XVI

ON QUATERNIONS, OR ON A NEW SYSTEM OF IMAGINARIES IN ALGEBRA; WITH SOME GEOMETRICAL ILLUSTRATIONS

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The Chair having been taken, *pro tempore*, by the Rev. J. H. Todd, D.D., V.P., the President gave an account of some additional researches in the theory of Quaternions, or of a new system of Imaginaries in Algebra.

In the theory which Sir William Hamilton submitted to the Academy in November 1843, the name *quaternion* was employed to denote a certain quadrinomial expression, of which one term was called (by analogy to the language of ordinary algebra) the *real part*, while the three other terms made up together a trinomial, which (by the same analogy) was called the *imaginary part* of the quaternion: the square of the former part (or term) being always a positive, but the square of the latter part (or trinomial) being always a negative quantity. More particularly, this imaginary trinomial was of the form $ix+jy+kz$, in which x, y, z were three real and independent coefficients, or *constituents*, and were, in several applications of the theory, constructed or represented by three rectangular coordinates; while i, j, k were certain *imaginary units*, or symbols, subject to the following *laws of combination* as regards their *squares and products*,

$$i^2=j^2=k^2=-1, \tag{A}$$

$$ij=k, \quad jk=i, \quad ki=j, \tag{B}$$

$$ji=-k, \quad kj=-i, \quad ik=-j, \tag{C}$$

but were entirely *free from any linear relation* among themselves; in such a manner, that to establish an equation between two such imaginary trinomials was to equate *each* of the three constituents, xyz, of the one to the corresponding constituent of the other; and to equate two quaternions was (in general) to establish FOUR separate and distinct equations between real quantities. *Operations* on such quaternions were performed, as far as possible, according to the analogies of ordinary algebra; the *distributive* property of multiplication, and another, which may be called the *associative* property of that operation, being, for example, retained: with one important departure, however, from the received rules of calculation, arising from the abandonment of the *commutative* property of multiplication, as *not* in general holding good for the *mixture* of the new imaginaries; since the product ji (for example) has, by its definition, a different sign from ij. And several constructions and conclusions, especially as respected the geometry of the sphere, were drawn from these principles, of which some have since been printed among the *Proceedings of the Academy* for the date already referred to.

The author has not seen cause, in his subsequent reflexions on the subject, to abandon any of the principles which have been thus briefly recapitulated; but he conceives that he has been enabled to present some of them in a clearer view, as regards their bearings on geometrical questions; and also to improve the algebraical method of applying them, or what may be called the CALCULUS OF QUATERNIONS.

Thus he has found it useful, in many applications, to dismiss the *separate* consideration of the three real constituents, x, y, z, of the imaginary trinomial $ix+jy+kz$, and to denote that trinomial by some *single* letter (taken often from the Greek alphabet). And on account of the facility with which this so-called *imaginary* expression, or square root of a negative quantity, is constructed by a *right line having direction in space*, and having x, y, z for its three rectangular components, or projections on three rectangular axes, he has been induced to call the trinomial expression itself, as well as the line which it represents, a VECTOR. A *quaternion* may thus be said to consist generally of a *real* part and a *vector*. The fixing a special attention on this last part, or element, of a quaternion, by giving it a special name, and denoting it in many calculations by a single and special sign, appears to the author to have been an improvement in his method of dealing with the subject: although the general notion of treating the constituents of the imaginary part as coordinates had occurred to him in his first researches.

Regarded from a geometrical point of view, this algebraically imaginary part of a quaternion has thus so natural and simple a signification or representation in space, that the difficulty is transferred to the algebraically real part; and we are tempted to ask what this last can denote in geometry, or what in space might have suggested it.

By the fundamental equations of definition for the squares and products of the symbols i, j, k, it is easy to see that any (so-called) real and positive quantity is to any vector whatever, as that vector is to a certain real and negative quantity; this being indeed only another mode of saying that, in this theory, *every vector has a negative square*. Again, the product of any two rectangular vectors is a third vector at right angles to both the factors (but having one or other of two opposite directions, according to the order in which those factors are taken); a relation which may be expressed by saying, that the fourth proportional to the real unit and to any two rectangular vectors is a third vector rectangular to both; or, conversely, that the *fourth proportional to any three rectangular vectors* is a quantity *distinct from every vector*, and of the kind called *real* in this theory, as contrasted with the kind called *imaginary*.

Now, in fact, what originally led the author of the present communication to conceive (in 1843) his theory of quaternions (though he had, at a date earlier by several years, speculated on *triplets* and *sets** of numbers, as an extension of the theory of *couples*, or of the ordinary imaginaries of algebra, and also as an additional illustration of his views respecting the Science of Pure Time), was a desire to form to himself a distinct conception, and to find a manageable algebraical expression, of a *fourth proportional to three rectangular lines*, when the DIRECTIONS of those lines were taken into account; as Mr Warren† and Mr Peacock‡ had shewn how to conceive and express the fourth proportional to any three lines having direction, but situated *in one common plane*. And it has since appeared to Sir William Hamilton that the subject of quaternions may be illustrated by considering more closely, though briefly, this question of the determination of a fourth proportional to three rectangular directions in space, rather in a geometrical than in an algebraical point of view.

Adopting the known results above referred to, for proportions between lines having direction

* See *Trans. Roy. Irish Acad.* vol. XVII (1837), pp. 293–422. [See I.]

† Treatise on the Geometrical Representations of the Square Roots of Negative Quantities, by the Rev. John Warren. Cambridge, 1828.

‡ Treatise on Algebra, by the Rev. George Peacock. Cambridge, 1830.

in a single plane (though varying a little the known manner of speaking on the subject), it may be said that, in the horizontal plane, 'West is to South as South is to East,' and generally as any direction is to one less advanced than itself in azimuth by ninety degrees. Let it be now assumed, as an extension of this view, that in *some analogous sense* there exists a fourth proportional to the three rectangular directions, West, South, and Up; and let this be called, provisionally, *Forward*, by contrast to the opposite direction, *Backward*, which must be assumed to be (in the same general sense) a fourth proportional to the directions of West, South, and Down. We shall then have, inversely, Forward to Up as South to West; and therefore, as West to North: if we admit, as it seems natural and almost necessary to do, that (for directions, as for lengths) the inverses of equal ratios are equal; and that ratios equal to the same ratio are equal to each other. But again, Up is to South as South to Down, and also as North to Up: and we can scarcely avoid admitting, or defining, that (in the present comparison of directions) ratios similarly compounded of equal ratios are to be considered as being themselves equal ratios. Compounding, therefore, on the one hand, the ratios of Forward to Up, and of Up to South; and on the other hand the respectively equal (or similar) ratios of West to North, and of North to Up, we are conducted to admit that Forward is to South as West to Up. By a reasoning exactly similar, we find that Forward is to West as Up to South; and generally that if X, Y, Z denote any three rectangular directions such that $A : X :: Y : Z$, A here denoting what we have expressed by the word Forward, then also $A : Y :: Z : X$ (and of course, for the same reason, $A : Z :: X : Y$); so that the *three* directions XYZ may be *all changed together* by advancing them in a *ternary cycle*, according to the formula just written, without disturbing the proportionality assumed. But also, by the principle respecting proportions of directions in one plane, we may cause *any two* of the three rectangular directions XYZ to revolve together round the third, as round an axis, without altering their ratio to each other. And by combining these two principles, it is not difficult to see that *because* Forward has been supposed to be to Up as South to West, *therefore* the same (as yet unknown) direction 'Forward' must be supposed to be *to any direction X whatever*, as *any* direction Y, perpendicular to X, is to *that third* direction Z which is perpendicular to both X and Y, and which is obtained from Y by a right-handed (and not by a left-handed) rotation, through a right angle, round X; in the same manner as (and because) the direction West was so chosen as to be to the right of South, with reference to Up as an axis of rotation. Conversely we must suppose that if *any three rectangular directions*, XYZ, be arranged, as to order of rotation, in the manner just now stated, then $Z : Y :: X : A$; or in other words, we must admit, if we reason in this way at all, that the direction called already Forward, will be the fourth proportional to ZYX. And if we vary the order, so as to have Z to the left, and not to the right of Y, with reference to X, then will the fourth proportional to ZYX become the direction which we have lately called Backward, as being the opposite to that named Forward.

Again, since Forward is to Up as South to West, that is in a ratio compounded of the ratios of South to East and of East to West, or in one compounded of the ratios of West to South, and of any direction to its own opposite; or, finally, in a ratio compounded of the ratios of Up to Forward and of Forward to Backward, that is, in the ratio of Up to Backward, we see that the third proportional to the directions Forward and Up is the direction Backward: and by an exactly similar reasoning, with the help of the conclusions recently obtained, we see that if X be ANY *direction in tridimensional space*, then $A : X :: X : B$; B here denoting, for shortness, the direction which has been above called Backward.

The geometrical study of the relations between directions in space, combined with a few very simple and guiding principles respecting the composition of relations generally, might therefore have led to the conception, or assumption, of a certain *pair of contrasted directions*, namely, those which we have called Forward and Backward, and denoted by the letters A and B. And these are such that if we conceive a quantitative element to be combined with each, and give the name of POSITIVE UNITY to the unit of magnitude measured in the direction of *Forward*, but that of *Negative Unity* to the same magnitude measured backward; and if we extend to *this* positive unity and to lines having direction in space the received definitions of multiplication, that 'Positive Unity is to Multiplier as Multiplicand is to Product,' and that 'the product of two equal factors is the square of either;' we may then consistently and naturally be led to assert the same results as those already enunciated from the theory of quaternions respecting the product of two vectors, in the two principal cases, first, where those two vectors are rectangular, and second, where they are coincident with each other. And thus may we justify, or at least interpret and explain, the fundamental definitions (A) (B) (C) of this theory, by regarding the symbols ijk as denoting three vector-units having three rectangular directions in space.

But farther, we derive from this view of the whole subject an illustration (if not a confirmation) of the remarkable conclusion that the so-called *real and positive unit* $+1$ is not (in this theory) to be confounded with *any* vector unit whatever, but is to be regarded as of a kind essentially distinct from every vector. For this positive unit $+1$ is in the direction above called Forward, and denoted by A. Now if this could coincide with a direction X in tridimensional space, then, whatever this latter direction might be supposed to be, we could always, by the general formula $A : X :: Y : Z$ (where X is arbitrary), deduce the inadmissible proportion $X : X :: Y :: Z$, in which the two directions in one ratio are identical, but those in the other are rectangular to each other. If then we resolve to retain the assumption of the existence of a fourth proportional A to three rectangular directions in space, as subject to be reasoned on at all in the way already described, and as determined in direction by its contrast to its own opposite B (corresponding to an opposite order of rotation in the system XYZ), we must think of these two opposite directions A and B as merely *laid down upon a scale*, but must abstain from attributing to this SCALE any one direction rather than another in tridimensional space, as having such or such a zenith distance, or such or such an azimuth, rather than such or such another. And the progression *on this scale* from negative to positive infinity, obtained by combining a quantitative element with the contrast between two opposite directions, corresponds less to the conception of *space* itself (though we have seen that considerations of space might have suggested it) than to the conception of *time*; the variety which it admits is not *tri-* but *uni-* dimensional; and it would, in the language of some philosophical systems, be said to appertain rather to the notion of *intensive* than of *extensive* magnitude. Though answering precisely to the progression of the quantities called *real* in algebra, it has, when viewed from the geometrical side, somewhat the same sort of *imaginariness*, and yet (it is believed) of *utility*, as compared with lines in space, which the square root of an ordinary negative has, when compared with positive and negative quantities. This analogy becomes still more complete when we observe that (in this theory) the fourth proportional to any direction X in space, and either of the two directions A or B upon the scale, is the direction opposite to X; so that, *if a vector-unit* in any determined direction *X had been taken for positive unity*, then each of the two *scalar* units in the directions A and B (in common, it is true, with every vector-unit perpendicular to X) *might have been called*, by the general nomenclature of multiplication, *a square root of negative one*.

It is, however, a peculiarity of the calculus of quaternions, at least as lately modified by the author, and one which seems to him important, that it selects *no one direction in space as eminent* above another, but treats them as all equally related to that *extra-spatial*, or simply SCALAR direction, which has been recently called 'Forward.' In this respect it differs in its processes from the Cartesian method of coordinates, and seems often to admit of being more simply and directly applied to the treatment of geometrical problems, because it requires no previous selection of axes, rectangular or other. The author is, indeed, aware that the cooperation of other and better analysts will be necessary in order to bring the method of quaternions to anything approaching to perfection. But he hopes that an instance or two of the facility with which *some* questions at least allow themselves to be treated by this method, even in its present state, may not be without interest to the Academy. And he conceives that two examples in particular, one relating to the composition of translations, and the other to the composition of rotations in space, may usefully be selected for statement on the present occasion.

As preliminary illustrations of the operations employed, it may be remarked that for any system of lines having direction in space, it is required by many analogies (and is, for lines in one plane included among the definitions or results of the theories of Mr Warren and Mr Peacock), that the *sum* should be regarded as being equal to that one line which constructs or represents the *total effect* of all the different rectilinear motions which are expressed by the different summands. *Vectors* are therefore to be *added* to each other by a certain geometrical composition, exactly analogous to the composition of motions or of forces, and following the same known rules. *Scalars*, on the other hand (that is to say, the so-called real parts of any proposed quaternions), admitting only of a progression in quantity, and of a change of sign, without any other changes of direction, are to be added among themselves by the known rules of algebra, for the addition of positive and negative numbers. The addition of a scalar and a vector to each other can be no otherwise performed, or rather indicated, than by writing their symbols with the sign + interposed; each being, as we have seen, in some sense, imaginary with respect to the other. These operations of addition are all of the commutative, and also of the associative kind; that is to say, the order of all the summands may be changed, and any group of them may be collected or associated into one partial sum.

Scalars are *multiplied*, as well as added, by the known rules of ordinary algebra, for the multiplication of real numbers, positive or negative; because the positive unity of the system has been assumed to be itself a scalar, and not a vector unit.

For the same reason, *to multiply any vector by any scalar a*, is in general to change its length in a known ratio, and to preserve or reverse its direction, according as a is $>$ or <0; the product is therefore a new vector which may be denoted by $a\alpha$. The same new vector is obtained, under the form αa, when we multiply the scalar a by the vector α. If $a+\alpha$ and $b+\beta$ be two quaternion factors, of which a and b are the scalar parts, and α, β the vectors, then with a view to preserving the distributive character of multiplication, it is natural to define that the product may be distributed into the four following parts:

$$(a+\alpha)(b+\beta)=ab+\alpha\beta+\alpha b+\alpha\beta.$$

And if the multiplicand vector β be decomposed into two parts, or summands, one $=\beta_1$ and in the direction of the multiplier α, or in a direction exactly opposite thereto, and the other $=\beta_2$, and in a direction perpendicular to the former (so that β_1 and β_2 are the projections of β on α itself, and on the plane perpendicular to α), then it may be farther defined that the *multiplication of any one vector β by another vector α* may be accomplished by the formula

$$\alpha\beta = \alpha(\beta_1 + \beta_2) = \alpha\beta_1 + \alpha\beta_2;$$

in which, by what has been shewn, the partial product $\alpha\beta_1$ is to be considered as equal to a scalar, namely, the product of the lengths of α and β_1, taken with the sign $-$ or $+$, according as the direction of β_1 coincides with, or is opposite to that of α; while the other partial product $\alpha\beta_2$ is a vector, of which the length is the product of the lengths of α and β_2, while its direction is perpendicular to both of theirs, being obtained from that of β_2, by making it revolve right-handedly through a right angle round α as an axis. These definitions, which are compatible with the formulae (A) (B) (C), and may serve to replace them, will be found sufficient to prove generally, and perhaps with somewhat greater geometrical clearness than those formulae, the distributive and associative properties of quaternion multiplication, which have been already stated to exist. They give easily the following corollaries, which are of very frequent use in this calculus:

$$\alpha\beta + \beta\alpha = 2\alpha\beta_1 = -2AB\cos(A, B); \tag{a}$$

$$\alpha\beta - \beta\alpha = 2\alpha\beta_2 = 2\gamma AB\sin(A, B); \tag{b}$$

A and B denoting here the *lengths* of the lines α and β, and (A, B) the *angle* between them; while γ is a vector-unit perpendicular to their plane, and such that a right-handed rotation, equal to the angle (A, B), performed round γ, would bring the direction of α to coincide with that of β. For example, when $\beta = \alpha$, then $B = A$, $(A, B) = 0$, and

$$\alpha\beta = \beta\alpha = \alpha^2 = -A^2,$$

so that the length A of any vector α, in this theory, may be expressed under the form

$$A = \sqrt{-\alpha^2}. \tag{c}$$

More generally we have the equation

$$\alpha\beta - \beta\alpha = 0, \tag{d}$$

when the lines α and β are coincident or opposite in direction; while, on the contrary, the condition for their being at right angles to each other is expressed by the formula

$$\alpha\beta + \beta\alpha = 0. \tag{e}$$

These simple principles suffice to give, in a new way, algebraical solutions of many geometrical problems, of various degrees of difficulty and importance. Thus, if it be required, as an easy instance, to determine the length of the resultant of several successive rectilinear motions, or the magnitude of the statical sum of several forces acting together at one point, as a function of the amounts of those successive motions, or of those component forces, and of their inclinations to each other, we have only to denote the components by the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$, and their sum by α, the corresponding magnitudes being $A_1, A_2, \ldots, A_n$, and A; and the equation

$$\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_n$$

will give, by being squared,

$$\alpha^2 = \alpha_1^2 + \alpha_2^2 + \ldots + \alpha_n^2 + \alpha_1\alpha_2 + \alpha_2\alpha_1 + \ldots + \alpha_1\alpha_n + \alpha_n\alpha_1 + \ldots;$$

that is, by the foregoing principles (after changing all the signs),

$$A^2 = A_1^2 + A_2^2 + \ldots + A_n^2 + 2A_1A_2\cos(A_1, A_2) + \ldots + 2A_1A_n\cos(A_1A_n) + \ldots;$$

a known result, it is true, but one which can scarcely be derived in any other way by so very short a process of *calculation*. For it is not *quite* so easy, on the *algebraical* side of the question, to see that

$$(\Sigma x)^2 + (\Sigma y)^2 + (\Sigma z)^2 = \Sigma(x^2 + y^2 + z^2) + 2\Sigma(xx' + yy' + zz').$$

however easy this may be, as it is to see that

$$(\Sigma\alpha)^2 = \Sigma(\alpha^2) + \Sigma(\alpha\alpha' + \alpha'\alpha): \qquad \text{(f)}$$

although the *geometrical interpretation* of the first of these two formulae is of course more obvious than that of the latter, to those who are familiar with the method of coordinates, and not with the method of quaternions.

Again, let us consider the more difficult problem of the composition of any number of successive rotations of a body, or, at first, of any one line thereof, round several successive axes, through any angles, small or large. Let the axis of the first of these rotations have the direction of the vector-unit α, $(\alpha^2 = -1)$, and let the amount of the positive rotation round this axis be denoted by a, which letter here represents still a scalar or real number. Let β be the revolving line, considered in its original position; β' the same line, after it has revolved through the angle a round the axis α. The part, or component, of β, which is in the direction of this axis, is that which was denoted lately by β_1; and the formula (a), when multiplied by $-\frac{1}{2}\alpha$, gives, as an expression for this part,

$$\beta_1 = \tfrac{1}{2}(\beta - \alpha\beta\alpha), \qquad \text{(g)}$$

because it has been supposed that $\alpha^2 = -1$. This part of β remains unaltered by the rotation. The other part, or component of β, is, in like manner, by (b),

$$\beta_2 = \tfrac{1}{2}(\beta + \alpha\beta\alpha); \qquad \text{(h)}$$

and this part is to be multiplied by $\cos a$, in order to find the part of β', which is perpendicular to α, but in the plane of α and β. Again, multiplying by α, we cause β_2 to turn through a right angle in the positive direction round α, and obtain, for the result of *this* rotation,

$$\alpha\beta_2 = \tfrac{1}{2}(\alpha\beta - \beta\alpha);$$

an expression which is the half of that marked (b), and which is to be multiplied by $\sin a$, in order to arrive at the remaining part of the sought line β', namely, the part which is perpendicular to the plane of α and β. Collecting, therefore, the three parts, or terms, which have been thus separately obtained, we find,

$$\begin{aligned}\beta' = \beta_1 + (\cos a + \alpha \sin a)\beta_2 &= \tfrac{1}{2}(\beta - \alpha\beta\alpha) + \tfrac{1}{2}\cos a(\beta + \alpha\beta\alpha) + \tfrac{1}{2}\sin a(\alpha\beta - \beta\alpha) \\ &= \left(\cos\frac{a}{2}\right)^2 . \beta - \left(\sin\frac{a}{2}\right)^2 . \alpha\beta\alpha + \cos\frac{a}{2}\sin\frac{a}{2} . (\alpha\beta - \beta\alpha);\end{aligned}$$

that is,

$$\beta' = \left(\cos\frac{a}{2} + \alpha\sin\frac{a}{2}\right)\beta\left(\cos\frac{a}{2} - \alpha\sin\frac{a}{2}\right); \qquad \text{(i)*}$$

* [*Note added during printing.*]—The printing of this abstract having been delayed, the Author desires to be permitted to append the following remarks:

If we should make, for abridgment

$$\alpha\tan\frac{a}{2} = -\gamma,$$

the formula (i) for any single rotation might be thus written,

$$\beta' = (1+\gamma)^{-1}\beta(1+\gamma). \qquad \text{(i')}$$

And if we then made

$$\beta = ix + jy = kz, \quad \beta' = ix_{,} + jy_{,} + kz_{,}, \quad \gamma = i\lambda + j\mu + k\nu,$$

i, j, k being the same three rectangular vectors, or imaginary units, as in the formulae (A) (B) (C), but x, y, z, x', y', z', λ, μ, ν, being nine real or scalar quantities, we should obtain the same general formula for the transformation of rectangular coordinates (with the same geometrical meanings of the coefficients λ, μ, ν) as that which Mr Cayley has deduced, with a similar view, but by a different process, and has published, with other 'Results relating to Quaternions,' *Phil. Mag.* vol. XXVI (1845), pp. 141–5.

The present writer desires to return his sincere acknowledgments to Mr Cayley for the attention which he has given to the Papers on Quaternions, published in the above-mentioned Magazine: and gladly the operations here indicated being thus sure to make no change in the part β_1, which is in the direction of the axis of rotation, but to cause the other part β_2 to revolve round that axis α through an angle $=a$. Again, let the same line β' revolve round a new axis of rotation denoted by a new vector unit α', through a new angle a', into a new position β''; we shall have, in like manner,

$$\beta''=\left(\cos\frac{a'}{2}+\alpha'\sin\frac{a'}{2}\right)\beta'\left(\cos\frac{a'}{2}-\alpha'\sin\frac{a'}{2}\right); \tag{j}$$

and so on, for any number n of rotations. Let the last position of β be denoted by β_n; and since it can easily be proved, by the theory of the multiplication of quaternions, that the continued products which present themselves admit of being thus transformed:

$$\left.\begin{aligned}\left(\cos\frac{a^{(n-1)}}{2}+\alpha^{(n-1)}\sin\frac{a^{(n-1)}}{2}\right)\ldots\left(\cos\frac{a'}{2}+\alpha'\sin\frac{a'}{2}\right)\left(\cos\frac{a}{2}+\alpha\sin\frac{a}{2}\right)=\cos\frac{a_n}{2}+\alpha_n\sin\frac{a_n}{2};\\ \left(\cos\frac{a}{2}-\alpha\sin\frac{a}{2}\right)\left(\cos\frac{a'}{2}-\alpha'\sin\frac{a'}{2}\right)\ldots\left(\cos\frac{a^{n-1}}{2}-\alpha^{(n-1)}\sin\frac{a^{(n-1)}}{2}\right)=\cos\frac{a_n}{2}-\alpha_n\sin\frac{a_n}{2};\end{aligned}\right\} \tag{k}$$

in which α_n is a new vector unit, and a_n a new real angle, we find that the result of all the n rotations is of the form

$$\beta_n=\left(\cos\frac{a_n}{2}+\alpha_n\sin\frac{a_n}{2}\right)\beta\left(\cos\frac{a_n}{2}-\alpha_n\sin\frac{a_n}{2}\right). \tag{l}$$

It conducts, therefore, to the same final position which would have been attained from the initial position β, by a *single rotation* $=a_n$, round the *single axis* α_n; the amount and axis of this resultant rotation being determined by either of the two equations of transformation (k), and being independent of the direction of the line β which was operated on, so that they are the same for all lines of the body.

If the present results be combined with the theorem marked (R), in the account, printed in the *Proceedings of the Academy*, of the remarks* made by the Author in November 1843, it will at once be seen that if the several axes of rotation be considered as terminating in the points of a spherical polygon, and if the angles of rotation be equal respectively to the doubles of the angles of this polygon (and be taken with proper signs or directions, determined by those angles), then the total effects of all these rotations will vanish; or, in other words, the body will at last be brought back to the position from which it set out.

Finally, it may be mentioned that the author is in possession of a general method for expressing by quaternions the tangent planes and normals to curved surfaces; and that in applying this method to find the cone of tangents enveloping a given sphere, and drawn from a given point, the geometrical impossibility of the problem, when the point is an internal one, is expressed by the square of a vector becoming in this case positive.

recognizes his priority, as respects the printing of the formula just now referred to. But while he conceives it to be very likely that Mr Cayley, who had previously published in the *Cambridge Mathematical Journal* some elegant researches on the rotation of bodies, may have perceived, not only independently, but at an earlier date than he did himself, the manner of applying quaternions to represent such a rotation; he yet hopes that he may be allowed to mention, that a formula differing only slightly in its notation from the formula (i) of the present abstract, with the corollaries there drawn respecting the composition of successive finite rotations, had been exhibited to his friend and brother Professor, the Rev. Charles Graves, of Trinity College, Dublin, in an early part of the month (October 1844), which preceded that communication to the Academy, of which an account is given above. [See VII, Note A.]

[See A. Cayley, *loc. cit.*, 'The discovery of the formula $q(ix+jy+kz)q^{-1}=ix'+jy'+kz'$ as expressing a rotation, was made by Sir W. R. Hamilton some months previous to the date of this paper.']

* [See V.]