[\log 2 - 1] = 4\log\frac{e}{2}.$$

Thus we have the result

$$4\log\frac{e}{2}=\frac{1}{1^2}+\frac{1}{2}\cdot\frac{1}{2^2}+\frac{1.3}{2.4}\cdot\frac{1}{3^2}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{4^2}+\dots \textit{ ad inf.}$$

2. $I=\int_0^1 \frac{x^2\log x}{\sqrt{1-x^2}}\,dx.$ [EULER, *Nov. Com. Petropol.*, xix., p. 30.]

Put $x=\sin\theta$, $I=\int_0^{\frac{\pi}{2}}\sin^2\theta\log\sin\theta\,d\theta=\int_0^{\frac{\pi}{2}}\frac{1-\cos 2\theta}{2}\log\sin\theta\,d\theta$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\log\sin\theta\,d\theta-\frac{1}{2}\left[\frac{\sin 2\theta}{2}\log\sin\theta\right]_0^{\frac{\pi}{2}}+\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$

$$=\frac{\pi}{4}\log\frac{1}{2}+\frac{\pi}{8}=\frac{\pi}{8}\log\frac{e}{4}.$$

3. Find the values of

$$I=\int_0^{\frac{\pi}{2}}\cos 2n\theta\log\sin\theta\,d\theta \quad\text{and}\quad I'=\int_0^{\frac{\pi}{2}}\sin^{2n}\theta\log\sin\theta\,d\theta.$$

Since

$$\sin\theta\frac{d}{d\theta}\Big\{\theta+\sin 2\theta+\frac{1}{2}\sin 4\theta+\frac{1}{3}\sin 6\theta+\dots$$
$$+\frac{1}{n-1}\sin(2n-2)\theta+\frac{1}{2n}\sin 2n\theta\Big\}=\sin 2n\theta\cos\theta,$$

we have

$$\int\sin 2n\theta\cot\theta\,d\theta=\theta+\sin 2\theta+\frac{\sin 4\theta}{2}+\dots+\frac{\sin(2n-2)\theta}{n-1}+\frac{\sin 2n\theta}{2n},$$

also $\int\sin 2n\theta\cot\theta\,d\theta=\sin 2n\theta\log\sin\theta-2n\int\cos 2n\theta\log\sin\theta\,d\theta.$

Hence $$I\equiv\int_0^{\frac{\pi}{2}}\cos 2n\theta\log\sin\theta\,d\theta=-\frac{\pi}{4n}\quad(n>0).$$

Again

$$\sin^{2n}\theta=\frac{1}{2^{2n}}\{{}^{2n}C_n-2\,{}^{2n}C_{n-1}\cos 2\theta+2\,{}^{2n}C_{n-2}\cos 4\theta-\dots+(-1)^n 2\cos 2n\theta\}.$$

$$\therefore\quad I'\equiv\int_0^{\frac{\pi}{2}}\sin^{2n}\theta\log\sin\theta\,d\theta$$

$$=\frac{1}{2^{2n}}\left\{{}^{2n}C_n\frac{\pi}{2}\log\frac{1}{2}-2\,{}^{2n}C_{n-1}\left(-\frac{\pi}{4}\right)+2\,{}^{2n}C_{n-2}\left(-\frac{\pi}{8}\right)-\dots+(-1)^n 2\left(-\frac{\pi}{4n}\right)\right\}$$

$$=\frac{\pi}{2^{2n+1}}\left\{{}^{2n}C_n\log\frac{1}{2}+{}^{2n}C_{n-1}-\frac{1}{2}\,{}^{2n}C_{n-2}+\frac{1}{3}\,{}^{2n}C_{n-3}-\dots+(-1)^{n-1}\frac{1}{n}\,{}^{2n}C_0\right\}.$$

Putting $\sin\theta=x$, we have the value of $\int_0^1\frac{x^{2n}\log x}{\sqrt{1-x^2}}\,dx.$

4. $$I=\int_0^1\frac{\log\frac{1}{x}}{\sqrt{1-x^2}}\,dx=\int_0^\infty y\left(e^{-y}+\frac{1}{2}e^{-3y}+\frac{1.3}{2.4}e^{-5y}+\dots\right)dy$$

$$=\frac{1}{1^2}+\frac{1}{2}\cdot\frac{1}{3^2}+\frac{1.3}{2.4}\cdot\frac{1}{5^2}+\dots.$$

Again putting $x=\sin\theta$,

$$\int_0^1 \frac{\log\frac{1}{x}}{\sqrt{1-x^2}}\,dx=-\int_0^{\frac{\pi}{2}}\log\sin\theta\,d\theta=\frac{\pi}{2}\log 2\,;$$

whence it appears that

$$\frac{\pi}{2}\log 2=\frac{1}{1^2}+\frac{1}{2}\cdot\frac{1}{3^2}+\frac{1\,.\,3}{2\,.\,4}\cdot\frac{1}{5^2}+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{1}{7^2}+\ldots.$$

1082. Group E. **Cases in which the Algebraic Factor is the Generating Function of a Recurring Series whose Coefficients are Powers of the Natural Numbers.**

1. $$\int_0^1 \frac{1+x}{(1-x)^3}\left(\log\frac{1}{x}\right)^3 dx=\int_0^\infty y^3(e^{-y}+e^{-2y})\left(1+3e^{-y}+\frac{3\,.\,4}{1\,.\,2}e^{-2y}+\ldots\right)dy$$
$$=\int_0^\infty y^3(e^{-y}+2^2e^{-2y}+3^2e^{-3y}+\ldots)dy$$
$$=3\,!\left(\frac{1^2}{1^4}+\frac{2^2}{2^4}+\frac{3^2}{3^4}+\ldots\right)=6\,.\,\frac{\pi^2}{6}=\pi^2.$$

Prove that

2. $$\int_0^1 \frac{1+x}{(1-x)^3}\left(\log\frac{1}{x}\right)^5 dx=\tfrac{4}{3}\pi^4,\quad \int_0^1 \frac{1+x}{(1-x)^3}\left(\log\frac{1}{x}\right)^7 dx=\tfrac{16}{3}\pi^6,$$
$$\int_0^1 \frac{1+x}{(1-x)^3}\left(\log\frac{1}{x}\right)^{2n+1} dx=\frac{2n+1}{2}(2\pi)^{2n}B_{2n-1}.$$

3. $$\int_0^1 \frac{1+6x^2+x^4}{(1-x^2)^3}\left(\log\frac{1}{x}\right)^3 dx=\frac{3\pi^2}{4},$$
$$\int_0^1 \frac{1+6x^2+x^4}{(1-x^2)^3}\left(\log\frac{1}{x}\right)^{2n+1} dx=\frac{2n+1}{2}(2^{2n}-1)\pi^{2n}B_{2n-1}.$$

4. $$\int_0^1 \frac{(1+x)(1+10x+x^2)}{(1-x)^5}\left(\log\frac{1}{x}\right)^{2n+3}dx$$
$$=\frac{(2n+3)(2n+2)(2n+1)}{2}(2\pi)^{2n}B_{2n-1}.$$

5. $$\int_0^1 \frac{1+x^2}{(1-x^2)^2}(\log x)^2\,dx=\frac{\pi^2}{4}.$$

6. $$\int_0^1 \frac{1+26x+66x^2+26x^3+x^4}{(1-x)^6}\left(\log\frac{1}{x}\right)^8 dx=2^6\,.\,7\,.\,\pi^4.$$

7. $$\int_0^1 \frac{x}{(1-x)^4}\left(\log\frac{1}{x}\right)^4 dx=\tfrac{2}{45}\pi^2(15-\pi^2),$$
$$\int_0^1 \frac{1+x^2}{(1-x)^4}\left(\log\frac{1}{x}\right)^4 dx=\frac{4\pi^2}{45}(15+2\pi^2).$$

8. $$\int_0^1 \frac{1+4x+x^2}{(1-x)^4}\left(\log\frac{1}{x}\right)^{2n+2} dx=(n+1)(2n+1)(2\pi)^{2n}B_{2n-1}.$$

9. $$\int_0^1 \frac{1+x^4}{(1-x)^6}\left(\log\frac{1}{x}\right)^6 dx=2\pi^2(1+\tfrac{7}{3}\pi^2+\tfrac{16}{105}\pi^4).$$

10. If $a_0, a_1, a_2, \dots a_{n-1}$ be defined by the equation

$$a_{s-1} = s^n - {}^{n+1}C_1(s-1)^n + {}^{n+1}C_2(s-2)^n - \dots + (-1)^{s-1}\,{}^{n+1}C_{s-1} \cdot 1^n$$

for all values of s from $s=1$ to $s=n$, then

$$\int_0^1 \frac{a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1}}{(1-x)^{n+1}} \left(\log\frac{1}{x}\right)^{n+2m-1} dx$$

$$= \frac{1}{2}\frac{(n+2m-1)!}{(2m)!}(2\pi)^{2m}B_{2m-1}.$$

It will be recognised that the several equations defining the letters $a_0, a_1, a_2, \dots a_{n-1}$, viz.

$$a_0 = 1^n, \quad a_1 = 2^n - (n+1)1^n, \quad a_2 = 3^n - (n+1)2^n + \frac{(n+1)n}{1 \cdot 2}1^n,$$

etc.,

$$a_{n-1} = n^n - (n+1)(n-1)^n + \frac{(n+1)n}{1 \cdot 2}(n-2)^n - \dots + (-1)^{n-1}\frac{(n+1)n}{1 \cdot 2}1^n,$$

are the results of equating coefficients in

$$a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} \equiv (1^n + 2^nx + 3^nx^2 + \dots \textit{ad. inf.})(1-x)^{n+1}$$

up to the coefficient of x^{n-1}. And it is known that

$$(n+r)^n - {}^{n+1}C_1(n+r-1)^n + \dots + (-1)^{n+1}\,{}^{n+1}C_{n+1}(r-1)^n$$

vanishes for all values of r from 1 to ∞, being the coefficient of x^n in

$$e^{(r-1)x}(e^x-1)^{n+1}, \quad \textit{i.e.} \text{ in } [1 + (r-1)x + \dots](x^{n+1} - \dots),$$

in which the term of lowest degree is x^{n+1}.

Hence $\dfrac{a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1}}{(1-x)^{n+1}}$ is the generating function of the recurring series $1^n + 2^nx + 3^nx^2 + \dots$.

Therefore $\displaystyle\int_0^1 \frac{a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1}}{(1-x)^{n+1}} \left(\log\frac{1}{x}\right)^{n+2m-1} dx$

$$= \int_0^\infty y^{n+2m-1}[1^ne^{-y} + 2^ne^{-2y} + 3^ne^{-3y} + \dots]dy$$

$$= (n+2m-1)!\left[\frac{1^n}{1^{n+2m}} + \frac{2^n}{2^{n+2m}} + \frac{3^n}{3^{n+2m}} + \dots\right]$$

$$= (n+2m-1)!\left[\frac{1}{1^{2m}} + \frac{1}{2^{2m}} + \frac{1}{3^{2m}} + \dots\right]$$

$$= (n+2m-1)!\,\frac{(2\pi)^{2m}}{2(2m)!}B_{2m-1}.$$

1083. Group F. **Gaps in the Development of the Algebraic Factor.**

Let α and β be any two prime numbers.

In the series formed by the development of

$$\frac{x}{1-x} - \frac{x^\alpha}{1-x^\alpha} - \frac{x^\beta}{1-x^\beta} \text{ in ascending powers of } x \quad (x<1),$$

the subtraction of $\dfrac{x^a}{1-x^a}$, *i.e.* $x^a+x^{2a}+x^{3a}+\dots$, from $\dfrac{x}{1-x}$, *i.e.* the complete series

$$x+x^2+x^3+\dots+x^a+x^{a+1}+\dots+x^{2a}+x^{2a+1}+\dots,$$

removes all terms whose indices are multiples of a.

The subsequent subtraction of $\dfrac{x^\beta}{1-x^\beta}$ removes all those terms which remain, and have indices multiples of β, restoring with the opposite sign such terms as have indices multiples of $a\beta$.

If we now add $\dfrac{x^{a\beta}}{1-x^{a\beta}}$ we are left with the complete series with all terms whose indices contain either a or β as a factor removed.

Exactly analogous to this is the effect of multiplying the series

$$S=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\frac{1}{5^p}+\dots, \quad (p>1),$$

$$(1) \text{ by } 1-\frac{1}{a^p}-\frac{1}{\beta^p}, \quad (2) \text{ by } \left(1-\frac{1}{a^p}\right)\left(1-\frac{1}{\beta^p}\right).$$

For $S-\dfrac{S}{a^p}-\dfrac{S}{\beta^p}\equiv$ the complete series S from which terms in which the denominators are multiples of a and β have been removed, but those whose denominators contain both a and β are restored with the opposite sign, whilst in the case $S\left(1-\dfrac{1}{a^p}\right)\left(1-\dfrac{1}{\beta^p}\right)$, no terms occur whose denominators contain either a or β as a factor.

Thus $$I=\int_0^1\left(\frac{x}{1-x}-\frac{x^a}{1-x^a}-\frac{x^\beta}{1-x^\beta}\right)\left(\log\frac{1}{x}\right)^{2n-1}\frac{dx}{x}$$

$$=(2n-1)!\left[\left\{\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\dots\right\}\right],$$

by putting $x=e^{-y}$ as usual, where the double bracket indicates that from the series included all terms have been removed which contain a and not β, or β and not a, as a factor, whilst terms with both a and β as a factor occur with the negative sign

$$=(2n-1)!\left(1-\frac{1}{a^p}-\frac{1}{\beta^p}\right)S_{2n}$$

$$=(2n-1)!\left(1-\frac{1}{a^p}-\frac{1}{\beta^p}\right)\frac{(2\pi)^{2n}}{2(2n)!}B_{2n-1}$$

$$=\left(1-\frac{1}{a^p}-\frac{1}{\beta^p}\right)\frac{(2\pi)^{2n}}{4n}B_{2n-1}.$$

And $$I'=\int_0^1\left(\frac{x}{1-x}-\frac{x^a}{1-x^a}-\frac{x^\beta}{1-x^\beta}+\frac{x^{a\beta}}{1-x^{a\beta}}\right)\left(\log\frac{1}{x}\right)^{2n-1}\frac{dx}{x}$$

$$=\left(1-\frac{1}{a^p}\right)\left(1-\frac{1}{\beta^p}\right)\frac{(2\pi)^{2n}}{4n}B_{2n-1}.$$

It may be noted that

$$\int_0^1 \frac{x^\alpha}{1-x^\alpha}\left(\log\frac{1}{x}\right)^{2n-1}\frac{dx}{x}=\frac{1}{\alpha^{2n}}\int_0^1 \frac{y}{1-y}\left(\log\frac{1}{y}\right)^{2n-1}\frac{dy}{y}$$

$$=\frac{1}{\alpha^{2n}}\int_0^1 \frac{x}{1-x}\left(\log\frac{1}{x}\right)^{2n-1}\frac{dx}{x},$$

and therefore

$$\int_0^1\left(\frac{x}{1-x}-\frac{Px^\alpha}{1-x^\alpha}-\frac{Qx^\beta}{1-x^\beta}+\frac{Rx^{\alpha\beta}}{1-x^{\alpha\beta}}\right)\left(\log\frac{1}{x}\right)^{2n-1}\frac{dx}{x}$$

$$=\left(1-\frac{P}{\alpha^{2n}}-\frac{Q}{\beta^{2n}}+\frac{R}{\alpha^{2n}\beta^{2n}}\right)\frac{(2\pi)^{2n}}{4n}B_{2n-1},$$

whatever numerical values may be assigned to P, Q, R.

And more generally, if $\alpha, \beta, \gamma, \ldots$ be any prime numbers, and if $F(x)$ be the function of x which would be formed by first developing

$$(1-A)(1-B)(1-C)(1-D)\ldots \text{ as } 1-(A+B+\ldots)+(AB+\ldots)-\text{etc.},$$

and then replacing

$$1 \text{ by } \frac{x}{1-x}, \quad A \text{ by } \frac{x^\alpha}{1-x^\alpha} \quad B \text{ by } \frac{x^\beta}{1-x^\beta}, \text{ etc.},$$

$$AB \text{ by } \frac{x^{\alpha\beta}}{1-x^{\alpha\beta}}, \quad ABC \text{ by } \frac{x^{\alpha\beta\gamma}}{1-x^{\alpha\beta\gamma}}, \text{ and so on,}$$

then $F(x)$ consists of such terms of the series $x+x^2+x^3+x^4+\ldots$ as are left when all those are removed which have α, β, γ or any combination of them as a factor of their indices; and then

$$\int_0^1 F(x)\left(\log\frac{1}{x}\right)^{2n-1}\frac{dx}{x}=\int_0^\infty y^{2n-1}(e^{-y}+e^{-2y}+\ldots)\,dy,$$

where the terms in the bracket are such that those whose indices are multiples of any of the primes $\alpha, \beta, \gamma, \ldots$ are missing,

$$=(2n-1)!\left(1-\frac{1}{\alpha^{2n}}\right)\left(1-\frac{1}{\beta^{2n}}\right)\left(1-\frac{1}{\gamma^{2n}}\right)\ldots S_{2n},$$

i.e. $$\int_0^1 F(x)\left(\log\frac{1}{x}\right)^{2n-1}\frac{dx}{x}=\frac{(2\pi)^{2n}}{4n}B_{2n-1}\left(1-\frac{1}{\alpha^{2n}}\right)\left(1-\frac{1}{\beta^{2n}}\right)\left(1-\frac{1}{\gamma^{2n}}\right)\ldots.$$

If we press the theorem further, and remove *all* the terms from $\frac{x}{1-x}$ except the first, then if $\alpha, \beta, \gamma, \ldots$ be all the prime numbers,

$$\int_0^1\left[\frac{x}{1-x}-\Sigma\frac{x^\alpha}{1-x^\alpha}+\Sigma\frac{x^{\alpha\beta}}{1-x^{\alpha\beta}}-\Sigma\frac{x^{\alpha\beta\gamma}}{1-x^{\alpha\beta\gamma}}+\Sigma\frac{x^{\alpha\beta\gamma\delta}}{1-x^{\alpha\beta\gamma\delta}}-\ldots \textit{ad. inf.}\right]\left(\log\frac{1}{x}\right)^{2n-1}\frac{dx}{x}$$

$$=(2n-1)!\left(1-\frac{1}{2^{2n}}\right)\left(1-\frac{1}{3^{2n}}\right)\left(1-\frac{1}{5^{2n}}\right)\left(1-\frac{1}{7^{2n}}\right)\ldots S_{2n}$$

$$=(2n-1)! \quad \text{(by Raabe's Theorem, \textit{Diff. Calc.}, p. 109, Ex. 29).}$$

And this result is *a priori* obvious, for the integral is merely

$$\int_0^1 x\left(\log\frac{1}{x}\right)^{2n-1}\frac{dx}{x}=\int_0^\infty e^{-y}y^{2n-1}\,dy=\Gamma(2n).$$

EXAMPLES.

1. Thus we have

$$\int_0^1 \left[\frac{x}{1-x}-\frac{x^2}{1-x^2}-\frac{x^3}{1-x^3}-\frac{x^5}{1-x^5}+\frac{x^6}{1-x^6}+\frac{x^{10}}{1-x^{10}}+\frac{x^{15}}{1-x^{15}}-\frac{x^{30}}{1-x^{30}}\right]\log\frac{1}{x}\cdot\frac{dx}{x}$$

$$=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\frac{(2\pi)^2}{4}B_1=\frac{3}{4}\cdot\frac{8}{9}\cdot\frac{24}{25}\cdot\frac{\pi^2}{6}=\frac{8\pi^2}{75}.$$

2. Prove that

(i) $\int_0^1 \frac{1+x}{1-x^3}\log\frac{1}{x}\,dx=\frac{4\pi^2}{27}$, (ii) $\int_0^1 \frac{1-x}{1+x^3}\log\frac{1}{x}\,dx=\frac{2\pi^2}{27}$,

(iii) $\int_0^1 \frac{1-x}{1+x^3}\left(\log\frac{1}{x}\right)^{2n-1}dx=\frac{(2^{2n-1}-1)(3^{2n}-1)}{2n\,3^{2n}}\pi^{2n}B_{2n-1}$,

(iv) $\int_0^1 \frac{1-x}{1+x^3}\left(\log\frac{1}{x}\right)^{2n}dx=(2n)!\left(1-\frac{1}{3^{2n+1}}\right)\left(1-\frac{1}{2^{2n}}\right)S_{2n+1}$.

3. Prove that $\int_0^1 \frac{1+x+x^2+x^3}{1-x^5}\left(\log\frac{1}{x}\right)^3dx=\frac{208}{3125}\pi^4$.

4. Show that $\int_0^1 \frac{1+x+x^2+x^3}{1-x^5}\left(\log\frac{1}{x}\right)^{2n-1}dx=\frac{1}{4n}\left(1-\frac{1}{5^{2n}}\right)(2\pi)^{2n}B_{2n-1}$.

5. Show that

$$\int_0^1 \frac{1-x^{p-1}}{(1-x)(1-x^p)}\left(\log\frac{1}{x}\right)^{2n-1}dx=\frac{1}{4n}\left(1-\frac{1}{p^{2n}}\right)(2\pi)^{2n}B_{2n-1},$$

where p is any prime number.

6. Show that

$$\int_0^1 \frac{1+x^2+x^4+x^5}{1-x^6}\left(\log\frac{1}{x}\right)^{2n-1}dx=\frac{1}{4n}\left(1-\frac{1}{2^{2n}}+\frac{1}{6^{2n}}\right)(2\pi)^{2n}B_{2n-1}.$$

1084. **Limits 0 to ∞.**

So far in this chapter the limits have been from 0 to 1. In some of the cases considered the integrations might have been taken from 0 to ∞; *e.g.* in the examples of Group B,

1. $\int_0^\infty \frac{\log\frac{1}{x}}{1-x^2}dx=\left(\int_0^1+\int_1^\infty\right)\frac{\log\frac{1}{x}}{1-x^2}dx.$ In the second integral put $x=\frac{1}{y}$.

$$\int_1^\infty \frac{\log\frac{1}{x}}{1-x^2}dx=\int_1^0 \frac{\log y}{1-y^{-2}}\left(-\frac{1}{y^2}\right)dy=\int_0^1 \frac{\log\frac{1}{x}}{1-x^2}dx;$$

$$\therefore \int_0^\infty \frac{\log\frac{1}{x}}{1-x^2}dx=2\int_0^1 \frac{\log\frac{1}{x}}{1-x^2}dx=2.\frac{\pi^2}{8}=\frac{\pi^2}{4}.$$

2. $\int_0^\infty \frac{\log\frac{1}{x}}{1+x^2}dx = \left(\int_0^1 + \int_1^\infty\right)\frac{\log\frac{1}{x}}{1+x^2}dx.$ The second integral is

$$\int_1^0 \frac{\log y}{1+y^{-2}}\left(-\frac{1}{y^2}\right)dy = \int_0^1 \frac{\log y}{1+y^2}dy = -\int_0^1 \frac{\log\frac{1}{x}}{1+x^2}dx\,;$$

$$\therefore \int_0^\infty \frac{\log\frac{1}{x}}{1+x^2}dx = 0.$$

3. $\int_0^\infty \frac{\left(\log\frac{1}{x}\right)^{2n-1}}{1-x^2}dx = \left(\int_0^1 + \int_1^\infty\right)\frac{\left(\log\frac{1}{x}\right)^{2n-1}}{1-x^2}dx = 2\int_0^1 \frac{\left(\log\frac{1}{x}\right)^{2n-1}}{1-x^2}dx$

$$= \frac{\pi^{2n}(2^{2n}-1)}{2n}B_{2n-1}.$$

4. $\int_0^\infty \frac{\left(\log\frac{1}{x}\right)^{2n}}{1+x^2}dx = \left(\int_0^1 + \int_1^\infty\right)\frac{\left(\log\frac{1}{x}\right)^{2n}}{1+x^2}dx = 2\int_0^1 \frac{\left(\log\frac{1}{x}\right)^{2n}}{1+x^2}dx$

$$= \left(\frac{\pi}{2}\right)^{2n+1}E_{2n},$$ and so on for other cases.

1085. Group G.

Integrals of the class

$$I \equiv \int_0^\infty \left(\frac{\log x}{x-1}\right)^n dx, \quad \text{i.e. } \int_0^\infty \frac{\left(\log\frac{1}{x}\right)^n}{(1-x)^n}dx, \quad n>1,$$

form a group of some interest. (Cf. Group C, Art. 1080.)

We have $I = \left(\int_0^1 + \int_1^\infty\right)\left(\frac{\log x}{x-1}\right)^n dx$, and putting $x = \frac{1}{y}$ in the second of these,

$$\int_1^\infty \left(\frac{\log x}{x-1}\right)^n dx = \int_1^0 \left(\frac{-\log y}{y^{-1}-1}\right)^n\left(-\frac{1}{y^2}\right)dy = \int_0^1 \left(\frac{\log y}{y-1}\right)^n y^{n-2}dy = \int_0^1 \left(\frac{\log x}{x-1}\right)^n x^{n-2}dx\,;$$

$$\therefore I = \int_0^1 \frac{1+x^{n-2}}{(x-1)^n}(\log x)^n dx\,;$$ and putting $x = e^{-z}$,

$$I = \int_0^\infty z^n\{1+e^{-(n-2)z}\}\left\{e^{-z} + ne^{-2z} + \frac{n(n+1)}{1.2}e^{-3z} + \dots\right\}dz,$$

the expansion being convergent as e^{-z} is <1 for all values of z between 0 and ∞;

$$\therefore I = \Gamma(n+1)\left[\frac{1}{1^{n+1}} + \frac{n}{1}\frac{1}{2^{n+1}} + \frac{n(n+1)}{1.2}\frac{1}{3^{n+1}} + \frac{n(n+1)(n+2)}{1.2.3}\frac{1}{4^{n+1}} + \dots\right.$$
$$\left. + \frac{1}{(n-1)^{n+1}} + \frac{n}{1}\frac{1}{n^{n+1}} + \frac{n(n+1)}{1.2}\frac{1}{(n+1)^{n+1}} + \frac{n(n+1)(n+2)}{1.2.3}\frac{1}{(n+2)^{n+1}} + \dots\right]$$
$$= n\left[\frac{\Gamma(n)}{1}\frac{1}{1^n} + \frac{\Gamma(n+1)}{1.2}\frac{1}{2^n} + \frac{\Gamma(n+2)}{1.2.3}\frac{1}{3^n} + \dots\right.$$
$$\left. + \frac{\Gamma(n-1)}{(n-1)^n} + \frac{\Gamma(n)}{1}\frac{1}{n^n} + \frac{\Gamma(n+1)}{1.2}\frac{1}{(n+1)^n} + \frac{\Gamma(n+2)}{1.2.3}\frac{1}{(n+2)^n} + \dots\right].$$

And if n be integral,

$$I = n\left[\frac{2.3\ldots(n-1)}{1^n} + \frac{3.4\ldots n}{2^n} + \frac{4.5\ldots(n+1)}{3^n} + \ldots\right.$$

$$\left. + \frac{1.2\ldots(n-2)}{(n-1)^n} + \frac{2.3\ldots(n-1)}{n^n} + \frac{3.4\ldots n}{(n+1)^n} + \ldots\right]$$

$$= n\sum_{r=1}^{r=\infty}\frac{(r+1)(r+2)\ldots(r+n-2)}{r^n} + n\sum_{r=n-1}^{r=\infty}\frac{(r-1)(r-2)\ldots\{r-(n-2)\}}{r^n}$$

$$= n\sum_{r=1}^{r=\infty}\frac{(r+1)(r+2)\ldots(r+\overline{n-2}) + (r-1)(r-2)\ldots(r-\overline{n-2})}{r^n}. \quad \ldots\ldots\text{(A)}$$

The case of this when n is even is given by Wolstenholme, [Prob. 1919].

If $S_p = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$ and P_p stand for the sum of the products p at a time of the first $n-2$ natural numbers, this result may obviously be written

$$I = 2n(S_2 + P_2S_4 + P_4S_6 + P_6S_8 + \ldots),$$

i.e.
$$= 2n\left(\frac{\pi^2}{6} + P_2\frac{\pi^4}{90} + P_4\frac{\pi^6}{945} + P_6\frac{\pi^8}{9450} + P_8\frac{\pi^{10}}{93555} + \ldots\right). \text{ (See Art. 879.)}$$

In the case when $n=1$,

$$\int_0^\infty \frac{\log x}{x-1}dx = \left(\int_0^1 + \int_1^\infty\right)\frac{\log x}{x-1}dx$$

$$= \int_0^1 \frac{\log x}{x-1}dx - \int_1^0 \frac{\log y}{y-1}\frac{dy}{y}, \quad \text{where } x = \frac{1}{y},$$

$$= \int_0^1 \frac{\log x}{x-1}dx + \int_0^1 \frac{\log x}{x(x-1)}dx$$

$$= \int_0^1 \frac{\log x}{x-1}dx + \int_0^1 \left(\frac{1}{x-1} - \frac{1}{x}\right)\log x\,dx$$

$$= 2\int_0^1 \frac{\log x}{x-1}dx - \frac{1}{2}\Big[(\log x)^2\Big]_0^1,$$

of which the second portion is infinite.

The first part is finite, viz. $2\,.\,\frac{\pi^2}{6} = \frac{\pi^2}{3}$.

Examples.

1. $$\int_0^\infty \left(\frac{\log x}{x-1}\right)^2 dx = \int_0^\infty z^2(2)\{e^{-z} + 2e^{-2z} + 3e^{-3z} + \ldots\}\,dz$$

$$= 4\left(\frac{1}{1^3} + \frac{2}{2^3} + \frac{3}{3^3} + \ldots\right) = \frac{2\pi^2}{3}.$$

2. Prove

$$\int_0^\infty \left(\frac{\log x}{x-1}\right)^3 dx = \pi^2, \qquad \int_0^\infty \left(\frac{\log x}{x-1}\right)^4 dx = 8\left(\frac{\pi^2}{6} + \frac{2\pi^4}{90}\right) = \frac{4}{3}\pi^2 + \frac{8}{45}\pi^4,$$

$$\int_0^\infty \left(\frac{\log x}{x-1}\right)^5 dx = \frac{5\pi^2}{3} + \frac{11\pi^4}{9}, \qquad \int_0^\infty \left(\frac{\log x}{x-1}\right)^6 dx = 2\pi^2 + \frac{14}{3}\pi^4 + \frac{32}{105}\pi^6,$$

and so on. (Cf. Examples 1, 7, 9, Group E, Art. 1082.)

1086. **A General Principle.**

More generally, it is an obvious principle that if $F(x)$ be any function of x which remains unaltered upon changing x into its reciprocal $\frac{1}{x}$, *i.e.* if $F(x)$ be a symmetric function of x and $\frac{1}{x}$, then, provided $\frac{F(x)}{x}$ remains finite from $x=0$ to $x=\infty$ inclusive,

$$\int_0^\infty F(x)\frac{dx}{x}=2\int_0^1 F(x)\frac{dx}{x}.$$

For $$\int_0^\infty F(x)\frac{dx}{x}=\left(\int_0^1+\int_1^\infty\right)F(x)\frac{dx}{x};$$

and changing x to $\frac{1}{y}$ in the second integral,

$$\int_1^\infty F(x)\frac{dx}{x}=\int_1^0 F\left(\frac{1}{y}\right)(-1)\frac{dy}{y}=\int_0^1 F(y)\frac{dy}{y}=\int_0^1 F(x)\frac{dx}{x}.$$

Hence $$\int_0^\infty F(x)\frac{dx}{x}=2\int_0^1 F(x)\frac{dx}{x}.$$

Similarly if $F\left(\frac{1}{x}\right)=-F(x)$, $\int_0^\infty F(x)\frac{dx}{x}=0.$

1087. Again, if the value of any definite integral of the above form, viz. $I\equiv\int_0^\infty F(x)\frac{dx}{x}$, has been found, $F(x)$ being a symmetric function of x and $\frac{1}{x}$, the value of $I'\equiv\int_0^\infty \frac{F(x)}{1+x^n}\frac{dx}{x}$ can be at once obtained, where n may have any value. For in this integral put $\frac{1}{y}$ for x.

Then $$I'=\int_\infty^0 \frac{y^n F\left(\frac{1}{y}\right)}{1+y^n}(-1)\frac{dy}{y}=\int_0^\infty \frac{x^n F(x)}{1+x^n}\frac{dx}{x};$$

$$\therefore\ 2I'=\int_0^\infty \frac{F(x)}{1+x^n}\frac{dx}{x}+\int_0^\infty \frac{x^n F(x)}{1+x^n}\frac{dx}{x}$$

$$=\int_0^\infty \frac{1+x^n}{1+x^n}F(x)\frac{dx}{x}=\int_0^\infty F(x)\frac{dx}{x}=I.$$

Hence $$I'=\tfrac{1}{2}I.$$

1088. Similarly, if $F(x)$ be a symmetric function of $\frac{x}{a}$ and $\frac{a}{x}$, so that

$$F(x)=F\left(a\,.\,\frac{x}{a}\right)=F\left(a\,.\,\frac{a}{x}\right)=F\left(\frac{a^2}{x}\right),$$

then putting $x=\frac{a^2}{y}$,

$$I\equiv\int_0^\infty \frac{F(x)}{a^n+x^n}\frac{dx}{x}=\int_\infty^0 \frac{F\left(\frac{a^2}{y}\right)}{a^n+\frac{a^{2n}}{y^n}}(-1)\frac{dy}{y}$$

$$=\frac{1}{a^n}\int_0^\infty \frac{y^n F(y)}{a^n+y^n}\frac{dy}{y}=\frac{1}{a^n}\int_0^\infty \frac{x^n F(x)}{a^n+x^n}\frac{dx}{x};$$

$$\therefore\ 2I=\int_0^\infty \frac{1+\frac{x^n}{a^n}}{a^n+x^n}F(x)\frac{dx}{x}=\frac{1}{a^n}\int_0^\infty F(x)\frac{dx}{x},$$

i.e.
$$\int_0^\infty \frac{F(x)}{a^n+x^n}\frac{dx}{x}=\frac{1}{2a^n}\int_0^\infty F(x)\frac{dx}{x}.$$

1089. Again, if $F(x)$ be symmetric in $\frac{x}{a}$ and $\frac{a}{x}$, so that $F(x)=F\left(\frac{a^2}{x}\right)$,

$$I\equiv\int_1^a F(x^2)\frac{dx}{x}=\int_1^a F(x)\frac{dx}{x}.$$

For writing $x^2=z$, we have

$$\int_1^a F(x^2)\frac{dx}{x}=\frac{1}{2}\int_1^{a^2} F(z)\frac{dz}{z}=\frac{1}{2}\left(\int_1^a+\int_a^{a^2}\right)F(z)\frac{dz}{z}.$$

Putting $z=\frac{a^2}{t}$ in the second,

$$\int_a^{a^2} F(z)\frac{dz}{z}=\int_a^1 F\left(\frac{a^2}{t}\right)(-1)\frac{dt}{t}=\int_1^a F(t)\frac{dt}{t}=\int_1^a F(z)\frac{dz}{z};$$

$$\therefore\ \int_1^a F(x^2)\frac{dx}{x}=\int_1^a F(z)\frac{dz}{z}=\int_1^a F(x)\frac{dx}{x}.$$

We note also that it is therefore proved that

$$\int_1^{a^2} F(x)\frac{dx}{x}=2\int_1^a F(x^2)\frac{dx}{x}=2\int_1^a F(x)\frac{dx}{x}.$$

Again, taking $\int_{a^2}^{a^3} F(x)\frac{dx}{x}$, if we put $x=\frac{a^2}{t}$, we have

$$\int_{a^2}^{a^3} F(x)\frac{dx}{x}=-\int_1^{\frac{1}{a}} F\left(\frac{a^2}{t}\right)\frac{dt}{t}=-\int_1^{\frac{1}{a}} F(t)\frac{dt}{t};$$

$$\therefore\ \int_{a^2}^{a^3} F(x)\frac{dx}{x}=\int_{\frac{1}{a}}^1 F(x)\frac{dx}{x},\text{ with other similar results.}$$

1090. Since $\int_0^\infty \frac{1}{x+\frac{1}{x}}\frac{dx}{x}=\int_0^\infty \frac{dx}{1+x^2}=\frac{\pi}{2}$, it follows that

$$\int_0^\infty \frac{dx}{(1+x^2)(1+x^n)}=\int_0^\infty \frac{1}{(x+x^{-1})(1+x^n)}\frac{dx}{x}=\frac{1}{2}\int_0^\infty \frac{1}{x+x^{-1}}\frac{dx}{x}=\frac{\pi}{4}.$$

Similarly, since

$$\int_0^\infty \frac{1}{\frac{a^2}{x^2}+\frac{x^2}{a^2}}\frac{dx}{x} \quad =\int_0^\infty \frac{a^2x\,dx}{a^4+x^4}=\frac{1}{2}\left[\tan^{-1}\frac{x^2}{a^2}\right]_0^\infty=\frac{\pi}{4},$$

we have $\int_0^\infty \frac{1}{a^n+x^n}\frac{1}{\frac{a^2}{x^2}+\frac{x^2}{a^2}}\frac{dx}{x}=\frac{1}{2a^n}\frac{\pi}{4},$

that is $\int_0^\infty \frac{x\,dx}{(a^4+x^4)(a^n+x^n)}=\frac{\pi}{8}\frac{1}{a^{n+2}}.$

1091. It follows from Art. 1087, that since the expression $\frac{2x}{1+x^2}$ is unaltered by writing $\frac{1}{x}$ for x, writing $x=\tan\frac{\theta}{2}$,

$$I\equiv\int_0^\infty \frac{F\left(\frac{2x}{1+x^2}\right)}{1+x^n}\frac{dx}{x}=\frac{1}{2}\int_0^\infty F\left(\frac{2x}{1+x^2}\right)\frac{dx}{x}$$
$$=\int_0^1 F\left(\frac{2x}{1+x^2}\right)\frac{dx}{x}=\int_0^{\frac{\pi}{2}} F(\sin\theta)\frac{d\theta}{\sin\theta},$$

a transformation given by Wolstenholme (*Educ. Times*, 9931).

We may also see the truth of this result by differentiation with regard to n, which gives

$$\frac{dI}{dn}=-\int_0^\infty \frac{F\left(\frac{2x}{1+x^2}\right)}{(1+x^n)^2}x^n\log x\frac{dx}{x}, \text{ and writing } \frac{1}{x} \text{ for } x,$$
$$=\int_0^\infty \frac{F\left(\frac{2x}{1+x^2}\right)}{(x^n+1)^2}x^n\log x\frac{dx}{x}=-\frac{dI}{dn}.$$

$\therefore \frac{dI}{dn}=0$, and I is therefore *independent of n*, and therefore the same as if $n=0$, *i.e.*

$$I\equiv\int_0^\infty \frac{F\left(\frac{2x}{1+x^2}\right)}{1+x^n}\frac{dx}{x}=\frac{1}{2}\int_0^\infty F\left(\frac{2x}{1+x^2}\right)\frac{dx}{x}=\text{ etc.}$$

Putting $\frac{x}{a}$ for x, it follows that

$$\int_0^\infty \frac{F\left(\frac{2ax}{a^2+x^2}\right)}{a^n+x^n}\frac{dx}{x}=\frac{1}{a^n}\int_0^{\frac{\pi}{2}} F(\sin\theta)\frac{d\theta}{\sin\theta}.$$

1092. Thus, if $F(z)=z$, we have

$$\int_0^\infty \frac{dx}{(a^2+x^2)(a^n+x^n)}=\frac{1}{2a^{n+1}}\frac{\pi}{2}=\frac{\pi}{4a^{n+1}},$$

or if $F(z)=z^p$, p being a positive integer,

$$\int_0^\infty \frac{x^{p-1}}{(a^2+x^2)^p(a^n+x^n)}\,dx=\frac{1}{2^p}\frac{1}{a^{p+n}}\int_0^{\frac{\pi}{2}}\sin^{p-1}\theta\,d\theta$$

$$=\frac{1}{2^p}\frac{1}{a^{p+n}}\frac{p-2}{p-1}\frac{p-4}{p-3}\cdots\frac{2}{3}\left(\text{or }\ldots\frac{1}{2}\frac{\pi}{2}\right),\text{ as } p \text{ is even or odd.}$$

1093. Consider next the value of $I_n\equiv\int_0^{\frac{\pi}{2}}(\log\tan\theta)^{2n}d\theta$, where n is any positive integer. Put $\tan\theta=x$.

Then $$I_n\equiv\int_0^\infty\frac{(\log x)^{2n}}{1+x^2}dx=\left(\int_0^1+\int_1^\infty\right)\frac{(\log x)^{2n}}{1+x^2}dx.$$

In the second integral put $x=\frac{1}{y}$,

$$\int_1^\infty\frac{(\log x)^{2n}}{1+x^2}dx=\int_1^0\frac{(-\log y)^{2n}}{1+\frac{1}{y^2}}\left(-\frac{1}{y^2}\right)dy=\int_0^1\frac{(\log x)^{2n}}{1+x^2}dx;$$

$$\therefore\ I=2\int_0^1\frac{(\log x)^{2n}}{1+x^2}dx;$$

$$\therefore\ I_n=2\int_\infty^0\frac{(-z)^{2n}}{1+e^{-2z}}(-e^{-z})\,dz,\text{ where } x=e^{-z}$$

$$=2\int_0^\infty z^{2n}(e^{-z}-e^{-3z}+e^{-5z}-e^{-7z}+\ldots)\,dz\quad(0<z<\infty)$$

$$=2\Gamma(2n+1)\left[\frac{1}{1^{2n+1}}-\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}-\ldots\right]$$

$$=2\Gamma(2n+1)\frac{E_{2n}\left(\frac{\pi}{2}\right)^{2n+1}}{2(2n)!},\text{ where } E_{2n} \text{ is the } n^{\text{th}} \text{ Eulerian number;}$$

$$\therefore\ \int_0^{\frac{\pi}{2}}(\log\tan\theta)^{2n}d\theta=\left(\frac{\pi}{2}\right)^{2n+1}E_{2n};$$

and the values of E_{2n} being successively

$E_2=1,\quad E_4=5,\quad E_6=61,\quad E_8=1385$, etc. (see Art. 1073),

we have

$$\int_0^{\frac{\pi}{2}}(\log\tan\theta)^2d\theta=\frac{\pi^3}{8};\qquad\int_0^{\frac{\pi}{2}}(\log\tan\theta)^4d\theta=\frac{5\pi^5}{32};$$

$$\int_0^{\frac{\pi}{2}}(\log\tan\theta)^6d\theta=\frac{61\pi^7}{128};\quad\int_0^{\frac{\pi}{2}}(\log\tan\theta)^8d\theta=\frac{1385\pi^9}{512},\text{ etc.}$$

1094. Since $E_{2n}=$ coef. of $\frac{z^{2n}}{(2n)!}$ in the expansion of $\sec z$, *i.e.* $\left[\frac{d^{2n}\sec z}{dz^{2n}}\right]_{z=0}$,

we have $$\int_0^{\frac{\pi}{2}}(\log\tan\theta)^{2n}d\theta=\left(\frac{\pi}{2}\right)^{2n+1}\left[\frac{d^{2n}}{dz^{2n}}\sec z\right]_{z=0}\quad\text{[Wolstenholme].}$$

1095. The integral $I \equiv \int_0^{\frac{\pi}{2}} (\log \tan \theta)^{2n+1} d\theta$ vanishes.

For putting $\theta = \frac{\pi}{2} - \phi, \quad I = -I; \quad \therefore I = 0.$

Hence $\int_0^{\frac{\pi}{2}} (\log \tan \theta)^p d\theta = \left(\frac{\pi}{2}\right)^{p+1} E_p$ or 0, according as p is even or odd.

Also $\log \cot \theta = -\log \tan \theta$;

$\therefore \int_0^{\frac{\pi}{2}} (\log \cot \theta)^p d\theta = \left(\frac{\pi}{2}\right)^{p+1} E_p$ or 0, according as p is even or odd.

Hence $\int_0^{\frac{\pi}{2}} (\log \tan \theta)^p d\theta$ and $\int_0^{\frac{\pi}{2}} (\log \cot \theta)^p d\theta$ have been computed for all positive integral values of p.

1096. Let $\quad I_1 \equiv \int_0^{\frac{\pi}{2}} (\log \sin \theta)^2 d\theta = \int_0^{\frac{\pi}{2}} (\log \cos \theta)^2 d\theta,$

and $\quad I_2 \equiv \int_0^{\frac{\pi}{2}} (\log \sin \theta)(\log \cos \theta)\, d\theta.$

Then

$$2I_1 + 2I_2 = \int_0^{\frac{\pi}{2}} (\log \sin \theta + \log \cos \theta)^2 d\theta = \int_0^{\frac{\pi}{2}} (\log \sin 2\theta - \log 2)^2 d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\log \sin 2\theta)^2 d\theta - 2 \log 2 \int_0^{\frac{\pi}{2}} \log \sin 2\theta \, d\theta + (\log 2)^2 \int_0^{\frac{\pi}{2}} 1 \, d\theta.$$

Writing $2\theta = \phi$,

$$\int_0^{\frac{\pi}{2}} (\log \sin 2\theta)^2 d\theta = \frac{1}{2} \int_0^{\pi} (\log \sin \phi)^2 d\phi = \int_0^{\frac{\pi}{2}} (\log \sin \theta)^2 d\theta = I_1,$$

and $\int_0^{\frac{\pi}{2}} \log \sin 2\theta \, d\theta \quad = \frac{1}{2} \int_0^{\pi} \log \sin \phi \, d\phi \quad = \int_0^{\frac{\pi}{2}} \log \sin \phi \, d\phi \quad = \frac{\pi}{2} \log \frac{1}{2};$

$$\therefore 2I_1 + 2I_2 = I_1 - 2 \log 2 \cdot \frac{\pi}{2} \log \frac{1}{2} + (\log 2)^2 \frac{\pi}{2},$$

i.e. $$I_1 + 2I_2 = \frac{3\pi}{2} (\log 2)^2. \quad \text{.............................(A)}$$

Again

$$2I_1 - 2I_2 = \int_0^{\frac{\pi}{2}} (\log \sin \theta - \log \cos \theta)^2 d\theta = \int_0^{\frac{\pi}{2}} (\log \tan \theta)^2 d\theta = \frac{\pi^3}{8}; \quad \text{....(B)}$$

$$\therefore \left. \begin{aligned} I_1 + 2I_2 &= \frac{3\pi}{2} (\log 2)^2, \\ I_1 - I_2 &= \frac{\pi^3}{16}; \end{aligned} \right\}$$

$$\therefore \text{ solving, } \left. \begin{aligned} I_1 &\equiv \int_0^{\frac{\pi}{2}} (\log \sin \theta)^2 d\theta = \int_0^{\frac{\pi}{2}} (\log \cos \theta)^2 d\theta = \frac{\pi}{2} (\log 2)^2 + \frac{\pi^3}{24}, \\ I_2 &\equiv \int_0^{\frac{\pi}{2}} \log \sin \theta \,.\, \log \cos \theta \, d\theta = \frac{\pi}{2} (\log 2)^2 - \frac{\pi^3}{48}. \end{aligned} \right\}$$

These results are due to the late Professor Wolstenholme.

Obviously it follows that

$$\int_0^{\frac{\pi}{4}} \log\sin\theta \, . \log\cos\theta \, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \log\sin\theta \log\cos\theta \, d\theta = \frac{\pi}{4}(\log 2)^2 - \frac{\pi^3}{96}.$$

1097. We may write the expression for cosec z in partial fractions (Hobson, *Trigonometry*, p. 335) as

$$\operatorname{cosec} z = \ldots + \frac{1}{z-2\pi} - \frac{1}{z-\pi} + \overset{*}{\frac{1}{z}} - \frac{1}{z+\pi} + \frac{1}{z+2\pi} - \ldots, \quad \ldots\ldots\ldots(A)$$

it being *understood that this doubly infinite series extends equal distances to infinity* on either side of the central term $\frac{1}{z}$ marked with an asterisk.

A similar expression for $\operatorname{cosec}^2 z$ is

$$\operatorname{cosec}^2 z = \ldots + \frac{1}{(z-2\pi)^2} + \frac{1}{(z-\pi)^2} + \overset{*}{\frac{1}{z^2}} + \frac{1}{(z+\pi)^2} + \frac{1}{(z+2\pi)^2} + \ldots, \quad \ldots.(B)$$

with the same understanding as before. [614, Wolstenholme's *Problems.*]

The latter is obtainable from a consideration of the factorisation of

$$\frac{\cosh x + \cos\theta}{2\cos^2\frac{\theta}{2}}, \quad \text{viz.} \quad \prod_{r=-\infty}^{r=\infty}\left\{1 + \frac{x^2}{(\overline{2r+1}\,\pi+\theta)^2}\right\}$$

[viz. equating coefficients of x^2 in the expansion and writing $\pi - 2z$ for θ].

Differentiating these expressions respectively $2r+1$ times and $2r$ times, and then putting $z = \frac{\pi}{n}$ in each, we have

$$\frac{1}{(2r+1)!}\left(\frac{\pi}{n}\right)^{2r+2}\left[\frac{d^{2r}}{dz^{2r}}\left(\frac{\cos z}{\sin^2 z}\right)\right]_{z=\frac{\pi}{n}}$$

$$= \ldots + \frac{1}{(2n-1)^{2r+2}} - \frac{1}{(n-1)^{2r+2}} + \overset{*}{\frac{1}{1^{2r+2}}} - \frac{1}{(n+1)^{2r+2}} + \frac{1}{(2n+1)^{2r+2}} - \ldots, \quad (A')$$

$$\frac{1}{(2r+1)!}\left(\frac{\pi}{n}\right)^{2r+2}\left[\frac{d^{2r}}{dz^{2r}}\left(\frac{1}{\sin^2 z}\right)\right]_{z=\frac{\pi}{n}}$$

$$= \ldots + \frac{1}{(2n-1)^{2r+2}} + \frac{1}{(n-1)^{2r+2}} + \overset{*}{\frac{1}{1^{2r+2}}} + \frac{1}{(n+1)^{2r+2}} + \frac{1}{(2n+1)^{2r+2}} + \ldots \quad \ldots(B')$$

Now consider the integral

$$I = \int_0^\infty \frac{\left(\log\frac{1}{x}\right)^{2r+1}}{1+x^n}\,dx = \left(\int_0^1 + \int_1^\infty\right)\frac{\left(\log\frac{1}{x}\right)^{2r+1}}{1+x^n}\,dx.$$

In the second integral write $x = \frac{1}{y}$.

Then

$$\int_1^\infty \frac{\left(\log\frac{1}{x}\right)^{2r+1}}{1+x^n}\,dx = \int_1^0 \frac{(\log y)^{2r+1}}{1+y^{-n}}\left(-\frac{1}{y^2}\right)dy = -\int_0^1 \frac{\left(\log\frac{1}{x}\right)^{2r+1}}{1+x^n}\,x^{n-2}dx$$

$$\therefore\ I=\int_0^1 \frac{1-x^{n-2}}{1+x^n}\left(\log\frac{1}{x}\right)^{2r+1}dx=\int_0^\infty y^{2r+1}\frac{e^{-y}-e^{-(n-1)y}}{1+e^{-ny}}dy, \text{ where } x=e^{-y},$$

$$=\int_0^\infty y^{2r+1}\{e^{-y}-e^{-(n-1)y}\}(1-e^{-ny}+e^{-2ny}-e^{-3ny}+\dots)dy$$

$$=\int_0^\infty y^{2r+1}\{\dots+e^{-\overline{2n-1}\,y}-e^{-\overline{n-1}\,y}+\overset{*}{e^{-y}}-e^{-\overline{n+1}\,y}+e^{-\overline{2n+1}\,y}-\dots\}dy$$

$$=(2r+1)!\left\{\dots+\frac{1}{(2n-1)^{2r+2}}-\frac{1}{(n-1)^{2r+2}}+\frac{\overset{*}{1}}{1^{2r+2}}-\frac{1}{(n+1)^{2r+2}}+\frac{1}{(2n+1)^{2r+2}}-\dots\right\}$$

$$=(2r+1)!\,.\,\frac{1}{(2r+1)!}\left(\frac{\pi}{n}\right)^{2r+2}\left[\frac{d^{2r}}{dz^{2r}}\left(\frac{\cos z}{\sin^2 z}\right)\right]_{z=\frac{\pi}{n}}=\left(\frac{\pi}{n}\right)^{2r+2}\left[\frac{d^{2r}}{dz^{2r}}\left(\frac{\cos z}{\sin^2 z}\right)\right]_{z=\frac{\pi}{n}}.$$

1098. Again, if

$$I'=\int_0^\infty \frac{\left\{\log\frac{1}{x}\right\}^{2r+1}}{1-x^n}dx=\left[\int_0^1+\int_1^\infty\right]\frac{\left(\log\frac{1}{x}\right)^{2r+1}}{1-x^n}dx,$$

putting $x=\frac{1}{y}$ in the second integral,

$$I'=\int_0^1 \frac{\left(\log\frac{1}{x}\right)^{2r+1}}{1-x^n}dx+\int_0^1 x^{n-2}\frac{\left(\log\frac{1}{x}\right)^{2r+1}}{1-x^n}dx$$

$$=\int_0^1\frac{1+x^{n-2}}{1-x^n}\left(\log\frac{1}{x}\right)^{2r+1}dx=\int_0^\infty\frac{e^{-y}+e^{-(n-1)y}}{1-e^{-ny}}y^{2r+1}dy, \text{ where } x=e^{-y},$$

$$=\int y^{2r+1}\{e^{-y}+e^{-(n-1)y}\}\{1+e^{-ny}+e^{-2ny}+\dots\}dy$$

$$=\int_0^\infty y^{2r+1}\{\dots+e^{-\overline{2n-1}\,y}+e^{-\overline{n-1}\,y}+\overset{*}{e^{-y}}+e^{-\overline{n+1}\,y}+e^{-\overline{2n+1}\,y}+\dots\}dy$$

$$=(2r+1)!\left\{\dots+\frac{1}{(2n-1)^{2r+2}}+\frac{1}{(n-1)^{2r+2}}+\frac{\overset{*}{1}}{1^{2r+2}}+\frac{1}{(n+1)^{2r+2}}+\frac{1}{(2n+1)^{2r+2}}+\dots\right\}$$

$$=(2r+1)!\frac{1}{(2r+1)!}\left(\frac{\pi}{n}\right)^{2r+2}\left[\frac{d^{2r}}{dz^{2r}}\left(\frac{1}{\sin^2 z}\right)\right]_{z=\frac{\pi}{n}}=\left(\frac{\pi}{n}\right)^{2r+2}\left[\frac{d^{2r}}{dz^{2r}}\left(\frac{1}{\sin^2 z}\right)\right]_{z=\frac{\pi}{n}}.$$

Thus

$$\left.\begin{aligned}\int_0^\infty\frac{\left(\log\frac{1}{x}\right)^{2r+1}}{1+x^n}dx&=\left(\frac{\pi}{n}\right)^{2r+2}\left[\frac{d^{2r}}{dz^{2r}}\left(\frac{\cos z}{\sin^2 z}\right)\right]_{z=\frac{\pi}{n}},\\ \int_0^\infty\frac{\left(\log\frac{1}{x}\right)^{2r+1}}{1-x^n}dx&=\left(\frac{\pi}{n}\right)^{2r+2}\left[\frac{d^{2r}}{dz^{2r}}\left(\frac{1}{\sin^2 z}\right)\right]_{z=\frac{\pi}{n}},\end{aligned}\right\}\text{provided } n>1.$$

These results are due to Wolstenholme.*

1099. Group H. **Legendre's Rule.**

$$I_p\equiv\int_0^1\left(\frac{x^m-1}{\log x}\right)^p x^{n-1}dx. \quad \textbf{(Euler.)}$$

Integrating the result $\int_0^1 x^n dx=\frac{1}{n+1}$ with regard to n between limits 0 and n, we obtain

$$\int_0^1\frac{x^n-1}{\log x}dx=\log(1+n). \qquad \dots\dots(1)$$

* *Problems*, 1919, 41 and 42.

Hence

$$\int_0^1 \frac{x^m - x^n}{\log x}\,dx = \int_0^1 \frac{(x^m-1)-(x^n-1)}{\log x}\,dx = \log\frac{1+m}{1+n} \quad \text{.............(2)}$$

and

$$\int_0^1 \frac{x^m-1}{\log x}\,x^{n-1}dx = \int_0^1 \frac{(x^{m+n-1}-1)-(x^{n-1}-1)}{\log x}\,dx = \log\left(1+\frac{m}{n}\right). \quad \text{...(3)}$$

If $F(x)$ be any polynomial in which the sum of the coefficients is zero,

$$\equiv A_0x^n + A_1x^{n-1} + A_2x^{n-2} + \ldots + A_{n-1}x + A_n, \quad \sum_0^n A_r = 0,$$

$$\equiv A_0(x^n-1) + A_1(x^{n-1}-1)\ldots + A_{n-1}(x-1).$$

Then

$$\int_0^1 \frac{F(x)}{\log x}\,dx = A_0\log(n+1) + A_1\log n + A_2\log(n-1) + \ldots + A_{n-1}\log 2$$

$$= \log (n+1)^{A_0} n^{A_1} (n-1)^{A_2} \ldots 2^{A_{n-1}}. \quad \text{.................................(4)}$$

Let Δ be an operative symbol defined by

$$\Delta v_n = v_{n+m} - v_n.$$

Then equation (3) may be written

$$I_1 = \Delta \log n. \quad \text{.......................................(5)}$$

Taking $I_2 = \int_0^1 \left(\frac{x^m-1}{\log x}\right)^2 x^{n-1}dx,$

$$\frac{dI_2}{dm} = 2\int_0^1 \frac{x^m-1}{\log x}\,x^{m+n-1}dx = 2[\log(2m+n) - \log(m+n)].$$

Integrating with regard to m from 0 to m,

$$I_2 = 2\left[\frac{2m+n}{2}\log(2m+n) - \frac{2m+n}{2}\right]_0^m - 2\Big[(m+n)\log(m+n) - (m+n)\Big]_0^m$$

$$= (2m+n)\log(2m+n) - 2(m+n)\log(m+n) + n\log n = \Delta^2 n \log n. \quad \text{...(6)}$$

Similarly $\quad I_3 = \frac{1}{2!}\Delta^3 n^2 \log n, \qquad I_4 = \frac{1}{3!}\Delta^4 n^3 \log n$, etc. ....(7)

Some of these integrals were established by Euler (*Calc. Int.*, iv., p. 271). The general rule was given by Legendre (*Exercises*, p. 372).

1100. Kummer's Integrals. (Crelle, T. xvii., p. 224.)

From equation (2) of the last article,

$$\text{(i)}\quad I \equiv \int_0^1 \frac{x^a - x^b}{1+x^c}\cdot\frac{1}{\log x}\,\frac{dx}{x} = \int_0^1 (x^{a-1} - x^{b-1})(1 - x^c + x^{2c} - \ldots)\frac{dx}{\log x}$$

$$= \log\frac{a}{b} - \log\frac{a+c}{b+c} + \log\frac{a+2c}{b+2c} - \ldots = \log\left(\frac{a}{b}\cdot\frac{b+c}{a+c}\cdot\frac{a+2c}{b+2c}\cdot\frac{b+3c}{a+3c}\ldots\right);$$

$$\text{(ii)}\quad I' = \int_0^1 \frac{x^a - x^b}{1-x^c}\cdot\frac{1}{\log x}\cdot\frac{dx}{x} = \log\left(\frac{a}{b}\cdot\frac{a+c}{b+c}\cdot\frac{a+2c}{b+2c}\cdot\frac{a+3c}{b+3c}\ldots\right),$$

in the same way.

Putting $c=1$ and $a+b=1$ in (i),

$$\int_0^1 \frac{x^a-x^{1-a}}{1+x}\cdot\frac{1}{\log x}\cdot\frac{dx}{x}$$

$$=\int_0^1 \frac{x^{a-1}-x^{-a}}{1+x}\cdot\frac{dx}{\log x}=\log\left(\frac{a}{1-a}\cdot\frac{2-a}{1+a}\cdot\frac{2+a}{3-a}\cdot\frac{4-a}{3+a}\cdots\right)$$

$$=Lt_{n=\infty}\left\{a\frac{1-\frac{a^2}{2^2}}{1-\frac{a^2}{1^2}}\cdot\frac{1-\frac{a^2}{4^2}}{1-\frac{a^2}{3^2}}\cdots\frac{1-\frac{a^2}{(2n)^2}}{1-\frac{a^2}{(2n-1)^2}}\cdot\frac{1}{1-\frac{a}{2n+1}}\times\frac{2^2.4^2\ldots(2n)^2}{1^2.3^2\ldots(2n-1)^2}\cdot\frac{1}{2n+1}\right\}$$

$$=\log\left(\frac{2}{\pi}\tan\frac{\pi a}{2}\times\frac{\pi}{2}\right)=\log\tan\frac{\pi a}{2}.$$

Examples.

1. Deduce the integral $\int_0^{\frac{\pi}{2}}\log\sin\theta\,d\theta=\frac{\pi}{2}\log\frac{1}{2}$ from the theorem

$$\frac{x^{2n}-1}{x^2-1}=\left(x^2-2x\cos\frac{\pi}{n}+1\right)\left(x^2-2x\cos\frac{2\pi}{n}+1\right)\ldots\left\{x^2-2x\cos\frac{(n-1)\pi}{n}+1\right\}.$$

[Leslie Ellis, *Cam. Math. Jour.*, vol. vii., p. 282.]

2. Show that $\int_0^{\frac{\pi}{2}}\sin\theta\log\sin\theta\,d\theta=\log_e\left(\frac{2}{e}\right)$.

3. Show that $\int_0^{\frac{\pi}{2}}\sin^2\theta\log\sin\theta\,d\theta=\frac{\pi}{8}\log_e\left(\frac{e}{4}\right)$.

[Euler, *Nov. Com. Petrop.*, vol. xix., p. 30.]

4. Prove that $\int_0^1\frac{\log(1+x)}{1+x^2}dx=\frac{\pi}{8}\log_e 2$. [Colleges β, 1890.]

5. Prove that $\int_0^{\frac{\pi}{4}}\log(1+\tan\theta)d\theta=\frac{\pi}{8}\log_e 2$. [Trinity, 1885.]

6. Prove that $\int_0^{\frac{\pi}{2}}\tan\theta\log\operatorname{cosec}\theta\,d\theta=\frac{\pi^2}{24}$. [Trinity, 1884.]

7. Prove that $\int_0^{\frac{\pi}{2}}\sin 2\theta\log(1+\cos\theta)\,d\theta=\frac{1}{2}$. [Trinity, 1885.]

8. Prove that if a be <1, $\int_0^1\log\frac{1+ax}{1-ax}\cdot\frac{dx}{x\sqrt{1-x^2}}=\pi\sin^{-1}a$.

[Oxford, II. P., 1888.]

9. Prove that $\int_0^1\left(\frac{\log x}{1-x}\right)^2dx=2\int_0^1\left(\frac{\log x}{1+x}\right)^2dx=\frac{\pi^2}{3}$. [St. John's, 1881.]

10. Prove that

$$\int_0^{\frac{\pi}{2}}\sin x\log\left(\frac{1+\sin\alpha\sin x}{1-\sin\alpha\sin x}\right)dx=\int_0^{\pi}\sin x\tan^{-1}(\tan\alpha\sin x)dx=\pi\tan\frac{\alpha}{2}.$$

[St. John's, 1881.]

11. Show that $\int_0^1 \frac{(\log x)^2}{1+x^2}dx = \frac{1}{2}\int_0^\infty \frac{(\log x)^2}{1+x^2}dx = \frac{\pi^3}{16}$.

12. Prove that $\int_1^0 \frac{1}{x}\log(1-x)dx = \frac{\pi^2}{6}$. [OXFORD, I. P., 1889.]

13. Prove that $\int_0^\infty \frac{\left(\log\frac{1}{x}\right)^3}{(1+x)^4}dx = \frac{\pi^2}{2}$. [COLLEGES δ, 1883.]

14. Prove that $\int_0^\infty \frac{\log\frac{1}{x}}{(1+x)^4}dx = \frac{1}{2}$. [COLLEGES γ, 1882.]

15. Prove that $\int_0^1 \log\frac{1+2x\cos a+x^2}{1-2x\cos a+x^2}\frac{dx}{x} = \pi\left(\frac{\pi}{2}-a\right)$ where $\pi > a > 0$. [COLLEGES γ, 1882.]

16. Prove that $\int_0^1 \log x \log(1-x)dx = 2 - \frac{\pi^2}{6}$. [ST. JOHN'S, 1885.]

17. Show that $\int_0^\infty f\left(x+\frac{1}{x}\right)\log x\frac{dx}{x} = 0$. [COLLEGES δ, 1881.]

18. Show that $\int_0^{\frac{\pi}{2}} \frac{\log\sec x}{\sin x}dx = \frac{\pi^2}{8}$. [COLLEGES ε, 1881.]

19. Show that $\int_0^{\frac{\pi}{2}} \log\frac{1+\cos^2\theta}{\sqrt{1+\frac{1}{8}\cos^2\theta}}d\theta = \frac{\pi}{4}\log_e 2$. [R.P.]

20. Show that $\int_0^{\frac{\pi}{4}} \tan\theta\sec 2\theta\log\cot\theta\, d\theta = \frac{\pi^2}{24}$. [ST. JOHN'S, 1882.]

1101. GROUP I. Derivations from

$$\int_0^\infty \frac{x^{a-1}}{1+x}dx = \int_0^1 \frac{x^{a-1}+x^{-a}}{1+x}dx = \pi\,\mathbf{cosec}\,a\pi, \quad (1>a>0), \text{ Art. 871.} \;\ldots(1)$$

Put $x = y^n$, $a = \frac{p}{n}$. Then

$$\int_0^\infty \frac{y^{p-1}}{1+y^n}dy = \int_0^1 \frac{y^{p-1}+y^{n-p-1}}{1+y^n}dy = \frac{\pi}{n}\operatorname{cosec}\frac{p\pi}{n}, \quad (n>p>0). \quad \ldots(2)$$

The case $n=2$ gives

$$\int_0^\infty \frac{x^{p-1}}{1+x^2}dx = \int_0^1 \frac{x^{p-1}+x^{1-p}}{1+x^2}dx = \frac{\pi}{2}\operatorname{cosec}\frac{p\pi}{2}, \quad (2>p>0). \quad \ldots(3)$$

Putting $p=m+1$, we have

$$\int_0^\infty \frac{x^m}{1+x^2}dx = \int_0^1 \frac{x^m+x^{-m}}{1+x^2}dx = \frac{\pi}{2}\sec\frac{m\pi}{2}, \quad (1>m>-1). \ldots(4)$$

Put $p=1$ in (2),

$$\int_0^\infty \frac{dx}{1+x^n} = \int_0^1 \frac{1+x^{n-2}}{1+x^n}dx = \frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n}, \quad (n>1). \;\ldots\ldots\ldots(5)$$

Put $y=\dfrac{x}{\sqrt[n]{1-x^n}}$ in (2),

$$\int_0^1 \frac{x^{p-1}}{(1-x^n)^{\frac{p}{n}}}\,dx = \frac{\pi}{n}\operatorname{cosec}\frac{p\pi}{n}, \quad (n>p>0). \quad \ldots(6)$$

Put $p=1$ in (6),

$$\int_0^1 \frac{dx}{\sqrt[n]{1-x^n}} = \frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n}, \quad (n>1). \quad \ldots\ldots\ldots(7)$$

From (4), $$\int_0^1 \frac{x^m+x^{-m}}{x+x^{-1}}\frac{dx}{x} = \frac{\pi}{2}\sec\frac{m\pi}{2}, \quad (1>m>-1).$$

This may be written as

$$\int_0^1 \frac{\cosh(m\log x)}{\cosh(\log x)}\frac{dx}{x} = \frac{\pi}{2}\sec\frac{m\pi}{2}, \quad (1>m>-1). \quad (8)$$

Put $x=e^{-qz}$, q positive; $mq=p$, and replace z by x,

$$\int_0^\infty \frac{\cosh px}{\cosh qx}\,dx = \frac{\pi}{2q}\sec\frac{p\pi}{2q}, \quad (q>p>-q). \quad (9)$$

Put $q=\pi$, $$\int_0^\infty \frac{\cosh px}{\cosh \pi x}\,dx = \frac{1}{2}\sec\frac{p}{2}, \quad (\pi>p>-\pi). \quad (10)$$

Put $x=\dfrac{y}{b}$ in (1),

$$\int_0^\infty \frac{y^{a-1}}{b+y}\,dy = \pi b^{a-1}\operatorname{cosec} a\pi, \quad (1>a>0). \quad (11)$$

Diff. $r-1$ times with respect to b,

$$\int_0^\infty \frac{y^{a-1}}{(b+y)^r}\,dy = \frac{(1-a)(2-a)\ldots(r-1-a)}{1.2\ldots(r-1)}\,\pi b^{a-r}\operatorname{cosec} a\pi, \quad (1>a>0). \quad (12)$$

Integrate (11) with regard to b from b_1 to b_2,

$$\int_0^\infty y^{a-1}\log\frac{b_2+y}{b_1+y}\,dy = \pi\,\frac{b_2{}^a-b_1{}^a}{a}\operatorname{cosec} a\pi, \quad (1>a>0). \quad \ldots(13)$$

Write $x=by$ in (1),

$$\int_0^\infty \frac{y^{a-1}}{1+by}\,dy = \pi b^{-a}\operatorname{cosec} a\pi, \quad (1>a>0). \quad \ldots(14)$$

Diff. $r-1$ times with respect to b,

$$\int_0^\infty \frac{y^{a+r-2}}{(1+by)^r}\,dy = \pi\,\frac{a(a+1)\ldots(a+r-2)}{1.2\ldots(r-1)}\,b^{-a-r+1}\operatorname{cosec} a\pi, (1>a>0). \quad (15)$$

Diff. (10) with regard to p,

$$\int_0^\infty x\,\frac{\sinh px}{\cosh \pi x}\,dx = \frac{1}{4}\sec\frac{p}{2}\tan\frac{p}{2}, \quad (\pi>p>-\pi). \quad (16)$$

Integrate (10) with regard to p between 0 and p,

$$\int_0^\infty \frac{\sinh px}{\cosh \pi x}\frac{dx}{x} = \log\tan\frac{\pi+p}{4}, \quad (\pi>p>-\pi). \quad (17)$$

Diff. (1) with regard to a,

$$\int_0^\infty \frac{x^a\log x}{1+x}\frac{dx}{x} = -\pi^2\operatorname{cosec} a\pi\cot a\pi, \quad (1>a>0), \quad \ldots(18)$$

etc. Thus obviously a large number of such results may be derived.

1102. GROUP J.

Next consider the **similar integral** $\int_0^\infty \frac{x^{a-1}}{1-x}\,dx \quad (1>a>0)$.

Here the integrand $\frac{x^{a-1}}{1-x}$ has infinities at $x=0$ and at $x=1$. At $x=0$, since a is positive and <1, the limit of $\int_0^{\epsilon_1} \frac{x^{a-1}}{1-x}\,dx$, when ϵ_1 is indefinitely diminished, is zero (Art. 348). We have to examine the behaviour of the integral in the neighbourhood of $x=1$. Consider the integral

$$\left(\int_0^{1-\epsilon} + \int_{1+\eta}^\infty\right) \frac{x^{a-1}}{1-x}\,dx \quad (1>a>0),$$

where ϵ and η are small positive and arbitrary quantities.

In the second integral put $x=\frac{1}{y}$.

Then

$$\int_{1+\eta}^\infty \frac{x^{a-1}}{1-x}\,dx = \int_{\frac{1}{1+\eta}}^0 \frac{y^{1-a}}{1-y^{-1}}(-y^{-2})\,dy = -\int_0^{\frac{1}{1+\eta}} \frac{x^{-a}}{1-x}\,dx$$

$$= -\left(\int_0^{1-\epsilon} + \int_{1-\epsilon}^{\frac{1}{1+\eta}}\right) \frac{x^{-a}}{1-x}\,dx.$$

And in the second of these let $x=1-\xi$.

$$\int_{1-\epsilon}^{\frac{1}{1+\eta}} \frac{x^{-a}}{1-x}\,dx = -\int_\epsilon^{\frac{\eta}{1+\eta}} \frac{(1-\xi)^{-a}}{\xi}\,d\xi$$

$$= -\int_\epsilon^{\frac{\eta}{1+\eta}} \left(\frac{1}{\xi} + a + \frac{a\,\overline{a+1}}{1.2}\xi + \dots\right) d\xi,$$

a convergent series, since $\xi<1$,

$$= -\log\frac{\eta}{\epsilon(1+\eta)} - a\left(\frac{\eta}{1+\eta} - \epsilon\right) - \dots;$$

and if η and ϵ are made ultimately zero *in a ratio of equality*, the limit of this portion is zero, otherwise it is of arbitrary value.

Hence we shall take $\eta=\epsilon$, and then

$$\left(\int_0^{1-\epsilon} + \int_{1+\eta}^\infty\right) \frac{x^{a-1}}{1-x}\,dx$$

is in the limit the same as

$$\int_0^{1-\epsilon}\frac{x^{a-1}}{1-x}dx-\int_0^{1-\epsilon}\frac{x^{-a}}{1-x}dx,$$

i.e. the Principal Value of

$$\int_0^{\infty}\frac{x^{a-1}}{1-x}dx \quad\text{is}\quad Lt_{\epsilon=0}\int_0^{1-\epsilon}\frac{x^{a-1}-x^{-a}}{1-x}dx,$$

the General Value being an arbitrary quantity depending upon the relative mode of approach of ϵ and η to their limits.

Now in $\int_0^1\frac{x^{a-1}-x^{-a}}{1-x}dx$, the limit of $\frac{x^{a-1}-x^{-a}}{1-x}$, when x is unity, is $-(2a-1)$, and is therefore finite, so that the last element of the integral when expressed as a summation from $x=0$ to $x=1$, contributes nothing.

Therefore $Lt_{\epsilon=0}\int_0^{1-\epsilon}\frac{x^{a-1}-x^{-a}}{1-x}dx=\int_0^1\frac{x^{a-1}-x^{-a}}{1-x}dx$

$$=\int_0^1(x^{a-1}-x^{-a})\left(1+x+x^2+x^3+\ldots+x^n+\frac{x^{n+1}}{1-x}\right)dx$$

$$=\left\{\begin{matrix}\frac{1}{a}-\frac{1}{1-a}-\frac{1}{2-a}-\frac{1}{3-a}-\ldots-\frac{1}{n-a}\\ +\frac{1}{1+a}+\frac{1}{2+a}+\frac{1}{3+a}+\ldots+\frac{1}{n+a}\end{matrix}\right\}-\frac{1}{n-a+1}+\int_0^1 x^{n+1}\frac{x^{a-1}-x^{-a}}{1-x}dx.$$

Now in the limit when n is infinite, the portion in the brackets is ultimately equal to $\pi\cot a\pi$.

The limit of the term $\frac{1}{n-a+1}$ is zero; and in the integral the subject of integration is ultimately zero for all values of $x<1$, *i.e.*

$$Lt_{n=\infty}\int_0^{1-\epsilon}x^{n+1}\frac{x^{a-1}-x^{-a}}{1-x}dx=0.$$

And for the remaining part of the integral

$$\int_0^1 x^{n+1}\frac{x^{a-1}-x^{-a}}{1-x}dx,\quad\text{viz.}\ \int_{1-\epsilon}^1 x^{n-1}\frac{x^{a-1}-x^{-a}}{1-x}dx,$$

we may remark that, the integrand being finite, if we take P and Q as its greatest and least values in the region between $1-\epsilon$ and 1, this integral lies between

$$P\int_{1-\epsilon}^1 1\,.\,dx \quad\text{and}\quad Q\int_{1-\epsilon}^1 1\,.\,dx,$$

i.e. between $P\epsilon$ and $Q\epsilon$, and therefore vanishes in the limit.

Hence, summing up, the Principal Value of the integral

$$\int_0^\infty \frac{x^{a-1}}{1-x}\,dx \quad \text{is} \quad \int_0^1 \frac{x^{a-1}-x^{-a}}{1-x}\,dx,$$

and is equal to $\qquad \pi\cot a\pi \qquad (1>a>0).$ ....(1)

1103. In the derived results which follow we shall regard all the integrals which occur as Principal Values.

Starting with Prin. Val. of

$$\int_0^\infty \frac{x^{a-1}}{1-x}\,dx=\int_0^1 \frac{x^{a-1}-x^{-a}}{1-x}\,dx=\pi\cot a\pi, \quad (1>a>0), \quad \text{.........(1)}$$

we proceed as in Art. 1101.

Put $x=y^n$, $a=\dfrac{p}{n}$. Then

$$\int_0^\infty \frac{y^{p-1}}{1-y^n}\,dy=\int_0^1 \frac{y^{p-1}-y^{n-p-1}}{1-y^n}\,dy \quad =\frac{\pi}{n}\cot\frac{p\pi}{n}, \quad (n>p>0). \text{ ....(2)}$$

The case $n=2$ gives

$$\int_0^\infty \frac{x^{p-1}}{1-x^2}\,dx=\int_0^1 \frac{x^{p-1}-x^{1-p}}{1-x^2}\,dx \quad =\frac{\pi}{2}\cot\frac{p\pi}{2}, \quad (2>p>0). \text{ ....(3)}$$

Putting $p=m+1$, we have

$$\int_0^\infty \frac{x^m}{1-x^2}\,dx=\int_0^1 \frac{x^m-x^{-m}}{1-x^2}\,dx \quad =-\frac{\pi}{2}\tan\frac{m\pi}{2}, \quad (1>m>-1). \text{ ......(4)}$$

Put $p=1$ in (2),

$$\int_0^\infty \frac{dx}{1-x^n} \quad =\int_0^1 \frac{1-x^{n-2}}{1-x^n}\,dx \quad =\frac{\pi}{n}\cot\frac{\pi}{n}, \quad (n>1), \text{ ....(5)}$$

From (4), $\displaystyle\int_0^1 \frac{x^m-x^{-m}}{x-x^{-1}}\frac{dx}{x} \quad =\frac{\pi}{2}\tan\frac{m\pi}{2}, \quad (1>m>-1).$ ......(6)

This may be written as

$$\int_0^1 \frac{\sinh(m\log x)}{\sinh(\log x)}\frac{dx}{x} \quad =\frac{\pi}{2}\tan\frac{m\pi}{2}, \quad (1>m>-1). \text{ ......(7)}$$

Put $x=e^{-qz}$, q positive ; $mq=p$, and replace z by x,

$$\int_0^\infty \frac{\sinh px}{\sinh qx}\,dx \quad =\frac{\pi}{2q}\tan\frac{p\pi}{2q}, \quad (q>p>-q). \text{ ......(8)}$$

Put $q=\pi$, $\displaystyle\int_0^\infty \frac{\sinh px}{\sinh \pi x}\,dx \quad =\frac{1}{2}\tan\frac{p}{2}, \quad (\pi>p>-\pi).$ ...(9)

Differentiate with regard to p,

$$\int_0^\infty x\frac{\cosh px}{\sinh \pi x}\,dx \quad =\frac{1}{4}\sec^2\frac{p}{2}, \quad (\pi>p>-\pi). \text{ ...(10)}$$

Integrate (9) with regard to p from 0 to 1,

$$\int_0^\infty \frac{\sinh^2\dfrac{px}{2}}{\sinh \pi x}\frac{dx}{x} \quad =\frac{1}{2}\log\sec\frac{p}{2}, \quad (\pi>p>-\pi), \text{ ...(11)}$$

or between b and a,

$$\int_0^\infty \frac{\cosh ax - \cosh bx}{\sinh \pi x}\,\frac{dx}{x} = \log\left(\frac{\cos\frac{b}{2}}{\cos\frac{a}{2}}\right), \quad (\pi > a > b > -\pi), \quad (12)$$

and it is as before obvious that many further deductions may be made.

1104. **Lemma.** We shall require the factorisation of $\cos u\pi + \cosh v\pi$.

$$\cos u\pi + \cosh v\pi = \cos u\pi + \cos \iota v\pi = 2\cos\frac{u+\iota v}{2}\pi \cos\frac{u-\iota v}{2}\pi$$

$$= 2\prod_0^\infty \left(1 - \frac{(u+\iota v)^2}{(2r+1)^2}\right)\left(1 - \frac{(u-\iota v)^2}{(2r+1)^2}\right)$$

$$= 2\prod_0^\infty [(2r+1+u)^2+v^2][(2r+1-u)^2+v^2]/(2r+1)^4.$$

Logarithmic differentiation with regard to u and v gives

$$(1)\quad \frac{-\pi \sin u\pi}{\cos u\pi + \cosh v\pi} = 2\sum_0^\infty \left(\frac{2r+1+u}{\overline{2r+1+u}|^2+v^2} - \frac{2r+1-u}{\overline{2r+1-u}|^2+v^2}\right),$$

$$(2)\quad \frac{\pi \sinh v\pi}{\cos u\pi + \cosh v\pi} = 2v\sum_0^\infty \left(\frac{1}{\overline{2r+1+u}|^2+v^2} + \frac{1}{\overline{2r+1-u}|^2+v^2}\right).$$

1105. Group K.

Type $I \equiv \int_0^\infty \frac{\cosh px}{\sinh qx}\sin mx\,dx$, **etc.** ($q$ **positive,** $p^2 \not> q^2$).

Here $I = \int_0^\infty (e^{px}+e^{-px})(e^{-qx}+e^{-3qx}+e^{-5qx}+\dots)\sin mx\,dx$, the integrand being finite for all positive values of x and the series convergent;

$$\therefore I = \int_0^\infty \sum_0^\infty \left[e^{-\{(2r+1)q+p\}x} + e^{-\{(2r+1)q-p\}x}\right]\sin mx\,dx$$

$$= \sum_0^\infty \left[\frac{m}{\{(2r+1)q+p\}^2+m^2} + \frac{m}{\{(2r+1)q-p\}^2+m^2}\right]$$

$$= \frac{\pi}{2q}\,\frac{\sinh\frac{m\pi}{q}}{\cos\frac{p\pi}{q}+\cosh\frac{m\pi}{q}}, \text{ by the Lemma, } \dots\dots\dots\dots\dots\dots \text{(A)}$$

q being positive and p intermediate between q and $-q$, inclusive.

Similarly

$$\int_0^\infty \frac{\sinh px}{\sinh qx}\cos mx\,dx = \int_0^\infty (e^{px}-e^{-px})(e^{-qx}+e^{-3qx}+\dots)\cos mx\,dx$$

$$= \int_0^\infty \sum_0^\infty \left[e^{-\{(2r+1)q-p\}x} - e^{-\{(2r+1)q+p\}x}\right]\cos mx\,dx$$

$$= \sum_0^\infty \left[\frac{(2r+1)q-p}{\{(2r+1)q-p\}^2+m^2} - \frac{(2r+1)q+p}{\{(2r+1)q+p\}^2+m^2}\right]$$

$$= \frac{\pi}{2q}\,\frac{\sin\frac{p\pi}{q}}{\cos\frac{p\pi}{q}+\cosh\frac{m\pi}{q}}. \dots\dots\dots\dots\dots\dots\dots\dots \text{(B)}$$

Writing $2x$ for x in (A) and $p+\frac{q}{2}, p-\frac{q}{2}$ in succession for p, and subtracting,

$$\int_0^\infty \frac{\cosh(2p+q)x-\cosh(2p-q)x}{2\sinh qx\cosh qx}\sin 2mx\,dx$$

$$=\frac{\pi}{4q}\left(\frac{\sinh\frac{m\pi}{q}}{-\sin\frac{p\pi}{q}+\cosh\frac{m\pi}{2}}-\frac{\sinh\frac{m\pi}{q}}{\sin\frac{p\pi}{q}+\cosh\frac{m\pi}{q}}\right),$$

$$\therefore \int_0^\infty \frac{\sinh 2px}{\cosh qx}\sin 2mx\,dx=\frac{\pi}{2q}\frac{\sin\frac{p\pi}{q}\sinh\frac{m\pi}{q}}{\cosh^2\frac{m\pi}{q}-\sin^2\frac{p\pi}{q}}$$

$$=\frac{\pi}{q}\frac{\sin\frac{p\pi}{q}\sinh\frac{m\pi}{q}}{\cos\frac{2p\pi}{q}+\cosh\frac{2m\pi}{q}}\quad\left(p^2 \not> \frac{q^2}{4}\right);$$

and replacing $2p$ and $2m$ by p and m,

$$\int_0^\infty \frac{\sinh px}{\cosh qx}\sin mx\,dx=\frac{\pi}{q}\frac{\sin\frac{p\pi}{2q}\sinh\frac{m\pi}{2q}}{\cos\frac{p\pi}{q}+\cosh\frac{m\pi}{q}}\quad (q \text{ positive}, p^2 \not> q^2). \quad \ldots\text{(C)}$$

Treating (B) in the same way,

$$\int_0^\infty \frac{\sinh(2p+q)x-\sinh(2p-q)x}{2\sinh qx\cosh qx}\cos 2mx\,dx$$

$$=\frac{\pi}{4q}\left(\frac{\cos\frac{p\pi}{q}}{-\sin\frac{p\pi}{q}+\cosh\frac{m\pi}{q}}-\frac{-\cos\frac{p\pi}{q}}{\sin\frac{p\pi}{q}+\cosh\frac{m\pi}{q}}\right),$$

$$\int_0^\infty \frac{\cosh 2px}{\cosh qx}\cos 2mx\,dx=\frac{\pi}{2q}\frac{\cos\frac{p\pi}{q}\cosh\frac{m\pi}{q}}{\cosh^2\frac{m\pi}{q}-\sin^2\frac{p\pi}{q}}$$

$$=\frac{\pi}{q}\frac{\cos\frac{p\pi}{q}\cosh\frac{m\pi}{q}}{\cos\frac{2p\pi}{q}+\cosh\frac{2m\pi}{q}}\quad\left(p^2 \not> \frac{q^2}{4}\right);$$

and replacing $2p$ and $2m$ by p and m,

$$\int_0^\infty \frac{\cosh px}{\cosh qx}\cos mx\,dx=\frac{\pi}{q}\frac{\cos\frac{p\pi}{2q}\cosh\frac{m\pi}{2q}}{\cos\frac{p\pi}{q}+\cosh\frac{m\pi}{q}}\quad (q \text{ positive}, p^2 \not> q^2). \quad \ldots\text{(D)}$$

We thus have $(p^2 \ngtr q^2)$

$$\int_0^\infty \frac{\cosh px}{\sinh qx}\sin mx\,dx=\frac{\pi}{2q}\,\frac{\sinh\dfrac{m\pi}{q}}{\cos\dfrac{p\pi}{q}+\cosh\dfrac{m\pi}{q}}, \quad\text{(A)}$$

$$\int_0^\infty \frac{\sinh px}{\sinh qx}\cos mx\,dx=\frac{\pi}{2q}\,\frac{\sin\dfrac{p\pi}{q}}{\cos\dfrac{p\pi}{q}+\cosh\dfrac{m\pi}{q}}, \quad\text{(B)}$$

$$\int_0^\infty \frac{\sinh px}{\cosh qx}\sin mx\,dx=\frac{\pi}{q}\,\frac{\sin\dfrac{p\pi}{2q}\sinh\dfrac{m\pi}{2q}}{\cos\dfrac{p\pi}{q}+\cosh\dfrac{m\pi}{q}}, \quad\text{(C)}$$

$$\int_0^\infty \frac{\cosh px}{\cosh qx}\cos mx\,dx=\frac{\pi}{q}\,\frac{\cos\dfrac{p\pi}{2q}\cosh\dfrac{m\pi}{2q}}{\cos\dfrac{p\pi}{q}+\cosh\dfrac{m\pi}{q}}. \quad\text{(D)}$$

1106. **Special Cases.**

(i) Put $q=\pi$, then $(p^2 \ngtr \pi^2)$,

$$\int_0^\infty \frac{\cosh px}{\sinh \pi x}\sin mx\,dx=\frac{1}{2}\,\frac{\sinh m}{\cos p+\cosh m},\qquad \int_0^\infty \frac{\sinh px}{\cosh \pi x}\sin mx\,dx=\frac{\sin\dfrac{p}{2}\sinh\dfrac{m}{2}}{\cos p+\cosh m},$$

$$\int_0^\infty \frac{\sinh px}{\sinh \pi x}\cos mx\,dx=\frac{1}{2}\,\frac{\sin p}{\cos p+\cosh m},\qquad \int_0^\infty \frac{\cosh px}{\cosh \pi x}\cos mx\,dx=\frac{\cos\dfrac{p}{2}\cosh\dfrac{m}{2}}{\cos p+\cosh m}.$$

(ii) Put $q=\dfrac{\pi}{2}$, then $(4p^2 \ngtr \pi^2)$

$$\int_0^\infty \frac{\cosh px}{\sinh\dfrac{\pi x}{2}}\sin mx\,dx=\frac{\sinh 2m}{\cos 2p+\cosh 2m},\qquad \int_0^\infty \frac{\sinh px}{\cosh\dfrac{\pi x}{2}}\sin mx\,dx=2\,\frac{\sin p\sinh m}{\cos 2p+\cosh 2m},$$

$$\int_0^\infty \frac{\sinh px}{\sinh\dfrac{\pi x}{2}}\cos mx\,dx=\frac{\sin 2p}{\cos 2p+\cosh 2m},\qquad \int_0^\infty \frac{\cosh px}{\cosh\dfrac{\pi x}{2}}\cos mx\,dx=2\,\frac{\cos p\cosh m}{\cos 2p+\cosh 2m}.$$

(iii) Put $p=0$ in (A) and (D),

$$\int_0^\infty \frac{\sin mx}{\sinh qx}\,dx=\frac{\pi}{2q}\tanh\frac{m\pi}{2q},\qquad \int_0^\infty \frac{\cos mx}{\cosh qx}\,dx=\frac{\pi}{2q}\operatorname{sech}\frac{m\pi}{2q}.$$

(iv) Putting $q=\pi$ in these results,

$$\int_0^\infty \frac{\sin mx}{\sinh \pi x}\,dx=\frac{1}{2}\tanh\frac{m}{2},\qquad \int_0^\infty \frac{\cos mx}{\cosh \pi x}\,dx=\frac{1}{2}\operatorname{sech}\frac{m}{2}.$$

(v) Putting $m=0$ in (B) and (D) $(p^2 \ngtr q^2)$,

$$\int_0^\infty \frac{\sinh px}{\sinh qx}\,dx=\frac{\pi}{2q}\tan\frac{p\pi}{2q},\qquad \int_0^\infty \frac{\cosh px}{\cosh qx}\,dx=\frac{\pi}{2q}\sec\frac{p\pi}{2q}.$$

(vi) Putting $q=\pi$ in these results $(p^2 \ngtr \pi^2)$,

$$\int_0^\infty \frac{\sinh px}{\sinh \pi x}\,dx=\frac{1}{2}\tan\frac{p}{2},\qquad \int_0^\infty \frac{\cosh px}{\cosh \pi x}\,dx=\frac{1}{2}\sec\frac{p}{2}.$$

(vii) Putting $q=\frac{\pi}{2}$ in (v) $(4p^2 \not> \pi^2)$,

$$\int_0^\infty \frac{\sinh px}{\sinh \frac{\pi x}{2}} dx = \tan p, \qquad \int_0^\infty \frac{\cosh px}{\cosh \frac{\pi x}{2}} dx = \sec p.$$

(viii) Putting $p=q$ in (A) and (C),

$$\int_0^\infty \coth qx \sin mx\, dx = \frac{\pi}{2q} \coth \frac{m\pi}{2q}, \qquad \int_0^\infty \tanh qx \sin mx\, dx = \frac{\pi}{2q} \operatorname{cosech} \frac{m\pi}{2q}.$$

(ix) Putting $q=\pi$ in the latter,

$$\int_0^\infty \coth \pi x \sin mx\, dx = \frac{1}{2} \coth \frac{m}{2}, \qquad \int_0^\infty \tanh \pi x \sin mx\, dx = \frac{1}{2} \operatorname{cosech} \frac{m}{2}.$$

1107. Other Modes of Derivation.

Besides such integrals as those indicated, which are merely particular cases of one or other of the four formulae A, B, C, D, many definite integrals may be obtained by differentiation or integration, between specified limits, with regard to one or other of the constants p, q or m.

EXAMPLES.

1. Taking $\int_0^\infty \frac{\sin mx}{\sinh \pi x} dx = \frac{1}{2} \tanh \frac{m}{2}$, write $2m$ for m and integrate with regard to m from 0 to m. Then

$$\int_0^\infty \operatorname{cosech} \pi x \left[-\frac{\cos 2mx}{2x} \right]_0^m dx = \frac{1}{2} \log \cosh m,$$

that is $\displaystyle\int_0^\infty \operatorname{cosech} \pi x \sin^2 mx \frac{dx}{x} = \frac{1}{2} \log \cosh m.$

2. Deduce from $\int_0^\infty \frac{\cos mx}{\cosh \pi x} dx = \frac{1}{2} \operatorname{sech} \frac{m}{2}$,

$$(a) \int_0^\infty \frac{x \sin mx}{\cosh \pi x} dx = \frac{1}{4} \tanh \frac{m}{2} \operatorname{sech} \frac{m}{2}, \quad (b) \int_0^\infty \frac{\sin mx}{\cosh \pi x} \frac{dx}{x} = \tan^{-1}\left(\sinh \frac{m}{2}\right).$$

3. Deduce from $\displaystyle\int_0^\infty \frac{\cosh px}{\sinh \pi x} \sin mx\, dx = \frac{1}{2} \frac{\sinh m}{\cos p + \cosh m}$,

$$(a) \int_0^\infty \operatorname{cosech} \pi x \cosh px \sin^2 \frac{mx}{2} \frac{dx}{x} = \frac{1}{4} \log \left(\frac{\cos p + \cosh m}{1 + \cos p} \right),$$

$$(b) \int_0^\infty x \frac{\sinh px}{\sinh \pi x} \sin mx\, dx = \frac{1}{2} \frac{\sinh m \sin p}{(\cos p + \cosh m)^2},$$

$$(c) \int_0^\infty x \frac{\cosh px}{\sinh \pi x} \cos mx\, dx = \frac{1}{2} \frac{1 + \cos p \cosh m}{(\cos p + \cosh m)^2}.$$

4. Deduce from $\int_0^\infty \frac{\sinh px}{\sinh \pi x} \cos mx\, dx = \frac{1}{2}\frac{\sin p}{\cos p + \cosh m}$,

$$\int_0^\infty \frac{\sinh px}{\sinh \pi x} \sin mx \frac{dx}{x} = \tan^{-1}\left(\tanh\frac{m}{2}\tan\frac{p}{2}\right).$$

And it will be obvious that a large number of such results may be obtained. The results of putting $m=0$ will in many cases lead to integrals obtained in a different manner earlier.

1108. Group L. **Poisson's Formulae.**

Let $f(x)$ be a function of x such that Taylor's Theorem gives convergent expansions for $f(a+u)$ and $f(a+u^{-1})$, where $u=e^{\iota\theta}$. Then expanding

$$f(a+u)+f(a+u^{-1})$$
$$=2\left[f(a)+f'(a)\cos\theta+\frac{1}{2!}f''(a)\cos 2\theta+\frac{1}{3!}f'''(a)\cos 3\theta+\dots\right].$$

Multiplying by

$$\frac{1-c^2}{1-2c\cos\theta+c^2}=1+2c\cos\theta+2c^2\cos 2\theta+\dots,\quad \text{if } c^2<1,$$

or by

$$\frac{c^2-1}{1-2c\cos\theta+c^2}=1+2c^{-1}\cos\theta+2c^{-2}\cos 2\theta+\dots,\ \text{if } c^2>1,$$

and integrating between 0 and π, we have

$$\int_0^\pi \frac{f(a+u)+f(a+u^{-1})}{1-2c\cos\theta+c^2}d\theta=\frac{2\pi}{1-c^2}\left\{f(a)+cf'(a)+\frac{c^2}{2!}f''(a)+\dots\right\}$$
$$=\frac{2\pi}{1-c^2}f(a+c),\quad \text{if } c^2<1,$$
$$\text{or}\quad =\frac{2\pi}{c^2-1}\left\{f(a)+c^{-1}f'(a)+\frac{c^{-2}}{2!}f''(a)+\dots\right\}$$
$$=\frac{2\pi}{c^2-1}f(a+c^{-1}),\quad \text{if } c^2>1.$$

Examples.

1. Show that, u standing for $e^{\iota\theta}$,

$$\int_0^\pi \sin\theta\frac{f(a+u)-f(a+u^{-1})}{1-2c\cos\theta+c^2}d\theta=\frac{\iota\pi}{c}\{f(a+c)-f(a)\}\quad (\text{if } c^2<1)$$
$$\text{or}\quad =\frac{\iota\pi}{c}\{f(a+c^{-1})-f(a)\}\quad (\text{if } c^2>1).$$

2. Show that

$$\int_0^\pi \frac{1-c\cos\theta}{1-2c\cos\theta+c^2}\{f(a+u)+f(a+u^{-1})\}d\theta=\pi\{f(a)+f(a+c)\}\quad (c^2<1)$$
$$\text{or}\quad =\pi\{f(a)-f(a+c^{-1})\}\quad (c^2>1).$$

3. Show that

$$\int_0^\pi \frac{\sin\theta}{(1-2c\cos\theta+c^2)^2}\{f(a+u)-f(a+u^{-1})\}\,d\theta=\frac{\pi\iota}{1-c^2}f'(a+c)\quad (c^2<1).$$

4. Taking $f(x)=x^n$, show that

$$\int_0^\pi \frac{(1+2a\cos\theta+a^2)^{\frac{n}{2}}}{1-2c\cos\theta+c^2}\cos\left(n\tan^{-1}\frac{\sin\theta}{a+\cos\theta}\right)d\theta=\frac{\pi}{1-c^2}(a+c)^n\quad (c^2<1).$$

5. Show that

$$\int_0^\pi \frac{\sin\theta}{1-2c\cos\theta+c^2}(1+2a\cos\theta+a^2)^{\frac{n}{2}}\sin\left(n\tan^{-1}\frac{\sin\theta}{a+\cos\theta}\right)d\theta$$
$$=\frac{\pi}{2c}\{(a+c)^n-a^n\}\quad (c^2<1).$$

6. Show that

$$\int_0^\pi \frac{\sin\theta}{(1-2c\cos\theta+c^2)^2}(1+2a\cos\theta+a^2)^{\frac{n}{2}}\sin\left(n\tan^{-1}\frac{\sin\theta}{a+\cos\theta}\right)d\theta$$
$$=\frac{\pi}{2(1-c^2)}n(a+c)^{n-1}\quad (c^2<1).$$

7. Deduce known results from 4, 5, 6 by putting $n=1$.

8. Prove $\displaystyle\int_0^\pi \frac{e^{k\cos x}\cos(k\sin x)}{1-2c\cos x+c^2}\,dx=\frac{\pi}{1-c^2}e^{kc}\quad (c^2<1).$

1109. Group M. **Abel's Formula.** (See Bertrand, *Calc. Int.*, p. 171.)

Supposing $F(c+\alpha)$ capable of expansion in a series of powers of $e^{-\alpha}$ in the form $A_0+A_1e^{-\alpha}+A_2e^{-2\alpha}+\dots$, whether α be real or imaginary, then putting $\iota\beta t$ for α, we have

$$A_0+A_1\cos\beta t+A_2\cos 2\beta t+\dots=\tfrac{1}{2}\{F(c+\iota\beta t)+F(c-\iota\beta t)\}.$$

It follows that

$$\int_0^\infty \frac{F(c+\iota\beta t)+F(c-\iota\beta t)}{b^2+t^2}\,dt$$
$$=2\int_0^\infty\left(\frac{A_0}{b^2+t^2}+\frac{A_1\cos\beta t}{b^2+t^2}+\frac{A_2\cos 2\beta t}{b^2+t^2}+\dots\right)dt$$
$$=\frac{\pi}{b}\{A_0+A_1e^{-b\beta}+A_2e^{-2b\beta}+\dots\}$$
$$=\frac{\pi}{b}F(c+b\beta).$$

In Abel's Formula b is taken as unity.

EXAMPLES.

1. Taking $F(z)=z^{-n}$,

$$F(c+\iota\beta t)+F(c-\iota\beta t)=(c^2+\beta^2t^2)^{-\frac{n}{2}}2\cos\left(n\tan^{-1}\frac{\beta t}{c}\right);$$

$$\therefore \int_0^\infty \frac{\cos\left(n\tan^{-1}\dfrac{\beta t}{c}\right)}{(c^2+\beta^2t^2)^{\frac{n}{2}}}\frac{dt}{b^2+t^2}=\frac{\pi}{2b}(c+b\beta)^{-n}.$$

2. Deduce the formulae

$$(a)\ \int_0^\infty \frac{dt}{(c^2+a^2t^2)(b^2+t^2)}=\frac{\pi}{2bc}\,\frac{1}{c+ab},$$

$$(b)\ \int_0^{\frac{\pi}{2}} \frac{\cos n\phi\cos^n\phi\,d\phi}{a^2\cos^2\phi+c^2\sin^2\phi}=\frac{\pi}{2a}\,\frac{c^{n-1}}{(c+a)^n}.$$ [BERTRAND.]

3. Show that $\int_0^\infty \frac{e^{c\cos(at)}\cos(c\sin(at))}{b^2+t^2}dt=\frac{\pi}{2b}e^{ce^{-ba}}$.

1110. GROUP N. **A Set mainly due to** CAUCHY.

The integrand of $\int_0^\infty \frac{dx}{a^2-x^2}$ $(a>0)$ has infinities at a and at $-a$. The latter lies outside the range of integration.

Now

$$\int_0^{a-\epsilon}\frac{dx}{a^2-x^2}+\int_{a+\eta}^\infty\frac{dx}{a^2-x^2}=\frac{1}{2a}\left[\log\frac{a+x}{a-x}\right]_0^{a-\epsilon}+\frac{1}{2a}\left[\log\frac{x+a}{x-a}\right]_{a+\eta}^\infty$$

$$=\frac{1}{2a}\log\frac{2a-\epsilon}{\epsilon}-\frac{1}{2a}\log\frac{2a+\eta}{\eta}=\frac{1}{2a}\log\frac{\eta}{\epsilon}\cdot\frac{2a-\epsilon}{2a+\eta}.$$

If η, ϵ be made to vanish in a ratio of equality, this vanishes; $\therefore$ the Principal Value of $\int_0^\infty\frac{dx}{a^2-x^2}$ is zero.

1111. Consider next the Principal Values of

$$I_1\equiv\int_0^\infty\frac{dx}{(a^2-x^2)(x^2+p^2)},\quad I_2\equiv\int_0^\infty\frac{x^2dx}{(a^2-x^2)(x^2+p^2)}.$$

$$I_1\equiv\frac{1}{a^2+p^2}\int_0^\infty\frac{dx}{a^2-x^2}+\frac{1}{a^2+p^2}\int_0^\infty\frac{dx}{x^2+p^2}=0+\frac{1}{a^2+p^2}\cdot\frac{\pi}{2p}=\frac{\pi}{2p}\,\frac{1}{a^2+p^2},$$

$$I_2\equiv\frac{a^2}{a^2+p^2}\int_0^\infty\frac{dx}{a^2-x^2}-\frac{p^2}{a^2+p^2}\int_0^\infty\frac{dx}{x^2+p^2}=0-\frac{p^2}{a^2+p^2}\cdot\frac{\pi}{2p}=-\frac{\pi}{2}\,\frac{p}{a^2+p^2}.$$

If then $\phi(x)$ be such a function as can be expressed in partial fractions of the form $\phi(x)=\Sigma\frac{A}{a^2-x^2}$, we have as Principal Values,

$$I_1' \equiv \int_0^\infty \frac{\phi(x)}{p^2+x^2}\,dx = \frac{\pi}{2p}\Sigma\frac{A}{a^2+p^2} = \frac{\pi}{2p}\phi(p\sqrt{-1}),$$

$$I_2' \equiv \int_0^\infty \frac{x^2\phi(x)}{p^2+x^2}\,dx = -\frac{\pi}{2p}\Sigma\frac{Ap^2}{a^2+p^2} = \frac{\pi}{2p}F(p\sqrt{-1}),$$

where $F(x)=x^2\phi(x)$, provided $Lt_{x=\infty}\dfrac{x^2\phi(x)}{p^2+x^2}$ be finite.

[The results obtained in the following articles to **1118** are all Principal Values of the several integrals discussed.]

1112. Thus, for instance, since we have

$$\frac{\tan ax}{x} = 8a\sum_1^\infty \frac{1}{(2r-1)^2\pi^2-4a^2x^2}, \quad \sec ax = 4\sum_1^\infty \frac{(-1)^{r-1}(2r-1)\pi}{(2r-1)^2\pi^2-4a^2x^2},$$

$$x\cot ax = \frac{1}{a}+2\sum_1^\infty \frac{ax^2}{a^2x^2-r^2\pi^2}, \quad x\operatorname{cosec} ax = \frac{1}{a}+\sum_1^\infty \frac{(-1)^r\, 2ax^2}{a^2x^2-r^2\pi^2},$$

it follows that, considering Principal Values,

$$\left.\begin{aligned}
&\text{(i)} \int_0^\infty \frac{\tan ax}{x}\,\frac{dx}{p^2+x^2} = \frac{\pi}{2p}\,\frac{\tan \iota ap}{\iota p} = \frac{\pi}{2p^2}\tanh ap,\\
&\text{(ii)} \int_0^\infty \sec ax\,\frac{dx}{p^2+x^2} = \frac{\pi}{2p}\sec \iota ap = \frac{\pi}{2p}\operatorname{sech} ap,\\
&\text{(iii)} \int_0^\infty x\cot ax\,\frac{dx}{p^2+x^2} = \frac{\pi}{2p}\,\iota p\cot \iota ap = \frac{\pi}{2}\coth ap,\\
&\text{(iv)} \int_0^\infty x\operatorname{cosec} ax\,\frac{dx}{p^2+x^2} = \frac{\pi}{2p}\,\iota p\operatorname{cosec} \iota ap = \frac{\pi}{2}\operatorname{cosech} ap.
\end{aligned}\right\}\ldots\ldots\text{(A)}$$

1113. Again, it is clear from the expressions for $\sin\theta$ and $\cos\theta$ in factors, that the fractions $(a<b)$

$$\frac{\sin ax}{\sin bx},\quad \frac{\cos ax}{\cos bx},\quad \frac{\sin ax}{x\cos bx},\quad \frac{x\sin ax}{\cos bx},\quad \frac{x\cos ax}{\sin bx},$$

are expressible as the sums of an infinite number of partial fractions with pure quadratic denominators (*e.g.* see Ex. 52, p. 169), and therefore, when $a<b$, we have immediately

$$\text{(i)} \int_0^\infty \frac{\sin ax}{\sin bx}\,\frac{dx}{p^2+x^2} = \frac{\pi}{2p}\,\frac{\sinh ap}{\sinh bp}, \qquad \text{(ii)} \int_0^\infty \frac{\cos ax}{\cos bx}\,\frac{dx}{p^2+x^2} = \frac{\pi}{2p}\,\frac{\cosh ap}{\cosh bp},$$

$$\text{(iii)} \int_0^\infty \frac{\sin ax}{\cos bx}\,\frac{dx}{x(p^2+x^2)} = \frac{\pi}{2p^2}\,\frac{\sinh ap}{\cosh bp}, \quad \text{(iv)} \int_0^\infty \frac{\sin ax}{\cos bx}\,\frac{x\,dx}{p^2+x^2} = -\frac{\pi}{2}\,\frac{\sinh ap}{\cosh bp},$$

$$\text{(v)} \int_0^\infty \frac{\cos ax}{\sin bx}\,\frac{x\,dx}{p^2+x^2} = \frac{\pi}{2}\,\frac{\cosh ap}{\sinh bp}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(B)}$$

1114. In the limit when $a=0$, we have cases (i), (iii), (iv) giving a zero result, but from (ii) and (v),

$$\int_0^\infty \frac{\sec bx}{p^2+x^2}\,dx=\frac{\pi}{2p}\operatorname{sech} bp \text{ and } \int_0^\infty \frac{x\operatorname{cosec} bx}{p^2+x^2}\,dx=\frac{\pi}{2}\operatorname{cosech} bp. \quad \ldots\text{(C)}$$

Also in the case when $a=b$, we have,

$$\left.\begin{aligned}
&\text{(i) and (ii) become } \int_0^\infty \frac{dx}{p^2+x^2}=\frac{\pi}{2p},\\
&\text{(iii) } \int_0^\infty \frac{\tan bx\,dx}{x(p^2+x^2)}=\frac{\pi}{2p^2}\tanh bp \quad \text{(from A (i))},\\
&\text{(iv) } \int_0^\infty \frac{x\tan bx}{p^2+x^2}\,dx=\int_0^\infty\left\{\frac{\tan bx}{x}-\frac{p^2\tan bx}{x(p^2+x^2)}\right\}dx=\frac{\pi}{2}-\frac{\pi}{2}\tanh bp\\
&\qquad\qquad \text{(see Art. 1007)},\\
&\text{(v) } \int_0^\infty \frac{x\cot bx}{p^2+x^2}\,dx=\frac{\pi}{2}\coth bp \quad \text{(from A (iii))}.
\end{aligned}\right\}\text{(D)}$$

1115. The cases in which $a > b$ can readily be obtained by means of the following identities. Let $a=2rb+c$, where r is an integer and c is positive or negative, but numerically less than b.

(1) $2\{\cos(a-b)x+\cos(a-3b)x+\ldots+\cos(a-\overline{2r-1}\,b)x\} = \dfrac{\sin ax}{\sin bx}-\dfrac{\sin cx}{\sin bx}$,

(2) $2\{\cos(a-b)x-\cos(a-3b)x+\ldots+(-1)^{r-1}\cos(a-\overline{2r-1}\,b)x\}=\dfrac{\cos ax}{\cos bx}-(-1)^r\dfrac{\cos cx}{\cos bx}$,

(3) $2\{\sin(a-b)x-\sin(a-3b)x+\ldots+(-1)^{r-1}\sin(a-\overline{2r-1}\,b)x\} =\dfrac{\sin ax}{\cos bx}-(-1)^r\dfrac{\sin cx}{\cos bx}$,

(4) $2\{\sin(a-b)x+\sin(a-3b)x+\ldots+\sin(a-\overline{2r-1}\,b)x\} =\dfrac{\cos cx}{\sin bx}-\dfrac{\cos ax}{\sin bx}$.

Now $\displaystyle\int_0^\infty \frac{\cos rx}{p^2+x^2}\,dx=\frac{\pi}{2p}e^{-pr}, \quad \int_0^\infty \frac{x\sin rx}{p^2+x^2}\,dx=\frac{\pi}{2}e^{-pr},$

$$\int \frac{\sin rx}{x(p^2+x^2)}\,dx=\frac{\pi}{2p^2}(1-e^{-pr}) \quad (r>0,\ p>0).$$

Therefore

$$\int_0^\infty \sum_1^r \cos(a-\overline{2r-1}\,b)x\frac{dx}{p^2+x^2} =\frac{\pi}{2p}\{e^{-(a-b)p}+e^{-(a-3b)p}+\ldots\text{ to } r \text{ terms}\}$$
$$=\frac{\pi}{4p}\,\frac{e^{-cp}-e^{-ap}}{\sinh bp},$$

$$\int_0^\infty \sum_1^r (-1)^{r-1}\cos(a-\overline{2r-1}\,b)x\frac{dx}{p^2+x^2} =\frac{\pi}{4p}\,\frac{e^{-ap}-(-1)^r e^{-cp}}{\cosh bp},$$

$$\int_0^\infty \sum_1^r (-1)^{r-1}\sin(a-\overline{2r-1}\,b)x\frac{dx}{x(p^2+x^2)}=\frac{\pi}{2p^2}\left\{\frac{1-(-1)^r}{1-(-1)}-\frac{e^{-ap}-(-1)^r e^{-cp}}{2\cosh bp}\right\},$$

$$\int_0^\infty \sum_1^r (-1)^{r-1}\sin(a-\overline{2r-1}\,b)x\frac{x\,dx}{p^2+x^2} =\frac{\pi}{4}\,\frac{e^{-ap}-(-1)^r e^{-cp}}{\cosh bp},$$

$$\int_0^\infty \sum_1^r \sin(a-\overline{2r-1}\,b)x\frac{x\,dx}{p^2+x^2} =\frac{\pi}{4}\,\frac{e^{-cp}-e^{-ap}}{\sinh bp}.$$

Hence, if $a=2rb+c$, $c^2<b^2$, we have

$$\int_0^\infty \frac{\sin ax}{\sin bx}\frac{dx}{p^2+x^2}=2\int_0^\infty \sum_1^r \cos(a-\overline{2r-1}\,b)x\frac{dx}{p^2+x^2}+\int_0^\infty \frac{\sin cx}{\sin bx}\frac{dx}{p^2+x^2}$$

$$=\frac{\pi}{2p}\frac{e^{-cp}-e^{-ap}}{\sinh bp}+\frac{\pi}{2p}\left(\frac{\sinh cp}{\sinh bp},\ 0 \text{ or } 1\right),$$

according as $0>c^2>b^2$, or $c=0$ or $c=b$.

1116. Thus we have the several cases:

(A) $$\int_0^\infty \frac{\sin ax}{\sin bx}\frac{dx}{p^2+x^2} = \frac{\pi}{2p}\frac{\sinh ap}{\sinh bp}, \qquad a<b,$$

$$=2\cdot\frac{\pi}{2p}\frac{e^{-cp}-e^{-ap}}{2\sinh bp}+\frac{\pi}{2p}\frac{\sinh cp}{\sinh bp}=\frac{\pi}{2p}\frac{\cosh cp-e^{-ap}}{\sinh bp}, \qquad a=2rb+c,$$

$$=2\cdot\frac{\pi}{2p}\frac{e^{-cp}-e^{-ap}}{2\sinh bp}+0 \qquad =\frac{\pi}{2p}\frac{1-e^{-ap}}{\sinh bp}, \qquad a=2rb,\ c=0,$$

$$=2\cdot\frac{\pi}{2p}\frac{e^{-cp}-e^{-ap}}{2\sinh bp}+\frac{\pi}{2p} \qquad =\frac{\pi}{2p}\frac{\cosh bp-e^{-ap}}{\sinh bp}, \qquad a=(2r+1)b,\ c=b.$$

(B) $$\int_0^\infty \frac{\cos ax}{\cos bx}\frac{dx}{p^2+x^2} = \frac{\pi}{2p}\frac{\cosh ap}{\cosh bp}, \qquad a<b,$$

$$=2\cdot\frac{\pi}{2p}\frac{e^{-ap}-(-1)^r e^{-cp}}{2\cosh bp}+(-1)^r\frac{\pi}{2p}\frac{\cosh cp}{\cosh bp}$$

$$=\frac{\pi}{2p}\frac{(-1)^r\sinh cp+e^{-ap}}{\cosh bp}, \qquad a=2rb+c,$$

$$=2\cdot\frac{\pi}{2p}\frac{e^{-ap}-(-1)^r e^{-cp}}{2\cosh bp}+(-1)^r\frac{\pi}{2p}\operatorname{sech} bp$$

$$=\frac{\pi}{2p}\frac{e^{-ap}}{\cosh bp}, \qquad a=2rb,\ c=0,$$

$$=2\cdot\frac{\pi}{2p}\frac{e^{-ap}-(-1)^r e^{-cp}}{2\cosh bp}+(-1)^r\frac{\pi}{2p}$$

$$=\frac{\pi}{2p}\frac{(-1)^r\sinh bp+e^{-ap}}{\cosh bp}, \qquad a=(2r+1)b,\ c=b.$$

(C) $$\int_0^\infty \frac{\sin ax}{\cos bx}\frac{dx}{x(p^2+x^2)}=\frac{\pi}{2p^2}\frac{\sinh ap}{\cosh bp}, \qquad a<b,$$

$$=2\cdot\frac{\pi}{2p^2}\left[\frac{1-(-1)^r}{2}-\frac{e^{-ap}-(-1)^r e^{-cp}}{2\cosh bp}\right]+(-1)^r\frac{\pi}{2p^2}\frac{\sinh cp}{\cosh bp}$$

$$=\frac{\pi}{2p^2}\left[1-(-1)^r+\frac{(-1)^r\cosh cp-e^{-ap}}{\cosh bp}\right], \qquad a=2rb+c,$$

$$=2\cdot\frac{\pi}{2p^2}\left[\frac{1-(-1)^r}{2}-\frac{e^{-ap}-(-1)^r e^{-cp}}{2\cosh bp}\right]+0$$

$$=\frac{\pi}{2p^2}\left[1-(-1)^r+\frac{(-1)^r-e^{-ap}}{\cosh bp}\right], \qquad a=2rb,\ c=0,$$

$$=2\cdot\frac{\pi}{2p^2}\left[\frac{1-(-1)^r}{2}-\frac{e^{-ap}-(-1)^r e^{-cp}}{2\cosh bp}\right]+(-1)^r\frac{\pi}{2p^2}\tanh bp$$

$$=\frac{\pi}{2p^2}\left[1-\frac{e^{-ap}}{\cosh bp}\right], \qquad a=(2r+1)b,\ c=b.$$

(D) $\int_0^\infty \frac{\sin ax}{\cos bx}\frac{x\,dx}{p^2+x^2} = -\frac{\pi}{2}\frac{\sinh ap}{\cosh bp}$, $\quad a<b$,

or $= 2\cdot\frac{\pi}{2}\frac{e^{-ap}-(-1)^r e^{-cp}}{2\cosh bp} - (-1)^r\frac{\pi}{2}\frac{\sinh cp}{\cosh bp}$

$= \frac{\pi}{2}\frac{e^{-ap}-(-1)^r\cosh cp}{\cosh bp}$, $\quad a=2rb+c$,

or $= 2\cdot\frac{\pi}{2}\frac{e^{-ap}-(-1)^r e^{-cp}}{2\cosh bp}+0 = \frac{\pi}{2}\frac{e^{-ap}-(-1)^r}{\cosh bp}$, $\quad a=2rb,\ c=0$.

or $= 2\cdot\frac{\pi}{2}\frac{e^{-ap}-(-1)^r e^{-cp}}{2\cosh bp}+(-1)^r\left(\frac{\pi}{2}-\frac{\pi}{2}\tanh bp\right)$

$= \frac{\pi}{2}\frac{e^{-ap}}{\cosh bp}$, $\quad a=(2r+1)b,\ c=b$.

(E) $\int_0^\infty \frac{\cos ax}{\sin bx}\frac{x\,dx}{p^2+x^2} = \frac{\pi}{2}\frac{\cosh ap}{\sinh bp}$, $\quad a<b$,

or $= -2\cdot\frac{\pi}{2}\frac{e^{-cp}-e^{-ap}}{2\sinh bp}+\frac{\pi}{2}\frac{\cosh cp}{\sinh bp}$

$= \frac{\pi}{2}\frac{\sinh cp+e^{-ap}}{\sinh bp}$, $\quad a=2rb+c$,

or $= -2\cdot\frac{\pi}{2}\frac{e^{-cp}-e^{-ap}}{2\sinh bp}+\frac{\pi}{2}\operatorname{cosech} bp = \frac{\pi}{2}\frac{e^{-ap}}{\sinh bp}$, $\quad a=2rb,\ c=0$.

or $= -2\cdot\frac{\pi}{2}\frac{e^{-cp}-e^{-ap}}{2\sinh bp}+\frac{\pi}{2}\coth bp$

$= \frac{\pi}{2}\frac{\sinh bp+e^{-ap}}{\sinh bp}$, $\quad a=(2r+1)b,\ c=b$.

1117. Adding the results of (D) to p^2 times those of (C),

$$\int_0^\infty \frac{\sin ax}{\cos bx}\frac{dx}{x}=0,\quad \frac{\pi}{2}\{1-(-1)^r\},\quad \frac{\pi}{2}\{1-(-1)^r\} \text{ or } \frac{\pi}{2} \quad \text{according as}$$
$$a<b,\quad a=2rb+c,\quad a=2rb \quad \text{or } a=(2r+1)b.$$

If $a=b$ we have $\int_0^\infty \frac{\tan ax}{x}dx=\frac{\pi}{2}$ as established in Art. 1007, and used above. The majority of these results are due to Cauchy [*Mém. des Savans Ét.*, T. I.].*

1118. Some of the general results above ($a<b$ or $a=2b+c$) may be derived from others by differentiation with regard to a; bearing in mind that if b be kept constant $da=dc$.

Differentiation with regard to b, p or p^2, or integration between specified limits, will furnish other results. For example, taking $a<b$ and starting with $\int_0^\infty \frac{\sin ax}{\sin bx}\frac{dx}{p^2+x^2}=\frac{\pi}{2p}\frac{\sinh ap}{\sinh bp}$ and integrating with regard to b between b_1 and b_2, we have

$$\int_0^\infty \sin ax \log\frac{\tan\frac{b_1x}{2}}{\tan\frac{b_2x}{2}}\frac{dx}{x(p^2+x^2)}=\frac{\pi}{2p^2}\sinh ap\log\left\{\frac{\tanh\frac{b_1p}{2}}{\tanh\frac{b_2p}{2}}\right\},$$

*See also Legendre, *Exercices*, vol. ii., p. 174; Gregory, *Ex.*, pp. 491-499.

or, differentiating with regard to p,

$$\int_0^\infty \frac{\sin ax}{\sin bx}\frac{dx}{(p^2+x^2)^2} = -\frac{\pi}{4p}\frac{d}{dp}\left(\frac{\sinh ap}{p\sinh bp}\right).$$

Again, since $\int_0^\infty \frac{x \operatorname{cosec} x}{p^2+x^2}\,dx = \frac{\pi}{2}\operatorname{cosech} p$ (from Art. 1114), we have

$$\int_0^\infty \operatorname{cosec} x\left[\tan^{-1}\frac{p}{x}\right]_{p_2}^{p_1} dx = \frac{\pi}{2}\int_{p_2}^{p_1}\frac{2e^p\,dp}{e^{2p}-1} = \frac{\pi}{2}\left[\log\frac{e^p-1}{e^p+1}\right]_{p_2}^{p_1},$$

i.e. $$\int_0^\infty\left(\tan^{-1}\frac{p_1}{x}-\tan^{-1}\frac{p_2}{x}\right)\frac{dx}{\sin x} = \frac{\pi}{2}\log\frac{\tanh\frac{p_1}{2}}{\tanh\frac{p_2}{2}}$$

or $$\int_0^\infty\left(\tan^{-1}\frac{x}{p_1}-\tan^{-1}\frac{x}{p_2}\right)\frac{dx}{\sin x} = \frac{\pi}{2}\log\frac{\coth\frac{p_1}{2}}{\coth\frac{p_2}{2}},$$

and so on for other cases.

1119. Since $$z\operatorname{cosech} z = 1 - \frac{2z^2}{z^2+\pi^2} + \frac{2z^2}{z^2+2^2\pi^2} - \frac{2z^2}{z^2+3^2\pi^2} + \dots,$$

$$\int_0^\infty \frac{z\operatorname{cosech} z}{z^2+b^2}\,dz = \frac{\pi}{2b} + 2\sum_1^\infty(-1)^r\int_0^\infty\frac{z^2\,dz}{(z^2+r^2\pi^2)(z^2+b^2)} = \frac{\pi}{2b} + \pi\sum_1^\infty\frac{(-1)^r}{(b+r\pi)},$$

and when b is an integral multiple of π, $=n\pi$ say, we have

$$\int_0^\infty \frac{z\operatorname{cosech} z}{z^2+n^2\pi^2}\,dz = \frac{1}{2n} - (-1)^n\left\{\log 2 - \frac{1}{1} + \frac{1}{2} - \frac{1}{3} + \dots + (-1)^n\frac{1}{n}\right\}.$$

1120. **Some Special Forms** given by LEGENDRE (*Exercices*, p. 243) and LANDEN (*Math. Mem.*, p. 112, etc.).

Taking $$-\frac{\pi^2}{6} = \int_0^1 \frac{\log(1-x)}{x}\,dx = \left(\int_0^a + \int_a^1\right)\frac{\log(1-x)}{x}\,dx,$$

write $1-x=y$ in the second integral. Then ($x<1$)

$$\begin{aligned}\int_a^1 \frac{\log(1-x)}{x}\,dx &= \int_0^{1-a}\frac{\log x}{1-x}\,dx \\ &= -\Big[\log(1-x)\log x\Big]_0^{1-a} + \int_0^{1-a}\frac{\log(1-x)}{x}\,dx \\ &= -\log a\log(1-a) + \int_0^{1-a}\frac{\log(1-x)}{x}\,dx.\end{aligned}$$

Hence $$\left(\int_0^a + \int_0^{1-a}\right)\frac{\log(1-x)}{x}\,dx = \log a\log(1-a) - \frac{\pi^2}{6};$$

and if $\phi(a) \equiv \int_0^a \frac{\log(1-x)}{x}\,dx$, we have

$$\phi(a) + \phi(1-a) = \log a\log(1-a) - \frac{\pi^2}{6}, \quad\text{.......................(i)}$$

and $$\phi(\tfrac{1}{2}) = \tfrac{1}{2}(\log\tfrac{1}{2})^2 - \frac{\pi^2}{12}, \quad (a=\tfrac{1}{2}). \quad\text{.......................(ii)}$$

Also $\phi'(x)=\{\log(1-x)\}/x$, and

$$\frac{d}{dx}\phi\left(\frac{-x}{1-x}\right)=\phi'\left(\frac{-x}{1-x}\right)\frac{-1}{(1-x)^2}=\frac{\log\left(1+\dfrac{x}{1-x}\right)}{\dfrac{-x}{1-x}}\,\frac{-1}{(1-x)^2}=\frac{1}{x(1-x)}\log\frac{1}{1-x};$$

$$\therefore\ \phi\left(\frac{-x}{1-x}\right)=-\int_0^x\left(\frac{1}{x}+\frac{1}{1-x}\right)\log(1-x)\,dx=-\phi(x)+\tfrac{1}{2}\{\log(1-x)\}^2.$$

Let $x=y/(1+y)$, then

$$\phi(-y)+\phi\left(\frac{y}{1+y}\right)=\tfrac{1}{2}\{\log(1+y)\}^2,$$

i.e.

$$\phi(-x)+\phi\left(\frac{x}{1+x}\right)=\tfrac{1}{2}\{\log(1+x)\}^2. \qquad\text{(iii)}$$

Again

$$\phi(x)=\int_0^x\frac{\log(1-x)}{x}dx=-\left(\frac{x}{1^2}+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\dots\right);$$

$$\therefore\ \phi(x)+\phi(-x)=-2\left(\frac{x^2}{2^2}+\frac{x^4}{4^2}+\frac{x^6}{6^2}+\dots\right)=\tfrac{1}{2}\phi(x^2); \qquad\text{(iv)}$$

$$\therefore\ -\phi(x)+\tfrac{1}{2}\phi(x^2)+\phi\left(\frac{x}{1+x}\right)=\tfrac{1}{2}\{\log(1+x)\}^2. \qquad\text{(v)}$$

(Legendre.)

In the case $\dfrac{x}{1+x}=x^2$, *i.e.* $x(x+1)=1$ or $x=\dfrac{\sqrt{5}-1}{2}=a$, say,

$$\tfrac{3}{2}\phi(a^2)-\phi(a)=\tfrac{1}{2}\{\log(1+a)\}^2,$$

i.e.

$$\tfrac{3}{2}\phi(1-a)-\phi(a)=\tfrac{1}{2}\left(\log\frac{1}{a}\right)^2=\tfrac{1}{2}(\log a)^2.$$

But

$$\phi(1-a)+\phi(a)=\log a\log(1-a)-\frac{\pi^2}{6}=\log a\log a^2-\frac{\pi^2}{6}$$
$$=2(\log a)^2-\frac{\pi^2}{6}.$$

Hence solving

$$\phi(1-a)=(\log a)^2-\frac{\pi^2}{15},\quad \phi(a)=(\log a)^2-\frac{\pi^2}{10},$$

where

$$a=\frac{\sqrt{5}-1}{2}=2\sin\frac{\pi}{10},\quad (1-a)=\sqrt{a^2}=\left(2\sin\frac{\pi}{10}\right)^2.$$

Thus

$$\int_0^{2\sin\frac{\pi}{10}}\frac{\log(1-x)}{x}dx=\left(\log 2\sin\frac{\pi}{10}\right)^2-\frac{\pi^2}{10},\quad \int_0^{4\sin^2\frac{\pi}{10}}\frac{\log(1-x)}{x}dx=\log\left(2\sin\frac{\pi}{10}\right)^2-$$

These curious results are due to Landen. They are quoted by Bertrand, *Calc. Int.*, pp. 216-217.

The series $\dfrac{x}{1^2}+\dfrac{x^2}{2^2}+\dfrac{x^3}{3^2}+\dfrac{x^4}{4^2}+\dfrac{x^5}{5^2}+\dots$ *ad inf.* is therefore summable in the four cases $x=\pm1$, $x=\tfrac{1}{2}$, $x=2\sin\dfrac{\pi}{10}$, $x=\left(2\sin\dfrac{\pi}{10}\right)^2$.

PROBLEMS.

Prove the following results

1. $\int_0^{\frac{\pi}{2}} \cot\theta(\log\sec\theta)^3\,d\theta = \frac{\pi^4}{240}$. 2. $\int_0^{\frac{\pi}{4}} \tan\theta(\log\cot\theta)^3\,d\theta = \frac{7\pi^4}{1920}$.

3. $\int_0^1 \frac{\left(\log\frac{1}{x}\right)^5}{1-x}\,dx = \frac{8\pi^6}{63}$. 4. $\int_0^1 \frac{x^2-x+1}{1-x}\log\frac{1}{x}\,dx = \frac{\pi^2}{6} - \frac{1}{4}$.

5. $\int_0^{\frac{\pi}{2}} (\cos^4\theta + \sin^2\theta\cos^2\theta + \sin^4\theta)\tan\theta\log\operatorname{cosec}\theta\,d\theta = \frac{2\pi^2-3}{48}$.

6. (i) $\int_0^1 \frac{1+x}{1-x}\log\frac{1}{x}\,dx = \frac{\pi^2-3}{3}$,

(ii) $\int_0^{\frac{\pi}{4}} \tan\theta\sec^2\theta\sec 2\theta\log\cot\theta\,d\theta = \frac{\pi^2-3}{12}$.

7. $\int_0^1 \frac{(1+x)^2}{1-x}\log\frac{1}{x}\,dx = \frac{8\pi^2-39}{12}$. 8. $\int_0^1 \frac{x^2-4}{x-1}\log\frac{1}{x}\,dx = \frac{2\pi^2+5}{4}$.

9. $\int_0^1 \frac{x^3}{1-x}\log\frac{1}{x}\,dx = \frac{6\pi^2-49}{36}$.

10. $\int_0^1 \frac{a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n}{1-x}\log\frac{1}{x}\,dx$

$$= \frac{\pi^2}{6}\sum_0^n a_r - \frac{1}{1^2}\sum_1^n a_r - \frac{1}{2^2}\sum_2^n a_r - \ldots - \frac{a_n}{n^2}.$$

11. $\int_0^1 \frac{1-x^n}{(1-x)^2}\log\frac{1}{x}\,dx = \frac{n\pi^2}{6} - \frac{n-1}{1^2} - \frac{n-2}{2^2} - \frac{n-3}{3^2} - \ldots - \frac{1}{(n-1)^2}$.

12. $\int_0^1 \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^3}\log\frac{1}{x}\,dx = \frac{n(n+1)\pi^2}{12} - \frac{(n-1)(n+2)}{2\,.\,1^2}$

$$- \frac{(n-2)(n+3)}{2\,.\,2^2} - \frac{(n-3)(n+4)}{2\,.\,3^2} - \ldots - \frac{1\,.\,2n}{2(n-1)^2}.$$

13. (1) $\int_0^{\frac{\pi}{2}} \log(\sec\theta)\frac{d\theta}{\sin\theta} = \frac{\pi^2}{8}$, (2) $\int_0^{\frac{\pi}{2}} (\log\sec\theta)^3\frac{d\theta}{\sin\theta} = \frac{\pi^4}{16}$,

(3) $\int_0^{\frac{\pi}{2}} (\log\sec\theta)^5\frac{d\theta}{\sin\theta} = \frac{\pi^6}{8}$, (4) $\int_0^{\frac{\pi}{2}} (\log\sec\theta)^7\frac{d\theta}{\sin\theta} = \frac{17\pi^8}{32}$.

14. $\int_0^1 x^{2n}\frac{\log\frac{1}{x}}{1-x^2}\,dx = \frac{\pi^2}{8} - \frac{1}{1^2} - \frac{1}{3^2} - \frac{1}{5^2} - \ldots - \frac{1}{(2n-1)^2}$.

15. $\int_0^1 \frac{a+bx^2+cx^4}{1-x^2}\log\frac{1}{x}\,dx=(a+b+c)\frac{\pi^2}{8}-\frac{a+b}{1^2}-\frac{c}{3^2}.$

16. $\int_0^1 \frac{1-x^6}{(1-x^2)^2}\log\frac{1}{x}\,dx=\frac{3\pi^2}{8}-\frac{19}{9}.$ 17. $\int_0^1 \frac{1+x^6}{1-x^4}\log\frac{1}{x}\,dx=\frac{9\pi^2-8}{72}.$

18. $\int_0^1 \frac{x^{2n}}{1+x^2}\left(\log\frac{1}{x}\right)^2 dx$

$$=2(-1)^n\left[\frac{\pi^3}{32}-\frac{1}{1^3}+\frac{1}{3^3}-\frac{1}{5^3}+\ldots+(-1)^n\frac{1}{(2n-1)^3}\right].$$

19. $\int_0^1 \frac{a+bx^2+cx^4}{1+x^2}\left(\log\frac{1}{x}\right)^2 dx=(a-b+c)\frac{\pi^3}{16}+2(b-c)+\frac{2c}{27}.$

20. $\int_0^1 \frac{1-x^6}{1-x^4}\left(\log\frac{1}{x}\right)^2 dx=\frac{\pi^3}{16}+\frac{2}{27}.$

21. $\int_0^1 \frac{1+x^6}{(1+x^2)^2}\left(\log\frac{1}{x}\right)^2 dx=\frac{3\pi^3}{16}-\frac{106}{27}.$

22. $\int_0^{\frac{\pi}{4}} (1+\tan^2\theta+\tan^4\theta)(\log\tan\theta)^2\,d\theta=\frac{\pi^3}{16}+\frac{2}{27}.$

23. (1) $\int_0^{\frac{\pi}{4}} (\log\cot\theta)^2\,d\theta=\frac{\pi^3}{16},$ (2) $\int_0^{\frac{\pi}{4}} (\log\cot\theta)^4\,d\theta=\frac{5\pi^5}{64}.$

(3) $\int_0^{\frac{\pi}{4}} (\log\cot\theta)^6\,d\theta=\frac{61\pi^7}{256}.$

24. Prove that $\int_0^{\frac{\pi}{4}} \frac{\log\cot\theta}{(\sin^n\theta+\cos^n\theta)^2}\sin^{n-1}2\theta\,d\theta=\frac{2^{n-1}}{n^2}\log 2.$

25. Establish the following results:

(1) $\int_0^{\frac{\pi}{2}} \left\{\frac{\log\tan\theta}{\sin\left(\theta-\frac{\pi}{4}\right)}\right\}^2 d\theta=\frac{4\pi^2}{3},$

(2) $\int_0^{\frac{\pi}{2}} \left\{\frac{\log\tan\theta}{\sin\left(\theta-\frac{\pi}{4}\right)}\right\}^3 \cos\theta\,d\theta=2\pi^2\sqrt{2},$

(3) $\int_0^{\frac{\pi}{2}} \left\{\frac{\log\tan\theta}{\sin\left(\theta-\frac{\pi}{4}\right)}\right\}^4 \cos^2\theta\,d\theta=\frac{16\pi^2}{3}+\frac{32\pi^4}{45},$

(4) $\int_0^{\frac{\pi}{2}} \left\{\frac{\log\tan\theta}{\sin\left(\theta-\frac{\pi}{4}\right)}\right\}^5 \cos^3\theta\,d\theta=\left(\frac{20\pi^2}{3}+\frac{44\pi^4}{9}\right)\sqrt{2}.$

26. Show that

$$\int_0^{\frac{\pi}{2}} \frac{1+4\sin^2\theta+\sin^4\theta}{\cos^6\theta}\,.\tan\theta(\log\operatorname{cosec}\theta)^{2n+2}\,d\theta = \frac{(n+1)(2n+1)}{8}\pi^{2n}B_{2n-1},$$

where B_{2n-1} is the n^{th} Bernoullian number.

27. Evaluate (1) $\int_0^{\frac{\pi}{2}} \frac{\cos^3\theta\sin^{n-1}\theta\,d\theta}{\log\operatorname{cosec}\theta}$, (2) $\int_0^{\frac{\pi}{2}} \frac{\cos^5\theta\sin^{n-1}\theta\,d\theta}{(\log\operatorname{cosec}\theta)^2}$,

(3) $\int_0^{\frac{\pi}{2}} \frac{\cos^7\theta\sin^{n-1}\theta\,d\theta}{(\log\operatorname{cosec}\theta)^3}$.

28. Show that $\int_0^\infty \frac{x^a\log x}{x-1}\frac{dx}{x} = \pi^2\operatorname{cosec}^2 a\pi \quad (0 < a < 1)$.

29. Establish the results (a) $\int_0^1 \frac{\log(1-x)}{2-x}dx = -\frac{\pi^2}{12}$,

(b) $\int_0^\infty \frac{\log x}{1-x^2}dx = \frac{\pi^2}{4}$, (c) $\int_0^1 \frac{(\log x)^7}{1-x^2}dx = -\frac{17\pi^8}{32}$.

30. Establish the results

(a) $\int_0^1 \frac{x^p - x^{-p}}{x^q - x^{-q}}\frac{dx}{x} = \frac{\pi}{2q}\tan\frac{p\pi}{2q} \quad (q > p > -q)$;

(b) $\int_0^1 \frac{x^p + x^{-p}}{x^q + x^{-q}}\frac{dx}{x} = \frac{\pi}{2q}\sec\frac{p\pi}{2q} \quad (q > p > -q)$.

31. Establish the result $\int_0^1 \frac{x^{a-1} - x^{1-a}}{1-x^2}dx = \frac{\pi}{2}\cot\frac{\pi a}{2} \quad (2 > a > 0)$.

32. Prove that $\int_0^\infty \frac{\sinh px}{\cosh \pi x}\frac{dx}{x} = \log\tan\frac{p+\pi}{4} \quad (\pi > p > -\pi)$,

33. Show that $(\pi > a > -\pi)$,

(1) $\int_0^\infty \frac{\cosh ax}{\sinh \pi x}\sin rx\,dx = \frac{1}{2}\,\frac{\sinh r}{\cosh r + \cos a}$,

(2) $\int_0^\infty \frac{\sinh ax}{\sinh \pi x}\cos rx\,dx = \frac{1}{2}\,\frac{\sin a}{\cosh r + \cos a}$,

(3) $\int_0^\infty \frac{\sin rx}{\sinh \pi x}dx = \frac{1}{2}\tanh\frac{r}{2}$, (4) $\int_0^\infty \frac{\sinh ax}{\sinh \pi x}dx = \frac{1}{2}\tan\frac{a}{2}$,

(5) $\int_0^\infty \frac{x\cos rx}{\sinh \pi x}dx = \frac{1}{4}\operatorname{sech}^2\frac{r}{2}$. [GREGORY, *Ex.*, p. 495.]

34. Show that

$$(1) \int_0^\infty \frac{\cosh ax}{\cosh \pi x} \cos rx\, dx = \frac{\cosh\frac{r}{2}\cos\frac{a}{2}}{\cosh r + \cos a} \quad (\pi > a > -\pi),$$

$$(2) \int_0^\infty \frac{\cos rx}{\cosh \pi x} dx = \frac{1}{2}\operatorname{sech}\frac{r}{2},$$

$$(3) \int_0^\infty \frac{\operatorname{sech} \pi x}{p^2 + x^2} dx = \frac{1}{2p}\int_0^\infty e^{-pr}\operatorname{sech}\frac{r}{2}\, dr = \frac{1}{p}\int_0^1 \frac{z^{p-\frac{1}{2}}}{1+z} dz,$$

$$(4) \int_0^\infty \frac{\operatorname{sech} \pi x}{\frac{1}{4} + x^2} dx = \log_e 4, \qquad (5) \int_0^\infty \frac{\operatorname{sech} \pi x}{1+x^2} dx = 2 - \frac{\pi}{2}.$$

[Gregory, *Ex.*, p. 496.]

35. Show that

$$\int_0^\infty \frac{a+bx+cx^2}{\sqrt{e^{2x}-1}} dx = \frac{\pi}{2}\left[a + b\log 2 + c(\log 2)^2 + \frac{\pi^2 c}{12}\right].$$ [a, 1891.]

36. Show that $\displaystyle\int_0^\infty \frac{\log\frac{1}{x}}{(1+x)^n} dx = \frac{1}{n-1}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-2}\right)$, n being a positive integer > 2.

37. Show that the integral $\displaystyle\int_0^\infty \frac{\sin ax}{\sin bx}\frac{dx}{1+x^2}$ has the value $\dfrac{\pi}{2}\dfrac{\sinh a}{\sinh b}$ if a be $< b$, but has the value $\dfrac{\pi}{2}\dfrac{\cosh c - e^{-a}}{\sinh b}$ if $a > b$ and $= 2rb + c$, where r is an integer and $c < b$. [R. P.]

38. Prove that the coefficient of x^n in the expansion of $\sec x$ in ascending powers of x is equal to

$$\frac{1}{n!}\left(\frac{2}{\pi}\right)^{n+1}\int_0^{\frac{\pi}{2}} (\log\tan x)^n dx.$$

[Math. Trip., Part I., 1888.]

39. Show that $\displaystyle\int_0^\infty \frac{1-3x}{(1+x)^5}(\log x)^4 dx = 2\pi^2$.

40. If $\chi(x) \equiv x + \dfrac{x^3}{3^2} + \dfrac{x^5}{5^2} + \dots$, show that

$$\text{(i)}\ \chi\left(\frac{1-x}{1+x}\right) = \int_0^1 \frac{\log x}{1-x^2} dx,$$

$$\text{(ii)}\ \chi(x) + \chi\left(\frac{1-x}{1+x}\right) = \frac{\pi^2}{8} + \tfrac{1}{2}\log x \,.\, \log\frac{1+x}{1-x},$$

$$\text{(iii)}\ \chi\left(\tan\frac{\pi}{8}\right) = \frac{\pi^2}{8} - \tfrac{1}{2}\left(\log\tan\frac{\pi}{8}\right)^2,$$

and that the value of the series $\chi(x)$ is known in the four cases

$$x = 1, \quad x = 2\sin\frac{\pi}{10}, \quad x = \sqrt{5} - 2, \quad x = \tan\frac{\pi}{8}.$$

[Legendre, *Ex.*, p. 247.]

41. If $\Lambda(x) \equiv x + \frac{x^2}{2^3} + \frac{x^3}{3^3} + \frac{x^4}{4^3} + \dots$, show that, $\phi(x)$ being as defined in Art. 1120,

(i) $\Lambda(x) + \Lambda(1-x) + \Lambda\left(-\frac{x}{1-x}\right)$

$$= \Lambda(1) - \log x \,.\, \phi(x) - \log(1-x)\,\phi(1-x) - \log\frac{x}{1-x}\cdot\phi\left(-\frac{x}{1-x}\right) + \log x \,.\, \log^2(1-x) - \tfrac{1}{3}\log^3(1-x),$$

(ii) $\frac{7}{8}\Lambda(1) = \Lambda(\tfrac{1}{2}) + \frac{\pi^2}{12}\log 2 - \tfrac{1}{6}(\log 2)^3$,

(iii) $\Lambda(1) = \frac{5}{4}\Lambda\left(4\sin^2\frac{\pi}{10}\right) - \frac{\pi^2}{6}\log\left(2\sin\frac{\pi}{10}\right) + \frac{5}{6}\log^3\left(2\sin\frac{\pi}{10}\right)$,

(iv) $\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \dots = \frac{\theta^2}{1^3} + \frac{\theta^4}{2^3} + \frac{\theta^6}{3^3} + \frac{\theta^8}{4^3} + \dots$, where $\theta = 2\sin\pi/10$.

[LANDEN, *Math. Mem.*]

42. Prove that

$$\int_0^{2-\sqrt{2}} \log\frac{1-x}{1-\frac{x}{2}}\,\frac{dx}{x} = \tfrac{1}{2}\log(\sqrt{2}-1)\log\{2(\sqrt{2}-1)\} - \tfrac{1}{8}(\log 2)^2 - \frac{\pi^2}{24}.$$

[MORLEY, *E.T.*, 9224.]

43. If $f(x) = f(0) + xf'(0) + \dots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n}{n!}f^{(n)}(\theta x)$ and r be a positive proper fraction, show that

$$\int_0^\infty \frac{f^{(n)}(\theta x)}{x^r}\,dx = \frac{\Gamma(n+1)\Gamma(r)}{\Gamma(n+r)}\int_0^\infty \frac{f^{(n)}(x)}{x^r}\,dx.$$

[M. TRIP., 1883.]

44. Prove that $\int_0^\infty \sin x^n\,dx = b\Gamma(1+1/n)$, $(n>1)$, where b is the real coefficient of the imaginary part of $(-1)^{\frac{1}{2n}}$, and hence find the value of the integral to four places of decimals when n is 2 or 3.

[SANJÁNA, *E.T.*, 13,609.]

45. Prove that

$$\int_0^{\frac{\pi}{2}}\int_0^1 \tan^{-1}\frac{2m\cos\theta}{1-m^2}\,d\theta\,dm = \frac{\pi^2}{4} - 2\log 2, \quad (0<m<1).$$

[SANJÁNA, *E.T.*, 13,636.]

46. Prove that $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}-\theta} \cos^4(\theta+\phi)\sec^2\phi\,d\theta\,d\phi = \tfrac{1}{4}$.

[W. J. C. MILLER, *E.T.*, 13,784.]

47. Prove that the value of

$$\iint x^{k-1}y^{-k}e^{x+y}\,dx\,dy \quad\text{is}\quad \frac{\pi}{\sin k\pi}(e^c - 1),$$

the integral being taken so as to give the variables all positive values consistent with the condition $x + y \not> c$; $(0<k<1)$. [OX. II. P., 1885.]

48. Show that $\iint\ldots\int\sqrt{\Delta}\,dx_1\,dx_2\ldots dx_n=\dfrac{a^{\frac{n}{r}}}{nr^{n-1}}\dfrac{\left\{\Gamma\left(\frac{1}{r}\right)\right\}^n}{\Gamma\left(\frac{n}{r}\right)}$,

where $x_1, x_2, \ldots x_n$ are the roots and Δ the discriminant of the equation

$$x^n+p_1x^{n-1}+\ldots+p_{n-1}x+p_n=0,$$

the integral being taken over all values of the variables such that the sum of the r^{th} powers of the coefficients in this equation, which are all positive, does not exceed a given quantity a.

[MATH. TRIP., 1884.]

49. If $I_m\equiv\int_0^a(\cos x-\cos a)^m\,dx$ and $J_m=\dfrac{1}{\sin^{2m+1}a}I_m$, prove

(i) $mI_m+(2m-1)\cos a\,I_{m-1}-(m-1)\sin^2 a\,I_{m-2}=0$,

(ii) $J_m=\dfrac{1}{m!}\left(\dfrac{1}{\sin a}\dfrac{d}{da}\right)^m\left(\dfrac{a}{\sin a}\right)$.

50. If $f(z)$ be an even function of z, and

$$I_{2n}=\int_0^\infty x^{2n}f\left(x-\frac{1}{x}\right)dx,\quad J_{2n}=\int_0^\infty x^{2n}f(x)\,dx,$$

show that $I_{2n}=J_0+\dfrac{(n+1)n}{1\,.\,2}J_2+\dfrac{(n+2)(n+1)n(n-1)}{1\,.\,2\,.\,3\,.\,4}J_4+\ldots+J_{2n}$.

[Use the expansion of $\dfrac{\cos m\theta}{\cos\theta}$ in powers of $\sin\theta$.] [CAUCHY.]

51. If $f(z)$ be an odd function of z, and

$$I'_{2n-1}=\int_0^\infty x^{2n-1}f\left(x-\frac{1}{x}\right)dx,\quad J'_{2n-1}=\int_0^\infty x^{2n-1}f(x)\,dx,$$

show that $I'_{2n-1}=\dfrac{n}{1}J_1'+\dfrac{(n+1)n(n-1)}{1\,.\,2\,.\,3}J_3'$

$$+\frac{(n+2)(n+1)n(n-1)(n-2)}{1\,.\,2\,.\,3\,.\,4\,.\,5}J_5'+\ldots+J'_{2n-1}.$$

[GLAISHER.*]

52. If $f(z)$ be an even function of z, show that

$$\int_0^\infty f\left(x-\frac{1}{x}\right)dx=\int_0^\infty f(x)\,dx;$$

show also that $\int_0^\infty f\left(x-\dfrac{a}{x}\right)dx$ is independent of a. [GLAISHER.]

* *Camb. Phil. Soc.*, 1876.