XLIX

INQUIRY INTO THE VALIDITY OF A METHOD RECENTLY PROPOSED BY GEORGE B. JERRARD, ESQ.,* FOR TRANSFORMING AND RESOLVING EQUATIONS OF ELEVATED DEGREES

[*British Assoc. Report*, 1837, pp. 295–348.]

[1]. It is well known that the result of the elimination of x, between the general equation of the m^{th} degree,

$$X = x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + Dx^{m-4} + Ex^{m-5} + \&c. = 0 \tag{1}$$

and an equation of the form $$y = f(x), \tag{2}$$

(in which $f(x)$ denotes any rational function of x, or, more generally, any function which admits of only one value for any one value of x,) is a new or transformed equation of the m^{th} degree, which may be thus denoted,

$$\{y - f(x_1)\}\{y - f(x_2)\} \dots \{y - f(x_m)\} = 0, \tag{3}$$

$x_1, x_2, \dots, x_m$ denoting the m roots of the proposed equation; or, more concisely, thus,

$$Y = y^m + A'y^{m-1} + B'y^{m-2} + C'y^{m-3} + D'y^{m-4} + E'y^{m-5} + \&c. = 0, \tag{4}$$

the coefficients A', B', C', &c., being connected with the values $f(x_1), f(x_2)$, &c., by the relations,

$$\left.\begin{aligned} -A' &= f(x_1) + f(x_2) + \&c. + f(x_m), \\ +B' &= f(x_1)f(x_2) + f(x_1)f(x_3) + f(x_2)f(x_3) + \&c. + f(x_{m-1})f(x_m), \\ -C' &= f(x_1)f(x_2)f(x_3) + \&c. \end{aligned}\right\} \tag{5}$$

And it has been found possible, in several known instances, to assign such a form to the function $f(x)$ or y, that the new or transformed equation, $Y = 0$, shall be less complex or easier to resolve, than the proposed or original equation $X = 0$. For example, it has long been known that by assuming

$$y = f(x) = \frac{A}{m} + x, \tag{6}$$

one term may be taken away from the general equation (1); that general equation being changed into another of the form $$Y = y^m + B'y^{m-2} + C'y^{m-3} + \&c. = 0, \tag{7}$$

in which there occurs no term proportional to y^{m-1}, the condition

$$A' = 0 \tag{8}$$

being satisfied; and Tschirnhausen discovered† that by assuming

$$y = f(x) = P + Qx + x^2, \tag{9}$$

* [See XLIX, p. 516, footnote.]

† [In 1683. See G. Salmon, *Lessons introductory to the Higher Algebra* (3rd ednn), Dublin, 1876, pp. 298–392, containing a section on the Tschirnhausen Transformation, and bibliography.]

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and by determining P and Q so as to satisfy two equations which can be assigned, and which are respectively of the first and second degrees, it is possible to fulfil the condition

$$B'=0, \tag{10}$$

along with the condition $A'=0$, (8)

and therefore to *take away two terms* at once from the general equation of the m^{th} degree; or, in other words, to change that equation (1) to the form

$$Y=y^m+C'y^{m-3}+D'y^{m-4}+\&c.=0, \tag{11}$$

in which there occurs no term proportional either to y^{m-1} or to y^{m-2}. But if we attempted to take away *three terms* at once, from the general equation (1), or to reduce it to the form

$$Y=y^m+D'y^{m-4}+E'y^{m-5}+\&c.=0, \tag{12}$$

(in which there occurs no term proportional to y^{m-1}, y^{m-2}, or y^{m-3},) by assuming, according to the same analogy,

$$y=P+Qx+Rx^2+x^3, \tag{13}$$

and then determining the three coefficients P, Q, R, so as to satisfy the three conditions

$$A'=0, \quad (8)$$

$$B'=0, \quad (10)$$

and

$$C'=0, \tag{14}$$

we should be conducted, by the law (5) of the composition of the coefficients A', B', C', to a system of three equations, of the 1st, 2nd, and 3rd degrees, between the three coefficients P, Q, R; and consequently, by elimination, in general, to a final equation of the 6th degree, which the known methods are unable to resolve. Still less could we take away, in the present state of algebra, *four terms* at once from the general equation of the m^{th} degree, or reduce it to the form

$$Y=y^m+E'y^{m-5}+\&c.=0, \tag{15}$$

by assuming an expression with four coefficients,

$$y=P+Qx+Rx^2+Sx^3+x^4; \tag{16}$$

because the four conditions,

$$A'=0, \quad (8)$$

$$B'=0, \quad (10)$$

$$C'=0, \quad (14)$$

and

$$D'=0, \tag{17}$$

would be, with respect to these four coefficients, P, Q, R, S, of the 1st, 2nd, 3rd, and 4th degrees, and therefore would in general conduct by elimination to an equation of the 24th degree. In like manner, if we attempted to take away the 2nd, 3rd, and 5th terms (instead of the 2nd, 3rd, and 4th) from the general equation of the m^{th} degree, or to reduce it to the form

$$y^m+C'y^{m-3}+E'y^{m-5}+\&c.=0, \tag{18}$$

so as to satisfy the three conditions (8), (10) and (17),

$$A'=0, \quad B'=0, \quad D'=0,$$

by assuming

$$y=P+Qx+Rx^2+x^3, \quad (13)$$

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we should be conducted to a final equation of the 8th degree; and if we attempted to satisfy these three other conditions

$$A'=0, \quad (8)$$

$$C'=0, \quad (14)$$

and

$$D'-\alpha B'^2=0, \tag{19}$$

(in which α is any known or assumed number,) so as to transform the general equation (1) to the following,

$$Y=y^m+B'y^{m-2}+\alpha B'^2y^{m-4}+E'y^{m-5}+\&c.=0, \tag{20}$$

by the same assumption (13), we should be conducted by elimination to an equation of condition of the 12th degree. It might, therefore, have been naturally supposed that each of these four transformations, (12), (15), (18), (20), of the equation of the m^{th} degree, was in general impossible to be effected in the present state of algebra. *Yet Mr Jerrard has succeeded in effecting them all*, by suitable assumptions of the function y or $f(x)$, without being obliged to resolve any equation higher than the fourth degree, and has even effected the transformation (12) without employing biquadratic equations. His method may be described as consisting in *rendering the problem indeterminate*, by assuming an expression for y with a number of disposable coefficients greater than the number of conditions to be satisfied; and in employing this indeterminateness to *decompose certain of the conditions* into others, for the purpose of *preventing that elevation of degree* which would otherwise result from the eliminations. This method is valid, in general, when the proposed equation is itself of a *sufficiently elevated degree*; but I have found that when the exponent m of that degree is *below a certain minor limit*, which is different for different transformations, (being $=5$ for the first, $=10$ for the second, $=5$ for the third, and $=7$ for the fourth of those already designated as the transformations (12), (15), (18) and (20),) the processes proposed by Mr Jerrard conduct in general to an expression for the new variable y which is *a multiple of the proposed evanescent polynome* X of the m^{th} degree in x; and that on this account these processes, although *valid as general transformations of the equation of the* m^{th} *degree*, become in general *illusory* when they are applied to *resolve equations of the fourth and fifth degrees*, by reducing them to the binomial form, or by reducing the equation of the fifth degree to the known solvible form of De Moivre. An analogous process, suggested by Mr Jerrard, for *reducing the general equation of the sixth to that of the fifth degree*, and a more general method of the same kind for resolving equations of higher degrees, appear to me to be in general, for a similar reason, illusory. Admiring the great ingenuity and talent exhibited in Mr Jerrard's researches, I come to this conclusion with regret, but believe that the following discussion will be thought to establish it sufficiently.

[2]. To begin with the transformation (12), or the taking away of the second, third and fourth terms at once from the general equation of the m^{th} degree, Mr Jerrard effects this transformation by assuming generally an expression with *seven* terms,

$$y=f(x)=\Lambda'x^{\lambda'}+\Lambda''x^{\lambda''}+\Lambda'''x^{\lambda'''}+\mathrm{M}'x^{\mu'}+\mathrm{M}''x^{\mu''}+\mathrm{M}'''x^{\mu'''}+\mathrm{M}^{\mathrm{IV}}x^{\mu^{\mathrm{IV}}} \tag{21}$$

the seven unequal exponents $\lambda'\ \lambda''\ \lambda'''\ \mu'\ \mu''\ \mu'''\ \mu^{\mathrm{IV}}$ being chosen at pleasure out of the indefinite line of integers

$$0,\ 1,\ 2,\ 3,\ 4,\quad \&c. \tag{22}$$

and the seven coefficients $\Lambda'\ \Lambda''\ \Lambda'''\ \mathrm{M}'\ \mathrm{M}''\ \mathrm{M}'''\ \mathrm{M}^{\mathrm{IV}}$, or rather their six ratios

$$\frac{\Lambda'}{\Lambda'''},\quad \frac{\Lambda''}{\Lambda'''},\quad \frac{\mathrm{M}'}{\mathrm{M}^{\mathrm{IV}}},\quad \frac{\mathrm{M}''}{\mathrm{M}^{\mathrm{IV}}},\quad \frac{\mathrm{M}'''}{\mathrm{M}^{\mathrm{IV}}},\quad \frac{\Lambda'''}{\mathrm{M}^{\mathrm{IV}}} \tag{23}$$

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being determined so as to satisfy the three conditions

$$A'=0, \quad (8)$$

$$B'=0, \quad (10)$$

$$C'=0, \quad (14)$$

without resolving any equation higher than the third degree, by a process which may be presented as follows.

In virtue of the assumption (21) and of the law (5) of the composition of the coefficients A', B', C', it is easy to perceive that those three coefficients are rational and integral and homogeneous functions of the seven quantities Λ' Λ'' Λ''' M' M'' M''' M^{IV}, of the dimensions one, two, and three respectively; and therefore that A' and B' may be developed or decomposed into parts as follows:

$$A'=A'_{1,0}+A'_{0,1}, \tag{24}$$

$$B=B'_{2,0}+B'_{1,1}+B'_{0,2}, \tag{25}$$

the symbol $A'_{h,i}$ or $B'_{h,i}$ denoting here a rational and integral function of Λ', Λ'', Λ''', M', M'', M''', M^{IV}, which is homogeneous of the degree h with respect to Λ', Λ'', Λ''', and of the degree i with respect to M', M'', M''', M^{IV}. If then we first determine the two ratios of Λ', Λ'', Λ''', so as to satisfy the two conditions

$$A'_{1,0}=0, \tag{26}$$

$$B'_{2,0}=0, \tag{27}$$

and afterwards determine the three ratios of M', M'', M''', M^{IV}, so as to satisfy the three other conditions

$$A'_{0,1}=0, \tag{28}$$

$$B'_{1,1}=0, \tag{29}$$

$$B'_{0,2}=0, \tag{30}$$

we shall have decomposed the two conditions (8) and (10), namely,

$$A'=0, \quad B'=0,$$

into five others, and shall have satisfied these five by means of the five first ratios of the set (23), namely

$$\frac{\Lambda'}{\Lambda'''}, \quad \frac{\Lambda''}{\Lambda'''}, \quad \frac{\mathrm{M}'}{\mathrm{M}^{\mathrm{IV}}}, \quad \frac{\mathrm{M}''}{\mathrm{M}^{\mathrm{IV}}}, \quad \frac{\mathrm{M}'''}{\mathrm{M}^{\mathrm{IV}}}, \tag{31}$$

without having yet determined the remaining ratio of that set, namely

$$\frac{\Lambda'''}{\mathrm{M}^{\mathrm{IV}}}; \tag{32}$$

which remaining ratio can then in general be chosen so as to satisfy the remaining condition

$$C'=0,$$

without our being obliged, in any part of the process, to resolve any equation higher than the third degree. And such, in substance, is Mr Jerrard's general process for taking away the second, third, and fourth terms at once from the equation of the m^{th} degree, although he has expressed it in his published *Researches* by means of a new and elegant *notation of symmetric functions*, which it has not seemed necessary here to introduce, because the argument itself can be sufficiently understood without it.

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[3]. On considering this process with attention, we perceive that it consists essentially of two principal parts, the one conducting to an expression of the form

$$y = f(x) = \Lambda''' \phi(x) + \mathrm{M}^{\mathrm{IV}} \chi(x), \tag{33}$$

which satisfies the two conditions $A' = 0, \quad B' = 0,$

the functions $\phi(x)$ and $\chi(x)$ being determined, namely,

$$\phi(x) = \frac{\Lambda'}{\Lambda'''} x^{\lambda'} + \frac{\Lambda''}{\Lambda'''} x^{\lambda''} + x^{\lambda'''}, \tag{34}$$

and
$$\chi(x) = \frac{\mathrm{M}'}{\mathrm{M}^{\mathrm{IV}}} x^{\mu'} + \frac{\mathrm{M}''}{\mathrm{M}^{\mathrm{IV}}} x^{\mu''} + \frac{\mathrm{M}'''}{\mathrm{M}^{\mathrm{IV}}} x^{\mu'''} + x^{\mu^{\mathrm{IV}}}, \tag{35}$$

but the multipliers Λ''' and M^{IV} being arbitrary, and the other part of the process determining afterwards the ratio of those two multipliers so as to satisfy the remaining condition

$$C' = 0.$$

And hence it is easy to see that if we would exclude those useless cases in which the ultimate expression for the new variable y, or for the function $f(x)$, is a multiple of the proposed evanescent polynome X of the m^{th} degree in x, we must, in general, exclude the cases in which the two functions $\phi(x)$ and $\chi(x)$, determined in the first part of the process, are connected by a relation of the form

$$\chi(x) = a\phi(x) + \lambda X, \tag{36}$$

a being any constant multiplier, and λX any multiple of X. For in all such cases the expression (33), obtained by the first part of the process, becomes

$$y = f(x) = (\Lambda''' + a\mathrm{M}^{\mathrm{IV}})\,\phi(x) + \lambda \mathrm{M}^{\mathrm{IV}} X; \tag{37}$$

and since this gives, by the nature of the roots $x_1, \ldots, x_m$,

$$f(x_1) = (\Lambda''' + a\mathrm{M}^{\mathrm{IV}})\,\phi(x_1), \ldots, f(x_m) = (\Lambda''' + a\mathrm{M}^{\mathrm{IV}})\,\phi(x_m), \tag{38}$$

we find, by the law (5) of the composition of the coefficients of the transformed equation in y,

$$C' = c(\Lambda''' + a\mathrm{M}^{\mathrm{IV}})^3, \tag{39}$$

the multiplier c being known, namely,

$$c = -\phi(x_1)\,\phi(x_2)\,\phi(x_3) - \phi(x_1)\,\phi(x_2)\,\phi(x_4) - \&\text{c}. \tag{40}$$

and being in general different from 0, because the three first of the seven terms of the expression (21) for y can only accidentally suffice to resolve the original problem; so that when we come, in the second part of the process, to satisfy the condition

$$C' = 0,$$

we shall, in general, be obliged to assume

$$(\Lambda''' + a\mathrm{M}^{\mathrm{IV}})^3 = 0, \tag{41}$$

that is,
$$\Lambda''' + a\mathrm{M}^{\mathrm{IV}} = 0; \tag{42}$$

and consequently the expression (37) for y reduces itself ultimately to the form which we wished to exclude, since it becomes

$$y = \lambda \mathrm{M}^{\mathrm{IV}} X. \tag{43}$$

Reciprocally, it is clear that the second part of the process, or the determination of the ratio of Λ''' to M^{IV} in the expression (33), cannot conduct to this useless form for y unless the two functions $\phi(x)$ and $\chi(x)$ are connected by a relation of the kind (36); because, when we equate the expression (33) to any multiple of X, we establish thereby a relation of that kind between

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those two functions. We must therefore endeavour to avoid those cases, and we need avoid those only, which conduct to this relation (36), and we may do so in the following manner.

[4]. Whatever positive integer the exponent ν may be, power x^ν may always be identically equated to an expression of this form,

$$x^\nu = s_0^{(\nu)} + s_1^{(\nu)} x + s_2^{(\nu)} x^2 + \ldots + s_{m-1}^{(\nu)} x^{m-1} + L^{(\nu)} X, \tag{44}$$

$s_0^{(\nu)}, s_1^{(\nu)}, s_2^{(\nu)}, \ldots, s_{m-1}^{(\nu)}$ being certain functions of the exponent ν, and of the coefficients $A, B, C, \ldots$ of the proposed polynome X, while $L^{(\nu)}$ is a rational and integral function of x, which is $=0$ if ν be less than the exponent m of the degree of that proposed polynome X, but otherwise is of the degree $\nu - m$. In fact, if we divide the power x^ν by the polynome X, according to the usual rules of the integral division of polynomes, so as to obtain an integral quotient and an integral remainder, the integral quotient may be denoted by $L^{(\nu)}$, and the integral remainder may be denoted by

$$s_0^{(\nu)} + s_1^{(\nu)} x + s_2^{(\nu)} x^2 + \ldots + s_{m-1}^{(\nu)} x^{m-1},$$

and thus the identity (44) may be established. It may be noticed that the m coefficients $s_0^{(\nu)}, s_1^{(\nu)}, \ldots, s_{m-1}^{(\nu)}$, may be considered as symmetric functions of the m roots $x_1, x_2, \ldots, x_m$ of the proposed equation $X=0$, which may be determined by the m relations,

$$\left.\begin{aligned} x_1^\nu &= s_0^{(\nu)} + s_1^{(\nu)} x_1 + s_2^{(\nu)} x_1^2 + \ldots + s_{m-1}^{(\nu)} x_1^{m-1}, \\ x_2^\nu &= s_0^{(\nu)} + s_1^{(\nu)} x_2 + s_2^{(\nu)} x_2^2 + \ldots + s_{m-1}^{(\nu)} x_2^{m-1}, \\ &\ldots\ldots\ldots \\ x_m^\nu &= s_0^{(\nu)} + s_1^{(\nu)} x_m + s_2^{(\nu)} x_m^2 + \ldots + s_{m-1}^{(\nu)} x_m^{m-1}. \end{aligned}\right\} \tag{45}$$

These symmetric functions of the roots possess many other important properties, but it is unnecessary here to develop them.

Adopting the notation (44), we may put, for abridgment,

$$\left.\begin{aligned} \Lambda' s_0^{(\lambda')} + \Lambda'' s_0^{(\lambda'')} + \Lambda''' s_0^{(\lambda''')} &= p_0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Lambda' s_{m-1}^{(\lambda')} + \Lambda'' s_{m-1}^{(\lambda'')} + \Lambda''' s_{m-1}^{(\lambda''')} &= p_{m-1}', \end{aligned}\right\} \tag{46}$$

$$\left.\begin{aligned} \mathrm{M}' s_0^{(\mu')} + \mathrm{M}'' s_0^{(\mu'')} + \mathrm{M}''' s_0^{(\mu''')} + \mathrm{M}^{\mathrm{IV}} s_0^{(\mu^{\mathrm{IV}})} &= p_0', \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mathrm{M}' s_{m-1}^{(\mu')} + \mathrm{M}'' s_{m-1}^{(\mu'')} + \mathrm{M}''' s_{m-1}^{(\mu''')} + \mathrm{M}^{\mathrm{IV}} s_{m-1}^{(\mu^{\mathrm{IV}})} &= p_{m-1}', \end{aligned}\right\} \tag{47}$$

$$\Lambda' L^{(\lambda')} + \Lambda'' L^{(\lambda'')} + \Lambda''' L^{(\lambda''')} = \Lambda, \tag{48}$$

$$\mathrm{M}' L^{(\mu')} + \mathrm{M}'' L^{(\mu'')} + \mathrm{M}''' L^{(\mu''')} + \mathrm{M}^{\mathrm{IV}} L^{(\mu^{\mathrm{IV}})} = \mathrm{M}, \tag{49}$$

$$\Lambda + \mathrm{M} = L \tag{50}$$

and then the two parts, of which the expression for y is composed, will take the forms

$$\Lambda' x^{\lambda'} + \Lambda'' x^{\lambda''} + \Lambda''' x^{\lambda'''} = p_0 + p_1 x + \ldots + p_{m-1} x^{m-1} + \Lambda X, \tag{51}$$

$$\mathrm{M}' x^{\mu'} + \mathrm{M}'' x^{\mu''} + \mathrm{M}''' x^{\mu'''} + \mathrm{M}^{\mathrm{IV}} x^{\mu^{\mathrm{IV}}} = p_0' + p_1' x + \ldots + p_{m-1}' x^{m-1} + \mathrm{M} X, \tag{52}$$

and the expression itself will become

$$y = f(x) = p_0 + p_0' + (p_1 + p_1') x + \ldots + (p_{m-1} + p_{m-1}') x^{m-1} L X. \tag{53}$$

At the same time we see that the case to be avoided, for the reason lately assigned, is the case

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of proportionality of $p'_0, p'_1, \ldots, p'_{m-1}$, to $p_0, p_1, \ldots, p_{m-1}$. It is therefore convenient to introduce these new abbreviations,

$$\frac{p'_{m-1}}{p_{m-1}} = p, \tag{54}$$

and
$$p'_0 - pp_0 = q_0, \quad p'_1 - pp_1 = q_1, \ldots, p'_{m-2} - pp_{m-2} = q_{m-2}; \tag{55}$$

for thus we obtain the expressions

$$p'_0 = q_0 + pp_0, \quad p'_1 = q_1 + pp_1, \ldots, p'_{m-2} = q_{m-2} + pp_{m-2}, \quad p'_{m-1} = pp_{m-1}, \tag{56}$$

and
$$y = f(x) = (1+p)(p_0 + p_1 x + \ldots + p_{m-1}x^{m-1}) + q_0 + q_1 x + \ldots + q_{m-2}x^{m-2} + LX; \tag{57}$$

and we have only to take care that the $m-1$ quantities, $q_0, q_1, \ldots, q_{m-2}$ shall not all vanish. Indeed, it is tacitly supposed in (54) that p_{m-1} does not vanish; but it must be observed that Mr Jerrard's method itself essentially supposes that the function $\Lambda' x^{\lambda'} + \Lambda'' x^{\lambda''} + \Lambda''' x^{\lambda'''}$ is not any multiple of the evanescent polynome X, and therefore that *at least some one* of the m quantities $p_0, p_1, \ldots, p_{m-1}$ is different from 0; now the spirit of the definitional assumptions here made, and of the reasonings which are to be founded upon them, requires only that *some one* such non-evanescent quantity p_i out of this set $p_0, p_1, \ldots, p_{m-1}$ should be made the denominator of a fraction like (54), $\dfrac{p'_i}{p_i} = p$, and that thus some one term $q_i x^i$ should be taken away out of the difference of the two polynomes $p'_0 + p'_1 x + \ldots$ and $p(p_0 x + p_1 x + \ldots)$; and it is so easy to make this adaptation, whenever the occasion may arise, that I shall retain in the present discussion, the assumptions (54) (55), instead of writing p_i for p_{m-1}.

The expression (57) for $f(x)$, combined with the law (5) of the composition of the coefficients A' and B', shows that these two coefficients of the transformed equation in y may be expressed as follows,

$$A' = (1+p)A''_{1,0} + A''_{0,1}, \tag{58}$$

and
$$B' = (1+p)^2 B''_{2,0} + (1+p)B''_{1,1} + B''_{0,2}; \tag{59}$$

$A''_{h,i}$ and $B''_{h,i}$ being each a rational and integral function of the $2m-1$ quantities $p_0, p_1, \ldots, p_{m-1}$, $q_0, q_1, \ldots, q_{m-2}$, which is independent of the quantity p and of the form of the function L, and is homogeneous of the dimension h with respect to $p_0, p_1, \ldots, p_{m-1}$, and of the dimension i with respect to $q_0, q_1, \ldots, q_{m-2}$. Comparing these expressions (58) and (59) with the analogous expressions (24) and (25), (with which they would of necessity identically coincide, if we were to return from the present to the former symbols, by substituting, for $p, p_0, p_1, \ldots, p_{m-1}, \ldots$ $q_0, q_1, \ldots, q_{m-2}$, their values as functions of Λ', Λ'', Λ''', M′ M″, M‴, M$^{\text{IV}}$, deduced from the equations of definition (54) (55) and (46) (47),) we find these identical equations:

$$A'_{1,0} = A''_{1,0}; \quad A'_{0,1} = pA''_{1,0} + A''_{0,1}; \tag{60}$$

and
$$B'_{2,0} = B''_{2,0}; \quad B'_{1,1} = 2pB''_{2,0} + B''_{1,1}; \quad B'_{0,2} = p^2 B''_{2,0} + pB''_{1,1} + B''_{0,2}; \tag{61}$$

observing that whatever may be the dimension of any part of A' or B', with respect to the m new quantities $p, q_0, q_1, \ldots, q_{m-2}$, the same is the dimension of that part, with respect to the four old quantities M′, M″, M‴, M$^{\text{IV}}$.

The system of the five conditions (26) (27) (28) (29) (30) may therefore be transformed to the following system,

$$A''_{1,0} = 0, \quad B''_{2,0} = 0, \tag{62}$$

$$A''_{0,1} = 0, \quad B''_{1,1} = 0, \quad B''_{0,2} = 0; \tag{63}$$

and may in general be treated as follows. The two conditions (62), combined with the m equations of definition (46), will in general determine the $m+2$ ratios of the $m+3$ quantities

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$p_0, p_1, \ldots, p_{m-1}$, Λ', Λ'', Λ'''; and then the three conditions (63), combined with the m equations of definition (47), and with the m other equations (56), will in general determine the $2m+3$ ratios of the $2m+4$ quantities $q_0, q_1, \ldots, q_{m-2}, pp_{m-1}, p'_0, p'_1, \ldots, p'_{m-1}$, M$'$, M$''$, M$'''$, M$^{\text{IV}}$; after which, the ratio of Λ''' to M$^{\text{IV}}$ is to be determined, as before, so as to satisfy the remaining condition $C'=0$. But because the last-mentioned system, of $2m+3$ homogeneous equations, (63) (56) (47), between $2m+4$ quantities, involves, as a part of itself, the system (63) of *three homogeneous equations* (rational and integral) *between* $m-1$ *quantities*, $q_0, q_1, \ldots, q_{m-2}$, we see that it will in general conduct to the result which we wished to exclude, namely, the simultaneous vanishing of all those quantities,

$$q_0 = 0, \quad q_1 = 0, \ \ldots, \ q_{m-2} = 0, \tag{64}$$

unless their number $m-1$ *be greater than* 3, that is, *unless the degree m of the proposed equation* (1) *be at least equal to the minor limit* FIVE. It results, then, from this discussion, that *the transformation by which Mr Jerrard has succeeded in taking away three terms at once from the general equation of the* m^{th} *degree, is not in general applicable when that degree is lower than the* 5*th*; in such a manner that it is in general inadequate to reduce the biquadratic equation

$$x^4 + Ax^3 + Bx^2 + Cx^3 + D = 0, \tag{65}$$

to the binomial form

$$y^4 + D' = 0, \tag{66}$$

except by the useless assumption

$$y = L(x^4 + Ax^3 + Bx^2 + Cx^3 + D), \tag{67}$$

which gives

$$y^4 = 0. \tag{68}$$

However, the foregoing discussion may be considered as *confirming the adequacy of the method to reduce the general equation of the* 5*th degree*,

$$x^5 + Ax^4 + Bx^3 + Cx^2 + Dx + E = 0, \tag{69}$$

to the trinomial form

$$y^5 + D'y + E' = 0; \tag{70}$$

and to effect the analogous transformation (12) *for equations of all higher degrees*: an unexpected and remarkable result, which is one of Mr Jerrard's principal discoveries.

[5]. Analogous remarks apply to the process proposed by the same mathematician for taking away the second, third and fifth terms at once from the general equation (1), so as to reduce that equation to the form (18). This process agrees with the foregoing in the whole of its first part, that is, in the assumption of the form (21) for $f(x)$, and in the determination of the five ratios (31) so as to satisfy the two conditions $A'=0$, $B'=0$, by satisfying the five others (26) (27) (28) (29) (30), into which those two may be decomposed; and the difference is only in the second part of the process, that is, in determining the remaining ratio (32) so as to satisfy the condition $D'=0$, instead of the condition $C'=0$, by resolving a biquadratic instead of a cubic equation. The discussion which has been given of the former process of transformation adapts itself therefore, with scarcely any change, to the latter process also, and shows that this process can only be applied with success, in general, to equations of the fifth and higher degrees. It is, however, a remarkable result that it can be applied generally to such equations, and especially that the general equation of the fifth degree may be brought by it to the following trinomial form,

$$y^5 + C'y^2 + E' = 0, \tag{71}$$

as it was reduced, by the former process, to the form

$$y^5 + D'y + E' = 0. \quad (70)$$

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Mr Jerrard, to whom the discovery of these transformations is due, has remarked that by changing y to $\frac{1}{z}$ we get two other trinomial forms to which *the general equation of the fifth degree* may be reduced; so that, *in any future researches respecting the solution of such equations, it will be permitted to set out with any one of these four trinomial forms*,

$$x^5+Ax^4+E=0, \quad x^5+Bx^3+E=0, \quad x^5+Cx^2+E=0, \quad x^5+Dx+E=0, \tag{72}$$

in which the intermediate coefficient A or B or C or D may evidently be made equal to unity, or to any other assumed number different from zero. We may, for example, consider the difficulty of resolving the *general* equation of the fifth degree as reduced by Mr Jerrard's researches to the difficulty of resolving an equation of the form

$$x^5+x+E=0; \tag{73}$$

or of this other form,

$$x^5-x+E=0. \tag{74}$$

It is, however, important to remark that the coefficients of these new or transformed equations will often be imaginary, even when the coefficients of the original equation of the form (69) are real.*

[6]. In order to accomplish the transformation (20), (to the consideration of which we shall next proceed,) Mr Jerrard assumes, in general, an expression with *twelve* terms,

$$y=f(x)=\Lambda' x^{\lambda'}+\Lambda'' x^{\lambda''}+\Lambda''' x^{\lambda'''}+\mathrm{M}' x^{\mu'}+\mathrm{M}'' x^{\mu''}+\mathrm{M}''' x^{\mu'''}+\mathrm{M}^{\mathrm{IV}} x^{\mu^{\mathrm{IV}}}$$
$$+\mathrm{N}' x^{\nu'}+\mathrm{N}'' x^{\nu''}+\mathrm{N}''' x^{\nu'''}+\mathrm{N}^{\mathrm{IV}} x^{\nu^{\mathrm{IV}}}+\mathrm{N}^{\mathrm{V}} x^{\nu^{\mathrm{V}}}; \tag{75}$$

the twelve unequal exponents,

$$\lambda', \lambda'', \lambda''', \mu', \mu'', \mu''', \mu^{\mathrm{IV}}, \nu', \nu'', \nu''', \nu^{\mathrm{IV}}, \nu^{\mathrm{V}}, \tag{76}$$

being chosen at pleasure out of the indefinite line of integers (22); and the twelve coefficients,

$$\Lambda', \Lambda'', \Lambda''', \mathrm{M}', \mathrm{M}'', \mathrm{M}''', \mathrm{M}^{\mathrm{IV}}, \mathrm{N}', \mathrm{N}'', \mathrm{N}''', \mathrm{N}^{\mathrm{IV}}, \mathrm{N}^{\mathrm{V}}, \tag{77}$$

or rather their eleven ratios, which may be arranged and grouped as follows,

$$\frac{\Lambda'}{\Lambda'''}, \quad \frac{\Lambda''}{\Lambda'''}, \tag{78}$$

$$\frac{\mathrm{M}'}{\mathrm{M}^{\mathrm{IV}}}, \quad \frac{\mathrm{M}''}{\mathrm{M}^{\mathrm{IV}}}, \quad \frac{\mathrm{M}'''}{\mathrm{M}^{\mathrm{IV}}}, \tag{79}$$

$$\frac{\mathrm{N}'}{\mathrm{N}^{\mathrm{V}}}, \quad \frac{\mathrm{N}''}{\mathrm{N}^{\mathrm{V}}}, \quad \frac{\mathrm{N}'''}{\mathrm{N}^{\mathrm{V}}}, \quad \frac{\mathrm{N}^{\mathrm{IV}}}{\mathrm{N}^{\mathrm{V}}}, \tag{80}$$

$$\frac{\mathrm{M}^{\mathrm{IV}}}{\mathrm{N}^{\mathrm{V}}}, \tag{81}$$

$$\frac{\Lambda'''}{\mathrm{N}^{\mathrm{V}}}, \tag{82}$$

being then determined so as to satisfy the system of the three conditions

$$A'=0, \quad (8)$$
$$C'=0, \quad (14)$$
$$D'-\alpha B'^2=0, \quad (19)$$

by satisfying another system, composed of eleven equations, which are obtained by decom-

* [See Sylvester, *Mathematical Papers*, vol. IV, p. 536, where it is proved that this depends solely on the intrinsic character of the original equation.]

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posing the condition (8) into three, and the condition (14) into seven new equations, as follows. By the law (5) of the formation of the four coefficients A', B', C', D', and by the assumed expression (75), those four coefficients are rational and integral and homogeneous functions, of the first, second, third, and fourth degrees, of the twelve coefficients (77); and therefore, when these latter coefficients are distributed into three groups, one group containing Λ', Λ'', Λ''', another group containing M', M'', M''', M^{IV}, and the third group containing N', N'', N''', N^{IV}, N^{V}, the coefficient or function A' may be decomposed into three parts,

$$A' = A'_{1,0,0} + A'_{0,1,0} + A'_{0,0,1}, \qquad (83)$$

and the coefficient or function C' may be decomposed in like manner into ten parts,

$$C' = C'_{3,0,0} + C'_{2,1,0} + C'_{2,0,1} + C'_{1,2,0} + C'_{1,1,1} + C'_{1,0,2} + C'_{0,3,0} + C'_{0,2,1} + C'_{0,1,2} + C'_{0,0,3}, \qquad (84)$$

in which each of the symbols of the forms $A'_{h,i,k}$ and $C'_{h,i,k}$ denotes a rational and integral function of the twelve quantities (77); which function ($A'_{h,i,k}$ or $C'_{h,i,k}$) is also homogeneous of the dimension h with respect to the quantities Λ', Λ'', Λ''', of the dimension i with respect to the quantities M', M'', M''', M^{IV}, and of the dimension k with respect to the quantities N', N'', N''', N^{IV}, N^{V}. Accordingly Mr Jerrard decomposes the conditions $A' = 0$ and $C' = 0$ into ten others, which may be thus arranged:

$$A'_{1,0,0} = 0, \quad C'_{3,0,0} = 0; \qquad (85)$$

$$A'_{0,1,0} = 0, \quad C'_{2,1,0} = 0, \quad C'_{1,2,0} = 0; \qquad (86)$$

$$A'_{0,0,1} = 0, \quad C'_{2,0,1} = 0, \quad C'_{1,1,1} = 0, \quad C'_{1,0,2} = 0; \qquad (87)$$

$$C'_{0,3,0} + C'_{0,2,1} + C'_{0,1,2} + C'_{0,0,3} = 0; \qquad (88)$$

nine of the thirteen parts of the expressions (83) and (84) being made to vanish separately, and the sum of the other four parts being also made to vanish. He then determines the two ratios (78), so as to satisfy the two conditions (85); the three ratios (79), so as to satisfy the three conditions (86); the four ratios (80), so as to satisfy the four conditions (87); and the ratio (81), so as to satisfy the condition (88); all which determinations can in general be successively effected, without its being necessary to resolve any equation higher than the third degree. The first part of the process is now completed, that is, the two conditions (8) and (14),

$$A' = 0, \quad C' = 0,$$

are now both satisfied by an expression of the form

$$y = f(x) = \Lambda'''\phi(x) + \mathrm{N}^{\mathrm{V}}\chi(x), \qquad (89)$$

which is analogous to (33), and in which the functions $\phi(x)$ and $\chi(x)$ are known, but the multipliers Λ''' and N^{V} are arbitrary; and the second and only remaining part of the process consists in determining the remaining ratio (82), of Λ''' to N^{V}, by resolving an equation of the fourth degree, so as to satisfy the remaining condition,

$$D' - \alpha B'^2 = 0. \quad (19)$$

[7]. Such, then, (the notation excepted,) is Mr Jerrard's general process for reducing the equation of the m^{th} degree,

$$X = x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + Dx^{m-4} + Ex^{m-5} + \&\text{c.} = 0, \quad (1)$$

to the form $$Y = y^m + B'y^{m-2} + \alpha B'^2 y^{m-4} + E'y^{m-5} + \&\text{c.} = 0, \quad (20)$$

without resolving any auxiliary equation of a higher degree than the fourth. But, on considering this remarkable process with attention, we perceive that if we would avoid its

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becoming illusory, by conducting to an expression for y which is a multiple of the proposed polynome X, we must, in general, (for reasons analogous to those already explained in discussing the transformation (12),) exclude all those cases in which the functions $\phi(x)$ and $\chi(x)$, in the expression (89), are connected by a relation of the form

$$\chi(x) = a\phi(x) + \lambda X; \tag{36}$$

because, in all the cases in which such a relation exists, the first part of the process conducts to an expression of the form

$$y = (\Lambda''' + a\mathrm{N}^{\mathrm{V}})\,\phi(x) + \lambda \mathrm{N}^{\mathrm{V}} X, \tag{90}$$

and then the second part of the same process gives in general

$$(\Lambda''' + a\mathrm{N}^{\mathrm{V}})^4 = 0, \tag{91}$$

that is

$$\Lambda''' + a\mathrm{N}^{\mathrm{V}} = 0, \tag{92}$$

and ultimately

$$y = \lambda \mathrm{N}^{\mathrm{V}} X. \tag{93}$$

On the other hand, the second part of the process cannot conduct to this useless form for y, unless the first part of the process has led to functions, $\phi(x), \chi(x)$, connected by a relation of the form (36). This consideration suggests the introduction of the following new system of equations of definition.

$$\left.\begin{array}{l} \mathrm{N}' s_0^{(\nu')} + \mathrm{N}'' s_0^{(\nu'')} + \mathrm{N}''' s_0^{(\nu''')} + \mathrm{N}^{\mathrm{IV}} s_0^{(\nu^{\mathrm{IV}})} + \mathrm{N}^{\mathrm{V}} s_0^{(\nu^{\mathrm{V}})} = p_0'', \\ \mathrm{N}' s_{m-1}^{(\nu')} + \mathrm{N}'' s_{m-1}^{(\nu'')} + \mathrm{N}''' s_{m-1}^{(\nu''')} + \mathrm{N}^{\mathrm{IV}} s_{m-1}^{(\nu^{\mathrm{IV}})} + \mathrm{N}^{\mathrm{V}} s_{m-1}^{(\nu^{\mathrm{V}})} = p_{m-1}'', \end{array}\right\} \tag{94}$$

$$\mathrm{N}' L^{(\nu')} + \mathrm{N}'' L^{(\nu'')} + \mathrm{N}''' L^{(\nu''')} + \mathrm{N}^{\mathrm{IV}} L^{(\nu^{\mathrm{IV}})} + \mathrm{N}^{\mathrm{V}} L^{(\nu^{\mathrm{V}})} = \mathrm{N}, \tag{95}$$

$$\frac{p_{m-1}''}{p_{m-1}} = p', \tag{96}$$

$$p_0'' - p'p_0 = q_0', \quad p_1'' - p'p_1 = q_1', \ldots, p_{m-2}'' - p'p_{m-2} = q_{m-2}' \tag{97}$$

to be combined with the definitions (46), (47), (48), (49), (54), (55), and with the following, which may now be conveniently used instead of the definition (50),

$$\Lambda + \mathrm{M} + \mathrm{N} = L. \tag{98}$$

In this notation we shall have, as before,

$$p_0' = q_0 + pp_0, \quad p_1' = q_1 + pp_1, \ldots, p_{m-2}' = q_{m-2} + pp_{m-2}, \; p_{m-1}' = pp_{m-1}, \tag{56}$$

and shall also have the analogous expressions

$$p_0'' = q_0' + p'p_0, \quad p_1'' = q_1' + p'p_1, \ldots, p_{m-2}'' = q_{m-2}' + p'p_{m-2}, \quad p_{m-1}'' = p'p_{m-1}; \tag{99}$$

the expression (75) for y will become

$$y = f(x) = p_0 + p_0' + p_0'' + (p_1 + p_1' + p_1'')x + \ldots + (p_{m-1} + p_{m-1}' + p_{m-1}'')x^{m-1} + LX, \tag{100}$$

that is, by (56) and (99),

$$y = f(x) = (1 + p + p')(p_0 + p_1 x + \ldots + p_{m-1}x^{m-1}) \\ + q_0 + q_0' + (q_1 + q_1')x + \ldots + (q_{m-2} + q_{m-2}')x^{m-2} + LX: \tag{101}$$

and the excluded case, or case of failure, will now be the case when the sums

$$p_0' + p_0'', \quad p_1' + p_1'', \; \ldots, \; p_{m-1}' + p_{m-1}''$$

are proportional to $p_0, p_1, \ldots, p_{m-1}$, that is, when

$$q_0 + q_0' = 0, \quad q_1 + q_1' = 0, \; \ldots, \; q_{m-2} + q_{m-2}' = 0. \tag{102}$$

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Indeed, it is here tacitly supposed that p_{m-1} does not vanish; but Mr Jerrard's method itself supposes tacitly that at least some one, such as p_i, of the m quantities $p_0, \ldots, p_{m-1}$, is different from 0, and it is easy, upon occasion, to substitute any such non-evanescent quantity p_i for p_{m-1}, and then to make the few other connected changes which the spirit of this discussion requires.

The expression (101) for $f(x)$, combined with the law (5) of the composition of the coefficients A' and C', gives, for those coefficients, expressions of the forms,

$$A' = (1+p+p')A''_{1,0,0} + A''_{0,1,0} + A''_{0,0,1}, \tag{103}$$

and
$$C' = (1+p+p')^3 C''_{3,0,0} + (1+p+p')^2 (C''_{2,1,0} + C''_{2,0,1}) + (1+p+p')(C''_{1,2,0} + C''_{1,1,1} + C''_{1,0,2}) + C''_{0,3,0} + C''_{0,2,1} + C''_{0,1,2} + C''_{0,0,3}, \tag{104}$$

$A''_{h,i,k}$ and $C''_{h,i,k}$ being rational and integral functions of the $3m-2$ quantities $p_0, p_1, \ldots, p_{m-1}$, $q_0, q_1, \ldots, q_{m-2}$, $q'_0, q'_1, \ldots, q'_{m-2}$, which functions are independent of p, p', and L, and are homogeneous of the dimension h with respect to $p_0, \ldots, p_{m-1}$, of the dimension i with respect to $q_0, \ldots, q_{m-2}$, and of the dimension k with respect to $q'_0, \ldots, q'_{m-2}$; they are also such that the sums

$$A''_{0,1,0} + A''_{0,0,1} \tag{105}$$

and
$$C''_{2,1,0} + C''_{2,0,1} \tag{106}$$

are homogeneous functions, of the 1st dimension, of the $m-1$ sums $q_0 + q'_0, \ldots, q_{m-2} + q'_{m-2}$; while the sum

$$C''_{1,2,0} + C''_{1,1,1} + C''_{1,0,2} \tag{107}$$

is a homogeneous function, of the 2nd dimension, and the sum

$$C''_{0,3,0} + C''_{0,2,1} + C''_{0,1,2} + C''_{0,0,3} \tag{108}$$

is a homogeneous function, of the 3rd dimension, of the same $m-1$ quantities. These new expressions, (103) and (104), for the coefficients A' and C', must identically coincide with the former expressions (83) and (84), when we return from the present to the former notation, by changing $p, p', p_0, p_1, \ldots, p_{m-1}, q_0, q_1, \ldots, q_{m-2}, q'_0, q'_1, \ldots, q'_{m-2}$, to their values as functions of Λ', Λ'', Λ''', M$'$, M$''$, M$'''$, M$^{\text{IV}}$, N$'$, N$''$, N$'''$, N$^{\text{IV}}$, N$^{\text{V}}$; and hence it is easy to deduce the following identical equations:

$$A'_{1,0,0} = A''_{1,0,0}; \quad A'_{0,1,0} = pA''_{1,0,0} + A''_{0,1,0}; \quad A'_{0,0,1} = p'A''_{1,0,0} + A''_{0,0,1}; \tag{109}$$

and
$$\left.\begin{aligned} &C'_{3,0,0} = C''_{3,0,0}; \quad C'_{2,1,0} = 3pC''_{3,0,0} + C''_{2,1,0}; \\ &C'_{2,0,1} = 3p'C''_{3,0,0} + C''_{2,0,1}; \quad C'_{1,2,0} = 3p^2C''_{3,0,0} + 2pC''_{2,1,0} + C''_{1,2,0}; \\ &C'_{1,1,1} = 6pp'C''_{3,0,0} + 2p'C''_{2,1,0} + 2pC''_{2,0,1} + C''_{1,1,1}; \\ &C'_{1,0,2} = 3p'^2C''_{3,0,0} + 2p'C''_{2,0,1} + C''_{1,0,2}; \\ &C'_{0,3,0} + C'_{0,2,1} + C'_{0,1,2} + C'_{0,0,3} = (p+p')^3 C''_{3,0,0} + (p+p')^2 (C''_{2,1,0} + C''_{2,0,1}) \\ &\qquad + (p+p')(C''_{1,2,0} + C''_{1,1,1} + C''_{1,0,2}) + C''_{0,3,0} + C''_{0,2,1} + C''_{0,1,2} + C''_{0,0,3}. \end{aligned}\right\} \tag{110}$$

The system of the ten conditions (85), (86), (87), (88), may therefore be transformed to the following:

$$A''_{1,0,0} = 0, \quad C''_{3,0,0} = 0; \tag{111}$$

$$A''_{0,1,0} = 0, \quad C''_{2,1,0} = 0, \quad C''_{1,2,0} = 0; \tag{112}$$

$$A''_{0,0,1} = 0, \quad C''_{2,0,1} = 0, \quad C''_{1,1,1} = 0, \quad C''_{1,0,2} = 0; \tag{113}$$

$$C''_{0,3,0} + C''_{0,2,1} + C''_{0,1,2} + C''_{0,0,3} = 0; \tag{114}$$

and may in general be treated as follows. The two conditions (111) may first be combined with

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the m equations of definition (46), and employed to determine the $m+2$ ratios of the $m+3$ quantities $p_0, \ldots, p_{m-1}$, Λ', Λ'', Λ'''; and therefore to give a result of the form

$$\Lambda' x^{\lambda'} + \Lambda'' x^{\lambda''} + \Lambda''' x^{\lambda'''} = \Lambda''' \phi(x), \tag{115}$$

the function $\phi(x)$ being known. The three conditions (112), combined with the $2m$ equations (47) and (56), may then be used to determine the $2m+3$ ratios of the $2m+4$ quantities $q_0, \ldots, q_{m-2}$, pp_{m-1}, $p'_0, \ldots, p'_{m-1}$, M$'$, M$''$, M$'''$, M$^{\text{IV}}$, and consequently to give

$$\mathrm{M}' x^{\mu'} + \mathrm{M}'' x^{\mu''} + \mathrm{M}''' x^{\mu'''} + \mathrm{M}^{\text{IV}} x^{\mu^{\text{IV}}} = \mathrm{M}^{\text{IV}} \psi(x), \tag{116}$$

$\psi(x)$ denoting a known function. The four conditions (113) may next be combined with the $2m$ equations (94) and (99), so as to determine the $2m+4$ ratios of the $2m+5$ quantities $q'_0, \ldots, q'_{m-2}$, $p'p_{m-1}$, $p''_0, \ldots, p''_{m-1}$, N$'$, N$''$, N$'''$, N$^{\text{IV}}$, N$^{\text{V}}$; and thus we shall have

$$\mathrm{N}' x^{\nu'} + \mathrm{N}'' x^{\nu''} + \mathrm{N}''' x^{\nu'''} + \mathrm{N}^{\text{IV}} x^{\nu^{\text{IV}}} + \mathrm{N}^{\text{V}} x^{\nu^{\text{V}}} = \mathrm{N}^{\text{V}} \omega(x), \tag{117}$$

the function $\omega(x)$ also being known; so that, at this stage, the expression (75) for y will be reduced to the form

$$y = f(x) = \Lambda''' \phi(x) + \mathrm{M}^{\text{IV}} \psi(x) + \mathrm{N}^{\text{V}} \omega(x), \tag{118}$$

the three functions $\phi(x)$, $\psi(x)$, $\omega(x)$ being known, but the three coefficients Λ''', M$^{\text{IV}}$, N$^{\text{V}}$, being arbitrary. The condition (114) will next determine the ratio of any one of the quantities $q_0, \ldots, q_{m-2}$ to any one of the quantities $q'_0, \ldots, q'_{m-2}$, and therefore also the connected ratio of M$^{\text{IV}}$ to N$^{\text{V}}$, and consequently will give

$$\mathrm{M}^{\text{IV}} \psi(x) + \mathrm{N}^{\text{V}} \omega(x) = \mathrm{N}^{\text{V}} \chi(x), \tag{119}$$

$\chi(x)$ being another known function; and thus we shall have accomplished, in a way apparently but not essentially different from that employed in the foregoing article, the first part of Mr Jerrard's process, namely, the discovery of an expression for y, of the form

$$y = f(x) = \Lambda''' \phi(x) + \mathrm{N}^{\text{V}} \chi(x), \quad (89)$$

which satisfies the two conditions $\quad A' = 0, \quad C' = 0,$

the functions $\phi(x)$ and $\chi(x)$ being determined and known, but the multipliers Λ''' and N$^{\text{V}}$ being arbitrary: after which it will only remain to perform the second part of the process, namely, the determination of the ratio of Λ''' to N$^{\text{V}}$, so as to satisfy the remaining condition

$$D' - \alpha B'^2 = 0,$$

by resolving a biquadratic equation.

[8]. The advantage of this new way of presenting the first part of Mr Jerrard's process is that it enables us to perceive, that if we would avoid the case of failure above mentioned, we must in general exclude those cases in which the ratios

$$\frac{q'_0}{q'_{m-2}}, \quad \frac{q'_1}{q'_{m-2}}, \quad \ldots, \quad \frac{q'_{m-3}}{q'_{m-2}}, \tag{120}$$

determined, as above explained, through the medium of the conditions (113), coincide with the ratios

$$\frac{q_0}{q_{m-2}}, \quad \frac{q_1}{q_{m-2}}, \quad \ldots, \quad \frac{q_{m-3}}{q_{m-2}}, \tag{121}$$

determined, at an earlier stage, through the medium of the conditions (112). In fact, when the ratios (120) coincide with the ratios (121), they necessarily coincide with the following ratios also,

$$\frac{q_0 + q'_0}{q_{m-2} + q'_{m-2}}, \quad \frac{q_1 + q'_1}{q_{m-2} + q'_{m-2}}, \quad \ldots, \quad \frac{q_{m-3} + q'_{m-3}}{q_{m-2} + q'_{m-2}}; \tag{122}$$

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and unless the ratios, thus determined, of the $m-1$ sums $q_0+q_0', \ldots, q_{m-2}+q_{m-2}'$, are accidentally such as to satisfy the condition (114), which had not been employed in determining them, then that condition, which is a rational and integral and homogeneous equation of the third degree between those quantities, will oblige them all to vanish, and therefore will conduct to the case of failure (102). Reciprocally, in that case of failure, the ratios (120) coincide with the ratios (121), because we have then

$$q_0' = -q_0, \quad q_1' = -q_1, \ldots, q_{m-2}' = -q_{m-2}. \tag{123}$$

The case to be excluded, in general, is therefore that in which the $m-1$ quantities $q_0', \ldots, q_{m-2}'$ are proportional to the $m-1$ quantities $q_0, \ldots, q_{m-2}$; and this consideration suggests the introduction of the following new symbols or definitions,

$$\frac{q_{m-2}'}{q_{m-2}} = q, \tag{124}$$

$$q_0' - qq_0 = r_0, \quad q_1' - qq_1 = r_1, \ldots, q_{m-3}' - qq_{m-3} = r_{m-3}; \tag{125}$$

because, by introducing these, we shall only be obliged to guard against the simultaneous vanishing of the $m-2$ quantities $r_0, r_1, \ldots, r_{m-3}$; that is, we shall have the following simplified statement of the general case of failure,

$$r_0 = 0, \quad r_1 = 0, \ldots, r_{m-3} = 0. \tag{126}$$

Adopting, therefore, the definitions (124) and (125), and consequently the expressions

$$q_0' = r_0 + qq_0, \quad q_1' = r_1 + qq_1, \ldots, q_{m-3}' = r_{m-3} + qq_{m-3}, \quad q_{m-2}' = qq_{m-2}, \tag{127}$$

which give

$$q_0 + q_0' = (1+q)q_0 + r_0, \quad q_1 + q_1' = (1+q)q_1 + r_1, \ldots, q_{m-3} + q_{m-3}' = (1+q)q_{m-3} + r_{m-3},$$
$$q_{m-2} + q_{m-2}' = (1+q)q_{m-2}, \tag{128}$$

we easily perceive that the three homogeneous functions (105) (106) (107), of these $m-1$ sums $q_0+q_0', \ldots, q_{m-2}+q_{m-2}'$, may be expressed in the following manner:

$$A''_{0,1,0} + A''_{0,0,1} = (1+q)A'''_{0,1,0} + A'''_{0,0,1}; \tag{129}$$

$$C''_{2,1,0} + C''_{2,0,1} = (1+q)C'''_{2,1,0} + C'''_{2,0,1}; \tag{130}$$

$$C''_{1,2,0} + C''_{1,1,1} + C''_{1,0,2} = (1+q)^2 C'''_{1,2,0} + (1+q)C'''_{1,1,1} + C'''_{1,0,2}; \tag{131}$$

the symbol $A'''_{h,i,k}$ or $C'''_{h,i,k}$ denoting here a rational and integral function of the $3m-3$ quantities $p_0, \ldots, p_{m-1}, q_0, \ldots, q_{m-2}, r_0, \ldots, r_{m-3}$, which is, like the function $A''_{h,i,k}$ or $C''_{h,i,k}$, homogeneous of the dimension k with respect to $p_0, \ldots, p_{m-1}$, and of the dimension i with respect to $q_0, \ldots, q_{m-2}$, but is homogeneous of the dimension k with respect to $r_0, \ldots, r_{m-3}$, and is independent of $q_0', \ldots, q_{m-2}'$ and of p, p', q; whereas $A''_{h,i,k}$ or $C''_{h,i,k}$ was homogeneous of the dimension k with respect to $q_0', \ldots, q_{m-2}'$, and was independent of $r_0, \ldots, r_{m-3}$. The three identical equations (129) (130) (131) may be decomposed into the seven following, which are analogous to (60) and (61):

$$A''_{0,1,0} = A'''_{0,1,0}; \quad A''_{0,0,1} = qA'''_{0,1,0} + A'''_{0,0,1}; \tag{132}$$

$$C''_{2,1,0} = C'''_{2,1,0}; \quad C''_{2,0,1} = qC'''_{2,1,0} + C'''_{2,0,1}; \tag{133}$$

$$C''_{1,2,0} = C'''_{1,2,0}; \quad C''_{1,1,1} = 2qC'''_{1,2,0} + C'''_{1,1,1}; \quad C''_{1,0,2} = q^2 C'''_{1,2,0} + qC'''_{1,1,1} + C'''_{1,0,2}; \tag{134}$$

and, in virtue of these, the seven conditions (112) and (113) may be put under the forms,

$$A'''_{0,1,0} = 0; \quad C'''_{2,1,0} = 0, \quad C'''_{1,2,0} = 0, \tag{135}$$

and

$$A'''_{0,0,1} = 0, \quad C'''_{2,0,1} = 0, \quad C'''_{1,1,1} = 0, \quad C'''_{1,0,2} = 0. \tag{136}$$

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The three conditions of the group (135) differ only in their notation from the three conditions (112), and are to be used exactly like those former conditions, in order to determine the ratios of $q_0, \ldots, q_{m-2}$, after the ratios of $p_0, \ldots, p_{m-1}$ have been determined, through the help of the conditions (111); but, in deducing the conditions (136) from the conditions (113), a real simplification has been effected (and not merely a change of notation) by suppressing several terms, such as $qA'''_{0,1,0}$, which vanish in consequence of the conditions (112) or (135). And since we have thus been led to perceive the existence of a group, (136), of four homogeneous equations (rational and integral) between the $m-2$ quantities $r_0, r_1, \ldots, r_{m-3}$, we see, at last, that we shall be conducted, in general, to the *case of failure* (126), in which all those quantities vanish, *unless their number $m-2$ be greater than four*; that is, *unless the degree of the proposed equation in x be at least equal to the minor limit* SEVEN. It results, then, from this analysis, that for equations of the *sixth* and *lower* degrees, Mr Jerrard's process for effecting the transformation (20), or for satisfying the three conditions (8) (14) and (19),

$$A' = 0, \quad C' = 0, \quad D' - \alpha B'^2 = 0,$$

will, in general, become *illusory*, by conducting to an useless expression, of the form (93), for the new variable y; so that it *fails*, for example, *to reduce the general equation of the fifth degree*,

$$x^5 + Ax^4 + Bx^3 + Cx^2 + Dx + E = 0, \quad (69)$$

to De Moivre's solvible form, $$y^5 + B'y^3 + \tfrac{1}{5}B'^2y + E' = 0, \tag{137}$$

except, by an useless assumption, of the form

$$y = L(x^5 + Ax^4 + Bx^3 + Cx^2 + Dx + E), \tag{138}$$

which gives, indeed, a very simple transformed equation, namely,

$$y^5 = 0, \tag{139}$$

but affords no assistance whatever towards resolving the proposed equation in x. Indeed, for equations of the *fifth* degree, the foregoing discussion may be considerably simplified, by observing, that, in virtue of the eight conditions (112) (113) (114), the four homogeneous functions (105) (106) (107) (108), of the $m-1$ sums $q_0 + q'_0, \ldots, q_{m-2} + q'_{m-2}$, are all $=0$, and therefore also (in general) those sums themselves must vanish (which is the case of failure (102),) when their number $m-1$ is not greater than four, that is, *when the proposed equation is not higher than the fifth degree*. But the foregoing discussion (though the great generality of the question has caused it to be rather long) has the advantage of extending even to equations of the *sixth* degree, and of showing that even for such equations the method generally fails, in such a manner that it will not in general reduce the equation

$$x^6 + Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F = 0 \tag{140}$$

to the form $$y^6 + B'y^4 + \alpha B'^2y^2 + E'y + F' = 0, \tag{141}$$

except by the assumption

$$y = L(x^6 + Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F); \tag{142}$$

which gives, indeed, a very simple result, namely,

$$y^6 = 0, \tag{143}$$

but does not at all assist us to resolve the proposed equation (140). However, this discussion may be regarded as *confirming the adequacy of the method to transform the general equation of the seventh degree*, $$x^7 + Ax^6 + Bx^5 + Cx^4 + Dx^3 + Ex^2 + Fx + G = 0, \tag{144}$$

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to another of the form $$y^7+B'y^5+\alpha B'^2y^3+E'y^2+F'y+G'=0, \qquad (145)$$

without assuming y = any multiple of the proposed evanescent polynome x^7+Ax^6+ &c.; and to effect the analogous transformation (20), for equations of all *higher* degrees; a curious and unexpected discovery, for which algebra is indebted to Mr Jerrard.

[9]. The result obtained by the foregoing discussion may seem, so far is it respects equations of the *sixth* degree, to be of very little importance; because the equation (141), to which it has been shown that the method fails to reduce the general equation (140), is not itself, in general, of any known solvible form, whatever value may be chosen for the arbitrary multiplier α. But it must be observed that if the method had in fact been adequate to effect that general transformation of the equation of the sixth degree, without resolving any auxiliary equation of a higher degree than the fourth, then it would also have been adequate to reduce the same general equation (140) of the sixth degree to this other form, which is obviously and easily solvible,

$$y^6+B'y^4+D'y^2+F'=0, \qquad (146)$$

by first assigning an expression of the form

$$y=f(x)=\Lambda'''\phi(x)+\mathrm{N}^{\mathrm{V}}\chi(x), \quad (89)$$

which should satisfy the two conditions

$$A'=0, \quad (8)$$

$$C'=0, \quad (14)$$

and by then determining the ratio of Λ''' to N^{V}, so as to satisfy this other condition,

$$E'=0, \qquad (147)$$

which could be done without resolving any auxiliary equation of a higher degree than the fifth; and this *reduction, of the difficulty of the sixth to that of the fifth degree*, would have been a very important result, of which it was interesting to examine the validity. The foregoing discussion, however, appears to me to prove that *this transformation also is illusory*; for it shows that, because the degree of the proposed equation is less than the minor limit 7, the functions $\phi(x)$ and $\chi(x)$ in (89) are connected by a relation of the form (36); on which account the expression (89) becomes

$$y=f(x)=(\Lambda'''+a\mathrm{N}^{\mathrm{V}})\,\phi(x)+\lambda\mathrm{N}^{\mathrm{V}}X, \quad (90)$$

and the condition $$E'=0, \quad (147)$$

gives, in general, $$(\Lambda'''+a\mathrm{N}^{\mathrm{V}})^5=0, \qquad (148)$$

that is, $$\Lambda'''+a\mathrm{N}^{\mathrm{V}}=0; \quad (92)$$

so that finally the expression for y becomes

$$y=\lambda\mathrm{N}^{\mathrm{V}}X, \quad (93)$$

that is, it takes in general the evidently useless form,

$$y=L(x^6+Ax^5+Bx^4+Cx^3+Dx^2+Ex+F). \quad (142)$$

[10]. Mr Jerrard has not actually stated, in his published *Researches*, the process by which he would effect in general the transformation (15), so as to *take away four terms* at once from the equation of the m^{th} degree, without resolving any auxiliary equation of a higher degree than

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the fourth; but he has sufficiently indicated this process, which appears to be such as the following. He would probably assume an expression with *twenty-one* terms for the new variable,

$$\begin{aligned} y=f(x) &= \Lambda' x^{\lambda'} + \Lambda'' x^{\lambda''} + \Lambda''' x^{\lambda'''} \\ &+ \mathrm{M}' x^{\mu'} + \mathrm{M}'' x^{\mu''} + \mathrm{M}''' x^{\mu'''} + \mathrm{M}^{\mathrm{IV}} x^{\mu^{\mathrm{IV}}} \\ &+ \mathrm{N}' x^{\nu'} + \mathrm{N}'' x^{\nu''} + \mathrm{N}''' x^{\nu'''} + \mathrm{N}^{\mathrm{IV}} x^{\nu^{\mathrm{IV}}} + \mathrm{N}^{\mathrm{V}} x^{\nu^{\mathrm{V}}} + \mathrm{N}^{\mathrm{VI}} x^{\nu^{\mathrm{VI}}} \\ &+ \Xi' x^{\xi'} + \Xi'' x^{\xi''} + \Xi''' x^{\xi'''} + \Xi^{\mathrm{IV}} x^{\xi^{\mathrm{IV}}} + \Xi^{\mathrm{V}} x^{\xi^{\mathrm{V}}} + \Xi^{\mathrm{VI}} x^{\xi^{\mathrm{VI}}} + \Xi^{\mathrm{VII}} x^{\xi^{\mathrm{VII}}} + \Xi^{\mathrm{VIII}} x^{\xi^{\mathrm{VIII}}}, \end{aligned} \tag{149}$$

and would develop or decompose the coefficients A', B', C', of the transformed equation in y, considered as rational and integral, and homogeneous functions of the twenty-one coefficients,

$$\Lambda', \quad \Lambda'', \quad \Lambda''', \tag{150}$$

$$\mathrm{M}', \quad \mathrm{M}'', \quad \mathrm{M}''', \quad \mathrm{M}^{\mathrm{IV}}, \tag{151}$$

$$\mathrm{N}', \quad \mathrm{N}'', \quad \mathrm{N}''', \quad \mathrm{N}^{\mathrm{IV}}, \quad \mathrm{N}^{\mathrm{V}}, \quad \mathrm{N}^{\mathrm{VI}}, \tag{152}$$

$$\Xi', \quad \Xi'', \quad \Xi''', \quad \Xi^{\mathrm{IV}}, \quad \Xi^{\mathrm{V}}, \quad \Xi^{\mathrm{VI}}, \quad \Xi^{\mathrm{VII}}, \quad \Xi^{\mathrm{VIII}}, \tag{153}$$

into the following parts:

$$A' = A'_{1,0,0,0} + A'_{0,1,0,0} + A'_{0,0,1,0} + A'_{0,0,0,1}; \tag{154}$$

$$\begin{aligned} B' &= B'_{2,0,0,0} + B'_{1,1,0,0} + B'_{1,0,1,0} + B'_{1,0,0,1} \\ &+ B'_{0,2,0,0} + B'_{0,1,1,0} + B'_{0,1,0,1} + B'_{0,0,2,0} + B'_{0,0,1,1} + B'_{0,0,0,2}; \end{aligned} \tag{155}$$

$$\begin{aligned} C' &= C'_{3,0,0,0} + C'_{2,1,0,0} + C'_{2,0,1,0} + C'_{2,0,0,1} + C'_{1,2,0,0} + C'_{1,1,1,0} + C'_{1,1,0,1} \\ &+ C'_{1,0,2,0} + C'_{1,0,1,1} + C'_{1,0,0,2} + C'_{0,3,0,0} + C'_{0,2,1,0} + C'_{0,2,0,1} \\ &+ C'_{0,1,2,0} + C'_{0,1,1,1} + C'_{0,1,0,2} + C'_{0,0,3,0} + C'_{0,0,2,1} + C'_{0,0,1,2} + C'_{0,0,0,3}; \end{aligned} \tag{156}$$

each part $A'_{h,i,k,l}$ or $B'_{h,i,k,l}$ or $C'_{h,i,k,l}$ being itself a rational and integral function of the twenty-one quantities (150) (151) (152) (153), and being also homogeneous of the degree h with respect to the three quantities (150), of the degree i with respect to the four quantities (151), of the degree k with respect to the six quantities (152), and of the degree l with respect to the eight quantities (153). He would then determine the two ratios of the two first to the last of the three quantities (150) (that is, the ratios of Λ' and Λ'' to Λ''') so as to satisfy the two conditions

$$A'_{1,0,0,0} = 0, \quad B'_{2,0,0,0} = 0; \tag{157}$$

the three ratios of the first three to the last of the four quantities (151), so as to satisfy the three conditions

$$A'_{0,1,0,0} = 0, \quad B'_{1,1,0,0} = 0, \quad B'_{0,2,0,0} = 0; \tag{158}$$

the ratio of the last of the quantities (150) to the last of the quantities (151), so as to satisfy the condition

$$C'_{3,0,0,0} + C'_{2,1,0,0} + C'_{1,2,0,0} + C'_{0,3,0,0} = 0; \tag{159}$$

the five ratios of the five first to the last of the six quantities (152), so as to satisfy the five conditions

$$\left.\begin{aligned} &A'_{0,0,1,0} = 0, && C'_{2,0,1,0} + C'_{1,1,1,0} + C'_{0,2,1,0} = 0, \\ &B'_{1,0,1,0} + B'_{0,1,1,0} = 0, && C'_{1,0,2,0} + C'_{0,1,2,0} = 0; \\ &B'_{0,0,2,0} = 0, && \end{aligned}\right\} \tag{160}$$

the seven ratios of the seven first to the last of the eight quantities (153), so as to satisfy the seven conditions

$$\left.\begin{aligned} &A'_{0,0,0,1} = 0, && C'_{2,0,0,1} + C'_{1,1,0,1} + C'_{0,2,0,1} = 0, \\ &B'_{1,0,0,1} + B'_{0,1,0,1} = 0, && C'_{1,0,1,1} + C'_{0,1,1,1} = 0, \\ &B'_{0,0,1,1} = 0, && C'_{1,0,0,2} + C'_{0,1,0,2} = 0; \\ &B'_{0,0,0,2} = 0, && \end{aligned}\right\} \tag{161}$$

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and the ratio of the last of the quantities (152) to the last of the quantities (153), so as to satisfy the condition

$$C'_{0,0,3,0}+C'_{0,0,2,1}+C'_{0,0,1,2}+C'_{0,0,0,3}=0: \qquad (162)$$

all which determinations could in general be successively effected, without its being necessary to resolve any equation of a higher degree than the fourth. The first part of the process would be now completed; that is, the assumed expression (149) for y would be reduced to the form

$$y=f(x)=\mathrm{M}^{\mathrm{IV}}\phi(x)+\Xi^{\mathrm{VIII}}\chi(x), \qquad (163)$$

the functions $\phi(x)$ and $\chi(x)$ being determined and known, but the multipliers M^{IV} and Ξ^{VIII} being arbitrary, and this expression (163) being such as to satisfy the three conditions (8) (10) and (14),

$$A'=0, \quad B'=0, \quad C'=0;$$

nineteen out of the twenty ratios of the twenty-one coefficients (150) (151) (152) (153) having been determined so as to satisfy the nineteen equations (157) (158) (159) (160) (161) (162), into which those three conditions had been decomposed. And the second and only remaining part of the process would consist in then determining the remaining ratio of M^{IV} to Ξ^{VIII}, so as to satisfy the remaining condition

$$D'=0, \quad (17)$$

and thereby to reduce the general equation of the m^{th} degree,

$$X=x^m+Ax^{m-1}+Bx^{m-2}+Cx^{m-3}+Dx^{m-4}+Ex^{m-5}+\&\text{c.}=0, \quad (1)$$

to the form

$$Y=y^m+E'y^{m-5}+\&\text{c.}=0. \quad (15)$$

It is possible, of course, that this may not be precisely the same as Mr Jerrard's unpublished process, but it seems likely that the one would not be found to differ from the other in any essential respect, notation being always excepted. It is, at least, a process suggested by the published researches of that author, and harmonizing with the discoveries which they contain.

But by applying to this new process the spirit of the former discussions, and putting, for abbreviation,

$$\left.\begin{array}{l}\Lambda's_0^{(\lambda')}+\Lambda''s_0^{(\lambda'')}+\Lambda'''s_0^{(\lambda''')}+\mathrm{M}'s_0^{(\mu')}+\ldots+\mathrm{M}^{\mathrm{IV}}s_0^{(\mu^{\mathrm{IV}})}=p_0,\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ \Lambda's_{m-1}^{(\lambda')}+\Lambda''s_{m-1}^{(\lambda'')}+\Lambda'''s_{m-1}^{(\lambda''')}+\mathrm{M}'s_{m-1}^{(\mu')}+\ldots+\mathrm{M}^{\mathrm{IV}}s_{m-1}^{(\mu^{\mathrm{IV}})}=p_{m-1},\end{array}\right\} \qquad (164)$$

$$\left.\begin{array}{l}\mathrm{N}'s_0^{(\nu')}+\ldots+\mathrm{N}^{\mathrm{VI}}s_0^{(\nu^{\mathrm{VI}})}=p'_0,\\ \cdots\cdots\cdots\cdots\\ \mathrm{N}'s_{m-1}^{(\nu')}+\ldots+\mathrm{N}^{\mathrm{VI}}s_{m-1}^{(\nu^{\mathrm{VI}})}=p'_{m-1},\end{array}\right\} \qquad (165)$$

$$\left.\begin{array}{l}\Xi's_0^{(\xi')}+\ldots+\Xi^{\mathrm{VIII}}s_0^{(\xi^{\mathrm{VIII}})}=p''_0,\\ \cdots\cdots\cdots\cdots\\ \Xi's_{m-1}^{(\xi')}+\ldots+\Xi^{\mathrm{VIII}}s_{m-1}^{(\xi^{\mathrm{VIII}})}=p''_{m-1},\end{array}\right\} \qquad (166)$$

$$\begin{array}{l}\Lambda'L^{(\lambda')}+\Lambda''L^{(\lambda'')}+\Lambda'''L^{(\lambda''')}+\mathrm{M}'L^{(\mu')}+\ldots+\mathrm{M}^{\mathrm{IV}}L^{(\mu^{\mathrm{IV}})}\\ \qquad +\mathrm{N}'L^{(\nu')}+\ldots+\mathrm{N}^{\mathrm{VI}}L^{(\nu^{\mathrm{VI}})}+\Xi'L^{(\xi')}+\ldots+\Xi^{\mathrm{VIII}}L^{(\xi^{\mathrm{VIII}})}=L,\end{array} \qquad (167)$$

we may change the expression (149) to the form (100), through the theorem and notation (44); and in order to avoid the case of failure, in which the functions $\phi(s)$ and $\chi(x)$ in (163) are connected by a relation of the form (36), we must avoid, as in the discussion given in the seventh article, the case where the m sums $p'_0+p''_0, \ldots, p'_{m-1}+p''_{m-1}$ are proportional to the m quantities $p_0, \ldots, p_{m-1}$, that is, the case

$$q_0+q'_0=0, \ldots, q_{m-2}+q'_{m-2}=0, \quad (102)$$

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if we adopt the definitions (54) (55) and (96) (97), so as to introduce the symbols $p, q_0, q_1, \ldots, q_{m-2}$, and $p', q'_0, q'_1, \ldots, q'_{m-2}$. With these additional symbols it is easy to transform the conditions (160) into others, which (when suitably combined with the equations of definition, and with the ratios of $p_0, \ldots, p_{m-1}$ already previously determined through the help of the conditions (157) (158) (159),) shall serve to determine the ratios (121) of $q_0, \ldots, q_{m-2}$; and then to determine, in like manner, with the help of the conditions (161), the ratios (120) of $q'_0, \ldots, q'_{m-2}$; after which, the condition (162) may be transformed into a rational and integral and homogeneous equation of the third degree between the sums $q_0 + q'_0, \ldots, q_{m-2} + q'_{m-2}$, and will in general oblige those sums to vanish, if their ratios (122) have been already determined independently of this condition (162), which will happen when the ratios (120) coincide with the ratios (121), that is, when the quantities $q'_0, \ldots, q'_{m-1}$ are proportional to the quantities $q_0, \ldots, q_{m-1}$. We must, therefore, in general avoid this last proportionality, in order to avoid the case of failure (102); and thus we are led to introduce the symbols $q, r_0, r_1, \ldots, r_{m-3}$, defined by the equations (124) (125), and to express the case of failure by the equations

$$r_0 = 0, \quad r_1 = 0, \ldots, r_{m-3} = 0. \quad (126)$$

With these new symbols we easily discover that the seven conditions (161) may be reduced to seven rational and integral and homogeneous equations between the quantities $r_0, r_1, \ldots, r_{m-3}$, which will in general oblige them all to vanish, and therefore will produce the case of failure (126), *unless the number* $m-2$ *of these quantities be greater than the number seven*, that is, *unless the exponent* m *of the degree of the proposed equation be at least equal to the minor limit* TEN. It results, then, from this discussion, that the process described in the present article *will not in general avail to take away four terms at once, from equations lower than the* TENTH *degree*, and, of course, that *it will not reduce the general equation of the fifth degree*,

$$x^5 + Ax^4 + Bx^3 + Cx^2 + Dx + E = 0, \quad (69)$$

to the binomial form

$$y^5 + E' = 0, \quad (168)$$

except by the useless assumption

$$y = L(x^5 + Ax^4 + Bx^3 + Cx^2 + Dx + E), \quad (138)$$

which gives

$$y^5 = 0. \quad (139)$$

[11]. A principal feature of Mr Jerrard's general method is to *avoid*, as much as possible, *the raising of degree in elimination*; and for that purpose to *decompose the equations of condition* in every question *into groups*, which shall *each* contain, if possible, *not more than one equation of a higher degree than the first*; although the occurrence of *two equations of the second degree* in one group is *not fatal* to the success of the method, because the final equation of such a group being *only elevated to the fourth degree*, can be resolved by the known rules. It might, therefore, have been more completely in the spirit of this general method, because it would have more completely avoided the elevation of degree by elimination, if, in order to take away four terms at once from the general equation of the m^{th} degree, we had assumed an expression with *thirty-three* terms, of the form

$$y = f(x) = \Lambda' x^{\lambda'} + \Lambda''^{\lambda''} + \Lambda''' x^{\lambda'''} + \mathrm{M}' x^{\mu'} + \ldots + \mathrm{M}^{\mathrm{IV}} x^{\mu^{\mathrm{IV}}} + \mathrm{N}' x^{\nu'} + \ldots + \mathrm{N}^{\mathrm{V}} x^{\nu^{\mathrm{V}}}$$
$$+ \Xi' x^{\xi'} + \ldots + \Xi^{\mathrm{VI}} x^{\xi^{\mathrm{VI}}} + \mathrm{O}' x^{o'} + \ldots + \mathrm{O}^{\mathrm{VII}} x^{o^{\mathrm{VII}}} + \Pi' x^{\varpi'} + \ldots + \Pi^{\mathrm{VIII}} x^{\varpi^{\mathrm{VIII}}}; \quad (169)$$

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and had determined the six ratios of Λ', Λ'', Λ''', M' ..., M^{IV}, and the twenty-five ratios of $\mathrm{N}', \ldots, \Pi^{\mathrm{VIII}}$, so as to satisfy the thirty-one conditions

$$A'_{1,0,0,0,0,0}=0, \quad B'_{2,0,0,0,0,0}=0, \tag{170}$$

$$A'_{0,1,0,0,0,0}=0 \quad B'_{1,1,0,0,0,0}=0, \quad B'_{0,2,0,0,0,0}=0, \tag{171}$$

$$C'_{3,0,0,0,0,0}+C'_{2,1,0,0,0,0}+C'_{1,2,0,0,0,0}+C'_{0,3,0,0,0,0}=0, \tag{172}$$

$$\left.\begin{array}{l} A'_{0,0,1,0,0,0}=0, \\ B'_{1,0,1,0,0,0}+B'_{0,1,1,0,0,0}=0, \quad B'_{0,0,2,0,0,0}=0, \\ C'_{2,0,1,0,0,0}+C'_{1,1,1,0,0,0}+C'_{0,2,1,0,0,0}=0, \end{array}\right\} \tag{173}$$

$$\left.\begin{array}{l} A'_{0,0,0,1,0,0}=0, \\ B'_{1,0,0,1,0,0}+B'_{0,1,0,1,0,0}=0, \quad B'_{0,0,1,1,0,0}=0, \quad B'_{0,0,0,2,0,0}=0, \\ C'_{2,0,0,1,0,0}+C'_{1,1,0,1,0,0}+C'_{0,2,0,1,0,0}=0, \end{array}\right\} \tag{174}$$

$$C'_{1,0,2,0,0,0}+C'_{1,0,1,1,0,0}+C'_{1,0,0,2,0,0}+C'_{0,1,2,0,0,0}+C'_{0,1,1,1,0,0}+C'_{0,1,0,2,0,0}=0, \tag{175}$$

$$\left.\begin{array}{l} A'_{0,0,0,0,1,0}=0, \\ B'_{1,0,0,0,1,0}+B'_{0,1,0,0,1,0}=0, \quad B'_{0,0,1,0,1,0}+B'_{0,0,0,1,1,0}=0, \quad B'_{0,0,0,0,0,2,0}=0, \\ C'_{2,0,0,0,1,0}+C'_{1,1,0,0,1,0}+C'_{0,2,0,0,1,0}=0, \quad C'_{1,0,1,0,1,0}+C'_{1,0,0,1,1,0}+C'_{0,1,1,0,1,0}+C'_{0,1,0,1,1,0}=0, \end{array}\right\} \tag{176}$$

$$\left.\begin{array}{l} A'_{0,0,0,0,0,1}=0, \\ B'_{1,0,0,0,0,1}+B'_{0,1,0,0,0,1}=0, \quad B'_{0,0,1,0,0,1}+B'_{0,0,0,1,0,1}=0, \quad B'_{0,0,0,0,1,1}=0, \quad B'_{0,0,0,0,0,2}=0, \\ C'_{2,0,0,0,0,1}+C'_{1,1,0,0,0,1}+C'_{0,2,0,0,0,1}=0, \quad C'_{1,0,1,0,0,1}+C'_{1,0,0,1,0,1}+C'_{0,1,1,0,0,1}+C'_{0,1,0,1,0,1}=0, \end{array}\right\} \tag{177}$$

$$C'_{1,0,0,0,2,0}+C'_{1,0,0,0,1,1}+C'_{1,0,0,0,0,2}+C'_{0,1,0,0,2,0}+C'_{0,1,0,0,1,1}+C'_{0,1,0,0,0,2}=0, \tag{178}$$

$$\begin{aligned} &C'_{0,0,3,0,0,0}+C'_{0,0,2,1,0,0}+C'_{0,0,2,0,1,0}+C'_{0,0,2,0,0,1}+C'_{0,0,1,2,0,0}+C'_{0,0,1,1,1,0}+C'_{0,0,1,1,0,1} \\ &\quad+C'_{0,0,1,0,2,0}+C'_{0,0,1,0,1,1}+C'_{0,0,1,0,0,2}+C'_{0,0,0,3,0,0}+C'_{0,0,0,2,1,0}+C'_{0,0,0,2,0,1}+C'_{0,0,0,1,2,0} \\ &\quad+C'_{0,0,0,1,1,1}+C'_{0,0,0,1,0,2}+C'_{0,0,0,0,3,0}+C'_{0,0,0,0,2,1}+C'_{0,0,0,0,1,2}+C'_{0,0,0,0,0,3}=0, \end{aligned} \tag{179}$$

into which the three conditions $A'=0, \quad B'=0, \quad C'=0,$

may be decomposed; the symbol $A'_{f,g,h,i,k,l}$, or $B'_{f,g,h,i,k,l}$, or $C'_{f,g,h,i,k,l}$ denoting here a rational and integral function of the thirty-three coefficients $\Lambda', \ldots, \Pi^{\mathrm{VIII}}$, which is homogeneous of the degree f with respect to $\Lambda', \Lambda'', \Lambda'''$, of the degree g with respect to $\mathrm{M}', \ldots, \mathrm{M}^{\mathrm{IV}}$, of the degree h with respect to $\mathrm{N}', \ldots, \mathrm{N}^{\mathrm{V}}$, of the degree i with respect to $\Xi', \ldots, \Xi^{\mathrm{VI}}$, of the degree k with respect to $\mathrm{O}', \ldots, \mathrm{O}^{\mathrm{VII}}$, and of the degree l with respect to $\Pi', \ldots, \Pi^{\mathrm{VIII}}$: while the remaining ratio of M^{IV} to Π^{VIII}, should afterwards be chosen so as to satisfy the remaining condition

$$D'=0.$$

But, upon putting, for abridgment,

$$\left.\begin{array}{l} \mathrm{N}'s_0^{(\nu')}+\ldots+\mathrm{N}^{\mathrm{V}}s_0^{(\nu^{\mathrm{V}})}+\Xi's_0^{(\xi')}+\ldots+\Xi^{\mathrm{VI}}s_0^{(\xi^{\mathrm{VI}})}=p'_0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \mathrm{N}'s_{m-1}^{(\nu')}+\ldots+\mathrm{N}^{\mathrm{V}}s_{m-1}^{(\nu^{\mathrm{V}})}+\Xi's_{m-1}^{(\xi')}+\ldots+\Xi^{\mathrm{VI}}s_{m-1}^{(\xi^{\mathrm{VI}})}=p'_{m-1}, \end{array}\right\} \tag{180}$$

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$$\left.\begin{array}{l} \mathrm{O}'s_0^{(o')}+\ldots+\mathrm{O}^{\mathrm{VII}}s_0^{(o^{\mathrm{VII}})}=p_0'', \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mathrm{O}'s_{m-1}^{(o')}+\ldots+\mathrm{O}^{\mathrm{VII}}s_{m-1}^{(o^{\mathrm{VII}})}=p_{m-1}'', \end{array}\right\} \tag{181}$$

$$\left.\begin{array}{l} \Pi's_0^{(\varpi')}+\ldots+\Pi^{\mathrm{VIII}}s_0^{(\varpi^{\mathrm{VIII}})}=p_0''', \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Pi's_{m-1}^{(\varpi')}+\ldots+\Pi^{\mathrm{VIII}}s_{m-1}^{(\varpi^{\mathrm{VIII}})}=p_{m-1}''', \end{array}\right\} \tag{182}$$

$$\left.\begin{array}{l} \Lambda'L^{(\lambda')}+\Lambda''L^{(\lambda'')}+\Lambda'''L^{(\lambda''')}+\mathrm{M}'L^{(\mu')}+\ldots+\mathrm{M}^{\mathrm{IV}}L^{(\mu^{\mathrm{IV}})} \\ \quad +\mathrm{N}'L^{(\nu')}+\ldots+\mathrm{N}^{\mathrm{V}}L^{(\nu^{\mathrm{V}})}+\Xi'L^{(\xi')}+\ldots+\Xi^{\mathrm{VI}}L^{(\xi^{\mathrm{VI}})} \\ \quad +\mathrm{O}'L^{(o')}+\ldots+\mathrm{O}^{\mathrm{VII}}L^{(o^{\mathrm{VII}})}+\Pi'L^{(\varpi')}+\ldots+\Pi^{\mathrm{VIII}}L^{(\varpi^{\mathrm{VIII}})}=L, \end{array}\right\} \tag{183}$$

$$\frac{p_{m-1}'''}{p_{m-1}}=p'', \tag{184}$$

$$p_0'''-p''p_0=q_0'', \ldots, p_{m-2}'''-p''p_{m-2}=q_{m-2}'', \tag{185}$$

$$\frac{q_{m-2}''}{q_{m-2}}=q', \tag{186}$$

$$q_0''-q'q_0=r_0', \ldots, q_{m-3}''-q'q_{m-3}=r_{m-3}', \tag{187}$$

$$\frac{r_{m-3}'}{r_{m-3}}=r, \tag{188}$$

$$r_0'-rr_0=t_0, \ldots, r_{m-4}'-rr_{m-4}=t_{m-4}, \tag{189}$$

and retaining the analogous expressions (164) (54) (55) (96) (97) (124) (125), we find, by a reasoning exactly analogous to that employed in the former discussions, that the final expression for y will in general be of the useless form

$$y=LX, \tag{190}$$

in the following case of failure,

$$t_0=0, \quad t_1=0, \ldots, t_{m-4}=0; \tag{191}$$

and on the other hand that the seven conditions (177) may be reduced to the form of seven rational and integral and homogeneous equations between these $m-3$ quantities $t_0, t_1, \ldots, t_{m-4}$; so that the case of failure will in general occur in the employment of the *present* process, *unless the number* $m-3$ *be greater than seven*, that is, *unless the degree* m *of the proposed equation in* x *be at least equal to the minor limit eleven.*

It must, however, be remembered that the less complex process described in the foregoing article, (since it contained no condition, nor group of conditions, in which the dimension, or the product of the dimensions, exceeded the number four,) agreed sufficiently with the spirit of Mr Jerrard's general method; and was adequate to take away four terms at once from the general equation of the *tenth*, or of any higher degree.

[12]. The various processes described in the 2nd, 5th, 6th and 11th articles of this communication, for *transforming the general equation of the* m^{th} *degree*, by satisfying certain systems of equations of condition, are connected with the solution of this far more general problem proposed by Mr Jerrard, 'to discover $m-1$ ratios of m disposable quantities,

$$a_1, a_2, \ldots, a_m, \tag{192}$$

which shall satisfy a given system of h_1 rational and integral and homogeneous equations of the first degree,

$$A'=0, \quad A''=0, \ldots, A^{(h_1)}=0; \tag{193}$$

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h_2 such equations of the second degree,

$$B'=0, \quad B''=0, \ldots, B^{(h_2)}=0; \tag{194}$$

h_3 of the third degree,

$$C'=0, \quad C''=0, \ldots, C^{(h_3)}=0; \tag{195}$$

and so on, as far as h_t equations of the t^{th} degree,

$$T'=0, \quad T''=0, \ldots, T^{(h_t)}=0, \tag{196}$$

without being obliged in any part of the process, to introduce any elevation of degree by elimination.' Mr Jerrard has not published his solution of this very general problem, but he has sufficiently suggested the method which he would employ, and it is proper to discuss it briefly here, with reference to the extent of its application, and the circumstances under which it fails; not only on account of the importance of such discussion in itself, but also because it is adapted to throw light on all the questions already considered.

If we assume

$$a_1=a_1'+a_1'', \quad a_2=a_2'+a_2'', \ldots, a_m=a_m'+a_m'', \tag{197}$$

that is, if we decompose each of the m disposable quantities $a_1, a_2, \ldots, a_m$ into two parts, we may then accordingly decompose every one of the h_1 proposed homogeneous functions of those m quantities, which are of the first degree, namely,

$$A', A'', \ldots, A^{(\alpha)}, \ldots, A^{(h_1)}; \tag{198}$$

every one of the h_2 proposed functions of the second degree,

$$B', B'', \ldots, B^{(\beta)}, \ldots, B^{(h_2)}; \tag{199}$$

every one of the h_3 functions of the third degree,

$$C', C'', \ldots, C^{(\gamma)}, \ldots, C^{(h_3)}; \tag{200}$$

and so on, as far as all the first $h_t - 1$ functions of the t^{th} degree,

$$T', T'', \ldots, T^{(\tau)}, \ldots, T^{(h_t-1)}, \tag{201}$$

(the last function $T^{(h_t)}$ being reserved for another purpose, which will be presently explained,) into other homogeneous functions, according to the general types,

$$\left.\begin{array}{l} A^{(\alpha)}=A^{(\alpha)}_{1,0}+A^{(\alpha)}_{0,1}, \\ B^{(\beta)}=B^{(\beta)}_{2,0}+B^{(\beta)}_{1,1}+B^{(\beta)}_{0,2}, \\ C^{(\gamma)}=C^{(\gamma)}_{3,0}+C^{(\gamma)}_{2,1}+C^{(\gamma)}_{1,2}+C^{(\gamma)}_{0,3}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ T^{(\tau)}=T^{(\tau)}_{t,0}+T^{(\tau)}_{t-1,1}+\ldots+T^{(\tau)}_{0,t}: \end{array}\right\} \tag{202}$$

each symbol of the class

$$A^{(\alpha)}_{p,q}, B^{(\beta)}_{p,q}, C^{(\gamma)}_{p,q}, \ldots, T^{(\tau)}_{p,q}, \tag{203}$$

denoting a rational and integral and homogeneous function of the $2m$ new quantities,

$$a_1', a_2', \ldots, a_m', \tag{204}$$

and

$$a_1'', a_2'', \ldots, a_m'', \tag{205}$$

which function is homogeneous of the degree p with respect to the quantities (204), and of the degree q with respect to the quantities (205). By this decomposition, we may substitute, instead of the problem first proposed, the system of the three following auxiliary problems. First, to satisfy, by ratios of the m quantities (204), an auxiliary system of equations, containing h_1 equations of the first degree, namely,

$$A'_{1,0}=0, \quad A''_{1,0}=0, \ldots, A^{(h_1)}_{1,0}=0; \tag{206}$$

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h_2 equations of the second degree,

$$B'_{2,0}=0,\quad B''_{2,0}=0,\ldots,B^{(h_2)}_{2,0}=0; \tag{207}$$

h_3 of the third degree, $$C'_{3,0}=0,\quad C''_{3,0}=0,\ \ldots,\ C^{(h_3)}_{3,0}=0; \tag{208}$$

and so on, as far as the following h_t-1 equations of the t^{th} degree,

$$T'_{t,0}=0,\quad T''_{t,0}=0,\ldots,T^{(h_t-1)}_{t,0}=0. \tag{209}$$

Second, to satisfy, by ratios of the m quantities (205), a system containing

$$h_1+h_2+h_3+\ldots+h_t-1$$

equations, which are of the first degree with respect to those m quantities, and are of the forms

$$A^{(\alpha)}_{0,1}=0,\quad B^{(\beta)}_{1,1}=0,\quad C^{(\gamma)}_{2,1}=0,\ldots,T^{(\tau)}_{t-1,1}=0; \tag{210}$$

$h_2+h_3+\ldots+h_t-1$ equations of the second degree, and of the forms

$$B^{(\beta)}_{0,2}=0,\quad C^{(\gamma)}_{1,2}=0,\ldots,T^{(\tau)}_{t-2,2}=0; \tag{211}$$

$h_3+\ldots+h_t-1$ equations of the third degree, and of the forms

$$C^{(\gamma)}_{0,3}=0,\ldots,T^{(\tau)}_{t-3,3}=0; \tag{212}$$

and so on, as far as h_t-1 equations of the t^{th} degree, namely,

$$T'_{0,t}=0,\quad T''_{0,t}=0,\ldots,T^{(h_t-1)}_{0,t}=0. \tag{213}$$

And third, to satisfy, by the ratio of any one of the m quantities (205) to any one of the m quantities (204), this one remaining equation of the t^{th} degree,

$$T^{(h_t)}=0. \tag{214}$$

For if we can resolve all these three auxiliary problems, we shall thereby have resolved the original problem also. And there is this advantage in thus transforming the question, that whereas there were h_t equations of the highest (that is of the t^{th}) degree, in the problem originally proposed, there are only h_t-1 equations of that highest degree, in each of the two first auxiliary problems, and only one such equation in the third. If, then, we apply the same process of transformation to each of the two first auxiliary problems, and repeat it sufficiently often, we shall get rid of all the equations of the t^{th} degree, and ultimately of all equations of degrees higher than the first; with the exception of certain equations, which are at various stages of the process set aside to be separately and singly resolved, without any such combination with others as could introduce an elevation of degree by elimination. And thus, at last, the original problem may doubtless be resolved, *provided that the number m, of quantities originally disposable, be large enough.*

[13]. But that some such condition respecting the magnitude of that number m is necessary, will easily appear, if we observe that when m is not large enough to satisfy the inequality,

$$m>h_1+h_2+h_3+\ldots+h_t, \tag{215}$$

then the original $h_1+h_2+h_3+\ldots+h_t$ equations, being rational and integral and homogeneous with respect to the original m quantities (192), will in general conduct to null values for all those quantities, that is, to the expressions

$$a_1=0,\quad a_2=0,\ldots,a_m=0, \tag{216}$$

and therefore to a result which we designed to exclude; because by the enunciation of the original problem it was by the $m-1$ *ratios* of those m quantities that we were to satisfy, if

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possible, the equations originally proposed. The same excluded case, or case of failure (216), will in general occur when the solution of the second auxiliary problem gives ratios for the m auxiliary quantities (205), which coincide with the ratios already found in resolving the first auxiliary problem for the m other auxiliary quantities (204); that is, when the two first problems conduct to expressions of the forms

$$a_1''=aa_1', \quad a_2''=aa_2', \ldots, a_m''=aa_m', \tag{217}$$

a being any common multiplier; for then these two first problems conduct, in virtue of the definitions (197), to a determined set of ratios for the m original quantities (192), namely,

$$\frac{a_1}{a_m}=\frac{a_1'}{a_m'}, \quad \ldots, \quad \frac{a_{m-1}}{a_m}=\frac{a_{m-1}'}{a_m'}; \tag{218}$$

and unless these ratios happen to satisfy the equation of the third problem (214), which had not been employed in determining them, that last homogeneous equation (214) will oblige all those m quantities (192) to vanish, and so will conduct to the case of failure (216). Now although, when the condition (215) is satisfied, the first auxiliary problem becomes indeterminate, because

$$m-1>h_1+h_2+h_3+\ldots+h_t-1,$$

so that the number $m-1$ of the disposable ratios of the m auxiliary quantities (204) is greater than the number of the homogeneous equations which those m quantities are to satisfy, yet whatever system of $m-1$ such ratios

$$\frac{a_1'}{a_m'}, \quad \frac{a_2'}{a_m'}, \quad \ldots, \quad \frac{a_{m-1}'}{a_m'}, \tag{219}$$

we may discover and employ, so as to satisfy the equations of the first auxiliary problem, it will always be possible to satisfy the equations of the second auxiliary problem also, by employing the same system of $m-1$ ratios for the m other auxiliary quantities (205), that is, by employing expressions for those quantities of the forms (217); and, reciprocally, it will in general be impossible to resolve the second auxiliary problem otherwise, unless the number of its equations be less than $m-1$. For if we put, for abridgment,

$$\frac{a_m''}{a_m'}=a, \tag{220}$$

and

$$a_1''-aa_1'=b_1, \quad a_2''-aa_2'=b_2, \ldots, a_{m-1}''-aa_{m-1}'=b_{m-1}, \tag{221}$$

we shall have, as a general system of expressions for the m quantities (205), the following,

$$a_1''=aa_1'+b_1, \quad a_2''=aa_2'+b_2, \ldots, a_{m-1}''=aa_{m-1}'+b_{m-1}, \quad a_m''=aa_m'; \tag{222}$$

and therefore by (197),

$$a_1=(1+a)\,a_1'+b_1, \ldots, a_{m-1}=(1+a)\,a_{m-1}'+b_{m-1}, \quad a_m=(1+a)\,a_m'; \tag{223}$$

so that the homogeneous functions $A^{(\alpha)}, B^{(\beta)}, \ldots, T^{(\tau)}$ may be, in general, decomposed in this new way,

$$\left.\begin{aligned} &A^{(\alpha)}=(1+a)\,A_{1,0}^{\backprime(\alpha)}+A_{0,1}^{\backprime(\alpha)};\\ &B^{(\beta)}=(1+a)^2 B_{2,0}^{\backprime(\beta)}+(1+a)\,B_{1,1}^{\backprime(\beta)}+B_{0,2}^{\backprime(\beta)};\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &T^{(\tau)}=(1+a)^t\,T_{t,0}^{\backprime(\tau)}+(1+a)^{t-1}\,T_{t-1,0}^{\backprime(\tau)}+\ldots+T_{0,t}^{\backprime(\tau)}: \end{aligned}\right\} \tag{224}$$

each symbol of the class

$$A_{p,q}^{\backprime(\alpha)}, \quad B_{p,q}^{\backprime(\beta)}, \ldots, T_{p,q}^{\backprime(\tau)} \tag{225}$$

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denoting a rational and integral function of the $2m-1$ quantities $a'_1, \ldots, a'_m, b_1, \ldots, b_{m-1}$, which is homogeneous of the dimension p with respect to the m quantities

$$a'_1,\ \ldots,\ a'_m, \quad (204)$$

and of the dimension q with respect to the $m-1$ quantities

$$b_1,\ \ldots,\ b_{m-1}, \tag{226}$$

but is independent of the multiplier a. And the identical equations obtained by comparing the expressions (202) and (224), resolve themselves into the following:

$$\left.\begin{array}{l}
A^{(\alpha)}_{1,0} = A'^{(\alpha)}_{1,0}; \quad A^{(\alpha)}_{0,1} = aA'^{(\alpha)}_{1,0} + A'^{(\alpha)}_{0,1};\\
B^{(\beta)}_{2,0} = B'^{(\beta)}_{2,0}; \quad B^{(\beta)}_{1,1} = 2aB'^{(\beta)}_{2,0} + B'^{(\beta)}_{1,1}; \quad B^{(\beta)}_{0,2} = a^2B'^{(\beta)}_{2,0} + aB'^{(\beta)}_{1,1} + B'^{(\beta)}_{0,2};\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
T^{(\tau)}_{t,0} = T'^{(\tau)}_{t,0}; \quad T^{(\tau)}_{t-1,1} = taT'^{(\tau)}_{t,0} + T'^{(\tau)}_{t-1,1};\\
T^{(\tau)}_{t-2,2} = \frac{t(t-1)}{2}a^2T'^{(\tau)}_{t,0} + (t-1)\,aT'^{(\tau)}_{t-1,1} + T'^{(\tau)}_{t-2,2};\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
T^{(\tau)}_{0,t} = a^tT'^{(\tau)}_{t,0} + a^{t-1}T'^{(\tau)}_{t-1,1} + \ldots + T'^{(\tau)}_{0,t};
\end{array}\right\} \tag{227}$$

so that the first system of auxiliary equations, (206)...(209), which are of the forms

$$A^{(\alpha)}_{1,0} = 0, \quad B^{(\beta)}_{2,0} = 0, \quad C^{(\gamma)}_{3,0} = 0, \ldots, T^{(\tau)}_{t,0} = 0, \tag{228}$$

may be replaced by the system

$$A'^{(\alpha)}_{1,0} = 0, \quad B'^{(\beta)}_{2,0} = 0, \quad C'^{(\gamma)}_{3,0} = 0, \ldots, T'^{(\tau)}_{t,0} = 0, \tag{229}$$

the change, so far, being only a change of notation; and, after satisfying this system by a suitable selection of the ratios of the quantities (204), the second system of auxiliary equations, (210)...(213), may then be transformed, with a real simplification, (which consists in getting rid of the arbitrary multiplier a, and in diminishing the number of the quantities whose ratios remain to be disposed of,) to another system of equations of the forms,

$$\left.\begin{array}{l}
A'^{(\alpha)}_{0,1} = 0, \quad B'^{(\beta)}_{1,1} = 0, \quad C'^{(\gamma)}_{2,1} = 0, \ldots, T'^{(\tau)}_{t-1,1} = 0;\\
B'^{(\beta)}_{0,2} = 0; \quad C'^{(\gamma)}_{1,2} = 0, \ldots, T'^{(\tau)}_{t-2,2} = 0;\\
C'^{(\gamma)}_{0,3} = 0, \ldots, T'^{(\tau)}_{t-3,3} = 0;\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
T'^{(\tau)}_{0,t} = 0;
\end{array}\right\} \tag{230}$$

which are rational and integral and homogeneous with respect to the $m-1$ quantities (226), and are independent of the multiplier a. Unless, then, the number of the equations of this transformed system (230), which is the same as the number of equations in the second auxiliary problem before proposed, be less than the number $m-1$ of the new auxiliary quantities (226), we shall have, in general, null values for all those quantities, that is, we shall have

$$b_1 = 0, \quad b_2 = 0,\ \ldots,\ b_{m-1} = 0; \tag{231}$$

and therefore we shall be conducted, by (222), to expressions of the forms (217), which will in general lead, as has been already shown, to the case of failure (216). We have therefore a *new*

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condition of inequality, which the number m must satisfy, in order to the general success of the method, namely the following,

$$m-1>h_1'+h_2'+h_3'+\ldots+h_t'; \tag{232}$$

in which, $h_1', h_2', h_3', \ldots, h_t'$ denote respectively the numbers of the equations of the first, second, third, ... and t^{th} degrees, in the second auxiliary problem; so that, by what has been already shown,

$$\left.\begin{aligned} h_t' &= h_t-1,\\ h_{t-1}' &= h_{t-1}+h_t-1,\\ h_{t-2}' &= h_{t-2}+h_{t-1}+h_t-1,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ h_2' &= h_2+\ldots+h_t-1,\\ h_1' &= h_1+h_2+\ldots+h_t-1. \end{aligned}\right\} \tag{233}$$

These last expressions give

$$h_1'+h_2'+h_3'+\ldots+h_t'=h_1+2h_2+3h_3+\ldots+th_t-t; \tag{234}$$

so that the new condition of inequality, (232), may be written as follows,

$$m-1>h_1+2h_2+3h_3+\ldots+t(h_t-1); \tag{235}$$

and therefore also thus,

$$m>h_1+h_2+h_3+\ldots+h_t+h_2+2h_3+\ldots+(t-1)(h_t-1). \tag{236}$$

It *includes*, therefore, in general, the old inequality (215); and may be considered as comprising in itself all the conditions respecting the magnitude of the number m, connected with our present inquiry: or, at least, as capable of furnishing us with all such conditions, if only it be sufficiently developed.

[14]. It must, however, be remembered, as a part of such development, that although, when this condition (232) or (235) or (236) is satisfied, the three auxiliary problems above stated are, in general, theoretically capable of being resolved, and of conducting to a system of ratios of the m original quantities (192), which shall satisfy the original system of equations, yet each of the two first auxiliary systems contains, in general, more than two equations of the second or higher degrees; and therefore that, in order to avoid any elevation of degree by elimination (as required by the original problem), the process must in general be *repeated*, and each of the two auxiliary systems themselves must be decomposed, and treated like the system originally proposed. These new decompositions introduce, in general, new conditions of inequality, analogous to the condition lately determined; but it is clear that the condition connected with the decomposition of the first of the auxiliary systems must be included in the condition connected with the decomposition of the second of those systems, because the latter system contains, in general, in each of the degrees $1, 2, 3, \ldots, t-1$, a greater number of equations than the former, while both contain, in the degree t, the same number of equations, namely, h_t-1. Conceiving, then, the second auxiliary system to be decomposed by a repetition of the process above described into two new auxiliary systems or groups of equations, and into one separate and reserved equation of the t^{th} degree, we are conducted to this new condition of inequality, analogous to (232),

$$m-2>h_1''+h_2''+h_3''+\ldots+h_t''; \tag{237}$$

$h_1'', h_2'', h_3'', \ldots, h_t''$ denoting, respectively, the numbers of equations of the first, second, third, ...

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and t^{th} degrees, in the second new group of equations; in such a manner that, by the nature of the process,

$$\left.\begin{aligned}
h''_t &= h'_t - 1,\\
h''_{t-1} &= h'_{t-1} + h'_t - 1,\\
h''_{t-2} &= h'_{t-2} + h'_{t-1} + h'_t - 1,\\
&\dots\dots\dots\dots\\
h''_1 &= h'_1 + h'_2 + \dots + h'_t - 1.
\end{aligned}\right\} \tag{238}$$

Repeating this process, we find, next, the condition,

$$m - 3 > h'''_1 + h'''_2 + h'''_3 + \dots + h'''_t, \tag{239}$$

and generally

$$m - i > h_1^{(i)} + h_2^{(i)} + h_3^{(i)} + \dots + h_t^{(i)}; \tag{240}$$

each new condition of this series including all that go before it, and the symbol $h_p^{(i)}$ being such that

$$h_p^{(o)} = h_p, \tag{241}$$

$$h_t^{(i+1)} - h_t^{(i)} = -1, \tag{242}$$

and

$$h_{t-n}^{(i+1)} - h_{t-n}^{(i)} = h_{t-n+1}^{(i+1)}. \tag{243}$$

Integrating these last equations as equations in finite differences, we find

$$\left.\begin{aligned}
h_t^{(i)} &= h_t - i;\\
h_{t-1}^{(i)} &= h_{t-1} + i\left(h_t - \frac{i+1}{2}\right);\\
h_{t-2}^{(i)} &= h_{t-2} + ih_{t-1} + i\,.\frac{i+1}{2}.\left(h_t - \frac{i+2}{3}\right);\\
h_{t-3}^{(i)} &= h_{t-3} + ih_{t-2} + i\frac{i+1}{2}h_{t-1} + i\frac{i+1}{2}\,\frac{i+2}{3}\left(h_t - \frac{i+3}{4}\right);\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
h_1^{(i)} &= h_1 + ih_2 + i\frac{i+1}{2}h_3 + i\frac{i+1}{2}\,\frac{i+2}{3}h_4 + \dots\\
&\quad + i\frac{i+1}{2}\,\frac{i+2}{3}\dots\frac{i+t-2}{t-1}\left(h_t - \frac{i+t-1}{t}\right).
\end{aligned}\right\} \tag{244}$$

And making, in these expressions,

$$i = h_t, \tag{245}$$

so as to have

$$h_t^{(i)} = 0, \tag{246}$$

and putting, for abridgment,

$$h_1^{(h_t)} = {}^{\backprime}h_1, \quad h_2^{(h_t)} = {}^{\backprime}h_2, \dots, h_{t-1}^{(h_t)} = {}^{\backprime}h_{t-1}, \tag{247}$$

we find that at the stage when all the equations of the t^{th} degree have been removed from the auxiliary groups of equations, we are led to satisfy, if possible, by the ratios of $m - h_t$ auxiliary quantities, a system containing ${}^{\backprime}h_1$ equations of the first degree, ${}^{\backprime}h_2$ of the second, ${}^{\backprime}h_3$ of the third, and so on as far as ${}^{\backprime}h_{t-1}$ of the degree $t-1$; in which

$$\left.\begin{aligned}
{}^{\backprime}h_{t-1} &= h_{t-1} + \tfrac{1}{2}h_t(h_t - 1),\\
{}^{\backprime}h_{t-2} &= h_{t-2} + h_t h_{t-1} + \tfrac{1}{3}(h_t + 1)\,h_t(h_t - 1),\\
{}^{\backprime}h_{t-3} &= h_{t-3} + h_t h_{t-2} + \tfrac{1}{2}(h_t + 1)\,h_t h_{t-1} + \tfrac{1}{8}(h_t + 2)\,(h_t + 1)\,h_t(h_t - 1),\\
{}^{\backprime}h_1 &= h_1 + h_t h_2 + \tfrac{1}{2}(h_t + 1)\,h_t h_3 + \tfrac{1}{6}(h_t + 2)\,(h_t + 1)\,h_t h_4 + \dots\\
&\quad + \frac{1}{2\,.\,3\,.\,4\dots(t-2)t}(h_t + t - 2)\,(h_t + t - 3)\dots h_t(h_t - 1);
\end{aligned}\right\} \tag{248}$$

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so that, at this stage, we arrive at the following condition of inequality,

$$m - h_t > {}^{\backprime}h_1 + {}^{\backprime}h_2 + {}^{\backprime}h_3 + \ldots + {}^{\backprime}h_{t-1}, \tag{249}$$

${}^{\backprime}h_1, {}^{\backprime}h_2, \ldots, {}^{\backprime}h_{t-1}$ having the meanings (248). In exactly the same way, we find the condition

$$m - h_t - {}^{\backprime}h_{t-1} > {}^{\backprime\backprime}h_1 + {}^{\backprime\backprime}h_2 + {}^{\backprime\backprime}h_3 + \ldots + {}^{\backprime\backprime}h_{t-2}, \tag{250}$$

in which,

$$\left.\begin{aligned} &{}^{\backprime\backprime}h_{t-2} = {}^{\backprime}h_{t-2} + \tfrac{1}{2}{}^{\backprime}h_{t-1}({}^{\backprime}h_{t-1} - 1), \\ &{}^{\backprime\backprime}h_{t-3} = {}^{\backprime}h_{t-3} + {}^{\backprime}h_{t-1}\,{}^{\backprime}h_{t-2} + \tfrac{1}{3}({}^{\backprime}h_{t-1} + 1)\,{}^{\backprime}h_{t-1}({}^{\backprime}h_{t-1} - 1), \\ &\&c., \end{aligned}\right\} \tag{251}$$

by clearing the auxiliary systems from all equations of the degree $t-1$; and again by clearing all such auxiliary groups from equations of the degree $t-2$, we obtain a condition of the form

$$m - h_t - {}^{\backprime}h_{t-1} - {}^{\backprime\backprime}h_{t-2} > {}^{\backprime\backprime\backprime}h_1 + \ldots + {}^{\backprime\backprime\backprime}h_{t-3}, \tag{252}$$

in which

$${}^{\backprime\backprime\backprime}h_{t-3} = {}^{\backprime\backprime}h_{t-3} + \tfrac{1}{2}{}^{\backprime\backprime}h_{t-2}({}^{\backprime\backprime}h_{t-2} - 1), \quad \&c. \tag{253}$$

so that at last we are conducted to a condition which may be thus denoted, and which contains the ultimate result of all the restrictions on the number m,

$$m - h_t - {}^{\backprime}h_{t-1} - {}^{\backprime\backprime}h_{t-2} - {}^{\backprime\backprime\backprime}h_{t-3} - \ldots - {}^{(t-2)}h_2 > {}^{(t-1)}h_1, \tag{254}$$

that is,

$$m > h_t + {}^{\backprime}h_{t-1} + {}^{\backprime\backprime}h_{t-2} + {}^{\backprime\backprime\backprime}h_{t-3} + \ldots + {}^{(t-2)}h_2 + {}^{(t-1)}h_1. \tag{255}$$

The number m, of quantities originally disposable, must therefore in general be at least equal to a certain minor limit, which may be thus denoted,

$$m(h_1, h_2, h_3, \ldots, h_t) = h_t + {}^{\backprime}h_{t-1} + {}^{\backprime\backprime}h_{t-2} + \ldots + {}^{(t-2)}h_2 + {}^{(t-1)}h_1 + 1, \tag{256}$$

in order that the method may succeed; and reciprocally, *the method will in general be successful when m equals or surpasses this limit.*

[15]. To illustrate the foregoing general discussion, let us suppose that

$$t = 2; \tag{257}$$

that is, let us propose to satisfy a system containing h_1 equations of the first degree,

$$A' = 0, \ldots, A^{(\alpha)} = 0, \ldots, A^{(h_1)} = 0, \quad (193)$$

and h_2 equations of the second degree,

$$B' = 0, \ldots, B^{(\beta)} = 0, \ldots, B^{(h_2)} = 0, \quad (194)$$

(but not containing any equations of higher degrees than the second,) by a suitable selection of the $m-1$ ratios of m quantities,

$$a_1, \ldots, a_m, \quad (192)$$

and without being obliged in any part of the process to introduce any elevation of degree by elimination. Assuming, as before,

$$a_1 = a_1' + a_1'', \ldots, a_m = a_m' + a_m'', \quad (197)$$

and employing the corresponding decompositions

$$A' = A'_{1,0} + A'_{0,1}, \ldots, A^{(h_1)} = A^{(h_1)}_{1,0} + A^{(h_1)}_{0,1}, \tag{258}$$

and

$$B' = B'_{2,0} + B'_{1,1} + B'_{0,2}, \ldots, B^{(h_2-1)} = B^{(h_2-1)}_{2,0} + B^{(h_2-1)}_{1,1} + B^{(h_2-1)}_{0,2}, \tag{259}$$

we shall be able to resolve the original problem, if we can resolve the system of the three following.

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First: to satisfy, by ratios of the m auxiliary quantities

$$a'_1, \ldots, a'_m, \quad (204)$$

an auxiliary system, containing the h_1 equations of the first degree

$$A'_{1,0}=0, \ldots, A^{(h_1)}_{1,0}=0, \quad (206)$$

and the h_2-1 equations of the second degree

$$B'_{2,0}=0, \ldots, B^{(h_2-1)}_{2,0}=0. \quad (260)$$

Second: to satisfy, by ratios of the m other auxiliary quantities

$$a''_1, \ldots, a''_m, \quad (205)$$

another auxiliary system, containing h_1+h_2-1 equations of the first degree,

$$A'_{0,1}=0, \ldots, A^{(h_1)}_{0,1}=0, \qquad B'_{1,1}=0, \ldots, B^{(h_2-1)}_{1,1}=0, \quad (261)$$

and h_2-1 equations of the second degree,

$$B'_{0,2}=0, \ldots, B^{(h_2-1)}_{0,2}=0. \quad (262)$$

Third: to satisfy, by the ratio of any one of the m quantities (205) to any one of the m quantities (204), this one remaining equation of the second degree

$$B^{(h_2)}=0. \quad (263)$$

The enunciation of the original problem supposes that

$$m>h_1+h_2; \quad (264)$$

since otherwise the original equations (193) and (194) would in general conduct to the excluded case, or case of failure,

$$a_1=0, \ldots, a_m=0. \quad (216)$$

In virtue of this condition (264), the first auxiliary problem is indeterminate, because

$$m-1>h_1+h_2-1. \quad (265)$$

But, by whatever system of ratios $\frac{a'_1}{a'_m}, \ldots, \frac{a'_{m-1}}{a'_m}$ (219)

we may succeed in satisfying the first auxiliary system of equations, (206) and (260), we may in general transform the second auxiliary system of equations, (261) and (262), into a system which may be thus denoted,

$$A'\,'_{0,1}=0, \ldots, A'\,^{(h_1)}_{0,1}=0, \quad B'\,'_{1,1}=0, \ldots, B'\,^{(h_2-1)}_{1,1}=0, \quad B'\,'_{0,2}=0, \ldots, B'\,^{(h_2-1)}_{0,2}=0, \quad (266)$$

and which contains h_1+h_2-1 equations of the first degree, and h_2-1 equations of the second degree, between the $m-1$ new combinations, or new auxiliary quantities following,

$$b_1=a''_1-\frac{a'_1}{a'_m}a''_m, \ldots, b_{m-1}=a''_{m-1}-\frac{a'_{m-1}}{a'_m}a''_m; \quad (267)$$

so that the solution of the second auxiliary problem will give, in general,

$$b_1=0, \ldots, b_{m-1}=0, \quad (231)$$

and therefore will give, for the m auxiliary quantities (205), a system of ratios coincident with the ratios (219),

$$\frac{a''_1}{a''_m}=\frac{a'_1}{a'_m}, \ldots, \frac{a''_{m-1}}{a''_m}=\frac{a'_{m-1}}{a'_m}, \quad (268)$$

unless

$$m-1>h_1+2(h_2-1). \quad (269)$$

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When, therefore, this last condition is not satisfied, the two first auxiliary problems will conduct, in general, to a system of determined ratios for the m original quantities (192), namely

$$\frac{a_1}{a_m}=\frac{a_1'}{a_m'},\ \ldots,\ \frac{a_{m-1}}{a_m}=\frac{a_{m-1}'}{a_m'}; \quad (218)$$

and unless these happen to satisfy the equation of the third auxiliary problem, namely

$$B^{(h_2)}=0, \quad (263)$$

which had not been employed in determining them, we shall fall back on the excluded case, or case of failure, (216). But, even when the condition (269) is satisfied, and when, therefore, the auxiliary equations are theoretically capable of conducting to ratios which shall satisfy the equations originally proposed, it will still be necessary, in general, to decompose each of the two first auxiliary systems of equations into others, in order to comply with the enunciation of the original problem, which requires that we should avoid all raising of degree by elimination, in every part of the process. Confining ourselves to the consideration of the second auxiliary problem, (which includes the difficulties of the first,) we see that the transformed auxiliary system (266) contains h_1' equations of the first degree, and h_2' of the second, if we put, for abridgment,

$$h_2'=h_2-1, \quad h_1'=h_1+h_2-1; \tag{270}$$

which new auxiliary equations are to be satisfied, if possible, by the ratios of $m-1$ new auxiliary quantities; so that a repetition of the former process of decomposition and transformation would conduct to a new auxiliary system, containing h_1'' equations of the first degree, and h_2'' of the second, in which

$$h_2''=h_2'-1, \quad h_1''=h_1'+h_2'-1, \tag{271}$$

and which must be satisfied, if possible, by the ratios of $m-2$ new auxiliary quantities; and thus we should arrive at this new condition, as necessary to the success of the method:

$$m-2>h_1'+2(h_2'-1); \tag{272}$$

or, more concisely,

$$m-2>h_1''+h_2''. \tag{273}$$

And so proceeding, we should find generally,

$$m-i>h_1^{(i)}+h_2^{(i)}, \tag{274}$$

the functions $h_1^{(i)}, h_2^{(i)}$ being determined by the equations

$$h_2^{(0)}=h_2, \quad h_1^{(0)}=h_1, \tag{275}$$

$$h_2^{(i+1)}-h_2^{(i)}=-1, \tag{276}$$

$$h_1^{(i+1)}-h_1^{(i)}=h_2^{(i+1)}; \tag{277}$$

which give, by integrations of finite differences,

$$h_2^{(i)}=h_2-i; \quad h_1^{(i)}=h_1+i\left(h_2-\frac{i+1}{2}\right). \tag{278}$$

Thus, making

$$i=h_2, \tag{279}$$

and putting, for abridgment,

$${}^{\backprime}h_1=h_1^{(h_2)}=h_1+\tfrac{1}{2}h_2(h_2-1), \tag{280}$$

we arrive at last at a stage of the process at which we have to satisfy a system of ${}^{\backprime}h_1$ equations of the first degree by the ratios of $m-h_2$ quantities; and now, at length, we deduce this final condition of inequality, to be satisfied by the number m, in order to the general success of the method (in the case $t=2$),

$$m-h_2>{}^{\backprime}h_1; \tag{281}$$

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that is, $$m > h_1 + \tfrac{1}{2}(h_2+1)h_2; \tag{282}$$

or, in other words, m must at least be equal to the following *minor limit*,

$$m(h_1, h_2) = h_1 + 1 + \tfrac{1}{2}(h_2+1)h_2. \tag{283}$$

For example, making $h_1 = 1$, and $h_2 = 2$, we find that a system containing *one* homogeneous equation of the first degree, and *two* of the second, can be satisfied, in general, without any elevation of degree by elimination, and therefore without its being necessary to resolve any equation higher than the second degree, by the ratios of m quantities, provided that this number m is not less than the minor limit *five*: a result which may be briefly thus expressed,

$$m(1,2) = 5. \tag{284}$$

[16]. Indeed, it might seem, that in the process last described, an advantage would be gained by stopping at that stage, at which, by making $i = h_2 - 1$ in the formulae (278), we should have

$$h_2^{(h_2-1)} = 1, \quad h_1^{(h_2-1)} = h_1 + \tfrac{1}{2}h_2(h_2-1), \tag{285}$$

and $$m - i = m - h_2 + 1; \tag{286}$$

that is, when we should have to satisfy, by the ratios of $m - h_2 + 1$ quantities, a system containing only *one* equation of the second degree, in combination with $h_1 + \frac{1}{2}h_2(h_2-1)$ of the first. For, the ordinary process of elimination, performed between the equations of this last system, would not conduct to any equation higher than the second degree; and hence, without going any further, we might perceive it to be sufficient that the number m should satisfy this condition of inequality,

$$m - h_2 + 1 > h_1 + \tfrac{1}{2}h_2(h_2-1) + 1. \tag{287}$$

But it is easy to see that this alteration of method introduces no real simplification; the condition (287) being really coincident with the condition (282) or (283). To illustrate this result, it may be worth observing, that, in general, instead of the ordinary mode of satisfying, by ordinary elimination, any system of rational and integral and homogeneous equations, containing n such equations of the first degree,

$$`A' = 0, \quad `A'' = 0, \; \ldots, \; `A^{(n)} = 0, \tag{288}$$

and one of the second degree, $$`B' = 0, \tag{289}$$

by the $n+1$ ratios of $n+2$ disposable quantities,

$$a_1, \; a_2, \; \ldots, \; a_{n+2}, \tag{290}$$

it is permitted to proceed as follows. Decomposing each of the first $n+1$ quantities into two parts, so as to put

$$a_1 = a_1' + a_1'', \quad a_2 = a_2' + a_2'', \; \ldots, \; a_{n+1} = a_{n+1}' + a_{n+1}'', \tag{291}$$

we may decompose each of the given functions of the first degree, such as $`A^{(\alpha)}$, into two corresponding parts, $`A^{(\alpha)}_{1,0}$ and $`A^{(\alpha)}_{0,1}$, of which the former, $`A^{(\alpha)}_{1,0}$, is a function of the first degree of the $n+2$ quantities,

$$a_1', \; a_2', \; \ldots, \; a_{n+1}', \; a_{n+2}', \tag{292}$$

while the latter, $`A^{(\alpha)}_{0,1}$, is a function of the first degree of the $n+1$ other quantities

$$a_1'', \; a_2'', \; \ldots, \; a_{n+1}''; \tag{293}$$

and then, after resolving in any manner the indeterminate problem, to satisfy the n equations of the first degree,

$$`A_{1,0}' = 0, \quad `A_{1,0}'' = 0, \; \ldots, \; `A^{(n)}_{1,0} = 0, \tag{294}$$

by a suitable selection of the $n+1$ ratios of the $n+2$ quantities (292), (excluding only the

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assumption $a_{n+2}=0$,) we may determine the n ratios of the $n+1$ quantities (293), so as to satisfy these n other equations of the first degree,

$$`A'_{0,1}=0,\ `A''_{0,1}=0,\ \ldots,\ `A^{(n)}_{0,1}=0; \tag{295}$$

after which it will only remain to determine the ratio of any one of these latter quantities (293) to any one of the former quantities (292), so as to satisfy the equation of the second degree (289), and the original problem will be resolved.

[17]. Again, let

$$t=3; \tag{296}$$

that is, let us consider a system containing h_1 equations of the first degree, such as those marked (193), along with h_2 equations of the second degree (194), and h_3 equations of the third degree (195), to be satisfied by the ratios of m disposable quantities (192). After exhausting, by the general process already sufficiently explained, all the equations of the third degree in all the auxiliary systems, we are conducted to satisfy, if possible, by the ratios of $m-h_3$ quantities, a system containing $`h_1$ equations of the first, and $`h_2$ of the second degree, in which,

$$`h_2=h_2+\tfrac{1}{2}h_3(h_3-1),\quad `h_1=h_1+h_3h_2+\tfrac{1}{3}(h_3+1)\,h_3(h_3-1); \tag{297}$$

and after exhausting, next, all the equations of the second degree in all the new auxiliary systems, we are conducted to satisfy, by the ratios of $m-h_3-`h_2$ quantities, a system of $``h_1$ equations of the first degree, in which,

$$``h_1=`h_1+\tfrac{1}{2}`h_2(`h_2-1). \tag{298}$$

We find, therefore, that the number m must satisfy the following condition of inequality,

$$m-h_3-`h_2>``h_1, \tag{299}$$

that is,

$$m>h_3+`h_2+``h_1. \tag{300}$$

On substituting for $``h_1$ its value (298), this last condition becomes,

$$m>h_3+\tfrac{1}{2}`h_2(`h_2+1)+`h_1; \tag{301}$$

that is, in virtue of the expressions (297),

$$m>h_1+\tfrac{1}{2}(h_2+1)\,h_2+\tfrac{1}{2}(h_2+1)\,(h_3+1)\,h_3 + \tfrac{1}{3}(h_3+1)\,h_3(h_3-1)+\tfrac{1}{8}(h_3+1)\,h_3(h_3-1)\,(h_3-2). \tag{302}$$

The number m must therefore equal or surpass a certain minor limit, which, in the notation of factorials, may be expressed as follows:

$$m<(h_1+1)+\tfrac{1}{2}[h_2+1]^2+\tfrac{1}{2}(h_2+1)\,[h_3+1]^2+\tfrac{1}{3}[h_3+1]^3+\tfrac{1}{8}[h_3+1]^4; \tag{303}$$

the symbol $[\eta]^n$ denoting the continued product,

$$[\eta]^n=\eta(\eta-1)\,(\eta-2)\ldots(\eta-n+1). \tag{304}$$

So that when we denote this minor limit of m by the symbol $m(h_1,h_2,h_3)$, we obtain, in general, the formula

$$m(h_1,h_2,h_3)=\eta_1+\tfrac{1}{2}[\eta_2]^2+\tfrac{1}{2}\eta_2[\eta_3]^2+\tfrac{1}{3}[\eta_3]^3+\tfrac{1}{8}[\eta_3]^4, \tag{305}$$

in which,

$$\eta_1=h_1+1,\quad \eta_2=h_2+1,\quad \eta_3=h_3+1. \tag{306}$$

For example,

$$m(1,1,1)=5. \tag{307}$$

[18]. When

$$t=4, \tag{308}$$

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that is, when some of the original equations are as high as the fourth degree, (but none more elevated,) then

$$\left.\begin{aligned}{}^{\backprime}h_3 &= h_3+\tfrac{1}{2}h_4(h_4-1),\\ {}^{\backprime}h_2 &= h_2+h_4h_3+\tfrac{1}{3}(h_4+1)h_4(h_4-1),\\ {}^{\backprime}h_1 &= h_1+h_4h_2+\tfrac{1}{2}(h_4+1)\,(h_4h_3+\tfrac{1}{8}(h_4+2)\,(h_4+1)h_4(h_4-1);\end{aligned}\right\}\tag{309}$$

$$\left.\begin{aligned}{}^{\backprime\backprime}h_2 &= {}^{\backprime}h_2+\tfrac{1}{2}{}^{\backprime}h_3({}^{\backprime}h_3-1),\\ {}^{\backprime\backprime}h_1 &= {}^{\backprime}h_1+{}^{\backprime}h_3\,{}^{\backprime}h_2+\tfrac{1}{3}({}^{\backprime}h_3+1)\,{}^{\backprime}h_3({}^{\backprime}h_3-1);\end{aligned}\right\}\tag{310}$$

$${}^{\backprime\backprime\backprime}h_1 = {}^{\backprime\backprime}h_1+\tfrac{1}{2}{}^{\backprime\backprime}h_2({}^{\backprime\backprime}h_2-1);\tag{311}$$

and the minor limit of m, denoted by the symbol $m(h_1, h_2, h_3, h_4)$, is given by the equation

$$m(h_1,h_2,h_3,h_4)=h_4+{}^{\backprime}h_3+{}^{\backprime\backprime}h_2+{}^{\backprime\backprime\backprime}h_1+1;\tag{312}$$

which may be thus developed,

$$\begin{aligned}m(h_1,h_2,h_3,h_4)=\eta_1&+\tfrac{1}{2}[\eta_2]^2+\tfrac{1}{2}\eta_2[\eta_3]^2+\tfrac{1}{3}[\eta_3]^3+\tfrac{1}{8}[\eta_3]^4+\eta_2\{\tfrac{1}{2}\eta_3[\eta_4]^2+\tfrac{1}{3}[\eta_4]^3+\tfrac{1}{8}[\eta_4]^4\}\\ &+\tfrac{1}{4}[\eta_3]^3\,[\eta_4]^2+[\eta_3]^2\,\{\tfrac{1}{2}[\eta_4]^2+\tfrac{2}{3}[\eta_4]^3+\tfrac{3}{16}[\eta_4]^4\}+\eta_3\{[\eta_4]^3+\tfrac{7}{4}[\eta_4]^4+\tfrac{2}{3}[\eta_4]^5+\tfrac{1}{16}[\eta_4]^6\}\\ &+\tfrac{3}{2}[\eta_4]^4+\tfrac{5}{2}[\eta_4]^5+\tfrac{79}{72}[\eta_4]^6+\tfrac{1}{6}[\eta_4]^7+\tfrac{1}{128}[\eta_4]^8,\end{aligned}\tag{313}$$

if we employ the notation of factorials, and put for abridgment,

$$\eta_1=h_1+1,\ \ldots,\ \eta_4=h_4+1.\tag{314}$$

In the notation of powers, we have

$$\begin{aligned}m(h_1,h_2,h_3,h_4)=1+h_1&+\tfrac{1}{24}h_2(12+10h_4+9h_4^2+2h_4^3+3h_4^4)+\tfrac{1}{2}h_2h_3(1+h_4+h_4^2)+\tfrac{1}{2}h_2h_3^2+\tfrac{1}{2}h_2^2\\ &+\tfrac{1}{48}h_3(20+22h_4+25h_4^2+9h_4^3+8h_4^4+5h_4^5+3h_4^6)\\ &+\tfrac{1}{48}h_3^2(18+10h_4+15h_4^2+14h_4^3+9h_4^4)+\tfrac{1}{12}h_3^3(1+3h_4+3h_4^2)+\tfrac{1}{8}h_3^4\\ &+\tfrac{1}{1152}(432h_4+364h_4^2+108h_4^3+169h_4^4+24h_4^5+34h_4^6+12h_4^7+9h_4^8).\end{aligned}\tag{315}$$

As examples, whichever formula we employ, we find

$$m(1,\,0,\,1,\,1)=7;\tag{316}$$

$$m(1,\,1,\,1,\,1)=11;\tag{317}$$

$$m(1,\,1,\,1,\,2)=47;\tag{318}$$

$$m(5,\,4,\,3,\,3)=922.\tag{319}$$

[19]. In general (by the nature of the process explained in the foregoing articles) the minor limit (256) of the number m, which we have denoted by the symbol

$$m(h_1,h_2,\ldots,h_t),$$

is a function such that

$$m(h_1,h_2,\ldots,h_t)=1+m(h_1',h_2',\ldots,h_t'),\tag{320}$$

$h_1',\ldots,h_t'$ being determined by the formulae (233). This equation in finite differences (320) may be regarded as containing the most essential element of the whole foregoing discussion; and from it the formulae already found for the cases $t=2$, $t=3$, $t=4$, might have been deduced in other ways. From it also we may perceive, that whenever the original system contains only one equation of the highest or t^{th} degree, in such a manner that

$$h_t=1,\tag{321}$$

then, whatever t may be, we have the formula

$$m(h_1,h_2,\ldots,h_{t-1},1)=1+m(h_1+h_2+\ldots+h_{t-1},h_2+\ldots+h_{t-1},\ldots,h_{t-1});\tag{322}$$

so that, for example,

$$m(1, 1, 1, 1, 1) = 1 + m(4, 3, 2, 1); \tag{323}$$

$$m(4, 3, 2, 1) = 1 + m(9, 5, 2) = 46; \tag{324}$$

$$m(1, 1, 1, 1, 1, 1) = 1 + m(5, 4, 3, 2, 1); \tag{325}$$

$$m(5, 4, 3, 2, 1) = 1 + m(14, 9, 5, 2) = 922; \tag{326}$$

and therefore

$$m(1, 1, 1, 1, 1) = 47, \tag{327}$$

and

$$m(1, 1, 1, 1, 1, 1) = 923. \tag{328}$$

[20]. The formula $m(1,1,1) = 5$, (307)

may be considered as expressing, generally, that in order to satisfy a system of three homogeneous equations, rational and integral, and of the forms

$$A' = 0, \quad B' = 0, \quad C' = 0, \tag{329}$$

that is, of the first, second, and third degrees, by a system of ratios of m disposable quantities

$$a_1, \ldots, a_m, \quad (192)$$

which ratios are to be discovered by Mr Jerrard's method of decomposition, without any elevation of degree by elimination, the number m ought to be at least equal to the minor limit *five*; a result which includes and illustrates that obtained in the 4th article of the present communication, respecting Mr Jerrard's process for *taking away three terms* at once from the general equation of the m^{th} degree: namely that this process is not generally applicable when m is less than *five*. Again, the process described in the eleventh article, for taking away, on Mr Jerrard's principles, *four terms* at once from the general equation of the m^{th} degree, without being obliged to eliminate between any two equations of condition of higher degrees than the first, was shown to require, for its success, in general, that m should be at least equal to the minor limit *eleven*; and this limitation is included in, and illustrated by, the result

$$m(1, 1, 1, 1) = 11, \quad (317)$$

which expresses generally a similar limitation to the analogous process for satisfying any four homogeneous equations of condition,

$$A' = 0, \quad B' = 0, \quad C' = 0, \quad D' = 0, \tag{330}$$

of the first, second, third, and fourth degrees, by the ratios of m disposable quantities, $a_1, a_2, \ldots, a_m$. In like manner it is shown by the result

$$m(1, 1, 1, 1, 1) = 47, \quad (327)$$

that Mr Jerrard's general method would not avail to satisfy the five conditions

$$A' = 0, \quad B' = 0, \quad C' = 0, \quad D' = 0, \quad E' = 0, \tag{331}$$

and so to take away *five terms* at once from the equation of the m^{th} degree, without any elevation of degree being introduced in the eliminations, unless m be at least $= 47$, that is, unless the equation to be transformed be at least of the 47th degree; and the result

$$m(1, 1, 1, 1, 1, 1) = 923, \quad (328)$$

shows that the analogous process for taking away *six terms* at once, or for satisfying the six conditions

$$A' = 0, \quad B' = 0, \quad C' = 0, \quad D' = 0, \quad E' = 0, \quad F' = 0, \tag{332}$$

is limited to equations of the 923rd and higher degrees.*

* [The numbers

$m(1) = 2$, $m(1, 1) = 3$, $m(1, 1, 1) = 5$, $m(1, 1, 1, 1) = 11$, $m(1, 1, 1, 1, 1) = 47$, $m(1, 1, 1, 1, 1, 1) = 923$

are the first six Hamiltonian numbers. See *Introduction*, Section 4].

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Finally, the result

$$m(1,0,1,1)=7, \quad (316)$$

and the connected result

$$m(1, 0, 1, 0, 1)=7, \quad (333)$$

show that it is not in general possible to satisfy, by the same method, a system of three conditions of the first, third, and fourth degrees, respectively, such as the system

$$A'=0, \quad C'=0, \quad D'-\alpha B'^2=0, \quad (334)$$

nor a system of 3 conditions of the first, third, and fifth degrees,

$$A'=0, \quad C'=0, \quad E'=0, \quad (335)$$

unless m be at least $=7$; which illustrates and confirms the conclusions before obtained respecting the inadequacy of the method to reduce the general equation of the fifth degree to De Moivre's solvible form,* or to reduce the general equation of the sixth to that of the fifth degree.

[21]. Indeed, if *some* elevation of degree be admitted in the eliminations between the auxiliary equations, the minor limit of the number m may sometimes be advantageously depressed. Thus, in the process for satisfying the system of equations (330), we first reduce the original difficulty to that of satisfying, by the ratios of $m-1$ quantities, a system containing three equations of the first degree, two of the second, and one of the third; and we next reduce this difficulty to that of satisfying, by the ratios of $m-2$ quantities, a system containing five equations of the first, and two of the second degree. Now, at this stage, it is advantageous to depart from the general method, and to have recourse to ordinary elimination; because we can thus resolve the last-mentioned auxiliary system, not indeed without *some* elevation of degree, but with an elevation which conducts no higher than a biquadratic equation; and by avoiding the additional decomposition which the unmodified method requires, we are able to employ a lower limit for m. In fact, the general method would have led us to a new transformation of the question, by which it would have been required to satisfy, by the ratios of $m-3$ new quantities, a system containing six new equations of the first, and one of the second degree; it would therefore have been necessary, in general, in employing that method, that $m-3$ should be greater than $6+1$, or in other words that m should be at least equal to the minor limit *eleven*; and accordingly we found

$$m(1, 1, 1, 1)=11. \quad (317)$$

But when we dispense with this last decomposition, we need only have $m-2>5+2$, and the process, by this modification, succeeds even for $m=10$. It was thus that Mr Jerrard's principles were shown, in the tenth article of this paper, to furnish a process for taking away four terms at once from equations as low as the tenth degree, provided that we employ (as we may) certain auxiliary systems of conditions, (160) and (161), of which each contains two equations of the second degree, but none of a degree more elevated. But it appears to be impossible, by any such mixture of ordinary elimination with the general method explained above, to depress so far that lower limit of m which has been assigned by the foregoing discussion, as to render the method available for *resolving* any general equation, by reducing it to any known solvible form. This *Method of Decomposition* has, however, conducted, in the hands of its inventor

* [See XLVII, p. 474, footnote.]

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Mr Jerrard, to several general *transformations* of equations, which must be considered as discoveries in algebra; and to the solution of an extensive class of problems in the analysis of *indeterminates*, which had not before been resolved: the *notation*, also, of *symmetric functions*, which has been employed by that mathematician, in his published *Researches** on these subjects, is one of great beauty and power.

† *Mathematical Researches*, by George B. Jerrard, A.B., Bristol; printed by William Strong, Clare Street; to be had of Longman and Co., London.

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