XXVIII

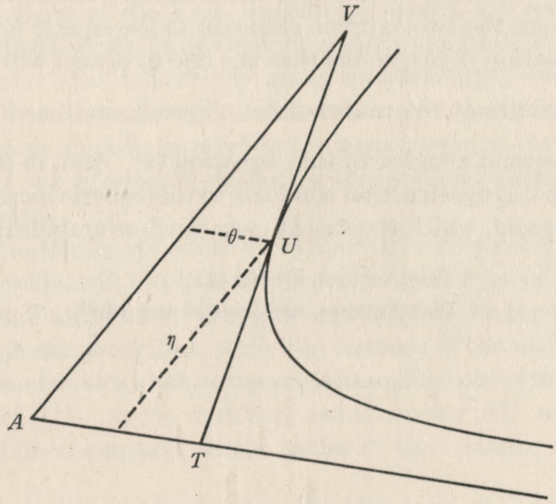
ON THE CONSTRUCTION OF THE ELLIPSOID BY TWO SLIDING SPHERES

Communicated 23 April 1849.

[*Proc. Roy. Irish Acad.* vol. IV (1850), pp. 341–2.]

The following extract of a letter from Sir William Rowan Hamilton to the Rev. Charles Graves was read to the Academy:

'If I had been more at leisure when last writing, I should have remarked that besides the construction of the ellipsoid by the two *sliding spheres*, which, in fact, led me last summer to an



equation nearly the same as that lately submitted to the Academy,* a simple interpretation may be given to the equation,

$$\mathrm{TV}\frac{\eta\rho-\rho\theta}{\mathrm{U}(\eta-\theta)}=\theta^2-\eta^2; \tag{1}$$

which may also be thus written,

$$\mathrm{TV}\frac{\rho\eta-\theta\rho}{\eta-\theta}=\frac{\theta^2-\eta^2}{\mathrm{T}(\eta-\theta)}. \tag{2}$$

'At an umbilic U, draw a tangent TUV to the focal hyperbola, meeting the asymptotes in T and V; then I can shew *geometrically*, as also in other ways—what, indeed, is likely enough to be known—that the sides of the triangle TAV are, as respects their *lengths*,

$$\overline{AV}=a+c; \quad \overline{AT}=a-c; \quad \overline{TV}=2b. \tag{3}$$

Now my η and θ are precisely the halves of the sides AV and AT of this triangle; or they are the two coordinates of the umbilic U, referred to the two asymptotes, when *directions* as well

* [See XXVI and XXII.]

as lengths are attended to. This explains several of my formulae, and accounts for the remarkable circumstance that we can pass to a *confocal surface*, by changing η and θ to $t^{-1}\eta$ and $t\theta$ respectively, where t is a scalar.

'Again, we have, identically,

$$\mathrm{V}\frac{\rho\eta-\theta\rho}{\eta-\theta}=\rho_1+\rho_2; \tag{4}$$

if for conciseness we write

$$\rho_1=(\eta-\theta)^{-1}\mathrm{S}.(\eta-\theta)\rho; \tag{5}$$

$$\rho_2=\mathrm{V}.(\eta-\theta)^{-1}\mathrm{V}.(\eta+\theta)\rho. \tag{6}$$

But ρ_1 is the perpendicular from the centre A of the ellipsoid on the plane of a circular section, through the extremity of any vector or semidiameter ρ; and ρ_2 may be shewn (by a process similar to that which I used to express Mac Cullagh's mode of generation)* to be a radius of that circular section, multiplied by the scalar coefficient $\mathrm{S}.(n-\theta)^{-1}(\eta+\theta)$, which is equal to

$$\frac{\theta^2-\eta^2}{-(\eta-\theta)^2}=\frac{\mathrm{T}\eta^2-\mathrm{T}\theta^2}{\mathrm{T}(\eta-\theta)^2}=\frac{ac}{b^2}. \tag{7}$$

If, then, from the foot of the perpendicular let fall, as above, on the plane of a circular section, we draw a right line in that plane, which bears to the radius of that section the constant ratio of the rectangle (ac) under the two extreme semiaxes to the square (b^2) of the mean semiaxis of the ellipsoid, the equation (2) expresses that the line so drawn will terminate on a spheric surface, which has its centre at the centre of the ellipsoid, and has its radius $=\frac{ac}{b}$; this last being the value of the second member of that equation (2). And, in fact, it is not difficult to prove *geometrically* that this construction conducts to this spheric locus, namely, to the sphere concentric with the ellipsoid, which touches at once the four umbilicar tangent planes.'†

* [See *Lectures*, article 441.]

† [See *Lectures*, articles 496 and 499.]