III

QUATERNIONS*

[Note-book 24·5, entry for 16 October 1843.]

October 16th, 1843

I, this morning, was led to what seems to me a theory of *quaternions*, which may have interesting developments. *Couples* being supposed known, and known to be representable by points in a plane, so that $\sqrt{-1}$ is perpendicular to 1, it is natural to conceive that there may be another sort of $\sqrt{-1}$, perpendicular to the plane itself. Let this new imaginary be j; so that $j^2=-1$, as well as $i^2=-1$. A point x, y, z in space may suggest the triplet $x+iy+jz$. The square of this triplet is on the one hand $x^2-y^2-z^2+2ixy+2jxz+2ijyz$; such at least it seemed to me at first, because I assumed $ij=ji$. On the other hand, if this is to represent the third proportional to $1, 0, 0$ and x, y, z, considered as *indicators of lines*, (namely the lines which end in the points having these coordinates, while they begin at the origin) and if this third proportional be supposed to have its length a third proportional to 1 and $\sqrt{x^2+y^2+z^2}$, and its distance twice as far angularly removed from $1, 0, 0$ as x, y, z; then its real part ought to be $x^2-y^2-z^2$ and its two imaginary parts ought to have for coefficients $2xy$ and $2xz$; thus the term $2ijyz$ appeared de trop, and I was led to assume at first $ij=0$. However I saw that this difficulty would be removed by supposing $ji=-ij$. I next considered the product

$$(a+iy+jz)(x+iy+jz)=ax-y^2-z^2+i(a+x)y+j(a+x)z+(ij+ji)yz.$$

The assumption $ji=-ij$ got rid of the last product; and I wished to know whether $ax-y^2-z^2$, $(a+x)y$, $(a+x)z$ were the coordinates of the end of the line which, on Warren's principles, is the fourth proportional to the three lines of which the ends are $1, 0, 0$; a, y, z; x, y, z. For this purpose I had†

$$\tan^{-1}\frac{\sqrt{y^2+z^2}}{a}+\tan^{-1}\frac{\sqrt{y^2+z^2}}{x}=\tan^{-1}\frac{(a+x)\sqrt{y^2+z^2}}{ax-y^2-z^2};$$

but
$$(ax-y^2-z^2)^2+(a+x)^2(y^2+z^2)=(a^2+y^2+z^2)(x^2+y^2+z^2).$$

$$\because\ \sqrt{a^2+y^2+z^2}\ \sqrt{x^2+y^2+z^2}\cos\tan^{-1}\ \&\text{c.}=ax-y^2-z^2;$$

$$\sqrt{\qquad}\ \sqrt{\qquad}\ \sin\tan^{-1}\ \&\text{c.}=(a+x)\sqrt{y^2+z^2}.$$

Therefore the question is resolved in the affirmative; and if we assume $ij=-ji$, $i^2=j^2=-1$, the product $(a+iy+jz)(x+iy+jz)$ comes out what we wished it to do, with respect to its geometrical interpretation. But, taking next

$$(a+ib+jc)(x+iy+jz)=ax-by-cz+i(ay+bx)+j(az+cx)+ij(bz-cy),$$

I saw that

$$(a^2+b^2+c^2)(x^2+y^2+z^2)=(ax-by-cz)^2+(ay+bx)^2+(az+cx)^2+(bz-cy)^2;$$

* [This extract has also been reproduced in *Proc. Roy. Irish Acad.* vol. L (1945), pp. 89–92.]

† [Strictly, the first equality should be mod π.]

and thus was led to conceive that the product ij might be equated to a *new imaginary*, say k; while $ji=-k$.

Still (and perhaps before) I thought it *possible* that ij might be equal to zero: and (trying to remember this evening my course of thought in the morning) I believe that I even thought it *likely*, though odd, that this equation $ij=0$ might turn out to be true; true, I mean, with reference to a desired agreement between the results of an algebraical process, (conducted according to analogy, as far as possible, with formerly known and usual processes of algebra), and a geometrical interpretation, thus far at least foreseen, that I expected the *modulus* of the product to be the product of the moduli.

I believe that I remember now the order of my thought. The equation $ij=0$ was recommended by the circumstance that

$$(ax-y^2-z^2)^2+(a+x)^2(y^2+z^2)=(a^2+y^2+z^2)(x^2+y^2+z^2).$$

I therefore *tried* whether it might not turn out to be true that

$$(a^2+b^2+c^2)(x^2+y^2+z^2)=(ax-by-cz)^2+(ay+bx)^2+(az+cx)^2;$$

but found that this equation required, in order to make it true, the addition of $(bz-cy)^2$ to the second member. This *forced* on me the non-neglect of ij; and *suggested* that it might be equal to k, a new imaginary.

For a while I thought it likely that k^2 might be equal to 1; but this supposition $k^2=1$, combined with $i^2=j^2=-1$, would give

$$(a+ib+jc+kd)(\alpha+i\beta+j\gamma+k\delta)=a\alpha-b\beta-c\gamma+d\delta+i(\&\text{c.})+j(\&\text{c.})+k(\&\text{c.});$$

in which the part multiplied by k *must* involve the terms $a\delta+d\alpha$, unless we choose to give up (or find ourselves compelled to do so) certain very simple and fundamental principles of ordinary algebra, extended now by analogy; but in the square of the modulus, (that is, in the sum of the 4 squares of the coefficients of $1, i, j, k$), this part of the coefficient of k would give the term $+2ad\alpha\delta$, while, if $k^2=1$, the coefficient of 1 would give the same term $2ad\alpha\delta$ repeated; thus there would be no mutual destruction of these two terms. But if we suppose $k^2=-1$, the real part (so to speak) is $a\alpha-b\beta-c\gamma-d\delta$; and this gives $-2ad\alpha\delta$, destroying $+2ad\alpha\delta$, which latter arises from $a\delta+d\alpha$.

Thus I was led to assume not only $i^2=-1$ and $j^2=-1$, but also $k^2=-1$, and $ij=k, ji=-k$. I then thought it likely that it might be proper to assume $ik=-j$, because $ik=iij$ and $i^2=-1$. If so, since $ji=-ij$, it seemed likely that $ki=-ik=j$ which might also follow from $k=-ji$. In like manner, it appeared possible (or at least natural to be assumed) that $kj=ijj=-i$, $jk=-jji=i$.

The *multiplication-assumptions*, or definitions, were therefore collected to be

$$i^2=j^2=k^2=-1; \quad ij=k,\ jk=i,\ ki=j; \quad ji=-k,\ kj=-i,\ ik=-j.$$

And thus we are led, or tempted, to assume as the formula for the multiplication of two quaternions the following

$$\begin{aligned}(a+ib+jc+kd)(\alpha+i\beta+j\gamma+k\delta)&=a\alpha-b\beta-c\gamma-d\delta\\&\quad+i(a\beta+b\alpha+c\delta-d\gamma)+j(a\gamma-b\delta+c\alpha+d\beta)+k(a\delta+b\gamma-c\beta+d\alpha).\end{aligned}$$

But if this is to agree with the assumed, or expected, principle, or rule, that the modulus of the product is to be equal to the product of the moduli, we ought then to have the following equation, of a most purely algebraic and ordinary form:

$$(a^2+b^2+c^2+d^2)(\alpha^2+\beta^2+\gamma^2+\delta^2)=(a\alpha-b\beta-c\gamma-d\delta)^2$$
$$+(a\beta+b\alpha+c\delta-d\gamma)^2+(a\gamma-b\delta+c\alpha+d\beta)^2+(a\delta+b\gamma-c\beta+d\alpha)^2.$$

The slightest inspection shows that it is sufficient to prove that the double products in the second member destroy each other; and this has already been proved for the case when $d=0$, $\delta=0$; it only remains then to prove that the double products involving d or δ or both, destroy each other. Now the corresponding products are the 18 following:

$$-ad\alpha\delta+bd\beta\delta+cd\gamma\delta+ac\beta\delta+bc\alpha\delta-ad\beta\gamma-bd\alpha\gamma$$
$$-cd\gamma\delta-ab\gamma\delta-bc\alpha\delta+ad\beta\gamma-bd\beta\delta+cd\alpha\beta+ab\gamma\delta$$
$$-ac\beta\delta+ad\alpha\delta+bd\alpha\gamma-cd\alpha\beta;$$

of which the sum equals zero. Thus the above equation is found in fact to be true; and therefore the *formula of multiplication of quaternions*, assumed above, is found to agree with the principle that the modulus of the product is to be equal to the product of the moduli.

Hence we may write, on the plan of my theory of couples,

$$(a,b,c,d)(\alpha,\beta,\gamma,\delta)=(a\alpha-b\beta-c\gamma-d\delta,\ a\beta+b\alpha+c\delta-d\gamma,\ a\gamma-b\delta+c\alpha+d\beta,\ a\delta+b\gamma-c\beta+d\alpha).$$

Hence $$(a,b,c,d)^2=(a^2-b^2-c^2-d^2,\ 2ab,\ 2ac,\ 2ad).$$

Thus $$(0,x,y,z)^2=-(x^2+y^2+z^2);\quad (0,x,y,z)^3=-(x^2+y^2+z^2)(0,x,y,z);$$
$$(0,x,y,z)^4=+(x^2+y^2+z^2)^2;\quad \&c.;$$

therefore

$$e^{(0,x,y,z)}=e^{ix+jy+kz}=1+\frac{ix+jy+kz}{1}-\frac{x^2+y^2+z^2}{1.2}-\&c.$$
$$=\cos\sqrt{x^2+y^2+z^2}+\frac{ix+jy+zk}{\sqrt{x^2+y^2+z^2}}\sin\sqrt{x^2+y^2+z^2},$$

and *the modulus of* $e^{(0,x,y,z)}=1$. [Like the modulus of $e^{(0,x)}$ or $e^{\sqrt{-1}x}$.]

Let $$\sqrt{x^2+y^2+z^2}=\rho,\quad x=\rho\cos\phi,\quad y=\rho\sin\phi\cos\psi,\quad z=\rho\sin\phi\sin\psi;$$

then $$e^{\rho(i\cos\phi+j\sin\phi\cos\psi+k\sin\phi\sin\psi)}=\cos\rho+(i\cos\phi+j\sin\phi\cos\psi+k\sin\phi\sin\psi)\sin\rho;$$

a theorem, which when $\phi=0$, becomes the well-known equation,

$$e^{i\rho}=\cos\rho+i\sin\rho,\quad i=\sqrt{-1}.$$