TRAVAUX MATHE'MATIQUES

PARTIE II TOPOLOGIE

ANDRZEJ GRANAS

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TOPOLOGIE

A. Topologie des espaces fonctionnels:

B. Theorie des points fixes:

A. Topologie des Espaces Fonctionnels

On local disconnection of Euclidean spaces

by

A. Granas (Warszawa)

1. Introduction. Let S_n be the n-dimensional sphere defined in the $(n+1)$-dimensional Euclidean space E_{n+1} by the equation

$$x_1^2+x_2^2+\ldots+x_{n+1}^2=1.$$

We consider a closed set $F\subset S_{n+1}$. Let U_a^ε denote the ε-neighbourhood of a point $a\in F$ in S_{n+1}, *i. e.*

$$U_a^\varepsilon=\underset{x\in S_{n+1}}{E}[|x-a|<\varepsilon].$$

Let us denote by $b_0^{\varepsilon,\eta}$, for every two positive numbers ε and η, $\varepsilon>\eta$, the number of components of $U_a^\varepsilon-F$ which have a common point with U_a^η. If $\eta<\eta'$, then $b_0^{\varepsilon,\eta}\leqslant b_0^{\varepsilon,\eta'}$, and consequently there exists

$$(1)\qquad \lim_{\eta\to 0} b_0^{\varepsilon,\eta}=b_0^\varepsilon.$$

Evidently $b_0^\varepsilon\geqslant b_0^{\varepsilon'}$ if $\varepsilon<\varepsilon'$. Consequently there exists a finite or infinite limit

$$(2)\qquad \lim_{\varepsilon\to 0} b_0^\varepsilon=b_0(a,S_{n+1}-F).$$

The number $b_0(a,S_{n+1}-F)$ will be called the *number of components in which F decomposes the $n+1$ dimensional sphere S_{n+1} at the point a.*

In 1933 E. Čech [1]) proved, using the notion of local Betti numbers, that the number $b_0(a,S_{n+1}-F)$ is a topological invariant. The purpose of this paper is to give an elementary proof of this fact without using any notion of algebraic topology. The method of proof is based on the notion of Borsuk's cohomotopy groups [2]) and Borsuk's theorem [3]) on the structure of the n-th cohomotopy group of closed subset F of S_{n+1}.

[1]) E. Čech [1].
[2]) K. Borsuk [2].
[3]) K. Borsuk [3].

2. Definitions and notations. Throughout the paper by space we understand a metric space and by a mapping a continuous transformation.

$X \times Y$ will denote the Cartesian product of two spaces X and Y *i. e.* the set of all ordered pairs (x,y) with $x \in X$, $y \in Y$, metrized by the formula

$$|(x,y)-(x',y')| = \sqrt{|x-x'|^2+|y-y'|^2}.$$

If X_0 is a subset of X and f a mapping with the range X, then $f|X_0$ will denote the *partial mapping* of f defined in X_0 *i. e.* the mapping f_0 defined in X_0 by the formula $f_0(x)=f(x)$. We shall say that f constitutes an *extension* of f_0 on X; we then write $f_0 \subset f$.

Y_n^X will denote the set of all mappings of X into a compact space Y_n. In the functional space Y_n^X we define a metric topology setting

$$|f-g| = \sup_{x \in X} |f(x)-g(x)| \quad \text{for every } f, g \in Y_n^X.$$

Two mappings $f, g \in Y_n^X$ are called *homotopic* (written $f \sim g$) if there exists a mapping $h \in Y_n^{X \times I}$ where I denotes the closed interval $0 \leqslant t \leqslant 1$, such that

$$\left.\begin{aligned} h(x,0)&=f(x), \\ h(x,1)&=g(x) \end{aligned}\right\} \quad \text{for every } x \in X.$$

The relation of homotopy, established in Y_n^X, is a relation of equivalence and thus the set of all mappings $f \in Y_n^X$ decomposes into disjoint classes of homotopic mappings. The class of all mappings homotopic with a mapping $f \in Y_n^X$ will be denoted by (f) and called the homotopy class of f. A mapping $f \in Y_n^X$ homotopic to a constant is said to be *unessential*; we then write $f \sim 1$.

If X_0 is a closed subset of a compact space X, then by $X\|X_0$ we denote the space obtained from X by identifying X_0 to a point q_{X_0}. It is known[4]) that for every space $X\|X_0$ there exists a natural mapping $\varphi \in (X\|X_0)^X$ which maps $X-X_0$ topologically onto $X\|X_0 - q_{X_0}$.

3. Cohomotopy groups. In this section we give the definition and some properties of Borsuk's cohomotopy groups needed in the sequel.

By a *product* of the mappings $f, g \in S_n^X$ we understand the mapping $f \times g \, (S_n \times S_n)^X$ defined by the formula

$$(f \times g)(x) = \big(f(x), g(x)\big) \quad \text{for every } x \in X.$$

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[4]) See for instance C. Kuratowski [4], p. 42.

It is known [5]) that:

(3) *If X is a compactum and* $\dim X < 2n$, *then for exery* $f \times g \in (S_n \times S_n)^X$ *there exists a mapping* $h \in (S_n \times S_n)^{X \times I}$ *satisfying the conditions*

$$h(x,0) = (f \times g)(x) \quad \text{for every } x \in X,$$
$$h(X,1) \subset b_0 \times S_n + S_n \times b_0,$$

where b_0 *is an arbitrary point of* S_n.

In this case we define the *sum* of the homotopy classes $(f),(g) \subset S_n^X$ in the following manner: Setting

$$\omega(x) = h(x,1) \quad \text{for} \quad x \in X,$$
$$\vartheta_n(y,b_0) = y \quad \text{for} \quad (y,b_0) \in S_n \times b_0,$$
$$\vartheta_n(b_0,y) = y \quad \text{for} \quad (b_0,y) \in b_0 \times S_n,$$
$$S_n \wedge S_n = b_0 \times S_n + S_n \times b_0,$$

we have $\omega \in (S_n \wedge S_n)^X$, $\vartheta_n \in S_n^{S_n \wedge S_n}$ and $\vartheta_n \omega \in S_n^X$. We define the *sum* $(f)+(g)$ of the homotopy classes (f), $(g) \subset S_n^X$ by setting

$$(f)+(g) = (\vartheta_n \omega).$$

It is known [6]) that:

(4) *If X is a compactum and* $\dim X < 2n-1$, *then the homotopy classes* $(f) \subset S_n^X$ *constitute an Abelian group with the operation defined as addition of homotopy classes.*

This group is called the n-th *Borsuk group* or n-th *cohomotopy group* of X and is denoted in the sequel by $B_n(X)$; the order of this group will be denoted by $b_n(X)$.

The *zero* element of $B_n(X)$ is the homotopy class which contains unessential mappings $f \in S_n^X$. An *inverse* element to $(f) \in B_n(X)$ is obtained in the following manner: Setting

$$\varrho_n(x_1,x_2,\dots,x_n,x_{n+1}) = (x_1,x_2,\dots,x_n,-x_{n+1}),$$

for every $(x_1,x_2,\dots,x_{n+1}) \in S_n$, we define the inverse element $-(f)$ as the homotopy class containing $\varrho_n f \in S_n^X$.

If X_0 is a closed subset of a compactum X (with $\dim X < 2n-1$), then

(5)
$$(f)+(g) = (h) \quad \text{implies} \quad (f|X_0)+(g|X_0) = (h|X_0),$$
$$(f_1) = -(f_2) \quad \text{implies} \quad (f_1|X_0) = -(f_2|X_0).$$

Let f be a mapping of X into Y. If $g \in S_n^Y$; then $gf \in S_n^X$, if $g_1 \sim g_2$, then $g_1 f \sim g_2 f$. Let us set, for every homotopy class $(g) \subset S_n^Y$, $\hat{f}[(g)] = (gf) \subset S_n^X$.

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[5]) See K. Borsuk [2] and E. Spanier [5].

[6]) See K. Borsuk [2] and E. Spanier [5], p. 211.

It is known[7]) that:

(6) *If* X *and* Y *are compacta and* $\dim X<2n-1$, $\dim Y<2n-1$, *then the mapping* $\hat{f}$ *(induced by* f*) is a homomorphism of* $B_n(Y)$ *into* $B_n(X)$.

Let X_0 be a closed subset of X. By $H_n(X,X_0)$ we denote the set of homotopy classes $(f)\subset S_n^X$ such that $f|X_0\sim 1$. Then:

If X *is a compactum and* $\dim X<2n-1$, *then the set* $H_n(X,X_0)$ *is a subgroup of* $B_n(X)$.

Proof. Let $(f),(g)\in H_n(X,X_0)$ and $(f)+(g)=(h)$. By (5) it is $(h|X_0)=(f|X_0)+(g|X_0)$. But $f|X_0\sim 1$ and $g|X_0\sim 1$, hence $h|X_0\sim 1$ and $(h)\in H_n(X,X_0)$. If $(f_1)\in H_n(X,X_0)$ and $(f_2)=-(f_1)$, then, by (5), $(f_2|X_0)=-(f_1|X_0)$. But $f_1|X_0\sim 1$, hence $f_2|X_0\sim 1$ and $(f_2)\in H_n(X,X_0)$.

The order of the group $H_n(X,X_0)$ will be denoted by $h_n(X,X_0)$.

4. Some lemmas. Let F be a proper closed subset of S_{n+1}. Let $G_0,G_1,\ldots,G_i,\ldots$ be a finite or infinite sequence of all components of $S_{n+1}-F$. In every component G_i we choose an arbitrary point p_i and a spherical $n+1$ dimensional element Q_i with centre p_i and boundary S_{ni}.

It is known[8]) that:

(7) *There exists a one-one correspondence between the set of all homotopy classes* $(f)\subset S_n^F$ *and the set of all sequences* $\{(f_i)\}$, *where* $(f_i)\subset S_n^{S_{ni}}$, $f_i=f|S_{ni}$, $i=1,2,\ldots$ *and* f_i *is unessential for almost all* i. *This correspondence is an isomorphism between the cohomotopy group* $B_n(F)$ *and the direct sum*[9]) $\sum_{i=1}^{\infty} B_n(S_{ni})$.

Now let $r^*_{p_0p_i}$ be for every $i\geqslant 1$ a mapping of S_{ni} onto S_n, homotopic to a homeomorphism of S_{ni} onto S_n. Let $r_{p_0p_i}$ be an extension of the mapping $r^*_{p_0p_i}\in S_n^{S_{ni}}$ on $S_{n+1}-(p_0)-(p_i)$. Then, for $i\neq j$, the mapping $r_{p_0p_i}|S_{nj}$ is unessential (because $r_{p_0p_i}|S_{nj}\subset r_{p_0p_i}|Q_{nj}\in S_n^{Q_{nj}}$), and for every $i=1,2,\ldots$ the homotopy class $(r_{p_0p_i}|S_{ni})\subset S_n^{S_{ni}}$ is a generator of the free cyclic group $B_n(S_{ni})$. From this, applying (7), we obtain the following

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[7]) See E. Spanier [5], p. 214.

[8]) See K. Borsuk [3], p. 227 and 240.

[9]) By the direct sum $\sum_i A_i$ of Abelian groups A_i, $i=1,2,\ldots$ we understand the Abelian group A constituted by all sequences $\{a_i\}$ with $a_i\in A_i$, where $a_i=0$ for almost all indices i and where the group operation is defined by the formula $\{a_i\}+\{a_i'\}=\{a_i+a_i'\}$. It is clear that if a_i is a generator of the free cyclic group A_i and $\delta^i_j=0$ for $i\neq j$, $\delta^i_j=1$ for $i=j$, then the sequence $\{\delta^1_i\cdot a_1\},\{\delta^2_i\cdot a_2\},\ldots,\{\delta^j_i\cdot a_j\},\ldots$ constitutes the basis of the group $A=\sum_i A_i$.

LEMMA 1. *The sequence of the homotopy classes*

$$(8)\qquad (r_{p_0p_1}|F),(r_{p_0p_2}|F),\ldots,(r_{p_0p_i}|F),\ldots\subset S_n^F$$

constitutes the basis of the n-th cohomotopy group of F.

LEMMA 2. *Let F be a closed subset of S_{n+1} $(F\neq S_{n+1})$ and G an open connected subset of S_{n+1} $(G\neq S_{n+1})$. If $G_0,G_1,\ldots,G_k$ are all components of $S_{n+1}-F$ such that $G\cdot G_i\neq 0$ for $i=0,1,\ldots,k$, then $k=h_n(F,F-G)$.*

Proof. Let us order all components of $S_{n+1}-F$ in a finite or infinite sequence $G_0,G_1,\ldots,G_k,G_{k+1},\ldots$ and choose in every component G_i a point p_i in such a manner that $p_0,p_1,\ldots,p_k\in G$. We infer by lemma 1 that the homotopy classes

$$(9)\qquad (r_{p_0p_1}|F),(r_{p_0p_2}|F),\ldots,(r_{p_0p_k}|F)\in B_n(F)$$

are linearly independent. Since $F-G$ does not disconnect S_{n+1} between any pair of points $p_0,p_1,\ldots,p_k$, then $r_{p_0p_i}|F-G\sim 1$ for every $i=1,2,\ldots,k$. Thus we have shown that there exist at least k linearly independent elements of the group $H_n(F,F-G)$.

Now let us have any mapping $f\in S_n^F$ such that $f|F-G\sim 1$. We shall prove that the homotopy class (f) is a linear combination of the classes (9). By lemma 1 the homotopy class (f) is a linear combination of a finite number of elements of the sequence (8):

$$(10)\qquad (f)=c_1(r_{p_0p_{i_1}}|F)+c_2(r_{p_0p_{i_2}}|F)+\ldots+c_m(r_{p_0p_{i_m}}|F).$$

Since the set $F-G$ disconnects S_{n+1} between every pair of the points of the sequence $p_0,p_{k+1},p_{k+2},\ldots$, we have

$$(11)\qquad (r_{p_0p_j}|F-G)\neq(\mathrm{const}\,|F-G)\qquad \text{for every } j>k.$$

From this and from $f|F-G\sim 1$ we infer that the linear combination (10) cannot contain any class $(r_{p_0p_j}|F)$ for $j>k$. Consequently the homotopy class (f) is a linear combination of the homotopy classes (9) and the proof of lemma 2 is completed.

LEMMA 3. *If H_1, H_2 and G are three open neighbourhoods of a point $a\in F\,(\dim F<2n-1)$ such that $H_1\subset H_2\subset G$, then*

$$(12)\qquad h_n[F\|F-G,(F\|F-G)-H_1]\leqslant h_n[F\|F-G,(F\|F-G)-H_2].$$

Proof. Let us set $F^*=F\|F-G$. For every $f\in S_n^{F^*}$ the relation $f|F^*-H_1\sim 1$ implies $f|F^*-H_2\sim 1$. It follows that $H_n[F^*,F^*-H_1]\subset H_n[F^*,F^*-H_2]$ and consequently also (12).

LEMMA 4. *If H, G_1 and G_2 are three open neighbourhoods of a point $a \in F(\dim F < 2n-1)$ such that $H \subset G_1 \subset G_2$, then*

$$h_n(F^*, F^*-H) \leqslant h_n[F^* \| F^*-G_1, (F^* \| F^*-G_1)-H], \tag{13}$$

where F^* denotes the set $F \| F-G_2$.

Proof. Let φ be a natural mapping of F^* onto $F^{**} = F^* \| F^*-G_1$ and $\hat{\varphi}$ the induced homomorphism of $B_n(F^{**})$ into $B_n(F^*)$. If $f \in S_n^{F^{**}}$ and $f|F^{**}-H \sim 1$, then $f|F^*-H \sim 1$. From this we infer that

$$\hat{\varphi}\{H_n(F^{**}, F^{**}-H)\} \subset H_n(F^*, F^*-H).$$

Now let $(g) \subset S_n^{F^*}$ and $g|F^*-H \sim 1$. Without loss of generality we can suppose that $g(F^*-H) = p_0 \in S_n$. It follows that $g(F^*-G_1) = p_0$. We define the mapping h of F^{**} into S_n as follows:

$$h(x) = \begin{cases} g[\varphi^{-1}(x)] & \text{for } x \in F^* \| F^*-G_1-(q_{F^*-G_1}), \\ p_0 & \text{if } x = q_{F^*-G_1}. \end{cases}$$

Evidently $h \in S_n^{F^{**}}$, $h|F^{**}-H \sim 1$ and $g(x) = h[\varphi(x)]$ for every $x \in F^*$. It follows that $\hat{\varphi}\{(h)\} = (g)$ and $\hat{\varphi}\{H_n[F^{**} \| F^{**}-H]\} = H_n(F^*, F^*-H)$. From this we infer the inequality (13).

5. The local cohomotopy numbers. Let a be an arbitrary point of a compactum F with $\dim F < 2n-1$. Let $U_a^\varepsilon(F)$ denote the ε-neighbourhood of a in F, *i. e.*

$$U_a^\varepsilon(F) = \underset{x \in F}{E}[|x-a| < \varepsilon].$$

Let us set

$$b_n^{\varepsilon,\eta}(a,F) = h_n\left[F \| F-U_a^\varepsilon(F), \left(F \| F-U_a^\varepsilon(F)\right)-U_a^\eta(F)\right] \quad \text{for} \quad 0 < \varepsilon < \eta. \tag{14}$$

By lemma 3, if $\eta < \eta'$, then $b_n^{\varepsilon,\eta} \leqslant b_n^{\varepsilon,\eta'}$. Consequently there exists

$$\lim_{\eta \to 0} b_n^{\varepsilon,\eta}(a,F) = b_n^\varepsilon(a,F). \tag{15}$$

By lemma 4, if $\varepsilon < \varepsilon'$, then $b_n^\varepsilon(a,F) \geqslant b_n^{\varepsilon'}(a,F)$. Consequently there exists a finite or infinite limit

$$\lim_{\varepsilon \to 0} b_n^\varepsilon(a,F) = b_n(a,F). \tag{16}$$

The number $b_n(a,F)$ will be called the *local cohomotopy number of F at the point $a \in F$.*

From the definition of $b_0^{\varepsilon,\eta}$ and $b_n^{\varepsilon,\eta}(a,F)$ and by lemma 2, we infer that in the case of $F \subset S_{n+1}$

$$b_0^{\varepsilon,\eta} = b_n^{\varepsilon,\eta}(a,F)+1. \tag{17}$$

From (17), (1), (2), (15) and (16) we obtain the following

THEOREM. *If $a \in F = \overline{F} \subset S_{n+1}$, then the number of components $b_0(a, S_{n+1}-F)$ in which the set F decomposes the $(n+1)$-dimensional sphere S_{n+1} at the point a is determined by the local cohomotopy number $b_n(a,F)$ of F at the point a by the formula*

$$b_0(a, S_{n+1}-F) = b_n(a,F)+1. \tag{18}$$

Since the number $b_n(a,F)$ is topologically invariant, we obtain the following

COROLLARY. *The number of components $b_0(a, S_{n+1}-F)$ in which a closed set $F \subset S_{n+1}$ decomposes the $(n+1)$-dimensional sphere S_{n+1} at the point $a \in F$ is topologically invariant.*

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К теории когомотопических групп Борсука

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В настоящей заметке рассматриваются некоторые подгруппы n-мерной когомотопической группы Борсука и устанавливаются соотношения между рангами рассматриваемых групп. В качестве простого следствия одного из доказанных соотношений выводится известная теорема Фрагмена-Брауэра о разбиении евклидовых пространств. Одномерный случай рассматривался Эйленбергом (см. [2]).

1. Введём сначала обозначения употребляемые в дальнейшем, а также напомним кратко определение когомотопической группы.

Пространство непрерывных отображений компакта X в компакт Y будем обозначать через Y^X. Метрика в Y^X определяется формулой:

$$\varrho(f,g)=\sup_{x\in X}\varrho\big(f(x),g(x)\big),\quad f,g\in Y^X.$$

Отображения $f,g\in S_n^X$ [1]) называем *гомотопными*, $f\sim g$, если существует отображение $h\in S_n^{X\times I}$ [2]) (I — замкнутый отрезок $\langle 0,1\rangle$), удовлетворяющее условию:

$$h(x,0)=f(x),\qquad h(x,1)=g(x)\qquad \text{для любого}\quad x\in X.$$

Совокупность отображений $g\in S_n^X$ гомотопных отображению $f\in S_n^X$ будем называть *гомотопическим классом* отображения f и обозначим через (f). Пространство S_n^X распадается благодаря соотношению гомотопии на непересекающиеся гомотопические классы. Если отображение $f\in S_n^X$ гомотопно отображению $y=\mathrm{const}$, то будем называть его *несущественным*, записывая $f\sim 1$.

Если A, A_0 два компакта $A_0\subset A$, $f_0, g_0\in S_n^{A_0}$, $f_0\sim g_0$, $f_0\subset f\in S_n^A$ [3]), то, в силу известной теоремы Борсука (см. [4], стр. 86), существует $g\in S_n^A$, такое что $g_0\subset g$ и $g\sim f$. Гомотопический класс $(f)\subset S_n^A$ будем называть *продолжением* гомотопического класса $(f_0)\subset S_n^{A_0}$ на A.

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[1]) Здесь S_n обозначает n-мерную сферу определяемую в $(n+1)$-ном евклидовом пространстве E_{n+1} уравнением $x_1^2+x_2^2+\ldots+x_{n+1}^2=1$.

[2]) $X\times Y$ обозначает топологическое произведение пространств X и Y.

[3]) Запись $f_0\subset f\in S_n^A$, где $f_0\in S_n^{A_0}$, $A_0\subset A$, означает, что f является продолжением f_0 на A, т. е. $f(x)=f_0(x)$ для всех $x\in A_0$.

Произведением $f\times g$ отображений $f,g\in S_n^X$ будем называть отображение $f\times g\in (S_n\times S_n)^X$, определённое формулой:

$$(f\times g)(x)=\big(f(x),g(x)\big) \quad \text{для любого} \quad x\in X.$$

Известно (см. [5], стр. 209-210), что *если размерность компакта X меньше $2n$, то для любого $(f\times g)\in (S_n\times S_n)^X$ существует $h\in (S_n\times S_n)^{X\times I}$, удовлетворяющее условию*:

$$h(X,1)\subset S_n\times b_0\cup b_0\times S_n,$$

$$h(x,0)=(f\times g)(x) \quad \textit{для любого} \quad x\in X$$

(здесь b_0 произвольно фиксированная точка сферы S_n).

Положим $\vartheta(b_0,y)=y$ для $(b_0,y)\in b_0\times S_n$, $\vartheta(\overline{y},b_0)=\overline{y}$ для $(\overline{y},b_0)\in S_n\times b_0$, $S_n\wedge S_n=b_0\times S_n\cup S_n\times b_0$, $\varphi(x)=h(x,1)$; тогда имеем $\vartheta\in S_n^{S_n\wedge S_n}$, $\varphi\in (S_n\wedge S_n)^X$, значит $\vartheta\varphi\in S_n^X$.

Сумму $(f)+(g)$ гомотопических классов $(f),(g)\subset S_n^X$ определяем формулой: $(f)+(g)=(\vartheta\varphi)$.

Известно (см. [5], стр. 210-214), что когда *размерность компакта X меньше $2n-1$, то совокупность гомотопических классов $(f)\subset S_n^X$ образует абелеву группу, если групповая операция определена как сложение гомотопических классов*.

Эту группу (n-мерную когомотопическую группу компакта X) будем обозначать символом $B_n(X)$, а её ранг [4]) — символом $b_n(X)$.

Нулём группы $B_n(X)$ является класс (const). Если определим отображение ϑ_n сферы S_n на себя формулой $\vartheta_n(x_1,x_2,\ldots,x_{n+1})=(x_1,x_2,\ldots,-x_{n+1})$, тогда элемент $-(f)$, для $(f)\in B_n(X)$, совпадает с гомотопическим классом $(\vartheta_n f)\subset S_n^X$.

2. Пусть X компакт и A его замкнутое подмножество. Обозначим символом $H_n(X,A)$ множество всех $(f)\subset S_n^X$ таких, что $f|A\sim 1$, а символом $G_n(A,X)$ множество всех $(f)\subset S_n^A$, которые можно продолжить на X. Если $\dim X<2n-1$, то $H_n(X,A)$ образует подгруппу группы $B_n(X)$ (см. [3], стр. 45). Аналогично легко проверяется, что если $\dim X<2n-1$, то $G_n(A,X)$ является подгруппой группы $B_n(A)$.

Пусть $h_n(X,A)$ ранг группы $H_n(X,A)$, а $g_n(A,X)$ ранг группы $G_n(A,X)$.

Теорема 1. *Между рангами групп $B_n(X)$, $H_n(X,A)$, $G_n(A,X)$ имеет место соотношение*: $b_n(X)=h_n(X,A)+g_n(A,X)$.

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[4]) Под *рангом* абелевой группы мы понимаем максимальное число линейно независимых элементов группы.

Доказательство. Ввиду неравенств $b_n(X) \geqslant g_n(A, X)$, $b_n(X) \geqslant h_n(X, A)$ можем предположить, что числа $h_n(X, A)$, $g_n(A, X)$ конечны. Пусть гомотопические классы:

(1) $$(g_1), (g_2), \ldots, (g_m) \subset S_n^A,$$

(2) $$(h_1), (h_2), \ldots, (h_k) \subset S_n^X,$$

образуют соответственно максимальную систему линейно независимых элементов группы $G_n(A, X)$ и группы $H_n(X, A)$, $m = g_n(A. X)$, $k = h_n(X, A)$.

Пусть $(\bar{g}_i) \subset S_n^X$ продолжение класса $(g_i) \subset S_n^A$ на X $(i = 1, 2, \ldots, m)$; покажем, что система

(3) $$(\bar{g}_1), (\bar{g}_2), \ldots, (\bar{g}_m), (h_1), (h_2), \ldots, (h_k) \subset S_n^X$$

есть максимальная система линейно независимых элементов группы $B_n(X)$.

Пусть имеет место соотношение

(4) $$\sum_{i=1}^{k} p_i(h_i) + \sum_{i=1}^{m} q_i(\bar{g}_i) = 0,$$

где $p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_m$ некоторые целочисленные коэффициенты. Рассматривая соотношение (4) на A, заключаем ввиду свойства системы (2), что все q_i равны нулю, затем, учитывая свойство системы (1), заключаем, что все p_i равны нулю, а это доказывает линейную независимость элементов системы (3).

Пусть (f) произвольный элемент группы $B_n(X)$. В силу свойства системы (1) существуют такие целочисленные, не равные одновременно нулю, коэффициенты $p_0, p_1, \ldots, p_m$, что $p_0(f|A) + \sum_{i=1}^{m} p_i(g_i) = 0$ $(p_0 \neq 0)$. Положим $(h) = p_0(f) + \sum_{i=1}^{m} p_i(\bar{g}_i)$; имеем $(h|A) = 0$, т. е. $h \in H_n(X, A)$. В силу свойства системы (2) найдутся такие целочисленные $q_0, q_1, \ldots, q_k$ не все равные нулю $(q_0 \neq 0)$, что $q_0(h) + \sum_{i=1}^{k} q_i(h_i) = 0$. Отсюда получаем соотношение

$$\sum_{i=1}^{k} q_i(h_i) + p_0 q_0(f) + \sum_{i=1}^{m} q_0 p_i(\bar{g}_i) = 0, \qquad \text{где} \quad \sum_{i=1}^{k} q_i^2 + \sum_{i=0}^{m} (p_i q_i)^2 \neq 0,$$

что доказывает максимальность системы (3). Теорема доказана.

3. Пусть A и B два компакта. Обозначим через $P_n(A, B)$ совокупность всех $(f) \subset S_n^{A \cup B}$ таких, что $f|A \sim 1$, $f|B \sim 1$. Если $\dim(A \cup B) < 2n - 1$, то из очевидного равенства $P_n(A, B) = H_n(A \cup B, A) \cap H_n(A \cup B, B)$ заключаем, что $P_n(A, B)$ является подгруппой группы $B_n(A \cup B)$. Обозначим через $p_n(A, B)$ ранг группы $P_n(A, B)$ и покажем, что имеет место следующая

Теорема 2. *Ранги групп* $H_n(A\cup B, A\cap B)$, $H_n(A, A\cap B)$, $H_n(B, A\cap B)$, $P_n(A,B)$ *связаны соотношением*:

$$h_n(A\cup B, A\cap B) = h_n(A, A\cap B) + h_n(B, A\cap B) + p_n(A,B).$$

Докажем предварительно следующую лемму:

Лемма. *Если* A *и* B *два компакта*, $f\in S_n^A$ *и* $f|A\cap B\sim 1$, *то существует продолжение* $\bar f$ *отображения* $f\subset \bar f\in S_n^{A\cup B}$ *на множество* $A\cup B$, *причём* $\bar f|B\sim 1$.

Действительно, $f|A\cap B\subset f^*\in S_n^B$ и $f^*\sim 1$. Определим $\bar f$ формулой: $\bar f(x) = f(x)$ для $x\in A$, $\bar f(x) = f^*(x)$ для $x\in B$.

Легко видеть, что функция $\bar f$ удовлетворяет условиям леммы.

Доказательство теоремы 2. Если $(g)\in H_n(B, A\cap B)$, то символом $(\bar g)$ будем обозначать существующее в силу леммы такое продолжение гомотопического класса $(g)\subset S_n^B$ на $A\cup B$, что $\bar g|A\sim 1$; $(\bar g)\in H_n(A\cup B, A\cap B)$. Аналогично если $(f)\in H_n(A, A\cap B)$, то символом $(\bar f)$ будем обозначать существующее в силу леммы такое продолжение класса $(f)\subset S_n^A$ на $A\cup B$, что $\bar f|B\sim 1$; $(\bar f)\in H_n(A\cup B, A\cap B)$. В силу неравенств $h_n(A, A\cap B)\leqslant h_n(A\cup B, A\cap B)$, $h_n(B, A\cap B)\leqslant h_n(A\cup B, A\cap B)$, $p_n(A,B)\leqslant h_n(A\cup B, A\cap B)$ можем предположить, что числа $p_n(A,B), h_n(A, A\cap B), h_n(B, A\cap B)$ конечны, ибо в противном случае теорема была бы доказана.

Пусть гомотопические классы:

$$(5)\qquad (f_1),(f_2),\ldots,(f_k)\subset S_n^A,$$

$$(6)\qquad (g_1),(g_2),\ldots,(g_l)\subset S_n^B,$$

$$(7)\qquad (h_1),(h_2),\ldots,(h_m)\subset S_n^{A\cup B}$$

образуют максимальную систему линейно независимых элементов соответственно групп $H_n(A, A\cap B), H_n(B, A\cap B), P_n(A,B)$; $k = h_n(A, A\cap B)$, $l = h_n(B, A\cap B)$, $m = p_n(A,B)$.

Рассмотрим гомотопические классы:

$$(8)\qquad (\bar f_1),(f_2),\ldots,(\bar f_k)\subset S_n^{A\cup B};\quad \bar f_i|B\sim 1\quad (i=1,2,\ldots,k),$$

$$(9)\qquad (\bar g_1),(\bar g_2),\ldots,(g_l)\subset S_n^{A\cup B};\quad \bar g_i|A\sim 1\quad (i=1,2,\ldots,l),$$

и покажем, что система

$$(10)\qquad (\bar f_1),(\bar f_2),\ldots,(\bar f_k),(\bar g_1),(\bar g_2),\ldots,(\bar g_l),(h_1),(h_2),\ldots,(h_m)$$

образует максимальную систему линейно независимых элементов группы $H_n(A\cup B, A\cap B)$. Рассмотрим произвольную равную нулю линейную комбинацию элементов системы (10)

$$(11)\qquad \sum_{i=1}^{k} p_i(\bar f_i) + \sum_{i=1}^{l} q_i(\bar g_i) + \sum_{i=1}^{m} r_i(h_i) = 0$$

и покажем, что все целочисленные коэффициенты p_i, q_i, r_i равны нулю. Действительно, рассматривая (11) на A, имеем $h_i|A\sim 1$, $\bar{g}_i|A\sim 1$, $\bar{f}_i|A=f_i$; отсюда заключаем, что $\sum_{i=1}^{k} p_i(f_i)=0$, а значит в силу свойства максимальности системы (5) все p_i равны нулю. Подобным образом, рассматривая (11) на B, заключаем (ввиду $h_i|B\sim 1$ $(i=1,2,\ldots,m)$, $\bar{f}_i|B\sim 1$ $(i=1,2,\ldots,k)$, $\bar{g}_i|B=g_i$ $(i=1,2,\ldots,l)$), что все q_i равны нулю и равенство (11) принимает вид $\sum_{i=1}^{m} r_i(h_i)=0$, но отсюда в силу свойства максимальности системы (7) следует, что все r_i также равны нулю и линейная независимость элементов (10) доказана.

Пусть $(f)\in H_n(A\cup B, A\cap B)$. На A имеем, в силу свойства максимальности системы (5), $p_0(f|A)+\sum_{i=1}^{k} p_i(f_i)=0$, где не все p_i равны нулю ($p_0\neq 0$), а на B имеем, в силу свойства максимальности системы (6), $q_0(f|B)+\sum_{i=1}^{l} q_i(g_i)=0$, где не все q_i равны нулю ($q_0\neq 0$).

Рассмотрим на $A\cup B$ элемент $(H)\subset S_n^{A\cup B}$, определённый формулой

$$(H)=p_0q_0(f)+\sum_{i=1}^{k} q_0p_i(\bar{f}_i)+\sum_{i=1}^{l} p_0q_i(\bar{g}_i). \tag{12}$$

В силу двух предыдущих соотношений имеем $(H)\in P_n(A,B)$, значит имеет соотношение $r_0(H)+\sum_{i=1}^{m} r_i(h_i)=0$, где не все r_i равны нулю ($r_0\neq 0$); отсюда, в силу соотношения (12), имеем

$$r_0p_0q_0(f)+\sum_{i=1}^{k} r_0q_0p_i(\bar{f}_i)+\sum_{i=1}^{l} r_0p_0q_i(\bar{g}_i)+\sum_{i=1}^{m} r_i(h_i)=0,$$

где не все целочисленные коэффициенты равны нулю ($r_0p_0q_0\neq 0$). Отсюда следует максимальность системы (10). Теорема 2 тем самым доказана.

4. Пусть $F=\bar{F}\subset S_{n+1}$ и $b_0(S_{n+1}\setminus F)$ обозначает число компонент, на которые множество F разбивает S_{n+1}. К. Борсук доказал (см. [1]), что число $b_0(S_{n+1}\setminus F)$ однозначно определяется рангом $b_n(F)$ когомотопической группы $B_n(F)$ множества F при помощи формулы:

$$b_0(S_{n+1}\setminus F)=b_n(F)+1. \tag{13}$$

Отсюда, используя теорему 2, выведем следующее предложение:

Теорема Фрагмена-Брауэра. *Пусть* $A=\bar{A}, B=\bar{B}, A\cup B\subset S_{n+1}$; *если* $\dim A\cap B\leqslant n-2$, *то* $b_0(S_{n+1}\setminus(A\cup B))=b_0(S_{n+1}\setminus A)+b_0(S_{n+1}\setminus B)-1$.

Доказательство. На основании (13) доказательство, как легко видеть, сводится к доказательству равенства:

(14) $$b_n(A\cup B) = b_n(A) + b_n(B).$$

Из $\dim A\cap B \leqslant n-2$ следует (см. [4], стр. 88), что если $f \in S_n^{A\cup B}$, $f|A\sim 1, f|B\sim 1$, то, $f\sim 1$ т. е. $P_n(A,B)=0$ $(p_n(A,B)=0)$. С другой стороны, предположение $\dim A\cap B \leqslant n-2$ влечёт за собой, что всякое $f \in S_n^{A\cup B}$ на $A\cap B$ несущественно (см. [4], стр. 124), а отсюда следуют равенства групп: $H_n(A\cup B, A\cap B) = B_n(A\cup B), H_n(A, A\cap B) = B_n(A), H_n(B, A\cap B) = B_n(B)$, а значит и рангов: $h_n(A\cup B, A\cap B) = b_n(A\cup B), h_n(A, A\cap B) = b_n(A)$, $h_n(B, A\cap B) = b_n(B)$. Отсюда в силу теоремы 2 следует равенство (14) и теорема Фрагмена-Брауэра тем самым доказана.

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МАТЕМАТИКА

А. ГРАНАС

О РАЗБИЕНИИ БАНАХОВЫХ ПРОСТРАНСТВ

Представлено К. БОРСУКОМ 16 мая 1959

1. Обозначим через R_n — n-мерное евклидово пространство, $\mathring{R}_n$ — пространство R_n без точки 0, а через X замкнутое и ограниченное множество, расположенное в R_n *).

В 1931 году К. Борсук доказал в работе [1] следующую теорему о разбиении евклидовых пространств:

Для того, чтобы множество X не разбивало пространства R_n, необходимо и достаточно, чтобы любые два отображения $f, g \in \mathring{R}_n^X$ были гомотопны **).

Основным результатом настоящей работы является теорема 2, которая представляет собой перенесение сформулированного предложения Борсука на случай банаховых пространств. Доказательство теоремы 2 элементарно (оно не использует понятий теории гомологий) и основано на применении теоремы Шаудера о неподвижной точке [7] и теоремы о продолжении гомотопии для банаховых пространств [4].

2. Введем некоторые обозначения, которыми будем пользоваться в дальнейшем: E_∞ — бесконечномерное банахово пространство, E_n — n-мерное подпространство пространства E_∞, P_∞ пространство E_∞ без точки 0, $P_n = E_n \setminus \{0\}$.

Пусть X какое-либо метрическое пространство. Непрерывное отображение $F\colon X \to E_\infty$ называем *вполне непрерывным* на X, если множество $F(X)$

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*) Для произвольных метрических пространств X, Y через Y^X обозначаем множество всех непрерывных отображений пространства X в Y, через $X \times Y$ топологическое произведение этих пространств. Гомотопия отображений $f, g \in Y^X$ означает существование функции $h \in Y^{X \times I}$ (I — замкнутый отрезок [0,1]), для которой при любом $x \in X$ имеем $h(x, 0) = f(x)$, $h(x, 1) = g(x)$. В дальнейшем запись $f_0 \subset f \in Y^X$, где $f_0 \in Y^{X_0}$, $X_0 \subset X$ означает, что f является продолжением f_0 на X, т. е. $f(x) = f_0(x)$ при любом $x \in X_0$; будем писать также $f_0 = f|X_0$.

**) Отметим, что для удобства мы несколько видоизменили оригинальную формулировку теоремы Борсука, заменив пространство отображений S_{n-1}^X множества X в $(n-1)$-мерную сферу S_{n-1} пространством $\mathring{R}_n^X$.

компактно в E_∞; если, кроме того, значения отображения F принадлежат некоторому подпространству $E_n \subset E_\infty$, т. е. $F\colon X \to E_n$, то F называем *конечномерным* отображением на X.

Имеет место следующее предложение о продолжении вполне непрерывных отображений:

(2,1). *Если X_0 замкнутое подмножество X и отображение $F_0\colon X_0 \to E_\infty$ вполне непрерывно, то существует вполне непрерывное отображение $F\colon X \to \mathrm{conv}(F_0(X_0))$ такое, что $F_0 \subset F$* *).

Пусть теперь $X \subset E_\infty$. Непрерывное отображение $f\colon X \to E$ будем называть *вполне непрерывным векторным полем на X*, если оно допускает представление вида:

$$f(x) = x - F(x),$$

где отображение $F\colon X \to E_\infty$ вполне непрерывно на X; если отображение F конечномерно, то f будем называть *конечномерным полем на X*.

Совокупность вполне непрерывных векторных полей на X обозначим через $\mathfrak{C}(E_\infty^X)$, совокупность же конечномерных полей на X через $\mathfrak{C}_0(E_\infty^X)$.

Множество $\mathfrak{C}(E_\infty^X)$ будем рассматривать как метрическое пространство, определяя расстояние $|f-g|$ элементов $f, g \in \mathfrak{C}(E_\infty^X)$ по формуле:

$$|f-g| = \sup_{x \in X} \|f(x) - g(x)\|.$$

Так как *любое вполне непрерывное отображение можно равномерно аппроксимировать конечномерными отображениями* (см. [5]), то мы получаем:

(2,2) *Множество $\mathfrak{C}_0(E_\infty^X)$ плотно в $\mathfrak{C}(E_\infty^X)$.*

Далее имеет место следующее предложение:

(2,3) *Если X замкнуто и $f \in \mathfrak{C}(E_\infty^X)$, то $f(X)$ также замкнуто.*

Предположим теперь, что множество $X \subset E_\infty$ замкнуто и положим:

$$\mathfrak{C}(P_\infty^X) = P_\infty^X \cap \mathfrak{C}(E_\infty^X); \qquad \mathfrak{C}_0(P_\infty^X) = P_\infty^X \cap \mathfrak{C}_0(E_\infty^X).$$

Элементы пространства $\mathfrak{C}(P_\infty^X)$ будем называть *неисчезающими вполне непрерывными полями на X*. Из (2,2) и (2,3) следует:

(2,4) *Множество $\mathfrak{C}_0(P_\infty^X)$ плотно в $\mathfrak{C}(P_\infty^X)$.*

3. Введем теперь основное для дальнейшего понятие гомотопии элементов пространства $\mathfrak{C}(P_\infty^X)$.

Будем говорить, что неисчезающие векторные поля $f, g \in \mathfrak{C}(P_\infty^X)$ *гомотопны* (пишем ($f \simeq g$ в $\mathfrak{C}(P_\infty^X)$)), если существует отображение $H\colon X \times I \to E_\infty$, которое удовлетворяет следующим условиям:

*) Это предложение является весьма частным случаем общей теоремы о продолжении непрерывных функций, доказанной Дугунди [2]; символ $\mathrm{conv}(F_0(X_0))$ обозначает наименьшее замкнутое выпуклое множество содержащее $F_0(X_0)$.

1°. отображение H вполне непрерывно на $X \times I$,

2°. функция $h(x,t) = x - H(x,t) \neq 0$ для любого $x \in X$ и $t \in I$,

3°. $h(x,0) = f(x)$, $h(x,1) = g(x)$ для любого $x \in X$.

Пространство $\mathfrak{C}(P_\infty^X)$ распадается благодаря соотношению гомотопии на непересекающиеся классы гомотопных полей.

Имеют место следующие предложения:

(3,1). *Если* $X_0 \subset X$ *и* $f, g \in \mathfrak{C}(P_\infty^X)$, *то из* $f \simeq g$ *в* $\mathfrak{C}(P_\infty^X)$ *следует* $f|X_0 \simeq g|X_0$ *в* $\mathfrak{C}(P_\infty^{X_0})$.

(3,2). *Если* $f \in \mathfrak{C}(P_\infty^X)$ *и число* ε *таково, что* $0 < \varepsilon < \varrho(0, f(X))$ *), *то из* $|f-g| < \varepsilon$, $g \in \mathfrak{C}(P_\infty^X)$ *следует* $f \simeq g$ *в* $\mathfrak{C}(P_\infty^X)$.

Из (2,4) и (3,2) следует:

(3,3). *Любое вполне непрерывное поле* $f \in \mathfrak{C}(P_\infty^X)$ *гомотопно некоторому конечномерному полю* $g \in \mathfrak{C}_0(P_\infty^X)$.

Пользуясь (3,3) доказываем следующее предложение:

(3,4) *Если* X — *какой-либо замкнутый шар пространства* E_∞, *то для любых* $f, g \in \mathfrak{C}(P_\infty^X)$ *имеем* $f \simeq g$ *в* $\mathfrak{C}(P_\infty^X)$.

Имеет место следующая:

Лемма 1 (о продолжении гомотопии). *Пусть* X_0 — *замкнутое подмножество множества* $X \subset E_\infty$, $f_0, g_0 \in \mathfrak{C}(P_\infty^{X_0})$ *причём* $f_0 \simeq g_0$ *в* $\mathfrak{C}(P_\infty^{X_0})$. *Если* $f_0 \subset f \in \mathfrak{C}(P_\infty^X)$, *то существует такое* $g \in \mathfrak{C}(P_\infty^X)$, *для которого* $g_0 \subset g$ *и* $f \simeq g$ *в* $\mathfrak{C}(P_\infty^X)$.

Доказательство леммы 1 приведено в работе автора [4].

4. Пусть X_0 — граница некоторой ограниченной замкнутой области $X \subset E_\infty$ и на X_0 задано векторное поле $f_0 \in \mathfrak{C}(P_\infty^{X_0})$.

Будем говорить, что поле $f_0 \in \mathfrak{C}(P_\infty^{X_0})$ *несущественно*, если для некоторого $f \in \mathfrak{C}(P_\infty^X)$ имеем $f_0 \subset f$; в противном случае будем говорить, что поле f_0 *существенно*.

Из теоремы Шаудера о неподвижной точке [7] легко выводится следующая:

Лемма 2. *Если* x_1 *является внутренней точкой области* X, *то рассматриваемое на* X_0 *неисчезающее вполне непрерывное поле* $x - x_1$ *является существенным*.

Мы говорим, что множество $X \subset E_\infty$ *отделяет* точки $x_1, x_2 \in E_\infty \setminus X$ в пространстве E_∞, если эти точки принадлежат различным компонентам дополнения $E_\infty \setminus X$.

Пусть X — замкнутое и ограниченное подмножество пространства E_∞ и $x_1, x_2 \in E_\infty \setminus X$.

*) $\varrho(0, f(X))$ — означает расстояние множества $f(X)$ до точки 0.

Теорема 1 (об отделении точек в банаховом пространстве). *Для того, чтобы множество X не отделяло точек $x_1, x_2 \in E_\infty \setminus X$ в пространстве E_∞ необходимо и достаточно, чтобы неисчезающие вполне непрерывные векторные поля $(x-x_1)|X$, $(x-x_2)|X$ были гомотопны в $\mathfrak{C}(P^X_\infty)$* *).

Теорема 1 доказывается при помощи леммы 1 и 2.

5. Мы говорим, что множество $X \subset E_\infty$ *разбивает* пространство E_∞, если дополнение $E_\infty \setminus X$ несвязно.

Пусть X — ограниченное и замкнутое подмножество пространства E_∞. Имеет место следующая:

Лемма 3. *Если множество X не разбивает пространства и X^* — какой-либо замкнутый шар пространства E_∞ содержащий X, то для любого поля $f \in \mathfrak{C}(P^X_\infty)$ существует поле $f^* \in \mathfrak{C}(P^{X^*}_\infty)$, для которого $f \subset f^*$.*

Основным результатом настоящей работы является следующая:

Теорема 2 (о разбиении банаховых пространств). *Для того, чтобы замкнутое и ограниченное множество $X \subset E_\infty$ не разбивало банахова пространства E_∞, необходимо и достаточно, чтобы любые два неисчезающие вполне непрерывные векторные поля $f, g \in \mathfrak{C}(P^X_\infty)$ были гомотопны в $\mathfrak{C}(P^X_\infty)$.*

Необходимость условия теоремы 2 вытекает из леммы 2 и предложений (3,1) и (3,4), достаточность же следует из теоремы 1.

6. Пусть теперь отображение $h \in \mathfrak{C}(E^X_\infty)$ будет гомеоморфизмом множества X на множество $X^* = h(X)$. Очевидно, что *если пространство $\mathfrak{C}(P^X_\infty)$ состоит из одного класса гомотопии, то этим же свойствот обладает пространство* $\mathfrak{C}(P^{X^*}_\infty)$. Отсюда и из Теор. 2 имеем:

Следствие. *Если множество X разбивает банахово пространство E_∞ и отображение $h \in \mathfrak{C}(E^X_\infty)$ является гомеоморфизмом, то образ $h(X)$ множества X при отображении h также разбивает пространство E_∞* **).

Полные доказательства приведенных теорем будут опубликованы в журнале „Fundamenta Mathematicae”.

УНИВЕРСИТЕТ ИМ. НИКОЛАЯ КОПЕРНИКА, ТОРУНЬ

ЦИТИРОВАННАЯ ЛИТЕРАТУРА

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*) Эта теорема в конечномерном случае известна как *критерий Борсука отделимости точек в евклидовом пространстве* (см. [3], стр. 302).

**) Приведенное следствие представляет собой качественную часть теоремы Жордана для банаховых пространств, доказанной Ж. Лере в работе [6]. Доказательство Лере основано на применении теории степени отображения [5].

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A. GRANAS. ON THE DISCONNECTION OF BANACH SPACES

Let E_∞ be a Banach space, $P_\infty = E_\infty \setminus \{0\}$. If $X, Y \subset E_\infty$ then we denote by $\mathfrak{C}(Y^X)$ the set of all continuous mappings $f\colon X \to Y$ which can be represented in the form

$$f(x) = x - F(x),$$

where the mapping $F\colon X \to E_\infty$ is completely continuous on X (i.e. $F(X)$ is compact in E_∞).

Two mappings $f, g \in \mathfrak{C}(P_\infty^X)$ are called *homotopic* ($f \simeq g$ in $\mathfrak{C}(P_\infty^X)$), if there exists a mapping $H\colon X \times I \to E_\infty$ (I-denotes the closed interval $[0,1]$) such that:

1°. the set $H(X \times I)$ is compact in E_∞,

2°. $h(x,t) = x - H(x,t) \neq 0$ for every $x \in X$ and $t \in I$,

3°. $h(x,0) = f(x)$, $h(x,1) = g(x)$.

If the points $x_1, x_2, \in X$ belong to the same component of the metric space X, then we shall write $x_1 \sim x_2$ in X.

Let X be a closed and bounded subset of the Banach space E_∞.

THEOREM 1. (Criterion of separation between two points). *Let $x_1, x_2 \in E_\infty \setminus X$. Then we have the equivalence*

$$\{x_1 \sim x_2 \text{ in } E_\infty \setminus X\} \equiv \{(x - x_1)|X \simeq (x - x_2)|X \text{ in } \mathfrak{C}(P_\infty^X)\}.$$

THEOREM 2. (On disconnection of Banach spaces). *The set $E_\infty \setminus X$ is connected, if and only if for every $f, g \in \mathfrak{C}(P_\infty^X)$ we have $f \simeq g$ in $\mathfrak{C}(P_\infty^X)$.*

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Dedicated to
Professor L. Lusternik
on his 60-th birthday

On the disconnection of Banach spaces

by

A. Granas (Toruń)

1. Introduction. For arbitrary metric spaces X and Y, we denote by Y^X the set of all continuous mappings of X into Y, and by $X\times Y$ — the Cartesian product of X and Y. If a mapping $f\in Y^X$, then we shall write also $f:X\to Y$.

If $X_0\subset X$ and $f\in Y^X$ then $f|X_0$ will denote the *partial* mapping of f, i. e. the mapping f_0, defined in X_0 by the formula $f_0(x)=f(x)$; we shall say that f is an *extension* of f_0 over X and then we shall write $f_0\subset f$.

Two mappings $f,g\in Y^X$ are called *homotopic* (written $f\simeq g$) if there exists a mapping $h\in Y^{X\times I}$ (I denotes the closed interval $[0,1]$) such that for each $x\in X$

$$h(x,0)=f(x)\,,\qquad h(x,1)=g(x)\,.$$

If $f\in Y^X$ is homotopic to a constant mapping (i. e. a mapping onto a single-point set in Y) then we shall write $f\simeq 1$.

If the points x_1,x_2 belong to the same component of the space X, then we shall write: $x_1\sim x_2$ *in* X.

Let X be a closed and bounded subset of the n-dimensional Euclidean space R^n and let 0 denote the origin of R^n.

In 1931 K. Borsuk proved in the paper [1] the following theorem:

The set $R^n\setminus X$ *is connected if and only if the functional space* $(R^n\setminus\{0\})^X$ *is connected or, which is the same, if any two mappings* $f,g\in(R^n\setminus\{0\})^X$ *are homotopic.*

The same author also gave a criterion concerning the separation of the Euclidean space between two points (see, for instance, [4], p. 302):

$$(x_1\sim x_2\ \ in\ \ R^n\setminus X)\equiv(x-x_1)|X\sim(x-x_2)|X\ \ in\ \ (R^n\setminus\{0\})^X\,;$$

that is to say:

The set X *does not separate the space* R^n *between two points* $x_1,x_2\in R^n\setminus X$ *if and only if the mappings* $(x-x_1)|X$ and $(x-x_2)|X$ *are homotopic in* $(R^n\setminus\{0\})^X$.

In this paper we shall give an extension of Borsuk's theorems to the case of arbitrary Banach spaces (Theorems 2 and 3). (1)

In this case the space $(R^n\setminus\{0\})^X$ is replaced by the space $\mathfrak{C}(P_\infty^X)$ consisting of all non-vanishing compact fields on X, where X is a bounded closed subset of the Banach space and the homotopy of two mappings $f, g \in (R^n\setminus\{0\})^X$ is replaced by a homotopy of two elements of the space $\mathfrak{C}(P_\infty^X)$.

The proof of Theorems 2 and 3 is based on Theorem 1, which is Borsuk's Extension Homotopy Theorem (2) formulated for Banach spaces.

The invariance of the disconnection property of Banach spaces under a certain class of homeomorphisms is deduced directly from Theorem 3. The proof of this does not refer to the Leray-Schauder notion of the degree of a mapping [8]; it is, as a matter of fact, a consequence of the well-known Schauder Fixed Points Theorem.

2. Preliminaries. We shall use the following notation: E_∞ — infinite-dimensional Banach space, E_n — a subspace of E_∞ of dimension n, P_∞ — the space E_∞ without the origin 0, P_n — the space E_n without 0. If Z is a subset of E_∞, we denote the closure of Z by $\bar{Z}$ and the convex closure (i. e. the smallest convex closed set containing Z) by $\operatorname{conv}(Z)$. We shall denote by $V_\infty(x_0, \varrho)$ an open spherical region in the space E_∞ with centre x_0 and radius ϱ and by $S_\infty(x_0, \varrho)$ its boundary; if $x_0 \in E_n$, then we shall put

$$V_n(x_0, \varrho) = V_\infty(x_0, \varrho) \cap E_n\,, \quad S_{n-1}(x_0, \varrho) = S_\infty(x_0, \varrho) \cap E_n\,.$$

In the sequel we shall use the following lemma:

2.1. *Let X be a closed bounded separable convex subset of E_∞. Then X is a retract of E_∞, i. e. there exists a mapping $r: E_\infty \to X$ such that $r(x) = x$ for every $x \in X$.*

Proof. For $x \in E_\infty\setminus X$ and $y \in E_\infty$ the function

$$p(x, y) = \min\left\{2 - \frac{\|x-y\|}{\inf\limits_{z\in X}\|x-z\|},\ 0\right\}$$

is continuous on the set $E_\infty\setminus X$ and we have $0 \leqslant p(x, y) \leqslant 2$. Hence if $\{y_k\}$ is a dense sequence of points in X, then the function

$$r(x) = \begin{cases} x, & x \in X\,, \\ \Big(\sum\limits_{k=1}^{\infty} 2^{-k} p(x, y_k)\Big)^{-1} \Big(\sum\limits_{k=1}^{\infty} 2^{-k} p(x, y_k)\, y_k\Big), & x \notin X\,, \end{cases}$$

is the required retraction of E_∞ onto X.

(1) These theorems were announced in [5].

(2) For Borsuk's Theorem, see [2] and [7], p. 86.

Let X be an arbitrary space. A mapping $F: X \to E_\infty$ is said to be *compact on* X if the image $F(X)$ is contained in some compact set.

Compact mappings will be denoted in the sequel by capital letters F, G, H.

A compact mapping $F: X \to E_\infty$ is said to be *finite-dimensional on* X if its values lie in some finite-dimensional subspace $E_n \subset E_\infty$ depending on F, i. e. $F: X \to E_n$.

The following theorem is due to J. Schauder and J. Leray [8]:

2.2. APPROXIMATION THEOREM. *Let* $F: X \to E_\infty$ *be a compact mapping on* X. *For every* $\varepsilon > 0$, *there exists a finite-dimensional mapping* $F_\varepsilon: X \to E_n$ *such that*

$$\|F(x) - F_\varepsilon(x)\| < \varepsilon \quad \textit{for each} \quad x \in X. \tag{1}$$

Proof. For a given $\varepsilon > 0$, we can find a finite subset $\{y_1, y_2, \ldots, y_k\}$ of E_∞ such that every point of the compact set $\overline{F(X)}$ is at a distance less than ε from at least one of the y_i. Let E_n be a finite-dimensional subspace of E_∞ which contains all the points y_i $(i = 1, 2, \ldots, k)$.

Let us put

$$F_\varepsilon(x) = \frac{\sum_{i=1}^{k} \lambda_i(x) y_i}{\sum_{i=1}^{k} \lambda_i(x)} \quad \text{for} \quad x \in X, \tag{2}$$

where

$$\lambda_i(x) = \max\{0, \varepsilon - \|F(x) - y_i\|\} \quad \text{for} \quad x \in X \ (i = 1, 2, \ldots, k). \tag{3}$$

The mapping F_ε defined by (2) is finite-dimensional on X, $F_\varepsilon: X \to E_n$, and satisfies inequality (1); thus the proof is complete.

2.3. *Every compact mapping* $F: X \to E_\infty$ *can be represented in the form*

$$F(x) = \sum_{n=0}^{\infty} F_n(x), \tag{4}$$

where the mappings F_n *are finite-dimensional on* X $(n = 0, 1, \ldots)$ *and*,

$$\|F_n(x)\| \leqslant \frac{1}{2^n} \quad \textit{for every} \quad x \in X \ \textit{and} \ n = 1, 2, \ldots \tag{5}$$

Proof. This is a simple consequence of the Approximation Theorem 2.2.

In the sequel we shall use the following theorem, which is a very special case of the theorem of Dugundji concerning extensions of continuous transformations [3]:

2.4. EXTENSION OF COMPACT MAPPINGS THEOREM. *Let* X_0 *be a closed subset of a metric space* X. *Then every compact mapping* $F: X_0 \to E_\infty$ *can be extended to a compact mapping* $\bar{F}: X \to \mathrm{conv}(F(X_0))$.

Proof. In the case when the mapping F is finite-dimensional our theorem is a simple consequence of lemma 2.1 and the well-known Tietze Extension Theorem ([7], p. 80). For the proof of our theorem in the general case let us consider the representation of F on X_0 given by formulas (4) and (5). Let $\bar{F}_n$ $(n = 0, 1, 2, \ldots)$ be an extension of the finite-dimensional mapping F_n from X_0 over X such that

$$\|\bar{F}_n(x)\| \leqslant \frac{1}{2^n} \quad \text{for} \quad x \in X \text{ and } n = 1, 2, \ldots$$

Denote by r a retraction of E_∞ on the set $\operatorname{conv}(F(X_0))$, which is obviously bounded and separable.

The mapping $\bar{F}$ defined on X by the formula

$$\bar{F}(x) = r\Big(\sum_{n=0}^{\infty} \bar{F}_n(x)\Big)$$

is the required extension of F from X_0 over X.

As a simple consequence of the Approximation Theorem 2.2 we shall prove the well-known Schauder Fixed Point Theorem [10], which will be used in the sequel:

2.5. *If X is a closed convex subset of E_∞ and F a compact mapping of X into itself, then F has a fixed point.*

Proof. By 2.1 for each $k = 1, 2, \ldots$ there exists a finite-dimensional mapping $F_{1/k}: X \to X \cap E_{n(k)}$ such that

$$(6) \qquad \|F(x) - F_{1/k}(x)\| \leqslant \frac{1}{k} \quad \text{for each} \quad x \in X.$$

By the Brouwer Fixed Point Theorem ([4]) the mapping $F_{1/k}$ has a fixed point $x_k = F_{1/k}(x_k)$ and hence by (6) we have

$$(7) \qquad \|F(x_k) - x_k\| \leqslant \frac{1}{k}.$$

Since F is a compact mapping, we can assume, without loss of generality, that there exists $\lim\limits_{k=\infty} F(x_k) = x^*$. On account of (7) we have $\lim\limits_{k=\infty} x_k = x^*$ and hence $\lim\limits_{k=\infty} F(x_k) = F(x^*)$, i. e. $x^* = F(x^*)$, which completes the proof.

3. The space $\mathfrak{C}(E_\infty^X)$ of compact fields in E_∞. Now let X be a subset of the Banach space E_∞.

A mapping $f: X \to E_\infty$ is said to be a *compact vector field on X* if it can be represented in the form

$$(8) \qquad f(x) = x - F(x),$$

where $F: X \to E_\infty$ is a compact mapping on the set X.

The set of all compact vector fields on X will be denoted by $\mathfrak{C}(E_\infty^X)$. A compact vector field $f \in \mathfrak{C}(E_\infty^X)$ is said to be *finite-dimensional* if the mapping F of formula (8) is finite-dimensional. The set of all finite-dimensional vector fields on X will be denoted by $\mathfrak{C}_0(E_\infty^X)$.

In the sequel we shall consider the set $\mathfrak{C}(E_\infty^X)$ as a metric space and define the distance $\varrho(f,g)$ by setting

$$(9)\qquad \varrho(f,g) = \sup_{x\in X}\|f(x)-g(x)\| \quad \text{for each} \quad f,g\in\mathfrak{C}(E_\infty^X).$$

From the Approximation theorem 2.2 we obtain:

3.1. *The set $\mathfrak{C}_0(E_\infty^X)$ is dense in the space $\mathfrak{C}(E_\infty^X)$.*

3.2. *If X is closed in E_∞ and $f\in\mathfrak{C}(E_\infty^X)$ then the set $f(X)$ is also closed in E_∞.*

Proof. Let $y_n\in f(X)$, $\lim\limits_{n=\infty} y_n = y_0$, $y_n = f(x_n) = x_n - F(x_n)$; without loss of generality, we can assume that there exists $\lim\limits_{n=\infty} F(x_n) = y^*$, $y^*\in E_\infty$. We have $\lim\limits_{n=\infty} x_n = y_0+y^*$, $\lim\limits_{n=\infty} f(x_n) = f(y_0+y^*)$, i. e. $y_0 = f(y_0+y^*)$, $y_0+y^*\in X$.

If X and Y are subsets of E_∞ then we shall put

$$\mathfrak{C}(Y^X) = \mathfrak{C}(E_\infty^X)\cap Y^X, \qquad \mathfrak{C}_0(Y^X) = \mathfrak{C}_0(E_\infty^X)\cap Y^X .$$

In the sequel we shall consider the space $\mathfrak{C}(P_\infty^X)$ of *non-vanishing compact fields* on X.

Let X be a closed subset of E_∞. From 3.1, 3.2 we infer that:

3.3. *The set $\mathfrak{C}_0(P_\infty^X)$ is dense in the space $\mathfrak{C}(P_\infty^X)$.*

4. Notion of homotopy in the space $\mathfrak{C}(P_\infty^X)$. Two non-vanishing compact vector fields $f,g\in\mathfrak{C}(P_\infty^X)$ are called *homotopic* in the space $\mathfrak{C}(P_\infty^X)$ (we shall write $f\simeq g$ in $\mathfrak{C}(P_\infty^X)$) if there exists a mapping $h\in P_\infty^{X\times I}$ which satisfies the following conditions:

1° $h(x,0) = f(x)$, $h(x,1) = g(x)$ for each $x\in X$;

2° a mapping h can be represented in the form

$$h(x,t) = x - H(x,t),$$

where the mapping $H: X\times I\to E_\infty$ is compact on $X\times I$.

The relation of homotopy established in the space $\mathfrak{C}(P_\infty^X)$ is a relation of equivalence and thus the set of all non-vanishing compact vector fields $f\in\mathfrak{C}(P_\infty^X)$ decomposes into disjoint classes of homotopic fields.

4.1. *Let X_0 be a subset of X and $f,g\in\mathfrak{C}(P_\infty^X)$; then $f\simeq g$ in $\mathfrak{C}(P_\infty^X)$ implies $f|X_0\simeq g|X_0$ in $\mathfrak{C}(P_\infty^{X_0})$.*

4.2. *For a given $f \in \mathfrak{C}(P_\infty^X)$, if a positive number ε is less than the distance* $\operatorname{dist}(f(X), 0)$ *then for every $g \in \mathfrak{C}(P_\infty^X)$ the condition $\varrho(f, g) < \varepsilon$ implies $f \simeq g$ in $\mathfrak{C}(P_\infty^X)$.*

Proof. Let $f(x) = x - F(x)$ and $g(x) = x - G(x)$. For each $x \in X$ we have $\|x - F(x)\| > \varepsilon$, $\|F(x) - G(x)\| < \varepsilon$. From these inequalities we infer that, for each $x \in X$ and $t \in I$, $x \neq H(x, t) = tF(x) + (1-t)G(x)$ and thus the mapping $h \in \mathfrak{C}(P_\infty^{X \times I})$ defined by $h(x, t) = x - H(x, t)$ (since H is a compact mapping on $X \times I$) is a homotopy between f and g.

Properties 3.3 and 4.2 imply:

4.3. *Every compact field $f \in \mathfrak{C}(P_\infty^X)$ is homotopic to some finite-dimensional field $g \in \mathfrak{C}_0(P_\infty^X)$.*

4.4. *Let $X = \overline{V_\infty(x_0, \varrho)}$ be a closed spherical region in E_∞. Then any two compact fields $f, g \in \mathfrak{C}(P_\infty^X)$ are homotopic $f \simeq g$ in $\mathfrak{C}(P_\infty^X)$.*

Proof. By 4.3 we can assume that the compact fields $f, g \in \mathfrak{C}(P_\infty^X)$ are finite-dimensional. Let $f(x) = x - F(x)$, $g(x) = x - G(x)$; we can assume that the values of F and G lie in the same finite-dimensional subspace $E_n \subset E_\infty$ and that the point x_0 belongs to E_n. Put $V_n = X \cap E_n$, $f_0 = f|V_n$, $F_0 = F|V_n$, $g_0 = g|V_n$, $G_0 = G|V_n$. We have $f_0, g_0 \colon V_n \to P_n$ and thus $f_0 \simeq g_0$. Let $h_0(x, t) = x - H_0(x, t)$ be a homotopy joining f_0 with g_0 in the space $P_n^{V_n}$; we have $H_0(x, 0) = F_0(x)$, $H_0(x, 1) = G_0(x)$ for each $x \in V_n$. We shall extend the mapping $H_0 \colon V_n \times I \to E_n$ over $X \times I$ to a compact mapping $H \colon X \times I \to E_\infty$ which satisfies the following conditions:

$$x \neq H(x, t) \quad \text{for each} \quad x \in X \text{ and } t \in I,$$

$$H(x, 0) = F(x),\ H(x, 1) = G(x) \quad \text{for each} \quad x \in X.$$

For this denote by $\{e_1, e_2, \ldots, e_n\}$, $e_k \in E_n$, a basis of E_n and by $\{l_1, l_2, \ldots, l_n\}$ the dual basis in the conjugate space to E_n; thus every element $z \in E_n$ can be written in the form

$$z = \sum_{i=1}^{n} l_i(z) e_i. \tag{10}$$

Let us consider the following closed subset of $X \times I$:

$$T_0 = (X \times \{0\}) \cup (V_n \times I) \cup (X \times l)$$

and define on T_0 a real-valued functions φ_i $(i = 1, 2, \ldots, n)$ as follows:

$$\varphi(x, t) = \begin{cases} l_i(F(x)) & \text{for} \quad x \in X \text{ and } t = 0, \\ l_i(G(x)) & \text{for} \quad x \in X \text{ and } t = 1, \\ l_i(H_0(x, t)) & \text{for} \quad x \in V_n \text{ and } t \in I. \end{cases} \tag{11}$$

Tietze Extension Theorem yields an extension $\widetilde{\varphi}_i(x,t)$ of $\varphi_i(x,t)$ over $X\times I$; since each function φ_i is bounded, we can assume that also each $\widetilde{\varphi}_i$ is bounded and thus the mapping $H: X\times I \epsilon E_n$ defined by

$$H(x,t)=\sum_{i=1}^{n}\widetilde{\varphi}_i(x,t)e_i \tag{12}$$

is compact. From (10) and (11) it follows that the mapping H defined by (12) is the desired extension of H_0 over $X\times I$ and thus the proof is complete.

5. Extension Homotopy Theorem. We shall consider the question of extending a non-vanishing compact field defined on a closed subset X_0 of $X\subset E_\infty$ to a non-vanishing compact field defined over the whole X. We shall prove that the existence of such extension depends only on the homotopy class of the given compact field.

THEOREM 1 (ON THE EXTENSION OF HOMOTOPY [5]). *Let X_0 be a closed subset of $X\subset E_\infty$ and $f_0, g_0 \in \mathfrak{C}(P_\infty^{X_0})$ two homotopic in $\mathfrak{C}(P_\infty^{X_0})$ compact fields. Then if there is an extension $f\in\mathfrak{C}(P_\infty^{X})$ of f_0 over X, there is also an extension $g\in\mathfrak{C}(P_\infty^{X})$ of g_0 over X with f and g homotopic in $\mathfrak{C}(P_\infty^{X})$.*

Proof. The homotopy of the non-vanishing compact fields

$$f_0(x)=x-F_0(x),\quad F_0: X_0\to E_\infty,$$
$$g_0(x)=x-G_0(x),\quad G_0: X_0\to E_\infty,$$

means that there exists a compact mapping $H_0: X\times I\to E_\infty$ satisfying the following conditions:

$$x\neq H_0(x,t)\quad\text{for each}\quad x\in X_0 \text{ and } t\in I,$$
$$H_0(x,0)=F_0(x),\quad H_0(x,1)=G_0(x)\quad\text{for each}\quad x\in X_0.$$

There exists, by hypotesis, an extension $f\in\mathfrak{C}(P_\infty^{X})$, $f(x)=x-F(x)$, of f_0 over X; thus $F_0\subset F: X\to E_\infty$.

Denote by T_0 the following subset of the Cartesian product $X\times I$:

$$T_0=(X_0\times I)\cup(X\times\{0\}),$$

and define the following mapping $H_0^*: T_0\to E_\infty$:

$$H_0^*(x,0)=F(x)\quad\text{for}\quad x\in X \text{ and } t=0,$$
$$H_0^*(x,t)=H_0(x,t)\quad\text{for}\quad x\in X_0 \text{ and } 0\leqslant t\leqslant 1.$$

The mapping H_0^* is compact on T_0 and hence by 2.4 it can be extended to a compact mapping $H^*: X\times I\to E_\infty$ over $X\times I$.

Let us define the set $X_1 \subset X$ by the condition:

$$(x \in X_1) \text{ if and only if } (x - H_0(x, t) = 0 \text{ for some } t \in I).$$

X_1 and X_0 are obviously disjoint closed subsets of X. Hence there is a continuous real-valued function $\lambda(x)$ defined over X whose range is between 0 and 1 and which is 0 on X_1 and 1 on X_0.

Now consider the mapping

$$H(x, t) = H^*(x, \lambda(x)t) \quad \text{for} \quad x \in X \text{ and } t \in I.$$

It is clear that H is a compact mapping on $X \times I$ and for each $x \in X$ and $t \in I$

$$x \neq H(x, t).$$

If we define $g(x)$ by

$$g(x) = x - G(x), \quad \text{where} \quad G(x) = H(x, 1), \; x \in X,$$

it is clear that $g(x)$ is an extension of $g_0(x)$ over X, and likewise that $H(x, 0) = F(x)$ for $x \in X$. Since $H(x, 1) = G(x)$ by definition, we conclude that the non-vanishing compact fields

$$f(x) = x - F(x) \quad \text{and} \quad g(x) = x - G(x) \quad (x \in X)$$

are homotopic in $\mathfrak{C}(P_\infty^X)$. The proof of Theorem 1 is complete.

6. Separation of the space between two points. Let X be a closed and bounded subset of E_∞.

THEOREM 2. *The set X does not separates the Banach space E_∞ between two points $x_1, x_2 \in E_\infty \setminus X$ if and only if the non-vanishing compact fields $(x - x_1)|X$, $(x - x_2)|X$ are homotopic in the space $\mathfrak{C}(P_\infty^X)$.*

The proof of theorem 1 is based on the following

LEMMA 1. *Let U be a bounded open set in E_∞, x_1 a point in U and Y the boundary of U. Then the non-vanishing compact field $(x - x_1)|Y$ cannot be extended to a non-vanishing compact field over $\overline{U} = U \cup Y$.*

Proof. Suppose it were possible to extend $(x - x_1)|Y$ over $\overline{U}$ to a non-vanishing compact field f, say, $f(x) = x - F(x)$, $f(x) = x - x_1$ for each $x \in Y$.

Let ϱ be so large that $\overline{U}$ and $F(\overline{U})$ are contained in the spherical region $\overline{V}_\infty = \overline{V_\infty(x_1, \varrho)}$ of radius ϱ and x_1 as centre.

The formulas

$$F^*(x) = x_1 \quad \text{for} \quad x \in \overline{V} \setminus U,$$

$$F^*(x) = F(x) \quad \text{for} \quad x \in U$$

would then define $F^*: \overline{V}_\infty \to \overline{V}_\infty$ as a compact mapping of $\overline{V}_\infty$ into itself without fixed points. This is a contradiction of the Schauder Fixed-Points Theorem 2.5. The proof is complete.

Proof of Theorem 2. Assuming first that x_1 and x_2 are not separated by X we shall prove that the compact fields $(x-x_1)|X$ and $(x-x_2)|X$ are not homotopic. We are given

$$E_\infty \setminus X = U \cup V,$$

U, V being disjoint sets which are open in E_∞, and $x_1 \in U$, $x_2 \in V$. One of the sets U, V, say U, is bounded. The non-vanishing compact field $(x-x_2)|X$ can be extended over $U \cup X$, in fact over $E_\infty \setminus \{x_2\} \supset U \cup X$. On the other hand, according to Lemma 1, it is not possible, in view of the boundary of U being contained in X, to extend $(x-x_1)|X$ over $U \cup X$ to a non-vanishing compact field. Hence $(x-x_1)|X$ and $(x-x_2)|X$ are not homotopic in the space $\mathfrak{C}(P_\infty^X)$ since Theorem 1 would be contradicted if they were.

Now let us assume that $x_1 \sim x_2$ in $E_\infty \setminus X$. Then one can join x_1 and x_2 by a continuous arc in $E_\infty \setminus X$, i. e. one can find a continuous function $r(t)$ of the real parameter t, $0 \leqslant t \leqslant 1$, with values in $E_\infty \setminus X$ such that

$$r(0) = x_1, \quad r(1) = x_2.$$

The mapping h: $X \times I \to P_\infty$, defined by

$$h(x, t) = x - r(t), \quad x \in X \text{ and } t \in I,$$

is obviously a homotopy joining $(x-x_1)|X$ and $(x-x_2)|X$ in $\mathfrak{C}(P_\infty^X)$. Hence Theorem 2 is proved.

7. The main theorem. For the proof of the main theorem we shall use the following lemmas.

Lemma 2. *Let $V_\infty = V_\infty(x_0, \varrho)$, $x_0 \in E_n \subset E_\infty$, $V_n = V_\infty \cap E_n$. Suppose that the mapping $F: \overline{V}_n \to E_n$ has a finite number of fixed points $x_1, x_2, \ldots, x_k \in \overline{V}_n$. Then there is a compact mapping $\overline{F}: \overline{V}_\infty \to E_n$ which has the same fixed points as F and which is an extension of F over $\overline{V}_\infty$.*

Proof. By 2.1 it follows that $\overline{V}_n$ is a retract of $\overline{V}_\infty$, i. e. there is a mapping $r: \overline{V}_\infty \to \overline{V}_n$ such that $r(x) = x$ for $x \in \overline{V}_n$.

Putting for each $x \in \overline{V}_\infty$

$$\overline{F}(x) = F(r(x))$$

we obviously obtain the desired compact mapping $\overline{F}: \overline{V}_\infty \to E_n$.

LEMMA 3. *Let X be a closed bounded subset of E_n and $f_0 \in P_n^X$. Then there exists a mapping $f \in E_n^{E_n}$ such that:*

1° *the set N of all roots of the equation*

$$f(x) = 0$$

is finite; if $x_1, x_2 \in N$ then x_1, x_2 belong to different components of $E_n \setminus X$,

2° *$f(x) = f_0(x)$ for each $x \in X$.*

The proof is a slight modification of the proof of a similar Lemma given in [4], p. 300.

LEMMA 4. *Suppose that a bounded and closed subset X of E_∞ does not disconnect E_∞ and that $V_\infty = V_\infty(x_0, \varrho)$ is a spherical region which contains X. Then every non-vanishing compact field $f_0 \in \mathfrak{C}(P_\infty^X)$ can be extended over $\overline{V}_\infty$ to a compact non-vanishing field $f \in \mathfrak{C}(P_\infty^{\overline{V}_\infty})$.*

Proof. By Theorem 1 and 4.3 we can assume, without loss of generality, that a compact field $f_0(x) = x - F_0(x)$ is finite-dimensional, i. e. $F_0: X \to E_n$.

Suppose that S_∞ is the boundary of V_∞ and that a point $x^* \in E_n$ does not belong to $\overline{V}_\infty$. Define the mapping $f_1 \in P_\infty^{X_1}$, $X_1 = S_\infty \cup X$ by

$$f_1(x) = x - F_1(x), \quad \text{where} \quad F_1(x) = \begin{cases} x^* & \text{for} \quad x \in S_\infty, \\ F_0(x) & \text{for} \quad x \in X. \end{cases}$$

Putting $X_1^* = X_1 \cap E_n$, $f_1^* = f_1 | X_1^*$ we have $f_1^* \in P_n^{X_1^*}$.

By Lemma 3 there exists a mapping $f_2^* \in E_n^{\overline{V}_n}$ such that the set N of all roots of the equation $f_2^*(x) = 0$ is finite, $N = \{x_1, x_2, \ldots, x_k\}$, and $f_2^*(x) = f_1^*(x)$ for every $x \in X_1^*$.

By Lemma 2 the mapping $f_2^* | (\overline{V}_n \setminus N)$ can be extended to a finite-dimensional field $f_2 \in \mathfrak{C}(P_\infty^{\overline{V}_\infty \setminus N})$.

Since the set $\overline{V}_\infty \setminus N$ is connected, the points $x_1, x_2, \ldots, x_k$ can be joined by a chain $V_\infty^1, V_\infty^2, \ldots, V_\infty^l \subset V_\infty \setminus X$ of open spherical regions such that $\overline{V}_\infty^i$ intersects $\overline{V}_\infty^j$ if and only if $|i-j| = 1$ $(i, j = 1, 2, \ldots, l)$.

Let E_m be a finite-dimensional subspace of E_∞ spanned by the centres of $\overline{V}_\infty^i$ $(i = 1, 2, \ldots, l)$ and containing E_n.

Let us put $T = \bigcup_{i=1}^{l} V_\infty^i$, $T^* = T \cap E_m$, $\overline{V}_m = \overline{V}_\infty \cap E_m$, $X_2 = \overline{V}_\infty \setminus T$, $X_2^* = \overline{V}_m \setminus T^*$, $f_2^* = f_2 | X_2^*$.

Since X_2^* is connected, it follows that the mapping $f^* \in P_m^{X_2^*}$ can be extended to a mapping $f_3^* \in P_m^{\overline{V}_m \setminus V_m'}$ over $\overline{V}_m \setminus V_m'$, where V_m' is a certain open spherical region contained in T^*. Since $f_3^*(x) = x - x^*$ for $x \in S_{m-1} = E_m \cap S_\infty$, we have $f_3 | S_{m-1} \simeq 1$ and consequently $f_3 | S_{m-1}' \simeq 1$, where S_{m-1}' is the boundary of V_m'. This implies that $f_3^* \subset f_4^* \in P_m^{\overline{V}_m}$.

By Lemma 2 we extend the mapping $f_4^*(x) = x - F_4^*(x)$, $x \in \overline{V}_m$, to a compact field $f \in \mathfrak{C}(P_\infty^{\overline{V}_\infty})$ and thus the proof of Lemma 4 is complete.

The main result of this paper is the following

THEOREM 3 (ON THE DISCONNECTION OF BANACH SPACES). *Let X be a bounded closed subset of the Banach space E_∞. The set $E_\infty \setminus X$ is connected if and only if any two non-vanishing compact fields $f, g \in \mathfrak{C}(P_\infty^X)$ are homotopic in the space $\mathfrak{C}(P_\infty^X)$.*

Proof. *Necessity.* Suppose that X does not disconnect E_∞, and $f, g \in \mathfrak{C}(P_\infty^X)$. By Lemma 4 compact fields f and g can be extended to non-vanishing compact fields $\bar{f}, \bar{g} \in \mathfrak{C}(P_\infty^{\overline{V}_\infty})$ over a closed spherical region $\overline{V}_\infty$ which contains X.

By 4.4 we have $\bar{f} \simeq \bar{g}$ in $\mathfrak{C}(P_\infty^{\overline{V}_\infty})$ and hence by 4.1 the compact fields f and g are homotopic in $\mathfrak{C}(P_\infty^X)$.

Sufficiency. Suppose that X disconnects E_∞. Then there certainly exist two points x_1 and x_2 separated by X. By Theorem 2 the non-vanishing compact fields $(x - x_1)|X$ and $(x - x_2)|X$ are not homotopic in $\mathfrak{C}(P_\infty^X)$ and thus the proof of Theorem 3 is complete.

8. Jordan separation theorem in Banach spaces. We shall say that two bounded and closed subsets X and Y of E_∞ are *homeomorphic in the narrow sense* if there exists a homeomorphism $h \in \mathfrak{C}(E_\infty^X)$ such that $Y = h(X)$.

It is clear that *if the closed and bounded subsets X and Y of E_∞ are homeomorphic in the narrow sense then the space $\mathfrak{C}(P_\infty^X)$ consists of one homotopy class if and only if the space $\mathfrak{C}(P_\infty^Y)$ consists of one homotopy class.*

From this we obtain the following, due to J. Leray [9]:

CORROLARY 1. *If a closed and bounded subset X of the Banach space E_∞ disconnects E_∞, then so does every subset of E_∞ which is homeomorphic to X in the narrow sense.*

As an obvious application of Corrolary 1 we obtain the following:

CORROLARY 2 (JORDAN SEPARATION THEOREM). *A subset of E_∞ which is homeomorphic in the narrow sense to a sphere $S_\infty(x_0, \varrho)$ disconnects E_∞.*

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TOPOLOGIE ALGÉBRIQUE. — *Sur la multiplication cohomotopique dans les espaces de Banach.* Note (*) de M. **Andrzej Granas**, présentée par M. Jean Leray.

Soit E_∞ un espace de Banach à dimension infinie. Nous désignons par P_0 l'espace E_∞ diminué du point O.

Soit X, $Y \subset E_\infty$. On appelle *champ vectoriel compact* sur X dans Y une application $f: X \to Y$ si elle peut être représentée sous la forme

$$f(x) = x - F(x), \qquad x \in X,$$

où $F: X \to E_\infty$ est une application compacte (¹). Nous notons $\mathfrak{C}(Y^X)$ l'ensemble des champs compacts sur X dans Y.

Soient $f, g: X \to Y$ deux champs vectoriels compacts. On dit que f et g sont *homotopes* s'il existe une application $h: X \times I \to Y$ telle que :

(i) $h(x, 0) = f(x)$ et $h(x, 1) = g(x)$ pour tout $x \in X$;

(ii) $h(x, t) = x - H(x, t)$ pour tout $(x, t) \in X \times I$,

où $H: X \times I \to E_\infty$ est une application compacte. Nous notons $[f]$ la *classe d'homotopie* du champ f (c'est-à-dire l'ensemble de tous les champs compacts homotopes à f). En classant les éléments de $\mathfrak{C}(Y^X)$ en classes d'homotopie on définit une décomposition de $\mathfrak{C}(Y^X)$ en classes disjointes. Dans le cas où $Y = P_\infty$ nous désignons la famille de ces classes par $\pi^\infty(X)$.

Soit X un sous-ensemble fermé et borné de E_∞, U une composante bornée de $E_\infty - X$, et f un champ compact sur X dans P_∞. On nomme U *composante inessentielle* pour f s'il existe un champ compact $\bar{f}$ sur $X \cup U$ dans P_∞ tel que $f \subset \bar{f}$ [c'est-à-dire $f(x) = \bar{f}(x)$ pour tout $x \in X$]. Sinon, on nomme U *composante essentielle* pour f.

Lemme. — *X étant un sous-ensemble fermé et borné de* E_∞, *soit f un champ compact sur* X *dans* P_∞. *Le nombre des composantes bornées de* $E_\infty - X$ *qui sont essentielles pour f est fini.*

Définition de multiplication cohomotopique dans $\pi^\infty(X)$. — X étant un sous-ensemble fermé et borné de E_∞, soient $\alpha, \beta \in \pi^\infty(X)$ et $f_\alpha \in \alpha$, $f_\beta \in \beta$, $f_\alpha(x) = x - F_\alpha(x)$, $f_\beta(x) = x - F_\beta(x)$, $x \in X$. Soit $U_1, U_2, \ldots, U_k$ la suite finie de composantes bornées de $E_\infty - X$ qui sont essentielles pour f_α ou pour f_β; soit $x_i \in U_i$ pour $i = 1, 2, \ldots, k$.

Sans restreindre la généralité on peut admettre que $F_\alpha: X \to E_n$, $F_\beta: X \to E_n$, où E_n est un sous-espace de E_∞ à n-dimensions tel que $x_1, x_2, \ldots, x_k \in E_n$. Soient

$$X_{(n)} = X \cap E_n, \qquad f_{\alpha(n)} = f_\alpha \mid X_{(n)}, \qquad f_{\beta(n)} = f_\beta \mid X_{(n)}.$$

On a

$$f_{\alpha(n)}, f_{\beta(n)}: \quad X_{(n)} \to E_n - \{0\} \qquad \text{alors} \qquad [f_{(\alpha n)}], [f_{(\beta n)}] \in \pi^{n-1}(X_{(n)})$$

où $\pi^{n-1}(X_{(n)})$ désigne le $(n-1)^{\text{ième}}$ groupe de cohomotopie de $X_{(n)}$ [2]. Soit $f_{\alpha\beta(n)} : X_{(n)} \to E_n - \{0\}$ une application telle que

$$f_{\alpha\beta(n)} \in [f_{\alpha(n)}].[f_{\beta(n)}] \in \pi^{n-1}(X_{(n)}).$$

L'application $f_{\alpha\beta(n)}$ se laisse étendre à un champ vectoriel compact $f_{\alpha\beta} : X \to P_\infty$ sur X dans P_∞, de façon que chaque composante bornée U de $E_\infty - X$ différente de $U_1, \ldots, U_k$ soit inessentielle pour $f_{\alpha\beta}$. Posons

$$\alpha.\beta = [f_\alpha].[f_\beta] = [f_{\alpha\beta}] \in \pi^\infty(X).$$

La classe d'homotopie $[f_{\alpha\beta}]$ est déterminée de façon univoque : elle ne dépend que des classes d'homotopie $[f_\alpha]$ et $[f_\beta]$. On appelle composition $(\alpha, \beta) \to \alpha.\beta$ la *multiplication cohomotopique* des classes.

Théorème 1. — *La multiplication cohomotopique des classes* $\alpha, \beta \in \pi^\infty(X)$ *définit sur* $\pi^\infty(X)$ *une loi du groupe abélien.*

Théorème 2 [3]. — X *étant un sous-ensemble fermé et borné de* E_∞ *soit* $U_1, U_2, \ldots, U_\tau, \ldots$ *la suite (finie ou transfinie) de toutes les composantes bornées de* $E_\infty - X$; *soit* $x_\tau \in U_\tau$. *Le groupe* $\pi^\infty(X)$ *est libre et a pour base les classes d'homotopie des translations*

$$[(x-x_1)\,|\,X],\quad [(x-x_2)\,|\,X],\quad \ldots,\quad [(x-x_\tau)\,|\,X],\quad \ldots.$$

Soient X et Y deux sous-ensembles bornés et fermés de E_∞. On dit que X et Y sont *homéomorphes au sens étroit* s'il existe une application bicontinue *h* de X sur Y telle que $h \in \mathfrak{C}(Y^X)$.

Théorème 3. — *Si* X *et* Y *sont homéomorphes au sens étroit on a l'isomorphisme*

$$\pi^\infty(X) \approx \pi^\infty(Y).$$

On en déduit le théorème d'invariance :

Corrolaire [4]. — *Soient* X *et* Y *deux sous-ensembles bornés et fermés de* E_∞ *qui sont homéomorphes au sens étroit. Alors la puissance de la famille des composantes de* $E_\infty - X$ *est égale à celle des composantes de* $E_\infty - Y$.

(*) Séance du 18 décembre 1961.

(1) C'est-à-dire $\overline{F(X)}$ est compact.

(2) *Voir* par exemple C. Kuratowski, *Topologie*, II, Appendice I, Warszawa, 1961.

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MATHEMATICS
(TOPOLOGY)

Algebraic Topology in Linear Normed Spaces. I. Basic Categories

by

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In the following sequence of notes we give some further development (see [4] and [2]) of the theory of compact vector fields. In particular we intend to include the treatment of cohomology functors, the representation theorem for such functors and Alexander—Pontriagin type of duality. One of the further papers will be devoted to the concept of a codimension. In this preliminary note we introduce basic categories and assemble the main facts and definitions which will enter in our discussion.

1. **Categorical preliminaries.** Let $\mathfrak{C}$ be a concrete category and $A, X \in \mathfrak{C}$. We write $A \subset X$ and call (X, A) to be a pair in $\mathfrak{C}$, if A is a subset of X and the inclusion map $i: A \to X$ is in $\mathfrak{C}$. If $f: X \to Y$ and $A \subset X$ we write $f' = f|A$ for the *restriction* of f to A; f is called an *extension* of f' over X and we write also $f' \subset f$.

Let $\mathfrak{L}$ be a subcategory of $\mathfrak{C}$. We say that a map $f': A \to Y$ is *inessential* (rel. $\mathfrak{L}$) provided each diagram

$$\Delta = \begin{array}{ccc} & & Y \\ & \nearrow f' & \uparrow f, \\ A & \longrightarrow & X \end{array}$$

where $A, X \in \mathfrak{L}$, can be completed in $\mathfrak{C}$. Otherwise f' is called *essential*.

An object $Y \in \mathfrak{C}$ is called an *extension* object for $X \in \mathfrak{L}$ provided for each $A \in \mathfrak{L}$, $A \subset X$ and $f': A \to Y$ the diagram Δ can be completed.

Example: If $\mathfrak{C} = \mathfrak{L}$ is the category of compact metrizable spaces, then the n-sphere S^n is an extension object for X if and only if $\dim X \leqslant n$ (Alexandroff Theorem).

An *h-category* $(\mathfrak{C}, \simeq)$ is a category $\mathfrak{C}$ such that for each pair of objects $A, B \in \mathfrak{C}$ there is defined in the set Map (A, B) an equivalence relation "$\simeq$" (called *homotopy*) satisfying the following (compositive) property:

$$f_1 \simeq f_2, g_1 \simeq g_2 \Rightarrow g_1 f_1 \simeq g_2 f_2.$$

If $f \in \mathrm{Map}(A, B)$, then by $[f]$ we denote the equivalence (*homotopy*) class containing f and we let $\pi(A, B)$ be the set of homotopy classes $[f]: A \to B$. By $\mathfrak{C}/\simeq$ we denote the category having the same objects as $(\mathfrak{C}, \simeq)$ and as maps the homotopy classes between objects.

A map $f: A \to B$ in $(\mathfrak{C}, \simeq)$ is called *invertible* (*h-invertible*) provided there is $f': B \to A$ such that $f'f = 1_A$, $ff' = 1_B$ ($f'f \simeq 1_A$, $ff' \simeq 1_B$). We write in the first case $A \sim B$ and in the second $A \underset{h}{\sim} B$.

A map $f: A \to B$ is an *r—map* (*r_h—map*) if there is a map $s: B \to A$ called an *s—map* for r (called *s_h—map* for r) such that $rs = 1_B$ ($rs \simeq 1_B$). We write $A \underset{r}{\sim} B$ ($A \underset{r_h}{\sim} B$) if there exist two r—maps (r_h—maps) $r_1: A \to B$, $r_2: B \to A$.

(1.1) *The relations* $\sim$, $\underset{h}{\sim}$, $\underset{r}{\sim}$, $\underset{r_h}{\sim}$ *defined in the class of objects of* $(\mathfrak{C}, \simeq)$ *are equivalence relations and we have*:

$$A \underset{h}{\sim} B \text{ in } (\mathfrak{C}, \simeq) \Leftrightarrow A \sim B \text{ in } \mathfrak{C}/\simeq,$$

$$A \underset{r_h}{\sim} B \text{ in } (\mathfrak{C}, \simeq) \Leftrightarrow A \underset{r}{\sim} B \text{ in } \mathfrak{C}/\simeq.$$

A functor $\lambda: \mathfrak{C}_1 \to \mathfrak{C}_2$ between h-categories is called an *h-functor* if it sends h-commutative diagrams in $\mathfrak{C}_1$ into h-commutative diagrams in $\mathfrak{C}_2$.

Example: Let $(\mathfrak{L}, \underset{1}{\simeq})$ be a subcategory of $(\mathfrak{C}, \underset{2}{\simeq})$ and U be a fixed object in $\mathfrak{C}$. For $f: X \to Y$ in $\mathfrak{L}$ we let $f^*: \pi(Y, U) \to \pi(X, U)$ be *the map induced by* f and defined by $[g] \to [gf]$. The assignments $X \to \pi(X, U)$, $f \to f^*$ define the contravariant h-functor $\pi(U)$ from $(\mathfrak{L}, \underset{1}{\simeq})$ to the category of sets $\mathfrak{C}_S$.

(1.2) *Let* $\lambda: \mathfrak{C}_1 \to \mathfrak{C}_2$ *be an h-functor. If* $\approx$ *is any one of the equivalence relations defined above, then we have*

$$X \approx Y \text{ in } \mathfrak{C}_1 \Rightarrow \lambda(X) \approx \lambda(Y) \text{ in } \mathfrak{C}_2.$$

2. The directed set $\mathcal{L}(E)$. Compact mappings. In what follows we denote by E an infinite dimensional linear normed space.

By $\mathcal{L}(E) = \mathcal{L} = \{L_\alpha, L_\beta, L_\gamma, \ldots\}$ we shall denote the directed set of *all* finite dimensional linear subspaces of E with the natural order relation $\leqslant$ defined by the condition $L_\alpha \leqslant L_\beta$ if and only if $L_\alpha \subset L_\beta$. For notational convenience we establish one-to-one correspondence $\alpha \Leftrightarrow L_\alpha$ between the symbols $\alpha, \beta, \gamma, \ldots$ and $L_\alpha, L_\beta, L_\gamma, \ldots$ and in the formulas to occur we replace frequently one kind of symbols by another. Thus, for example, we shall write $\alpha \leqslant \beta$ instead of $L_\alpha \leqslant L_\beta$. For $X, Y \subset E$ and $f: X \to Y$ we let $X_\alpha = X \cap L_\alpha$ and f_α be the restriction of $f: X_\alpha$.

A mapping $F: X \to E$ from a metric space X is *compact* provided the closure $\overline{F(X)}$ is compact. A compact mapping $F: X \to E$ is said to be an *α-mapping* (*finite dimensional mapping*) provided $F(X) \subset L_\alpha$ ($F(X) \subset L_\beta$ for some $\beta \in \mathcal{L}$).

The following facts concerning compact mappings are of fundamental importance:

(2.1) (Schauder's Approximation Theorem) *Every compact mapping is the uniform limit of a sequence of finite dimensional mappings.*

For the proof see e.g. [4] p. 24.

(2.2) (Extension of Compact Mappings Theorem). *If A is closed in X and $F: A \to E$ is a compact mapping, then there is a compact extension $F^*: X \to E$ of F over X such that the image $F^*(X)$ is contained in the convex hull* $\operatorname{conv}(F(A))$.

This, in view of Mazur's Lemma [5], is a consequence of the Dugundji Extension Theorem [1].

3. Compact and α-fields. The Leray-Schauder category $\mathfrak{C}$. Given $X, Y \subset E$ and a mapping $f: X \to Y$ we denote by the same but capital letter the mapping $F: X \to E$ defined by $F(x) = x - f(x)$, $x \in X$.

Let X and Y be in E and α be an element in L. A mapping $f: X \to Y$ is said to be a *compact vector field* (*α-field*) provided $F: X \to E$ is a compact mapping (α-mapping).

The set of all compact fields (α-fields) with domain X and range Y will be denoted by $\mathfrak{C}(X, Y)$ ($\mathfrak{C}_\alpha(X, Y)$).

(3.1) *If $f: X \to Y$ and $g: Y \to Z$ are compact fields (α-fields) then so is their composition $gf: X \to Z$. The identity mapping $1_X: X \to X$ is a compact field (α-field) for each $X \subset E$.*

It follows that the subsets of E and the compact vector fields form a category. This category will be denoted by $\mathfrak{C}(E)$ (or simply $\mathfrak{C}$) and called the *Leray–Schauder category*.

In addition the subsets of E and α-fields form a subcategory of $\mathfrak{C}$ denoted by $\mathfrak{C}_\alpha(E)$ (or simply by $\mathfrak{C}_\alpha$).

The union of all categories $\mathfrak{C}_\alpha$ ($\alpha \in L$) in $\mathfrak{C}$ will be denoted by $\mathfrak{C}_0$. The maps of the category $\mathfrak{C}_0$ will be called *finite dimensional* fields.

Remark. Some of the sets $\mathfrak{C}(X, Y)$ might be empty. For instance, the set $\mathfrak{C}(E, E - \{0\})$ is empty because, in view of the Schauder Fixed Point Theorem, there is no compact field $f: E \to E$ such that $f(x) \neq 0$ for all $x \in E$.

Let $\mathfrak{L}$ (resp. $\mathfrak{L}_\alpha$ and $\mathfrak{L}_0$) be the full subcategory of $\mathfrak{C}$ (resp. $\mathfrak{C}_\alpha$ and $\mathfrak{C}_0$) generated by the closed bounded subsets of E. The category $\mathfrak{L}$ will be called the *main category*.

Let $\mathfrak{D}$ be any of the categories introduced above. By $\mathfrak{D}^*$ we shall denote the category of based objects in $\mathfrak{D}$ and by $\overline{\mathfrak{D}}$ the category of pairs of objects in $\mathfrak{D}$. Thus, for example, $\overline{\mathfrak{L}}$ has as objects the closed bounded pairs (X, A), $(Y, B) \subset E$ and as maps the compact fields between such pairs; we write $\mathfrak{L}(X, A; Y, B)$ for such a set of maps. Similarly, $\mathfrak{L}^*$ has as objects the based closed bounded subsets $X, Y \subset E$ and as maps the compact fields preserving base points.

4. Compact and α-homotopies. $\mathfrak{C}$ and $\mathfrak{C}_\alpha$ as h-categories. Given $X, Y \subset E$ and a homotopy $h_t: X \to Y$ ($0 \leqslant t \leqslant 1$) we denote by the capital H the mapping from $X \times I$ ($I - [0, 1]$) to E defined by $H(x, t) = x - h_t(x)$, $(x, t) \in X \times I$.

Let $X, Y \subset E$. A family of compact fields (α-fields) $h_t: X \to Y$ is called a *compact homotopy* (*α-homotopy*) provided $H: X \times I \to E$ is a compact mapping (α-mapping).

Given a compact homotopy (α-homotopy) $h_t: X \to Y$ and a subset $A \subset X$ we let $h'_t = h_t \mid A$ to be the *partial compact homotopy* (α-homotopy); in this case we shall write also $h'_t \subset h_t$ and say that h_t is an extension of $h'_t: A \to Y$ over X.

Two compact fields (α-fields) $f_0, f_1: X \to Y$ are called homotopic, $f_0 \simeq f_1$, (α-homotopic, $f_0 \underset{\alpha}{\simeq} f_1$) provided there is a compact homotopy (α-homotopy) $h_t: X \to Y$ such that $h_0 = f_0$, $h_1 = f_1$.

(4.1) *The relation of homotopy (α-homotopy) is an equivalence relation in each of the sets $\mathfrak{C}(X, Y)$ ($\mathfrak{C}_\alpha(X, Y)$). If the fields ($\alpha$-fields) $f_1, f_2: X \to Y$ and $g_1, g_2: Y \to Z$ are homotopic (α-homotopic), then so are their compositions $g_1 f_1$, $g_2 f_2: X \to Z$.*

It follows that the relations $\simeq$ and $\underset{\alpha}{\simeq}$ convert the Leray—Schauder category $\mathfrak{C}$ and the category $\mathfrak{C}_\alpha$ into h-categories ($\mathfrak{C}$, $\simeq$) and ($\mathfrak{C}_\alpha$, $\underset{\alpha}{\simeq}$) respectively.

The set Map (X, Y) in $\mathfrak{C}/\simeq$ ($\mathfrak{C}_\alpha/_\alpha$) will be denoted by $\pi(X, Y)$ ($\pi_\alpha(X, Y)$) and $[f]$ ($[f]_\alpha$) will stand for the homotopy (α-homotopy) class containing the compact field (α-field) $f: X \to Y$.

It is evident that the main category $\mathfrak{L}$ admits a structure of an h-category ($\mathfrak{L}$, $\simeq$) with the relation of homotopy induced by that in $\mathfrak{C}$. Similarly, the relation $\underset{\alpha}{\simeq}$ in $\mathfrak{C}_\alpha$ converts $\mathfrak{L}_\alpha$ into an h-category ($\mathfrak{L}_\alpha$, $\underset{\alpha}{\simeq}$).

Let $\mathfrak{D}$ be any one of the categories defined in the previous paragraph. By considering relative compact homotopies (or relative α-homotopies) we convert the category of pairs $\overline{\mathfrak{D}}$ into an h-category. Similarly, by considering based compact homotopies (or based α-homotopies) we convert $\mathfrak{D}^*$ into an h-category.

5. The extension problem in $\mathfrak{C}$ and $\mathfrak{C}_\alpha$. Given a pair (X, A) in $\mathfrak{L}$ and a map $f: A \to U$ in $\mathfrak{C}$ (in $\mathfrak{C}_\alpha$) we consider the extension problem for the map f, i.e. the problem of extending f over X in $\mathfrak{C}$ (in $\mathfrak{C}_\alpha$). It is of importance that this problem, under quite general hypotheses, depends only on the homotopy (α-homotopy) class of the given map.

(5.1) (Homotopy Extension Theorem). *Let (X, A) be a closed pair in E, U an open set in E and let $h'_t: A \to U$ be a compact homotopy (α-homotopy). If $h'_0 \subset h_0 \in \mathfrak{C}(X, U)$ ($h_0 \in \mathfrak{C}_\alpha(X, U)$), then there is a compact homotopy (α-homotopy) $h_t: X \to U$ such that $h'_t \subset h_t$.*

For the proof see [3].

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MATHEMATICS
(TOPOLOGY)

Algebraic Topology in Linear Normed Spaces. II. The Functor $\pi(X, U)$

by

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In this Note*) the main category $\mathfrak{L}$ being of primary interest, we are concerned with a contravariant h-functor $\pi(U): \mathfrak{L} \to \mathfrak{C}_S^*$, which is associated with an object U in $\mathfrak{C}$. We show that for some objects U (called admissible), this functor can be converted into an h-functor from $\mathfrak{L}$ to the category $\mathfrak{C}_S^*$ of based sets.

If U is admissible, we associate with an object X in $\mathfrak{L}$ two directed (homotopy) systems of based sets $\{\pi_\alpha(X, U), i_{\alpha\beta}^*\}$ and $\{\pi(X_\alpha, U_\alpha), j_{\alpha\beta}^*\}$. It turns out that these systems are naturally equivalent and the based set $\pi(X, U)$ is a direct limit of the first of them.

1. Fields with admissible range U. Call an object $U \in \mathfrak{C}$ *admissible*, provided U is open in E and its complement is contained in some finite dimensional subspace of E. In the rest of this Note U will stand for an arbitrary but fixed admissible set and $W = E - U$ for its complement. The objects of the main category $\mathfrak{L}$ will be simply called objects and denoted by X, Y, A.

For a given object X we denote by $\mathcal{L}_X^U = \mathcal{L}_X$ the cofinal subset of $\mathcal{L}$ defined by the condition

$$\alpha \in \mathcal{L}_X \Leftrightarrow \begin{cases} \text{(i)} & W \subset L_\alpha \\ \text{(ii)} & U_\alpha = L_\alpha - W \text{ is connected} \\ \text{(iii)} & X_\alpha = X \cap L_\alpha \text{ is not empty} \end{cases}$$

The elements of $\mathcal{L}_X$ are said to be *admissible* with respect to X. We assume that the elements of $\mathcal{L}$ which appear in the sequel are admissible with respect to the objects under consideration.

For notational convenience (U being fixed admissible object) we shall use the following abbreviations:

$$\mathfrak{C}(X) = \mathfrak{C}(X, U); \qquad \pi(X) = \pi(X, U),$$
$$\mathfrak{C}_\alpha(X) = \mathfrak{C}_\alpha(X, U); \qquad \pi_\alpha(X) = \pi_\alpha(X, U),$$
$$\mathfrak{C}(X_\alpha) = \mathfrak{C}(X_\alpha, U_\alpha); \qquad \pi(X_\alpha) = \pi(X_\alpha, U_\alpha).$$

*) For the definitions and notation used in this Note see [1].

Now we establish certain lemmas concerning compact fields and compact homotopies with admissible range U.

(1.1) *Let $h_t : X \to U$ be a compact homotopy. For each ε satisfying $0 < \varepsilon \leqslant \leqslant \operatorname{dist}\big(h(X \times I), W\big)$ there is an α-homotopy $h_t' : X \to U$, such that $\|h_t(x) - h_t'(x)\| < \varepsilon$, $(x \in X,\ t \in I)$.*

Let $H' : X \times I \to L_\alpha$ be an ε-approximation of $H : X \times I \to E$. Assuming (without loss of generality) that $W \subset L_\alpha$ and putting $h_t'(x) = x - H'(x, t)$, $(x \in X,\ t \in I)$ we obtain the required α-homotopy.

As a consequence we have

(1.2) *Let $f : X \to U$ be a compact field. There exists an α-field $f^* : X \to U$, such that $f \simeq f^*$.*

(1.3) *Let $f' : X \to L_\alpha - W$ be a mapping. There exists an α-field $f : X \to E - W$ such that $f' = f_\alpha$.*

Let $F : X \to L_\alpha$ be a compact extension of $F' : X_\alpha \to L_\alpha$ over X and let $f(x) = x - F(x)$, $(x \in X)$. Evidently, $f_\alpha = f'$ and we claim that $f : X \to U$. For, suppose to the contrary that $f(x) \in W$ for some $x \in X$. Since $W \subset L_\alpha$ and $F(X) \subset L_\alpha$ we conclude that $x \in L_\alpha$. But then $f(x) = f'(x) \in W$. This contradiction completes the proof.

As a consequence we obtain

(1.4) *The set $\mathfrak{C}_\alpha(X)$ and hence the sets $\pi_\alpha(X)$ and $\pi(X)$ are not empty.*

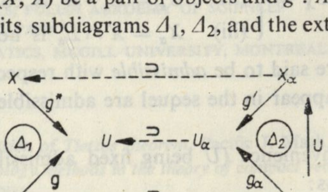
(1.5) *Let $f, g : X \to U$ be two α-fields and $h_t' : X \to L_\alpha - W$ a homotopy between f_α and g_α. There exists an α-homotopy $h_t : X \to U$ between f and g such that $h_t' \subset h_t$.*

Define on $T = (X \times \{0\}) \cup (X_\alpha \times I) \cup (X \times \{1\})$ an α-mapping H^* by

$$H^*(x, t) = \begin{cases} F(x) & \text{if} \quad x \in X \quad \text{and} \quad t = 0, \\ H'(x, t) & \text{if} \quad (x, t) \in X_\alpha \times I, \\ G(x) & \text{if} \quad x \in X \quad \text{and} \quad t = 1. \end{cases}$$

Let $H : X \times I \to L_\alpha$ be a compact extension of H^* over $X \times I$. Putting $h_t(x) = x - H(x, t)$, $(x \in X,\ t \in I)$, we obtain the required α-homotopy $h_t : X \to U$.

2. Inessential fields. Let (X, A) be a pair of objects and $g : A \to U$ an α-field. Consider the diagram (below), its subdiagrams Δ_1, Δ_2, and the extension problem for g.



Diagram

(2.1) *If an α-field g^* completes the diagram Δ_1, then the mapping g' defined by $g' = g_\alpha^*$ completes the diagram Δ_2. Conversely, if a mapping g' completes Δ_2, then there exists an α-field g^*, which completes the whole diagram Δ, i.e., $g^* \mid A = g$ and $g_\alpha^* = g'$.*

Let $T = A \cup X_\alpha$ and define $\bar{G} : T \to L_\alpha$ by

$$\bar{G}(x) = \begin{cases} G'(x) & \text{for } x \in X_\alpha, \\ G(x) & x \in A. \end{cases}$$

Evidently, if $\bar{g}(x) = x - \bar{G}(x)$, $x \in T$, then $\bar{g}(x) \in U$. Since T is closed in X, in view of the Extension of Compact Mappings Theorem, there is a compact extension $G^* : X \to L$ of $\bar{G}$ over X. Putting $g^*(x) = x - G^*(x)$, $x \in X$ we obtain the required α-field $g^* : X \to U$.

Using the same argument we obtain the following proposition

(2.2) *Let $g_1', g_2' : X_\alpha \to U_\alpha$ be two homotopic mappings. If each of them completes the diagram Δ_2, then there exist two α-homotopic α-fields g_1^* and g_2^* such that each of them completes the whole diagram Δ.*

For an object X there is defined a non-empty set $\pi(X)$ of homotopy classes of compact fields $f : X \to U$. We are going to show now that this set contains a distinguished element 0 called the zero homotopy class. Similarly, each set $\pi_\alpha(X)$ contains a distinguished element 0_α called the zero α-homotopy class.

(2.3) *The set $\mathfrak{C}(X)$ $\mathfrak{C}_\alpha(X)$ contains an inessential compact field (α-field).*

In view of the Homotopy Extension Theorem it is sufficient to prove the existence of an inessential α-field in $\mathfrak{C}_\alpha(X)$. Let $g' : X_\alpha \to U_\alpha$ be a constant map. In view of (2.1) there is an α-field $g : X \to U$ such that $g_\alpha = g'$. By the Homotopy Extension Theorem and (2.2) we conclude that the field g is inessential.

(2.4) *Any two inessential compact fields (α-fields) $g', g'' : X \to U$ are homotopic (α-homotopic).*

In view of the Homotopy Extension Theorem it is sufficient to prove the second part of our proposition. We may assume now without loss of generality that X is a closed ball. Then $g_\alpha' \sim g_\alpha''$, since X_α is contractible and $L_\alpha - W = U_\alpha$ is connected. Applying (1.5), we conclude that $g' \underset{\alpha}{\sim} g''$.

As a corollary we obtain the following proposition.

(2.5) *The set of all inessential fields (α-fields) from X into U constitutes a homotopy class (an α-homotopy class) called the 0-element (0_α-element) of $\pi(X)$ $\big(\pi_\alpha(X)\big)$.*

3. The functor π. The homotopy system $\{\pi_\alpha(X), i_{\alpha\beta}^*\}$. Let us denote by π the function which assigns to an object X the based set $\pi(X, U)$ and to each compact field $f : X \to Y$ the induced map $f^* : \pi(Y) \to \pi(X)$.

As a consequence of results of the previous section we obtain the following

THEOREM 1. *The function π is a contravariant h-functor from the main category $\mathfrak{L}$ to the category of based sets $\mathfrak{C}_S^*$.*

Assume now that X is fixed and for each α consider $\pi_\alpha(X)$ as the set with the 0_α-homotopy class as the based element. For each relation $\alpha \leqslant \beta$ let

$$i_{\alpha\beta}^* : \pi_\alpha(X) \to \pi_\beta(X)$$

be the map induced by the inclusion

$$i_{\alpha\beta} : \mathfrak{C}_\alpha (X) \to \mathfrak{C}_\beta (X).$$

The family $\mu (X) = \{\pi_\alpha (X), i^*_{\alpha\beta}\}$ indexed by α will be called the *homotopy system* of X.

(3.1) *The homotopy system of X is a directed system of based sets over the directed set $\mathcal{L}_X$.*

This follows immediately from the definitions.

THEOREM 2. *The homotopy set $\pi (X)$ is a direct limit of the homotopy system of X, i.e., in $\mathfrak{C}^*_S$ we have*

$$\pi (X) \simeq \underset{\longrightarrow}{\mathrm{Lim}} \{\pi_\alpha (X), i^*_{\alpha\beta}\}.$$

The Theorem follows from the following two lemmas

(3.2) *Let $f, g \in \mathfrak{C} (X)$ be two homotopic fields. There exist two α-fields $f^*, g^* \in \mathfrak{C}_\alpha (X)$ such that* (i) $f^* \underset{\alpha}{\sim} g^*$, (ii) $f^* \simeq f$ *and* $g^* \simeq g$.

This lemma follows from (1.1).

(3.3) *Let $f \in \mathfrak{C}_\alpha (X)$ be an α-field and $g \in \mathfrak{C}_\beta (X)$ be a β-field. If f and g are homotopic, then there is a γ with $\alpha \leqslant \gamma$, $\beta \leqslant \gamma$ and such that $f \underset{\gamma}{\sim} g$.*

This again follows from (1.1).

4. The homotopy system $\{\pi (X_\alpha), j^*_{\alpha\beta}\}$. The restriction map $\varkappa^*$. For an object X and $\alpha \in \mathcal{L}_X$ let us denote by $\varkappa^*_\alpha (X)$, or simply by $\varkappa^*_\alpha$, the map

$$\varkappa^*_\alpha : \pi_\alpha (X_\alpha) \to \pi (X_\alpha)$$

defined by the correspondence $[f]_\alpha \to [f_\alpha]$; for $\alpha \leqslant \beta$ we let

$$\varkappa^*_{\alpha\beta} : \pi_\alpha (X_\beta) \to \pi (X_\beta)$$

be the map $\varkappa^*_{\alpha (X_\beta)}$.

(4.1) $\varkappa^*_\alpha$ *and* $\varkappa^*_{\alpha\beta}$ $(\alpha \leqslant \beta)$ *are bijective based maps.*

This is a consequence of (1.5).

Consider now for each relation $\alpha \leqslant \beta$ the map

$$j^*_{\alpha\beta} : \pi (X_\alpha) \to \pi (X_\beta)$$

defined to be the composition

$$\pi (X_\alpha) \to \pi_\alpha (X_\beta) \to \pi (X_\beta)$$

of $(\varkappa^*_{\alpha\beta})^{-1}$ and the map given by $[f]_\alpha \to [f]$. The family $\mu^* (X) = \{\pi (X_\alpha), j^*_{\alpha\beta}\}$ indexed by α will be called the *restricted homotopy system* of X and the family $\varkappa^* = \{\varkappa^*_\alpha\}$ will be called the *restriction map* from $\mu (X)$ to $\mu^* (X)$.

(4.2) *The restricted homotopy system of X is a directed system of based sets over $\mathcal{L}_X$.*

(4.3) *The restriction map $\varkappa^*$ is a natural transformation, i.e., for each relation $\alpha \leqslant \beta$ the commutativity holds in the diagram*:

$$\begin{array}{ccc} \pi_\alpha(X) & \xrightarrow{i^*_{\alpha\beta}} & \pi_\beta(X) \\ \downarrow{\scriptstyle \varkappa^*_\alpha} & & \downarrow{\scriptstyle \varkappa^*_\beta} \\ \pi(X_\alpha) & \xrightarrow{j^*_{\alpha\beta}} & \pi(X_\beta) \end{array}$$

As a consequence of (4.1) and (4.3) we obtain the following theorem.

THEOREM 3. *The restriction map $\varkappa^*$ is a natural equivalence from the homotopy system $\mu(X)$ to the restricted homotopy system $\mu^*(X)$.*

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MATHEMATICS
(TOPOLOGY)

Algebraic Topology in Linear Normed Spaces. III. The Cohomology Functors $\overline{H}^{\infty-n}$

by

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Presented by K. BORSUK on December 9, 1966

In this Note we drop the assumption that the space E is complete.*) Let $\mathfrak{L}_0 = (\mathfrak{L}_0, \sim)$ be the h-category whose objects are closed bounded subsets of E, whose maps are finite dimensional fields and in which the relation $\sim$ means the finite dimensional homotopy of fields. For each integer $n = 1, 2, \ldots$ and a coefficient group G we define the cohomology h-functor $H^{\infty-n}$ from the category $\mathfrak{L}_0$ to the category Ab of Abelian groups. This construction leads to the Alexander—Pontriagin type of isomorphism in a normed space E: the $(\infty - n)$-th cohomology group $H^{\infty-n}(X; G)$, where X is a closed bounded subset of E turns out to be isomorphic to the $(n-1)$-th singular homology group $H_{n-1}(E - X; G)$ of the complement of X in E.

1. Notation. By R^{∞} we denote the normed space consisting of all sequences $x = (x_1, x_2, \ldots)$ of real numbers such that $x_i = 0$ for all but finite set of i, with the norm $\|x\| = \sqrt{\Sigma x_i^2}$ and we let:

$$R^k = \{x \in R^{\infty};\ x_i = 0 \text{ for } i \geqslant k+1\},$$
$$R_+^k = \{x \in R^k;\ x_k \geqslant 0\},$$
$$R_-^k = \{x \in R^k;\ x_k \leqslant 0\},$$
$$S^k = \{x \in R^{k+1};\ \|x\| = 1\},$$
$$E_+^k = \{x \in S^k;\ x_{k+1} \geqslant 0\},$$
$$E_-^k = \{x \in S^k;\ x_{k+1} \leqslant 0\}.$$

We let $s_0 = (1, 0, 0, \ldots)$ and by $\sigma_k : R^k \to S^k$ we denote the inverse of the stereographic projection from $S^k - \{s_0\}$ onto R^k. We note that $\sigma_{k+1}(x) = \sigma_k(x)$ for $x \in R^k$ and

$$\sigma_k(R_+^k) = E_+^k - \{s_0\},\ \sigma_k(R_-^k) = E_-^k.$$

*) In [4] this assumption was implicitly made in the proof of the Extension of Compact Mapping. Theorem. The results in [4] are proved for the complete space E.

2. Preliminaries on Alexander—Pontriagin duality. Let G be an arbitrary but fixed Abelian group and n a positive integer. By H^* (resp. H_*) we shall denote the Čech cohomology (resp. the singular homology) with coefficients in G. By H^0 and H_0 we shall denote the reduced zero-dimensional groups.

A diagram in the category Ab

$$\begin{array}{ccc} A & \xrightarrow{g_1} & B \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ C & \xrightarrow{f_2} & D \end{array}$$

will be called *commutative up to q*, where q is an integer, provided $f_2 f_1 = q g_2 g_1$.

Let (K, L) be a polyhedral pair in S^k $(k \geqslant n)$ and $(L^*. K^*)$ be its k-dual polyhedral pair (for definitions see [1] and [5]).

Denote by

$$D_k : H_{n-1}(K^*) \to H^{k-n}(K),$$
$$D_k : H_n(L^*, K^*) \to H^{k-n}(K, L)$$

the Alexander—Pontriagin isomorphism as defined in [5]. For an arbitrary closed subset $X \subset S^k$ we shall denote by the same letter D_k the isomorphism

$$D_k : H_{n-1}(S^k - X) \to H^{k-n}(X)$$

obtained by passing to the limit.

For an arbitrary subset $A \subset S^{k+1}$ we let

$$A_+ = A \cap E_+^{k+1}, \quad A_- = A \cap E_-^{k+1}$$

Let X be a closed subset of S^{k+1}; we denote by Δ_* (resp. Δ^*) the Mayer—Vietoris homomorphism of the triad (X, X_+, X_-). Thus

$$\Delta_* : H_n(X) \to H_{n-1}(X_0) \quad \text{and} \quad \Delta^* : H^{n-1}(X_0) \to H^n(X),$$

where $X_0 = X \cap S^k$.

2.1. *Let X be a closed subset of S^{k+1}, $X_0 = X \cap S^k$ and $i : (S^k - X_0) \to$ $\to (S^{k+1} - X)$ be the inclusion. Then the following diagram is commutative up to $(-1)^{k+n+1}$.*

$$\begin{array}{ccc} H_{n-1}(S^k - X_0) & \xrightarrow{i_*} & H_{n-1}(S^{k+1} - X) \\ {\scriptstyle D_k}\downarrow & & \downarrow{\scriptstyle D_{k+1}} \\ H^{k-n}(X_0) & \xrightarrow{\Delta^*} & H^{k+1-n}(X). \end{array}$$

Proof. In view of the definition of D_k it is sufficient to prove this in the case $(X, X_0) = (K, K_0)$, where (K, K_0) is a polyhedral pair in S^{k+1}. We may assume, without loss of generality, that K and K_0 are complete subcomplexes of some triangulation T of S^{k+1}. We may assume also that T is such that the suspension of an arbitrary subcomplex of S^k is a subcomplex of S^{k+1}.

Let L and M be the complements of K and K_0 in the triangulation T. Then L and M are $(k+1)$-dual to K and K_0, respectively. Moreover, $L_0 = L \cap S^k = M \cap S^k$ is k-dual to K_0 and the inclusion $SL_0 \subset M$ is a homotopy equivalence. It follows from the definition of L and M that $L_1 = L_+ \cup M_-$ and $L_2 = L_- \cup M_+$ are $(k+1)$-dual to K_+ and K_-, respectively. Denote by $S: H_{n-1}(L_0) \to H_n(SL_0)$ the suspension isomorphism as defined in [1]. Note that if

$$C_+ = (SL_0)_+, C_- = (SL_0)_- \quad \text{then} \quad S^{-1} = (-1)^n \Delta_*,$$

where Δ_* is the Mayer-Vietoris homomorphism of the triad (SL_0, C_+, C_-).

Let us consider the following diagram, in which all horizontal arrows denote homomorphisms from the sequences of corresponding pairs

$$\begin{array}{ccccccccc}
H_n(SL_0) & \longrightarrow & H_n(M) & \longrightarrow & H_n(M, L_1) & \longrightarrow & H_n(L_2, L) & \longrightarrow & H_{n-1}(L) \\
\uparrow S & \text{I} & \downarrow D_{k+1} & \text{II} & \downarrow D_{k+1} & \text{III} & \downarrow D_{k+1} & \text{IV} & \downarrow D_{k+1} \\
H_{n-1}(L_0) & \xrightarrow{D_k} & H^{k-n}(K_0) & \longrightarrow & H^{k+1-n}(K_+, K_0) & \longrightarrow & H^{k+1-n}(K, K_-) & \longrightarrow & H^{k+1-n}(K)
\end{array}$$

In the above diagram the square I is commutative up to $(-1)^{k-n+1}$ by the theorem 4.5 in [1], the square II is commutative (th. 3.5 in [1]), the square III, in which horizontal arrows represent the excision isomorphisms, is commutative by 2.6 in [1] and the square IV is commutative up to $(-1)^n$ by th. 3.5 in [1]. The composition of the bottom row homomorphisms is $\Delta^* D_k$; if we denote by Q the composition of the upper row of homomorphisms then

$$\Delta^* D_k = (-1)^{k+1} D_{k+1} QS.$$

To prove $QS = (-1)^n i_*$, let us consider the following diagram in which all non-labelled homomorphisms are induced by the corresponding inclusions

$$\begin{array}{ccccccc}
H_n(SL_0) & \longrightarrow & H_n(SL_0, C_-) & \longrightarrow & H_n(C_+, L_0) & \xrightarrow{\partial} & H_{n-1}(L_0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_n(M) & \longrightarrow & H_n(M, L_1) & \longrightarrow & H_n(L_2, L) & \xrightarrow{\partial} & H_{n-1}(L).
\end{array}$$

In the above diagram all the squares are commutative, the composition of the upper row of homomorphisms is Δ_* and the composition of the first horizontal homomorphism with the bottom row of homomorphisms is Q, thus $i_* \Delta_* = Q$. Therefore

$$\Delta^* D_k = (-1)^{k+1} D_{k+1} QS = (-1)^{k+1} D_{k+1} i_* \Delta_* S = (-1)^{k+n+1} D_{k+1} i_*$$

and the proof is completed.

Now, let $\{a_p\}$ be a sequence of numbers defined inductively

$$a_0 = 1, \quad a_{p+1} = (-1)^{p+1} a_p.$$

For a closed set $X \subset S^k$ we define the isomorphism

$$\mathcal{D}_k : H^{k-n}(X) \longrightarrow H_{n-1}(S^k - X)$$

by putting

$$\mathcal{D}_k = a_{k+n} D_k^{-1}.$$

In view of **2.1.**, we have

$$a_{k+n} \Delta^* D_k = (-1)^{k+n+1} a_{k+n} D_{k+1} i_* = a_{k+n+1} D_{k+1} i_*$$

and hence, taking into account the definition of $\mathcal{D}_k$, we obtain the following proposition.

2.2. *Let X, X_0, Δ^* and i be as in* **2.1.** *Then* $\mathcal{D}_{k+1} \Delta^* = i_* \mathcal{D}_k$.

Now let X be a compact subset of R^k and $U = R^k - X$. For $p \leqslant k-2$ the homomorphism

$$(\sigma_k|U) : H_p(U) \to H_p(S^k - \sigma_k(X))$$

is an isomorphism. We shall denote by the same letter $\mathcal{D}_k$ the isomorphism $\mathcal{D}_k$ "transferred" by σ_k from S^k to R^k, i.e., the isomorphism which completes the diagram

$$\begin{array}{ccc} H^{k-n}(\sigma_k(X)) & \xrightarrow{(\sigma_k|X)^*} & H^{k-n}(X) \\ \downarrow \mathcal{D}_k & & \downarrow \mathcal{D}_k \\ H_{n-1}(S^k - \sigma_k(X)) & \xleftarrow[(\sigma_k|U)_*]{} & H_{n-1}(U). \end{array}$$

Note that $\mathcal{D}_k$ is defined for $n \leqslant k-1$.

From **2.2** we have immediately the following proposition:

2.3. *Let X be a compact subset of R^{k+1}, $X_0 = X \cap R^k$ and let Δ^* be the Mayer—Vietoris homomorphism of the triad $(X, X \cap R_+^{k+1}, X \cap R_-^{k+1})$. Then the following diagram commutes*

$$\begin{array}{ccc} H^{k-n}(X_0) & \xrightarrow{\Delta^*} & H^{k-n+1}(X) \\ \downarrow \mathcal{D}_k & & \downarrow \mathcal{D}_{k+1} \\ H_{n-1}(R^k - X_0) & \xrightarrow{i_*} & H_{n-1}(R^{k+1} - X) \end{array}$$

($i : R^k - X_0 \to R^{k+1} - X$ *denotes the inclusion*).

3. An orientation in *E*. In what follows, given an element α of the directed set $\mathcal{L}$, we let $d(\alpha)$ be the dimension of α.

Call two linear isomorphisms $l_1, l_2 : L_\alpha \to R^{d(\alpha)}$ equivalent, $l_1 \sim l_2$, provided $l_1 l_2^{-1} \in GL_+(d(\alpha))$, i.e., the determinant of the corresponding matrix is positive. With respect to the relation $\sim$, the set of all linear isomorphisms from L_α to $R^{d(\alpha)}$ decomposes into exactly two equivalent classes. An arbitrary choice of one of these classes will be called an *orientation* of L_α.

Let us choose now an orientation O_α of L_α and call the family $O = \{O_\alpha\}$ an *orientation in E*. Given $\alpha < \beta$ with $d(\beta) - d(\alpha) = 1$ and $l_\alpha \in O_\alpha$, there exists $l_\beta \in O_\beta$ such that $l_\beta(\alpha) = l_\alpha(x)$ for all $x \in L_\alpha$.

We let

$$L_\beta^+ = l_\beta^{-1}(R_+^{d(\beta)}) \quad \text{and} \quad L_\beta^- = l_\beta^{-1}(R_+^{d(\beta)}).$$

Clearly, the definition of L_β^+ and L_β^- depends only on the orientation of L_α and L_β.

Given an object X we let

$$X_\beta^+ = X \cap L_\beta^+ \quad \text{and} \quad X_\beta^- = X \cap L_\beta^-.$$

3.1. *Let X and Y be two objects, $f: X\,Y$ be an α_0-field, and let α, β be two elements of $\mathcal{L}_X$ with $\alpha_0 \leqslant \alpha < \beta$. Then $f(X_\alpha) \subset Y_\alpha$ and $f_\beta: X_\beta \to Y_\beta$ maps the triad $(X_\beta, X_\beta^+, X_\beta^-)$ into the triad $(Y_\beta, Y_\beta^+, Y_\beta^-)$.*

4. $(\infty - n)$-th cohomology system of X. Let n be a positive integer, X be an object of $\mathfrak{L}_0$ and $U = E - X$. Let $O = \{O_\alpha\}$ be an orientation in E and $l_\alpha \in O_\alpha$.

For each $\alpha \in \mathcal{L}_X$ with $d(\alpha) \geqslant n$ define

$$\mathcal{D}_\alpha : H^{d(\alpha)-n}(X_\alpha) \to H_{n-1}(U_\alpha)$$

to be the Alexander—Pontriagin isomorphism "transferred" by l_α^{-1} from $R^{d(\alpha)}$ to L_α, i.e., the map that completes the diagram:

$$\begin{array}{ccc} H^{d(\alpha)-n}(X_\alpha) & \dashrightarrow & H_{n-1}(U_\alpha) \\ \downarrow (l_\alpha)^* & & \downarrow (l_\alpha)_* \\ H^{d(\alpha)-n}(i_\alpha(X_\alpha)) & \xrightarrow{\mathcal{D}_{d(\alpha)}} & H_{n-1}(l_\alpha(U_\alpha)). \end{array}$$

4.1. *Let X be an object in $\mathcal{L}_0$, $U = E - X$ and $\alpha, \beta \in \mathcal{L}_X$ with $d(\beta) - d(\alpha) = 1$. Let $i_{\alpha\beta}: U_\alpha \to U_\beta$ be the inclusion and*

$$\Delta_{\alpha\beta} : H^{d(\alpha)-n}(X_\alpha) \to H^{d(\beta)-n}(X_\beta)$$

be the Mayer—Vietoris homomorphism of the triad $(X_\beta, X_\beta^+, X_\beta^-)$. Then the following diagram commutes

$$\begin{array}{ccc} H^{d(\alpha)-n}(X_\alpha) & \xrightarrow{\Delta_{\alpha\beta}} & H^{d(\beta)-n}(X_\beta) \\ \downarrow \mathcal{D}_\alpha & & \downarrow \mathcal{D}_\beta \\ H_{n-1}(U_\alpha) & \xrightarrow{(i_{\alpha\beta})_*} & H_{n-1}(U_\beta). \end{array}$$

This follows from the definition of $\Delta_{\alpha\beta}$ and proposition 2.3.

Let $\alpha, \beta \in \mathcal{L}_X$, with $\alpha < \beta$, and let $\alpha = \alpha_0 < \alpha_1 < \ldots < \alpha_k = \beta$ be a chain of elements of $\mathcal{L}_X$ such that $d(\alpha_{i+1}) - d(\alpha_i) = 1$ $(i = 0, 1, \ldots, k-1)$. Define the homomorphism

$$\Delta_{\alpha\beta} : H^{d(\alpha)-n}(X_\alpha) \longrightarrow H^{d(\beta)-n}(X_\beta)$$

to be the composition of the homomorphisms $\Delta_{\alpha\alpha_1}, \Delta_{\alpha_1\alpha_2}, \ldots, \Delta_{\alpha_{k-1}\beta}$. It follows from 4.1. that the definition of $\Delta_{\alpha\beta}$ does not depend on the choice of the chain $\alpha_1, \ldots, \alpha_{k-1}$ joining α and β.

For an object $X \in \mathfrak{L}_0$ consider the groups $H^{d(\alpha)-n}(X_\alpha)$ together with the homomorphisms $\Delta_{\alpha\beta}$. The family $\{H^{d(\alpha)-n}(X_\alpha), \Delta_{\alpha\beta}\}$ indexed by $\alpha \in \mathcal{L}_X$ will be called the $(\infty - n)$-th *cohomology system of* X corresponding to the orientation O in E.

4.2. *The $(\infty - n)$-cohomology system of X is a direct system of Abelian groups isomorphic to the direct system of homology groups* $\{H_{n-1}(U_\alpha), (i_{\alpha\beta})_*\}$ $\alpha \in \mathcal{L}_X$.

This follows immediately from **4.1** and the definition of $\Delta_{\alpha\beta}$.

4.3. *Let* $\{H^{d(\alpha)-n}(X_\alpha), \Delta_{\alpha\beta}\}$, $\{H^{d(\alpha)-n}(X_\alpha), \bar{\Delta}_{\alpha\beta}\}$ *be two $(\infty - n)$-cohomology systems of X corresponding to different orientations $O = \{O_\alpha\}$ and $\bar{O} = \{\bar{O}_\alpha\}$ in E. Then the above systems are isomorphic.*

This follows immediately from **4.2**.

4.4. *Let $X, Y \in \mathfrak{L}_0$, $\alpha_0 \in \mathcal{L}_X$ and $f: X \to Y$ be an α_0-field. Then for each $\alpha, \beta \in \mathcal{L}_X$ with $\alpha_0 \leqslant \alpha < \beta$ the following diagram commutes:*

$$\begin{array}{ccc} H^{d(\alpha)-n}(X_\alpha) & \xrightarrow{f_\alpha^*} & H^{d(\alpha)-n}(Y_\alpha) \\ \downarrow \Delta_{\alpha\beta} & & \downarrow \Delta_{\alpha\beta} \\ H^{d(\beta)-n}(X_\beta) & \xrightarrow{f_\beta^*} & H^{d(\beta)-n}(Y_\beta). \end{array}$$

This follows from the definition of $\Delta_{\alpha\beta}$, the proposition **3.1** and the properties of the Mayer—Vietoris homomorphism (e.g. [3], p. 41).

5. The functor $\bar{H}^{\infty-n}: \mathfrak{L}_0 \to Ab$. Let X be an object in $\mathfrak{L}_0$. We define the Abelian group

$$H^{\infty-n}(X) = \underset{\overrightarrow{\alpha}}{\operatorname{Lim}} \{H^{d(\alpha)-n}(X_\alpha), \Delta_{\alpha\beta}\}$$

to be the direct limit of the $(\infty - n)$-th cohomology system of X.

Let X, Y be two objects and let $f: X \to Y$ be an α_0-field. It follows from **4.4** that f induces a map $\{f_\alpha^*\}$ from the $(\infty - n)$-th cohomology system of Y into the $(\infty - n)$-th cohomology system of X, and therefore determines a map

$$f^* = \underset{\overrightarrow{\alpha}}{\operatorname{Lim}} \{f_\alpha^*\}$$

from $H^{\infty-n}(Y)$ to $H^{\infty-n}(X)$.

5.1. *The induced homomorphism f^* satisfies the following properties:*

(a) *if* 1 *is an identity in $\mathfrak{L}_0$ then so is the* 1^* *in* Ab;

(b) *for any two composable fields f and g we have*

$$(gf)^* = f^* g^*;$$

(c) *if the fields f and g are homotopic then* $f^* = g^*$.

This follows from the properties of the direct limits (e.g. [3] p. 223) and from the properties of the Mayer—Vietoris boundary homomorphism (e.g. [3], th. I. 15. 4c).

Let us denote now by $\bar{H}^{\infty-n}$ the function which assigns to an object $X \in \mathfrak{L}_0$ the $(\infty - n)$-th cohomology group $H^{\infty-n}(X)$ and to a finite-dimensional field $f : X \to Y$ the induced homomorphism $f^* : H^{\infty-n}(Y) \to H^{\infty-n}(X)$.

As a sequence of proposition **5.1** we obtain the following

THEOREM 1. *The function* $\bar{H}^{\infty-n}$ *is a contravariant h-functor from the category* $\mathfrak{L}_0$ *to the category of Abelian groups Ab.*

5. The Alexander—Pontriagin duality in E. In view of **4.2** the $(\infty - n)$-th cohomology system of an object X is isomorphic to the direct system of homology groups $\{H_{n-1}(U_\alpha), (i_{\alpha\beta})_*\}$, where $U = E - X$. We have clearly

$$H_{n-1}(U) = \varinjlim_{\alpha} \{H_{n-1}(U_\alpha), (i_{\alpha\beta})_*\}.$$

As a consequence, in view of the definition of the group $H^{\infty-n}(X)$, we obtain the following theorem:

THEOREM 2. *For every object X in* $\mathfrak{L}_0$ *we have an isomorphism*

$$H^{\infty-n}(X) \approx H_{n-1}(E - X)$$

between the $(\infty - n)$*-th cohomology group of X and the* $(n - 1)$*-th singular homology group of the complement of X in E.*

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MATHEMATICS
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Algebraic Topology in Linear Normed Spaces. IV. The Alexander—Pontriagin Invariance Theorem in E

by

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In this Note*) the cohomology h-functor $\overline{H}^{\infty-n}:\mathfrak{L}_0\to Ab$, defined in the previous paper [3], is extended onto the main category $\mathfrak{L}$. As a consequence we obtain the Alexander—Pontriagin Invariance Theorem in infinite dimensional normed space E: the equivalence or h-equivalence of objects X and Y in $\mathfrak{L}$ implies for each n the isomorphism between the n-th homology groups of $E-X$ and $E-Y$.

1. Algebraic preliminaries. Given a directed set $\mathcal{L}=\{\alpha,\beta,\gamma,...\}$, denote by the same letter $\mathcal{L}$ the category having as objects the elements of $\mathcal{L}$ and as maps the relations $\alpha\leqslant\beta$ in $\mathcal{L}$. For a small category $\mathfrak{D}$, denote by $(\mathcal{L},\mathfrak{D})$ the category of covariant functors from $\mathcal{L}$ to $\mathfrak{D}$, i.e., the category of direct systems of objects of $\mathfrak{D}$ over $\mathcal{L}$.

By $\underset{\overrightarrow{\alpha}}{\mathrm{Lim}}:(\mathcal{L},\mathfrak{D})\to\mathfrak{D}$ we shall denote the "direct limit" functor, i.e., the left-adjoint to the constant functor from $\mathfrak{D}$ to $(\mathcal{L},\mathfrak{D})$.

Let $\mathcal{N}=\{k,l,m,...\}$ and $\mathcal{L}=\{\alpha,\beta,\gamma,...\}$ be two directed sets. Denote by $\mathcal{N}\times\mathcal{L}$ the corresponding product category.

Given a direct system of Abelian groups

$$\varkappa\in(\mathcal{N}\times\mathcal{L},Ab)$$

let us put

$$\varkappa(\alpha,k)=H_\alpha^k,$$
$$\varkappa(\alpha\leqslant\beta,k\leqslant k)=\Delta_{\alpha\beta}^k,$$
$$\varkappa(\alpha\leqslant\alpha,k\leqslant l)=i_{k,l}^\alpha.$$

*) For the definitions and the notation used in the sequel see [2].

For any relations $k \leqslant l$ and $\alpha \leqslant \beta$ we have the commutative diagram

$$\begin{array}{ccc} H_\alpha^k & \xrightarrow{i_{kl}^\alpha} & H_\alpha^l \\ {\scriptstyle \Delta_{\alpha\beta}^k}\downarrow & & \downarrow{\scriptstyle \Delta_{\alpha\beta}^l} \\ H_\beta^k & \xrightarrow{i_{kl}^\beta} & H_\beta^l \end{array}$$

Clearly, each double system of Abelian groups $\{H_\alpha^k, \}$ indexed by $\mathcal{N}\times\mathcal{L}$ together with the maps $\{i_{kl}^\alpha\}$, $\{\Delta_{\alpha\beta}^k\}$, (satisfying the natural functorial properties), may be identified with a functor $\varkappa \in (\mathcal{N}\times\mathcal{L}, Ab)$. We shall write simply $\varkappa = \{H_\alpha^k, \}$.

1.1. *Let $\mathcal{L} = \{\alpha, \beta, \gamma, \ldots\}$ and $\mathcal{N} = \{k, l, m, \ldots\}$ be two directed sets, $\mathcal{N}\times\mathcal{L}$ be the product category, and let*

$$\lambda : (\mathcal{N}\times\mathcal{L}, Ab) \to (\mathcal{N}, (\mathcal{L}, Ab)),$$

$$\mu : (\mathcal{N}\times\mathcal{L}, Ab) \to (\mathcal{L}, (\mathcal{N}, Ab))$$

be the natural isomorphisms between the corresponding categories. Then, the following diagram commutes

$$(\mathcal{D}) \qquad \begin{array}{ccc} (\mathcal{N}, (\mathcal{L}, Ab)) & \xrightarrow[\ k\ ]{\mathrm{Lim}} & (\mathcal{L}, Ab) \\ {\scriptstyle \lambda}\uparrow{\scriptstyle \sim} & & \downarrow \underrightarrow{\mathrm{Lim}}_{\alpha} \\ (\mathcal{N}\times\mathcal{L}, Ab) & & Ab \\ {\scriptstyle \mu}\downarrow{\scriptstyle \sim} & & \uparrow \underrightarrow{\mathrm{Lim}}_{k} \\ (\mathcal{L}, (\mathcal{N}, Ab)) & \xrightarrow[\ \alpha\ ]{\mathrm{Lim}} & (\mathcal{N}, Ab) \end{array}$$

The word "commutativity" stands for the natural equivalence of functors.

The commutativity of the diagram $(\mathcal{D})$ follows from the fact that the left-adjoint functor commutes with direct limits [4].

We shall restate now a part of the proposition (1.1) in equivalent but slightly more convenient terms.

Let us denote by

$$\tau : \underrightarrow{\mathrm{Lim}}_{k} \cdot \underrightarrow{\mathrm{Lim}}_{\alpha} \cdot \mu \to \underrightarrow{\mathrm{Lim}}_{\alpha} \cdot \underrightarrow{\mathrm{Lim}}_{k} \cdot \lambda$$

the natural equivalence between the corresponding functors. In order to simplify the notation given $\varkappa \in (\mathcal{N}\times\mathcal{L}, Ab)$, let us denote by the same letter $\varkappa$ the direct systems $\lambda(\varkappa)$ and $\mu(\varkappa)$.

1.2. *For any double direct system of Abelian groups $\varkappa = \{H_\alpha^k, \}$ indexed by $\mathcal{N}\times\mathcal{L}$, we have a natural isomorphism between the limit groups*

$$\tau_\varkappa : \underrightarrow{\mathrm{Lim}}_{\alpha}\ \underrightarrow{\mathrm{Lim}}_{k} \{H_\alpha^k, \} \to \underrightarrow{\mathrm{Lim}}_{k}\ \underrightarrow{\mathrm{Lim}}_{\alpha} \{H_\alpha^k, \}.$$

1.3. *Let $\varkappa = \{H_\alpha^k, \}$ and $\bar{\varkappa} = \{\bar{H}_\alpha^k, \}$ be two double directed systems of Abelian groups over $\mathcal{N}\times\mathcal{L}$ and let $\{f_{\alpha k}^*\}: \{H_\alpha^k, \} \to \{\bar{H}_\alpha^k, \}$ be a map from $\varkappa$ to $\bar{\varkappa}$. Then the following diagram commutes:*

$$\begin{array}{ccc}
\varinjlim_{\alpha} \varinjlim_{k} \{H_\alpha^k, \} & \xrightarrow{\ \tau_\varkappa\ } & \varinjlim_{k} \varinjlim_{\alpha} \{H_\alpha^k, \} \\
\Big\downarrow {\scriptstyle \varinjlim_{\alpha} \varinjlim_{k} \{f_{\alpha k}^*\}} & & \Big\downarrow {\scriptstyle \varinjlim_{k} \varinjlim_{\alpha} \{f_{\alpha k}^*\}} \\
\varinjlim_{\alpha} \varinjlim_{k} \{\bar{H}_\alpha^k, \} & \xrightarrow{\ \tau_{\bar{\varkappa}}\ } & \varinjlim_{k} \varinjlim_{\alpha} \{\bar{H}_\alpha^k, \}
\end{array}$$

2. Approximating sequences. For an object X and a natural number k let

$$X^{(k)} = \left\{x \in E,\ \varrho(x, X) \leqslant \frac{1}{k}\right\}.$$

We shall say that a sequence of objects $\{X_k\}$ is an approximating sequence or an (a)-*sequence* for an object X provided

(i) $X_k \supset X_{k+1}$ for each $k = 1, 2, \ldots$,

(ii) $X = \bigcap_{k=1}^{\infty} X_k$.

2.1. *Let $\{X_k\}$ be an* (a)-*sequence for an object X. Let us put for each k,*

$$\tilde{X}_k = (X_k)^k = \left\{x \in E;\ \varrho(x, X_k) \leqslant \frac{1}{k}\right\}.$$

Then the enlarged sequences $\{\tilde{X}_k\}$ is also an (a)-*sequence for X.*

2.2. *Let $\{X_k\}$ be an* (a)-*sequence for X and let α be an arbitrary element of $\mathcal{L}$ such that $X_\alpha = L_\alpha \cap X$ is non empty. Then the sequences $\{X_{k\alpha}\}$, $\{\tilde{X}_{k\alpha}\}$ are* (a)-*sequences for X_α.*

2.3. *Let $\{X_k\}$ and $\{Y_k\}$ be two* (a)-*sequence for X and Y, respectively. Then $\{X_k \cup Y_k\}$ is an* (a)-*sequence for $X \cup Y$.*

2.4. *Let $\{X_k\}$ be an* (a)-*sequence for X and let $f: X_1 \to E$ be a compact field. Then $\{f(X_k)\}$ is an* (a)-*sequence for $f(X)$.*

It is sufficient to prove the inclusion

$$\bigcap_{k=1}^{\infty} f(X_k) \subset f\left(\bigcap_{k=1}^{\infty} X_k\right).$$

Let $y \in \bigcap_{k=1}^{\infty} f(X_k)$; we have $y = f(x_k)$, where $x_k \in X_k$ and thus $y = x_k - F(x_k)$. Since F is compact we may assume without loss of generality that $\lim_{k=\infty} x_k = x$. Consequently, $y = \lim_{k=\infty} f(x_k) = f(x)$. Since $x \in \bigcap_{k=1}^{\infty} X_k$, thisc ompletes the proof.

3. The continuity of the functor $\bar{H}^{\infty-n}$. We shall establish first a property of the functor $\bar{H}^{\infty-n}$, which is analogous to the property of continuity of the Čech cohomology theory.

Let Y be an object, $\alpha \in \mathcal{L}_Y$ and let $\{Y_k\}$ be an (a)-sequence for Y. We let $Y_\alpha^k = Y_k \cap L_\alpha$. Consider the following inclusions, all of them being finite dimensional fields:

$$i_{k,l}: Y_l \to Y_k, \quad i_{k,l}^\alpha: Y_\alpha^l \to Y_\alpha^k, \quad (k \leqslant l);$$

$$j_k: Y \to Y_k, \quad j_{k\alpha}: Y_\alpha \to Y_\alpha^k.$$

To the commutative diagram in $\mathfrak{L}_0$

$$\begin{array}{ccc} Y & & \\ \downarrow{\scriptstyle j_k} & \searrow{\scriptstyle j_l} & \\ Y_k & \xleftarrow{\ i_{k,l}\ } & Y_l \end{array}$$

corresponds the commutative diagram of homomorphisms

$$\begin{array}{ccc} H^{\infty-n}(Y) & & \\ \uparrow{\scriptstyle j_k^*} & \nwarrow{\scriptstyle j_l^*} & \\ H^{\infty-n}(Y_k) & \xrightarrow{\ i_{k,l}^*\ } & H^{\infty-n}(Y_l) \end{array}$$

THEOREM 1. *The direct family $\{j_k^*\}$ represents the group $H^{\infty-n}(Y)$ as the direct limit of the direct system $\{H^{\infty-n}(Y_k), i_{k,l}^*\}$, i.e., the map*

$$\underset{\overrightarrow{k}}{\mathrm{Lim}}\{j_k^*\}: \underset{\overrightarrow{k}}{\mathrm{Lim}}\{H^{\infty-n}(Y_k), i_{k,l}^*\} \to H^{\infty-n}(Y)$$

is an isomorphism.

Consider over $\mathcal{N} \times \mathcal{L}$ the following double direct systems of Abelian groups $\varkappa = \{H^{d(\alpha)-n}(Y_{\alpha k}), \}$ and $\bar{\varkappa} = \{H^{d(\alpha)-n}(Y_\alpha), \}$. Clearly $\{j_{k\alpha}^*\}$ is a map from $\varkappa$ to $\bar{\varkappa}$.

In view of **2.2** and the continuity of the Čech cohomology (e.g., [1] p. 322), the map $\underset{\overrightarrow{k}}{\mathrm{Lim}}\{j_{k\alpha}^*\}$ is an isomorphism for each α, and therefore so is the map, $\underset{\overrightarrow{\alpha}}{\mathrm{Lim}}\,\underset{\overrightarrow{k}}{\mathrm{Lim}}\{j_{k\alpha}^*\}$. Consequently, in view of **1.3**, the map

$$\underset{\overrightarrow{k}}{\mathrm{Lim}}\{j_k^*\} = \underset{\overrightarrow{k}}{\mathrm{Lim}}\,\underset{\overrightarrow{\alpha}}{\mathrm{Lim}}\{j_{k\alpha}^*\}$$

is also an isomorphism and the proof of Theorem 1 is completed.

4. Approximating systems. Let X, Y be two objects and let $f: X \to Y$ be a compact field. A sequence $\{Y_k, f_k\}$ of objects Y_k and α_k-fields $f_k : X \to Y_k$ is called an *approximating system* or (a)-*system* for f, provided:

(i) $\{Y_k\}$ is an (a)-sequence for Y,

(ii) $f_k \sim j_k f$ in $\mathfrak{L}$, where $j_k : Y \to Y_k$ is the inclusion,

(iii) $f_k \sim i_{kl} f_l$ in $\mathfrak{L}_0$, where $i_{kl} : Y_l \to Y_k$ is the inclusion $(k < l)$.

4.1. *Let $f: X \to Y$ be a compact field, $Y_k = Y^{(k)}$, and let $f_k : X \to Y_k$ be an α_k-field such that*

$$\|f(x) - f_k(x)\| \leqslant \frac{1}{k} \quad \text{for all} \quad x \in X.$$

Then $\{Y_k, f_k\}$ is an (a)-*system for f.*

In what follows any system $\{Y_k, f_k\}$ as in **4.1** will be called a *standard* (a)-*system* for f.

Let $\{Y_k, f_k\}$ be an arbitrary (a)-system for a compact field $f: X \to Y$ and consider the triangle $\mathcal{D}_{k,l}$ for $k \leqslant l$

$$\begin{array}{ccc} H^{\infty-n}(Y_k) & \xrightarrow{\;i^*_{k,l}\;} & H^{\infty-n}(Y_l) \\ & {\scriptstyle f^*_k}\searrow \quad \swarrow{\scriptstyle f^*_l} & \\ & H^{\infty-n}(X) & \end{array}$$

Since for each $k \leqslant l$, $f_k \sim i_{kl} f_l$ in $\mathfrak{L}_0$, it follows that $\mathcal{D}_{k,l}$ commutes in Ab. Consequently, $\{f^*_k\}$ is a direct family of maps and

$$\varinjlim_k \{f^*_k\} : \varinjlim_k \{H^{\infty-n}(Y_k), i^*_{kl}\} \to H^{\infty-n}(X).$$

4.2. *Let $\{Y_k, f_k\}$ be an* (a)-*system for a field $f: X \to Y$ and $\{\overline{Y}_k\}$ be an* (a)-*sequence for Y such that, for each k, $Y_k \subset \overline{Y}_k$ with $l_k : Y_k \to \overline{Y}_k$ standing for the inclusion. Then $\{\overline{Y}_k, \overline{f}_k\}$, where $\overline{f}_k = l_k f_k$ is an* (a)-*system for f.*

4.3. *Let $\{Y_k, f_k\}$ and $\{\overline{Y}_k, \overline{f}_k\}$ be as in* **4.2** *and let $j_k : Y \to Y_k$, $i_k : Y \to \overline{Y}_k$ denote the corresponding inclusions. Then we have*

$$\varinjlim_k \{f^*_k\} (\varinjlim_k \{j^*_k\})^{-1} = \varinjlim_k \{\overline{f}^*_k\} (\varinjlim_k \{i^*_k\})^{-1}.$$

From the commutative diagram

$$\begin{array}{ccccc} & & Y_k & & \\ & {\scriptstyle j_k}\nearrow & & \nwarrow{\scriptstyle f_k} & \\ Y & & \downarrow{\scriptstyle l_k} & & X \\ & {\scriptstyle i_k}\searrow & & \swarrow{\scriptstyle \overline{f}_k} & \\ & & \overline{Y}_k & & \end{array}$$

we have

$$\varinjlim_k \{\bar{f}_k^*\} = \varinjlim_k \{f_k^*\} \varinjlim_k \{l_k^*\} = \varinjlim_k \{f_k^*\} (\varinjlim_k \{j_k^*\})^{-1} \varinjlim_k \{i_k^*\}.$$

This implies our assertion.

4.4. *Let $\{Y_k, f_k\}$, $\{Y_k, \bar{f}_k\}$ be two* (a)*-systems for a field $f: X \to Y$ with the same sequence $\{Y_k\}$. Consider the enlarged sequence $\{W_k\} = \{\tilde{Y}_k\}$, denote by $i_k : Y \to W_k$, $l_k : Y_k \to W_k$ the corresponding inclusions and put $g_k = l_k f_k$, $\bar{g}_k = l_k \bar{f}_k$. Then $\{W_k, g_k\}$ and $\{W_k, \bar{g}_k\}$ are again* (a)*-systems tor f and we have*:

$$\varinjlim_k \{g_k^*\} (\varinjlim_k \{j_k^*\})^{-1} = \varinjlim_k \{\bar{g}_k^*\} (\varinjlim_k \{j_k^*\})^{-1}.$$

In view of the definition, we have $f_k \sim \bar{f}_k$ in $\mathfrak{L}$. Let $h_t^k : X \to Y$ be the corresponding compact homotopy and let $\bar{h}_t^k : X \to E$ be a γ_k-homotopy such that

$$\|h_t^k(x) - \bar{h}_t^k(x)\| \leqslant \frac{1}{k} \quad \text{for all} \quad (x, t) \in X \times I.$$

Clearly $\bar{h}_t^k(x) \in W_k$ for all $(x, t) \in X \times I$ and consequently $\bar{h}_t^k$ may be viewed as a γ_k-homotopy $\bar{h}_t^k : X \to W_k$. Assuming, without loss of generality, that $f_k, \bar{f}_k$ are γ_k-fields, we evidently have $g_k \underset{\gamma_k}{\sim} \bar{h}_0^k$, $\bar{g}_k \underset{\gamma_k}{\sim} \bar{h}_1^k$ and therefore $g_k \underset{\gamma_k}{\sim} \bar{g}_k$.

This implies $g_k^* = \bar{g}_k^*$ for each k, which completes the proof.

4.5. *Let $\{Y_k, f_k\}$ and $\{\bar{Y}_k, \bar{f}_k\}$ be two arbitrary* (a)*-systems for a field $f: X \to Y$, and denote by $j_k : Y \to Y_k$, $i_k : Y \to \bar{Y}_k$ the corresponding inclusions. Then*

$$\varinjlim_k \{f_k^*\} (\varinjlim_k \{j_k^*\})^{-1} = \varinjlim_k \{\bar{f}_k^*\} (\varinjlim_k \{i_k^*\})^{-1}.$$

Let us put $W_k = (Y_k \cup \bar{Y}_k)^{(k)}$. In view of **2.3** and **2.1**, $\{W_k\}$ is an (*a*)-sequence for Y. Denote by $l_k : Y_k \to W_k$, $\bar{l}_k : \bar{Y}_k \to W_k$ the corresponding inclusions and define $g_k, \bar{g}_k : X \to W_k$ by putting $g_k = l_k f_k$, $\bar{g}_k = \bar{l}_k \bar{f}_k$. Clearly, $\{W_k, g_k\}$ and $\{W_k, \bar{g}_k\}$ are (a)-systems for f. Now the pairs $\{W_k, g_k\}$, $\{Y_k, f_k\}$ and $\{W_k, \bar{g}_k\}$, $\{\bar{Y}_k, \bar{f}_k\}$ satisfy the assumptions of **4.3** and therefore our assertion follows from **4.4**.

5. The functor $H^{\infty-n}$. Given a compact field $f: X \to Y$, let $\{Y_k, f_k\}$ be an (a)-system for f. We define the induced homomorphism

$$f^* : H^{\infty-n}(Y) \to H^{\infty-n}(X)$$

by the formula

$$f^* = \varinjlim_k \{f_k^*\} (\varinjlim_k \{j_k^*\})^{-1}.$$

In view of the results of the previous section, the definition of f^* does not depend on the choice of $\{Y_k, f_k\}$.

Now define the function $H^{\infty-n}$ from the main category $\mathfrak{L}$ to the category of abelian groups Ab by putting $H^{\infty-n}(X) = \overline{H}^{\infty-n}(X)$, $H^{\infty-n}(f) = f^*$. If $f: X \to Y$ is an α-field, it is easily seen (by taking $\{Y_k, f_k\}$ with $Y_k = Y, f_k = f$) that $\overline{H}^{\infty-n}(f) = H^{\infty-n}(f)$, i.e., $H^{\infty-n}$ extends $\overline{H}^{\infty-n}$ over $\mathfrak{L}$.

THEOREM 2. *The induced map f^* satisfies the following properties*: (*a*) *the homotopy $f \sim g$ implies $f^* = g^*$*; (*b*) $(gf)^* = f^* \circ g^*$. *In other words, $H^{\infty-n}$ is an h-functor from the main category $\mathfrak{L}$ to the category ot Abelian groups Ab.*

Proof of the property (a). Let $f, g: X \to Y$ be two compact fields and $h_t: X \to Y$ be a compact homotopy such that $f = h_0$, $g = h_1$. Let $h_t^{(k)}: X \to Y^{(k)}$ be an α_k-homotopy satisfying

$$\|h_t^{(k)}(x) - h_t(x)\| \leqslant \frac{1}{k} \quad \text{for all} \quad (x, t) \in X \times I.$$

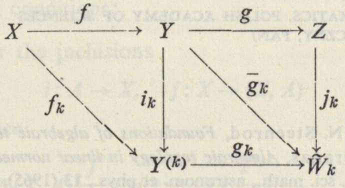
Let $f_k = h_0^{(k)}$, $g_k = h_1^{(k)}$. Evidently, $\{Y^{(k)}, f_k\}$ and $\{Y^{(k)}, g_k\}$ are (a)-systems for f and g respectively. Thus, since $f_k \underset{\gamma_k}{\sim} g_k$ we have $\varinjlim_k \{f_k^*\} = \varinjlim_k \{g_k^*\}$ and therefore $f^* = g^*$.

Proof of the property (b) (special case). Let $f: X \to Y$ be a compact field, $g: Y \to Z$ be an α_0-field and let $h = gf$. We shall prove that $h^* = f^* \circ g^*$.

Take a standard (a)-system $\{Y^{(k)}, f_k\}$ for f and take an arbitrary finite dimensional extension $\bar{g}: Y^{(1)} \to E$ of g over $Y^{(1)}$. The existence of $\bar{g}$ follows from the Tietze Extension Theorem.

Let us put $W_k = \bar{g}(Y^{(k)}) \cup Z$ for each k and consider the enlarged (a)-sequence $\{\widetilde{W}_k\}$ for Z.

In the diagram



define g_k by putting $g_k(y) = \bar{g}(y)$ for all $y \in Y^{(k)}$ and let $\bar{g}_k = g_k i_k$, $h_k = g_k f_k$. It is easily seen that $\{\widetilde{W}_k, h_k\}$, $\{\widetilde{W}_k, g_k\}$ are (a)-systems for h and g, respectively. We have

$$\varinjlim_k \{h_k^*\} = \varinjlim_k \{f_k^*\} \varinjlim_k \{g_k^*\} = \varinjlim_k \{f_k^*\} (\varinjlim_k \{i_k^*\})^{-1} \varinjlim_k \{\bar{g}_k^*\}$$

and thus

$$\varinjlim_k \{h_k^*\} = f^* (\varinjlim_k \{\bar{g}_k^*\}).$$

This implies $h^* = f^* \circ g^*$, and the proof is completed.

Proof of the property (b) (general case). Let $f: X \to Y$ and $g: Y \to Z$ be two compact fields and let $h = gf$.

Let $\{Z^{(k)}, h_k\}$ and $\{Z^{(k)}, g_k\}$ be two standard (a)-systems for h and g, respectively. From the inequalities

$$\|h_k(x) - h(x)\| \leqslant \frac{1}{k} \quad \text{for all} \quad x \in X,$$

$$\|g_k f(x) - h(x)\| \leqslant \frac{1}{k} \quad \text{for all} \quad x \in X,$$

it follows that the fields $g_k f, h_k: X \to Z^{(k)}$ are homotopic in $\mathfrak{L}$ for each $k = 1, 2, \ldots$. This implies, in view of the property (a), that $h_k^* = (g_k f)^*$. Since each g_k is finite dimensional we have $(g_k f)^* = f^* g_k^*$ and thus we obtain

$$\varinjlim_k \{h_k^*\} = \varinjlim_k \{f^* g_k^*\} = f^* \varinjlim_k \{g_k^*\}.$$

This implies $h^* = f^* g^*$ and the proof of Theorem 2 is completed.

6. The Alexander—Pontriagin Invariance Theorem in E. It follows from Theorem 1 that if the objects X and Y are equivalent or h-equivalent in $\mathfrak{L}$, then the $(\infty - n)$-th cohomology groups $H^{\infty-n}(X; G)$ and $H^{\infty-n}(Y; G)$ with the coefficients in G are isomorphic. From this, taking into account Theorem 2 in [3], we obtain the following

THEOREM 2. *Let X and Y be two equivalent or h-equivalent objects of the category $\mathfrak{L}$. Then for each $n = 1, 2, \ldots$ and a coefficient group G, the singular homology groups $H_{n-1}(E - X; G)$ and $H_{n-1}(E - Y; G)$ of $E - X$ and $E - Y$ are isomorphic.*

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MATHEMATICS
(TOPOLOGY)

Algebraic Topology in Linear Normed Spaces. V. Cohomology Theory on $\mathfrak{C}(E)$

by

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1. Introduction. Let E be an arbitrary but fixed normed space and $\mathfrak{C}=\mathfrak{C}(E)$ be the main category, i.e., the h-category whose objects are closed bounded subsets of E, whose maps are compact vector fields and in which the relation of homotopy means the homotopy of compact fields. Let $\mathfrak{C}^{(2)}$ be the h-category of pairs of $\mathfrak{C}$ and Ab be the category of Abelian groups. Let $\vartheta:\mathfrak{C}^{(2)}\to\mathfrak{C}^{(2)}$ be the functor defined by

$$\vartheta(X,A)=A \quad (=(A,\emptyset)),$$

$$\vartheta(f)=f|_A : A\to B \text{ for any } f:(X,A)\to(Y,B) \text{ in } \mathfrak{C}^{(2)}.$$

By a cohomology theory on $\mathfrak{C}$ we shall understand a sequence of contravariant h-functors

$$H^{\infty-n}:\mathfrak{C}^{(2)}\to Ab \qquad n=0,1,\ldots$$

together with a sequence of natural transformations

$$\delta^{\infty-n}: H^{\infty-n}\circ\vartheta\to H^{\infty-(n-1)}$$

satisfying the following conditions:

I. (Exactness). For the inclusions

$$i:A\to X, \quad j:X\to(X,A)$$

the cohomology sequence

$$\ldots\xrightarrow{\delta^{\infty-(n+1)}_{(X,A)}} H^{\infty-n}(X,A)\xrightarrow{j^*} H^{\infty-n}(X)\xrightarrow{i^*}$$
$$\xrightarrow{i^*} H^{\infty-n}(A)\xrightarrow{\delta^{\infty-n}_{(X,A)}} H^{\infty-(n-1)}(X,A)\to\ldots$$

is exact.

II. (Excision). Let $(X;A,B)$ be a triad in $\mathfrak{C}$ with $X=A\cup B$ and let $k:(A,A\cap B)\to(X,B)$ be the inclusion. Then, for all $n\geqslant 0$, the induced map

$$k^*: H^{\infty-n}(X,B)\to H^{\infty-n}(A,A\cap B)$$

is an isomorphism.

In this note, using the cohomology functors defined in our previous papers [2] and [3], we construct a cohomology theory on $\mathfrak{C}$, which satisfies the following condition:

III. (Dimension). Let $S^{\infty-n}$ be the unit sphere in a subspace $E^{\infty-n+1}$ of E of codimension $n-1$. Then

$$H^{\infty-k}(S^{\infty-n}) = \begin{cases} 0 & \text{for} \quad k \neq n, \\ G & \text{for} \quad k = n, \end{cases}$$

The constructed theory corresponds in finite dimensional case to the ordinary Čech cohomology for compact spaces.

2. Preliminaries. We recall first the definition and the properties of the cohomology functor $H^{\infty-n}: \mathfrak{C} \to Ab\,(n \geqslant 1)$. For details see [2] and [3].

Denote by $\mathcal{L} = \mathcal{L}(E) = \{L_\alpha, L_\beta, L_\gamma, \ldots\} = \{\alpha, \beta, \gamma, \ldots\}$ the directed set of all finite dimensional subspaces of E. Given $\alpha \in \mathcal{L}$, let $d(\alpha)$ be the dimension of L_α.

In the set of all linear isomorphisms from L_α to $R^{d(\alpha)}$ introduce an equivalence relation $\sim$ by the condition

$$l_1 \sim l_2 \Leftrightarrow l_1 l_2^{-1} \in GL_+(d(\alpha)).$$

Call an orientation of L_α an arbitrary choice of one of the corresponding two equivalence classes. Choosing for each $\alpha \in \mathcal{L}$ an orientation O_α in L_α we define an orientation $O = \{O_\alpha\}$ in E. Let us fix now an orientation $\{O_\alpha\}$ in E.

Given $\alpha < \beta$ with $d(\beta) = d(\alpha)+1$ the orientations of L_α and L_β determine the triad $(L_\beta, L_\beta^+, L_\beta^-)$ with $L_\alpha = L_\beta^+ \cap L_\beta^-$.

Putting for an object X in $\mathfrak{C}$

$$X_\beta = X \cap L_\beta, \quad X_\beta^+ = X \cap L_\beta^+, \quad X_\beta^- = X \cap L_\beta^-,$$

denote by $\mathcal{L}_X$ the cofinal subset of $\mathcal{L}$ consisting of elements $\alpha \in \mathcal{L}$ for which X_α is non empty.

Denote by $H^* = \{H^n, \delta^n\}$ the ordinary Čech cohomology theory over the fixed group of coefficients G.

Given $\alpha, \beta \in \mathcal{L}_X$ with $\alpha < \beta$ and $d(\beta) = d(\alpha)+1$ denote by

$$\Delta_{\alpha\beta}^{(n)}: H^{d(\alpha)-n}(X_\alpha) \to H^{d(\beta)-n}(X_\beta)$$

the Mayer–Vietoris homomorphism of the triad $(X_\beta, X_\beta^+, X_\beta^-)$. Given an arbitrary pair $\alpha, \beta \in \mathcal{L}_X$ with $\alpha < \beta$, take a chain $\alpha = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_{k+1} = \beta$ of elements of $\mathcal{L}_X$ satisfying $d(\alpha_{i+1}) = d(\alpha_i)+1$ $(i = 0, 1, \ldots, k)$ and define $\Delta_{\alpha\beta}^{(n)}$ to be the composition $\Delta_{\beta\alpha_k}^{(n)} \circ \ldots \circ \Delta_{\alpha_2\alpha_1}^{(n)} \circ \Delta_{\alpha_1\alpha}^{(n)}$.

We obtain the direct system of Abelian groups $\{H^{d(\alpha)-n}(X_\alpha); \Delta_{\alpha\beta}^{(n)}\}$ over $\mathcal{L}_X$ and define

$$H^{\infty-n}(X) = \varinjlim_{\alpha} \{H^{d(\alpha)-n}(X_\alpha); \Delta_{\alpha\beta}^{(n)}\}.$$

(2.1) (Alexander-Pontriagin Duality). *For every X in $\mathfrak{C}$ the group $H^{\infty-n}(X)$ is isomorphic to the $(n-1)$-th singular homology group $H_{n-1}(E-X, G)$ of the complement of X in E.*

Denote by $(\mathfrak{C}_0, \approx)$ the h-subcategory of $(\mathfrak{C}, \sim)$ whose objects are the same as in $\mathfrak{C}$, whose maps are finite dimensional fields and in which the relation $\approx$ means the finite dimensional homotopy. We recall that $f: X \to Y$ is in $\mathfrak{C}_0$ provided $F(x) = x - f(x)$ has values in L_α (for some α); in this case, for each $\beta > \alpha, f(X_\beta) \subset Y_\beta$ and therefore f determines the map $f_\beta: X_\beta \to Y_\beta$.

Given a finite dimensional field $f: X \to Y$, f induces a map $\{f_\alpha^*\}$ between the direct systems $\{H^{d(\alpha)-n}(Y_\alpha); \cdot\}$ and $\{H^{d(\alpha)-n}(X_\alpha); \cdot\}$ and therefore determines the induced map

$$f^* = \varinjlim_{\alpha} \{f_\alpha^*\}: H^{\infty-n}(Y) \to H^{\infty-n}(X).$$

We have the usual properties (i) $1_X^* = 1_{H^{\infty-n}(X)}$, (ii) $(gf)^* = f^* g^*$, (iii) $f \approx g$ in $\mathfrak{C}_0$ implies $f^* = g^*$.

Now we recall the definition of f^* for an arbitrary f in $\mathfrak{C}$.

Let Y be in $\mathfrak{C}$ and $\{Y_k\}$ be an approximative sequence for Y, i.e., a decreasing sequence of objects in $\mathfrak{C}$ such that $Y = \bigcap_{k=1} Y_k$. Consider the inclusions $j_k: Y \to Y_k$, $i_{kl}: Y_l \to Y_k$ ($k \leqslant l$) all being in $\mathfrak{C}_0$, the corresponding direct system of Abelian groups $\{H^{\infty-n}(Y_k); i_{kl}^*\}$ and the family $\{j_k^*\}$ of the induced maps $j_k^*: H^{\infty-n}(Y_k) \to H^{\infty-n}(Y)$.

(2.2) (Continuity of $H^{\infty-n}$). *The family* $\{j_k^*\}$ *is a direct family of homomorphisms and the map*

$$\varinjlim_{k} \{j_k^*\}: \varinjlim_{k} \{H^{\infty-n}(Y_k); i_{kl}^*\} \to H^{\infty-n}(Y)$$

is an isomorphism.

Now let $f: X \to Y$ be an arbitrary compact field. Take an approximating system $\{Y_k, f_k\}$ for f. This means that $\{f_k: X \to Y_k\}$ is a sequence of maps in $\mathfrak{C}_0$ such that (i) $\{Y_k\}$ is an approximating sequence for Y, (ii) $f_k \sim j_k f$ in $\mathfrak{C}$, (iii) $f_k \approx i_{kl} f_l$ in $\mathfrak{C}_0$.

It follows that the sequence $\{f_k^*: H^{\infty-n}(Y_k) \to H^{\infty-n}(X)\}$ of the induced maps is a direct family of homomorphisms and therefore

$$\varinjlim_{k} \{f_k^*\}: \varinjlim_{k} \{H^{\infty-n}(Y_k); i_{kl}^*\} \to H^{\infty-n}(X).$$

Taking into account (2.2), we define the induced map

$$H^{\infty-n}(f) = f^*: H^{\infty-n}(Y) \to H^{\infty-n}(X)$$

by the formula

$$f^* = \varinjlim_{k} \{f_k^*\} (\varinjlim_{k} \{j_k^*\})^{-1}.$$

The definition of f^* does not depend on the choice of $\{Y_k, f_k\}$.

3. The functors $\bar{H}^{\infty-n}: \bar{\mathfrak{C}} \to Ab$ and $\varrho: \mathfrak{C}^{(2)} \to \bar{\mathfrak{C}}$. In what follows we denote by $\bar{E}$ the direct product $E \oplus R$ of E and the real line R; we shall consider E as a one-codimensional subspace of $\bar{E}$. Further we let

$$\mathfrak{C} = \mathfrak{C}(E), \quad \mathfrak{C}_0 = \mathfrak{C}_0(E), \quad \bar{\mathfrak{C}} = \mathfrak{C}(\bar{E}) \quad \text{and} \quad \bar{\mathfrak{C}}_0 = \mathfrak{C}_0(\bar{E}).$$

Evidently, $\mathfrak{C}$ is a subcategory of $\bar{\mathfrak{C}}$; similarly $\mathfrak{C}_0 \subset \bar{\mathfrak{C}}_0$.

Let $x^+ = (0,1)$, $x^- = (0,-1)$.

By $\mathcal{L} = \mathcal{L}(E)$ and $\bar{\mathcal{L}} = \mathcal{L}(\bar{E})$ we denote the directed set of finite dimensional subspaces of E and $\bar{E}$, respectively. We let $\bar{\mathcal{L}}_0 = \{\alpha \in \bar{\mathcal{L}}, x^+ \in L_\alpha\}$; $\bar{\mathcal{L}}_0$ is a cofinal subset of $\bar{\mathcal{L}}$. For $\alpha \in \bar{\mathcal{L}}_0$ we let $\alpha' = L_\alpha \cap E$.

Let $\{O_\alpha\}$ be an orientation in $\bar{E}$. If $\alpha \in \bar{\mathcal{L}}_0$ then $d(\alpha') = d(\alpha) - 1$ and the orientations of L_α and $L_{\alpha'}$ determine the triad $(L_\alpha, L_\alpha^+, L_\alpha^-)$. We say that O_α and $O_{\alpha'}$ are *compatible* if $x^+ \in L_\alpha^+$.

Let us fix now an orientation $\{O_\alpha\}$ in $\bar{E}$ such that the orientations O_α and $O_{\alpha'}$ are compatible for every $\alpha \in \bar{\mathcal{L}}_0$.

For this particular orientation $\{O_\alpha\}$ in $\bar{E}$ and given $n \geqslant 1$ apply the construction of the preceding section to the space $\bar{E}$.

Let us denote by $\bar{H}^{\infty-n}: \bar{\mathfrak{C}} \to Ab$ the corresponding cohomology functor. The functors $H^{\infty-n}: \mathfrak{C}^{(2)} \to Ab$ of a cohomology theory on $\mathfrak{C}(E)$ will be defined later on with the aid of this functor.

We define now the cone functor C^+ from $\mathfrak{C}$ to $\bar{\mathfrak{C}}$. For $A \in \mathfrak{C}$ we let

$$C^+A = \{x \in \bar{E};\quad x = t \cdot a + (1-t) \cdot x^+;\quad a \in A,\quad 0 \leqslant t \leqslant 1\},$$

and for $f: A \to B$ in $\mathfrak{C}$ we define

$$f^+ = C^+ f: C^+A \to C^+B$$

by the formula

$$f^+\big(ta + (1-t)x^+\big) = t \cdot f(a) + (1-t) \cdot x^+.$$

Clearly, f^+ is in $\bar{\mathfrak{C}}$.

(3.1) *We have* (i) $(g \circ f)^+ = g^+ \circ f^+$, (ii) $f \sim g \Rightarrow f^+ \sim g^+$. *In other words, C^+ is an h-functor from $\mathfrak{C}$ to $\bar{\mathfrak{C}}$. Moreover, $C^+(\mathfrak{C}_0) \subset \bar{\mathfrak{C}}_0$.*

Define a functor $\varrho: \mathfrak{C}^{(2)} \to \bar{\mathfrak{C}}$ as follows:
for $(X, A) \in \mathfrak{C}^{(2)}$ we put

$$\varrho(X, A) = \begin{cases} X \cup CA & \text{if} \quad A \neq \theta, \\ X, & \text{if} \quad A = \theta, \end{cases}$$

and for a compact field $f: (X, A) \to (Y, B)$ define

$$\varrho(f) = \tilde{f}: X \cup CA \to Y \cup CB$$

by the formula

$$\tilde{f}(x) = \begin{cases} f^+(x) & \text{for all} \quad x \in CA, \\ f(x) & \text{for all} \quad x \in X. \end{cases}$$

(3.2) *We have* (i) $\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}$, (ii) $f \sim g$ *implies* $\tilde{f} \sim \tilde{g}$. *In other words, ϱ is an h-functor from $\mathfrak{C}^{(2)}$ to $\bar{\mathfrak{C}}$. Moreover, $\varrho(\mathfrak{C}_0^{(0)}) \subset \bar{\mathfrak{C}}_0$.*

4. Some lemmas. In this section, we shall prove some lemmas which will be used in defining the coboundary transformation $\delta^{\infty-n}$.

For an object X in $\mathfrak{C}$ denote by $\overline{\mathfrak{L}}_X^+$ the cofinal subset of $\overline{\mathfrak{L}}_X$ defined by $\overline{\mathfrak{L}}_X^+ = \overline{\mathfrak{L}}_X \cap \overline{\mathfrak{L}}_0$. If $\alpha \in \overline{\mathfrak{L}}_X^+$ then $X_{\alpha'} = X_\alpha$ and thus $H^{d(\alpha)-n}(X_\alpha) = H^{d(\alpha')-n+1}(X_{\alpha'})$.

Let $\alpha, \beta \in \overline{\mathfrak{L}}_X^+$ with $\beta > \alpha$ and $d(\beta) = d(\alpha)+1$. Since $O_\alpha, O_{\alpha'}$ and O_β, $O_{\beta'}$ are compatible we have

$$\Delta_{\alpha\beta}^{(n)} = -\Delta_{\alpha'\beta'}^{(n-1)} : H^{d(\alpha)-n}(X_\alpha) \to H^{d(\beta)-n}(X_\beta).$$

From this we obtain

(4.1) *Let $\alpha, \beta \in \overline{\mathfrak{L}}_X^+$ with $\beta > \alpha$. Then*

$$\Delta_{\alpha\beta}^{(n)} = (-1)^{d(\beta)-d(\alpha)} \Delta_{\alpha'\beta'}^{(n-1)} : H^{d(\alpha)-n}(X_\alpha) \to H^{d(\beta)-n}(X_\beta).$$

Let $(X, A) \in \mathfrak{C}^{(2)}$ and let $\alpha \in \overline{\mathfrak{L}}_A^+$. Denote by

$$\tilde{\delta}_\alpha^{(n)} : H^{d(\alpha)-n-1}(A_\alpha) \to H^{d(\alpha)-n}(X_\alpha \cup CA_\alpha)$$

the Mayer–Vietoris homomorphism of the triad $(X_\alpha \cup CA_\alpha, CA_\alpha, X_\alpha)$.

Let us put

$$\delta_\alpha^{(n)} = (-1)^{d(\alpha)} \tilde{\delta}_\alpha^{(n)}.$$

Since $X \subset E$, $(X \cup CA)_\alpha = X_\alpha \cup CA_\alpha$. Thus if $\beta > \alpha$ then

$$\Delta_{\alpha\beta}^{(n)} : H^{d(\alpha)-n}(X_\alpha \cup CA_\alpha) \to H^{d(\beta)-n}(X_\beta \cup CA_\beta).$$

(4.2) *Let $(X, A) = \mathfrak{C}^{(2)}$. Let $\alpha, \beta \in \overline{\mathfrak{L}}_A^+$, $\beta > \alpha$. Then the following diagram commutes*

$$\begin{array}{ccc} H^{d(\alpha)-n-1}(A_\alpha) & \xrightarrow{\delta_\alpha^{(n)}} & H^{d(\alpha)-n}(X_\alpha \cup CA_\alpha) \\ \downarrow \Delta_{\alpha\beta}^{(n+1)} & & \downarrow \Delta_{\alpha\beta}^{(n)} \\ H^{d(\beta)-n-1}(A_\beta) & \xrightarrow{\delta_\beta^{(n)}} & H^{d(\beta)-n}(X_\beta \cup CA_\beta). \end{array}$$

Proof. Define $\eta : E \to R$ by $\eta(x) = \inf\{\|y-x\|; y \in A\}$; clearly, η is continuous. Define $\varkappa : X \to E$ by $\varkappa(x) = x + \eta(x) \cdot x^-$ and let $\tilde{X} = \varkappa(X)$. Clearly, $A \subset \tilde{X}$ and $(\tilde{X} \cup CA)_\alpha = \tilde{X}_\alpha \cup CA_\alpha$. Since $A \subset E$, $A_{\alpha'} = A_\alpha$ and thus

$$H^{d(\alpha)-1-n}(A_\alpha) = H^{d(\alpha')-n}(A_{\alpha'}).$$

Therefore $\Delta_{\alpha'\beta'}^{(n)} : H^{d(\alpha)-1-n}(A_\alpha) \to H^{d(\beta)-1-n}(A_\beta)$. Consider the following commutative diagram:

$$\begin{array}{ccc} H^{d(\alpha)-1-n}(A_\alpha) & \xrightarrow{\Delta_{\alpha'\alpha}^{(n)}} & H^{d(\alpha)-n}(\tilde{X}_\alpha \cup CA_\alpha) \\ \downarrow \Delta_{\alpha'\beta'}^{(n)} & & \downarrow \Delta_{\alpha\beta}^{(n)} \\ H^{d(\beta)-1-n}(A_\beta) & \xrightarrow{\Delta_{\beta'\beta}^{(n)}} & H^{d(\beta)-n}(\tilde{X}_\beta \cup CA_\beta). \end{array}$$

Since O_α, $O_{\alpha'}$ and O_β, $O_{\beta'}$ are compatible, the horizontal homomorphisms of the above diagram are the Mayer–Vietoris homomorphisms of the triads $(\tilde{X}_\alpha \cup \cup CA_\alpha, CA_\alpha, \tilde{X}_\alpha)$ and $(\tilde{X}_\beta \cup CA_\beta, CA_\beta, \tilde{X}_\beta)$, respectively. Consider the map $\tilde{\varkappa}: X \cup CA \to \tilde{X} \cup CA$. For $\alpha \in \overline{\mathfrak{L}}_A^+$

$$\tilde{\varkappa}_\alpha : (X_\alpha \cup CA_\alpha, CA_\alpha, X_\alpha) \to (\tilde{X}_\alpha \cup CA_\alpha, CA_\alpha, \tilde{X}_\alpha)$$

is a homeomorphism of the triads. Applying $(\tilde{\varkappa}_\alpha^{-1})^*$ and $(\tilde{\varkappa}_\beta^{-1})^*$ to the last diagram we obtain a commutative diagram

$$\begin{array}{ccc} H^{d(\alpha)-1-n}(A_\alpha) & \xrightarrow{\tilde{\delta}_\alpha^{(n)}} & H^{d(\alpha)-n}(X_\alpha \cup CA_\alpha) \\ \downarrow \Delta_{\alpha'\beta'}^{(n)} & & \downarrow \Delta_{\alpha\beta}^{(n)} \\ H^{d(\beta)-1-n}(A_\beta) & \xrightarrow{\tilde{\delta}_\beta^{(n)}} & H^{d(\beta)-n}(X_\beta \cup CA_\beta). \end{array}$$

Now our assertion follows easily from (4.1) and the definition of $\delta_\alpha^{(n)}$.

The following two lemmas are easy consequences of the definition of $\delta_\alpha^{(n)}$.

(4.3) *Let* $(X, A) \in \mathfrak{C}^{(2)}$ *and let* $i: A \to X$, $k: X \to X \cup CA$ *denote the inclusion maps. Then, for every* $\alpha \in \overline{\mathfrak{L}}_A^+$ *the following sequence is exact*:

$$\ldots \xrightarrow{k_\alpha^*} H^{d(\alpha)-n}(X_\alpha) \xrightarrow{i_\alpha^*} H^{d(\alpha)-n}(A_\alpha) \xrightarrow{\delta_\alpha^{(n-1)}} H^{d(\alpha)-n+1}(X_\alpha \cup CA_\alpha) \xrightarrow{k_\alpha^*} \ldots.$$

(4.4) *Let* $f: (X, A) \to (Y, B)$ *be in* $\mathfrak{C}_0^{(2)}$, *and* $\alpha, \beta \in \overline{\mathfrak{L}}_A^+$ *with* $\beta > \alpha$. *Then the following diagram commutes*:

$$\begin{array}{ccc} H^{d(\alpha)-n-1}(A_\alpha) & \xrightarrow{\delta_\alpha^{(n)}} & H^{d(\alpha)-n}(X_\alpha \cup CA_\alpha) \\ \uparrow (f_{|A})_\alpha^* & & \uparrow (\tilde{f}_\alpha)^* \\ H^{d(\alpha)-n-1}(B_\alpha) & \xrightarrow{\delta_\alpha^{(n)}} & H^{d(\alpha)-n}(Y_\alpha \cup CB_\alpha). \end{array}$$

5. The definition of $H^{\infty-n}$ and the coboundary transformation $\delta^{\infty-n}$. Let $n \geq 0$. We define the relative cohomology functor $H^{\infty-n}$ from the category $\mathfrak{C}^{(2)}$ of pairs in $\mathfrak{C}$ to the category of Abelian groups by putting

$$H^{\infty-n} = \bar{H}^{\infty-(n+1)} \circ \varrho.$$

Thus, for (X, A) in $\mathfrak{C}^{(2)}$

$$H^{\infty-n}(X, A) = \begin{cases} \bar{H}^{\infty-(n+1)}(X \cup CA) & \text{for } A \neq \theta, \\ \bar{H}^{\infty-(n+1)}(X) & \text{for } A = \theta \end{cases}$$

and for $f: (X, A) \to (Y, B)$ in $\mathfrak{C}^{(2)}$

$$H^{\infty-n}(f) = (\tilde{f})^*.$$

It follows from the definition that $H^{\infty-n}$ is a contravariant h-functor.

Let $(X, A) \in \mathfrak{C}^{(2)}$. From (4.2), we conclude that the family $\{\delta_\alpha^{(n)}\}$ is the direct family of homomorphisms.

Therefore, taking into account the definition of $H^{\infty-n}$ and putting

$$\delta^{\infty-n}_{(X,A)} = \varinjlim_{a} \{\delta^{(n)}_a\}$$

we obtain the coboundary homomorphism

$$\delta^{\infty-n}_{(X,A)} : H^{\infty-n-1}(A) \to H^{\infty-n}(X, A).$$

(5.1) *The family $\delta^{\infty-n} = \{\delta^{\infty-n}_{(X,A)}\}$ indexed by the pairs $(X, A) \in \mathfrak{C}^{(2)}$ is a natural transformation from $H^{\infty-n} \circ \vartheta$ to $H^{\infty-n+1}$.*

Proof. We have to prove that for a map $f: (X, A) \to (Y, B)$ in $\mathfrak{C}^{(2)}$ the following diagram is commutative:

$$\begin{array}{ccc} H^{\infty-n-1}(A) & \xrightarrow{\delta^{\infty-n}_{(X,A)}} & H^{\infty-n}(X \cup CA) \\ \uparrow (f_A)^* & & \uparrow (\tilde{f})^* \\ H^{\infty-n-1}(B) & \xrightarrow{\delta^{\infty-n}_{(Y,B)}} & H^{\infty-n}(Y \cup CB). \end{array}$$

Assume first that f is in $\mathfrak{C}^{(2)}_0$. In this special case our assertion follows from the commutativity of the diagram in proposition (4.4).

Consider now the general case and take an approximating system $\{f^{(k)}: (X, A) \to (Y_k, B_k)\}$ for f. The definition and the proof of the existence of such a system is similar to that in the absolute case. It follows from (3.2) that the sequence

$$\{\tilde{f}^{(c)}: X \cup kA \to Y_k \cup CB_k\}$$

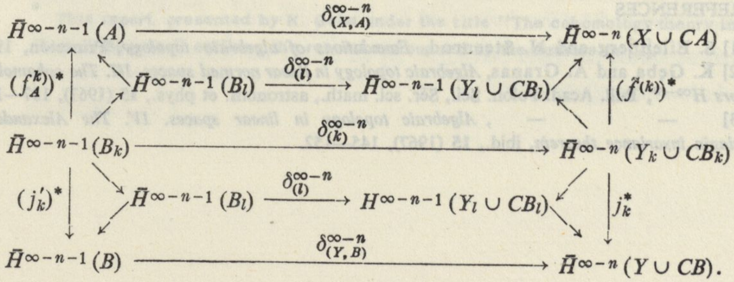
forms an approximating system for $\tilde{f}$.

Consider the inclusions

$$j_k: Y \cup CB \to Y_k \cup CB_k, \quad j'_k: B \to B_k$$

$$i_{kl}: Y_l \cup CB_l \to Y_k \cup CB_k, \quad i'_{kl}: B_l \to B_k \quad (k \leqslant l).$$

By the special case of our assertion the following diagram commutes for each pair $k \leqslant l$:



Applying the direct limit functor to the corresponding commutative diagram in the category of direct systems of Abelian groups we obtain the following commutative diagram:

$$\begin{array}{ccc}
\bar{H}^{\infty-n-1}(A) & \xrightarrow{\delta^{\infty-n}_{(X,A)}} & \bar{H}^{\infty-n}(X\cup CA) \\
\uparrow \varinjlim_k \{(f^k_A)^*\} & & \uparrow \varinjlim_k \{(\bar{f}^{(k)})^*\} \\
\varinjlim_k \{\bar{H}^{\infty-n-1}(B_k);\ \cdot\} & \xrightarrow{\varinjlim_k \{\delta^{\infty-n}_{(k)}\}} & \varinjlim_k \{\bar{H}^{\infty-n}(Y_k\cup CB_k);\ \cdot\} \\
\downarrow \varinjlim_k \{(j'_k)^*\} & & \downarrow \varinjlim_k \{j^*_k\} \\
\bar{H}^{\infty-n-1}(B) & \xrightarrow{\delta^{\infty-n}_{(Y,B)}} & \bar{H}^{\infty-n}(Y\cup CB)\,.
\end{array}$$

By (2.2) the homomorphisms $\varinjlim_k \{(j'_k)^*\}$ and $\varinjlim_k \{j^*_k\}$ are invertible. This, in view of the definition of the induced map, implies our assertion and thus the proof is completed.

We state now the main result of the paper

THEOREM. *The sequence of functors* $\{H^{\infty-n}\}$ *together with the sequence of coboundary transformations* $\{\delta^{\infty-n}\}$ *satisfies the conditions* I, II *and* III.

Proof. (Exactness). This follows from the exactness of the sequence in proposition (4.3) and the definition of $H^{\infty-n}$.

(Excision). Let $(X; A, B)$ be a triad in $\mathfrak{C}$ with $X = A\cup B$. If $k:(A, A\cap B)\to$ $\to (X, B)$ is the inclusion then so is the map $\tilde{k}: A\cup C(A\cap B)\to X\cup CB$. Since $(\tilde{k}_a)^*$ in an isomorphism for each a and $(\tilde{k})^* = \varinjlim_a \{(\tilde{k}_a)^*\}$ it follows that $H^{\infty-n}(k)$ is an isomorphism.

(Dimension). Since $S^{\infty-n}$ is the unit sphere in a subspace of $\bar{E}$ of codimension n, this follows from the Alexander—Pontriagin duality (2.1) in the space $\bar{E}$ and the definition of $H^{\infty-n}$.

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ON THE COHOMOLOGY THEORY IN LINEAR NORMED SPACES *

KAZIMIERZ GEBA AND ANDRZEJ GRANAS

Let E be a linear normed space and $(\mathfrak{L}^{\sim}, \sim)$ be the h-category whose objects are closed bounded subsets of E, whose morphisms are compact vector fields and in which the relation $\sim$ means the compact homotopy of fields. This category, introduced in 1933 by Leray and Schauder [4] in their study of topological degree plays a basic role in topology of an infinite dimensional normed space E.

The main theorem which we intend to present here is the following: for each $n \geq 1$ and a coefficient group G, there exists an h-functor $H^{\infty - n}$ from the category $\mathfrak{L}^{\sim}$ to the category *Ab* of abelian groups such that the group $H^{\infty - n}(X; G)$ is isomorphic to the (n − 1)-th singular homology $H_{n-1}(E - X; G)$ of $E - X$.

As an immediate consequence we obtain the Alexander-Pontriagin Invariance Theorem in E: if the objects X and Y are equivalent or homotopically equivalent in $\mathfrak{L}^{\sim}$, then for each $n \geq 0$ the singular homology groups $H_n(E - X; G)$ and $H_n(E - Y; G)$ are isomorphic. A special case of this theorem (when $n = 0$ and $G = Z$) was proved for a complete E by J. Leray in [4].

We also note that the Leray-Schauder theory of topological degree follows readily from the special case of the main theorem.

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* This report, presented by K. Geba under the title "The cohomology theory in Banach Spaces," outlines the results announced by the authors in [2].

§1. PRELIMINARIES

The directed set $\mathfrak{L}$. **Let E be an arbitrary but fixed infinite dimensional normed space. We shall denote by** $\mathfrak{L} = \{L_\alpha, L_\beta, L_\gamma, \ldots\}$ **the directed set of all finite dimensional subspaces of E with the natural order relation $\leq$ defined by the condition**

$$L_\alpha \leq L_\beta \iff L_\alpha \subset L_\beta .$$

For notational convenience, we establish 1-1 correspondence $\alpha \iff L_\alpha$ between the symbols $\alpha, \beta, \gamma, \ldots$ and $L_\alpha, L_\beta, L_\gamma, \ldots$ and in the formulas to occur we replace frequently one kind of symbols by another. Given $X \subset E$, we denote by $\mathfrak{L}_X$ the cofinal subset of $\mathfrak{L}$, consisting of those α for which $X_\alpha = X \cap L_\alpha$ is non-empty.

Compact fields. **A mapping $F: X \to E$ from a metric space X is *compact* provided the closure of F(X) is compact. A compact mapping F: $X \to E$ is an *α-mapping* (resp. *finite dimensional mapping*) provided $F(X) \subset L_\alpha$ (resp. $F(X) \subset L_\beta$ for some β).**

The following Schauder Approximation Theorem is of fundamental importance: every compact mapping can be uniformly approximated by finite dimensional mappings.

Given $X, Y \subset E$ and a mapping $f: X \to Y$, denote by the same but capital letter the mapping $F: X \to E$ defined by $F(x) = x - f(x)$ for $x \in X$. Then $f: X \to Y$ is said to be a *compact field* (resp. *α-field*) provided the corresponding $F: X \to E$ is a compact mapping (resp. α-mapping).

The subsets of E and the compact fields form a category, denoted by $\mathcal{C}$ and called the *Leray-Schauder category*. The subsets of E and the α-fields form the subcategory $\mathcal{C}_\alpha \subset \mathcal{C}$. The union of all categories $\mathcal{C}_\alpha$ is the subcategory $\mathcal{C}_0$ of $\mathcal{C}$. The morphisms of the category $\mathcal{C}_0$ will be called the *finite dimensional fields*.

Let $\mathfrak{L}^\sim$ (resp. $\mathfrak{L}^\sim_\alpha$ and $\mathfrak{L}^\sim_0$) be the full subcategory of $\mathcal{C}$ (resp. $\mathcal{C}_\alpha$ and $\mathcal{C}_0$) generated by the closed and bounded subsets of E. The category $\mathfrak{L}^\sim$ will be called the *main category*. By an object we shall understand in what follows an object of $\mathfrak{L}^\sim$.

Compact homotopies. Given $X, Y \subset E$ and a homotopy $h_t: X \to Y$, $(0 \leq t \leq 1)$, denote by the capital H the mapping from $X \times I$ to E defined by $H(x, t) = x - h_t(x)$ for $(x, t) \in X \times I$. Then, $h_t: X \to Y$ is called a *compact homotopy* (resp. α-homotopy) provided $H: X \times I \to E$ is a compact mapping (resp. an α-mapping).

Two compact fields (resp. α-fields) $f_0, f_1: X \to Y$ are called *homotopic*, $f_0 \sim f_1$, (resp. *α-homotopic*, $f_0 \overset{\sim}{\alpha} f_1$) provided there is a compact homotopy (resp. α-homotopy) $h_t: X \to Y$ such that $h_0 = f_0$, $h_1 = f_1$. Each of the above relations is clearly an equivalence relation. If the fields (resp. α-fields) $f_1, f_2: X \to Y$ and $g_1, g_2: Y \to Z$ are homotopic (resp. α-homotopic), then so are their compositions $g_1 \circ f_1, g_2 \circ f_2: X \to Z$.

It follows that the relations $\sim$ and $\overset{\sim}{\alpha}$ convert the Leray-Schauder category $\mathcal{C}$ and the category $\mathcal{C}_\alpha$ into the h-categories $(\mathcal{C}, \sim)$ and $(\mathcal{C}_\alpha, \overset{\sim}{\alpha})$ respectively. Similarly, we define the relation $\tilde{0}$ of finite dimensional homotopy and convert the category $\mathcal{C}_0$ into the h-category $(\mathcal{C}_0, \tilde{0})$.

It is evident that the main category $\mathcal{L}^{\sim}$ admits a structure of an h-category $(\mathcal{L}^{\sim}, \sim)$ with the relation of homotopy induced by that in $\mathcal{C}$. Similarly, the relation $\tilde{0}$ in $\mathcal{C}_0$ converts $\mathcal{L}_0^{\sim}$ into an h-category $(\mathcal{L}_0^{\sim}, \tilde{0})$.

An orientation in E. Denote by R^∞ the normed space consisting of all sequences $x = (x_1, x_2, \dots)$ of real numbers such that $x_i = 0$ for all but a finite set of i, with the norm $\|x\| = \sqrt{\Sigma\, x_i^2}$ and put:

$$R^k = \{x \in R^\infty;\ x_i = 0 \text{ for } i \geq k+1\},$$

$$R_+^k = \{x \in R^k;\ x_k \geq 0\},$$

$$R_-^k = \{x \in R^k;\ x_k \leq 0\}.$$

Let α be an element of the directed set $\mathcal{L}$ and let $d(\alpha)$ be the dimension of α. Call two linear isomorphisms $l_1, l_2: L_\alpha \to R^{d(\alpha)}$ *equivalent*, $l_1 \sim l_2$, provided $l_1 l_2^{-1} \in GL_+(d(\alpha))$, i.e., the determinant of the corresponding matrix is positive. With respect to the relation $\sim$, the set of all linear isomorphisms from L_α to $R^{d(\alpha)}$ decomposes into exactly two equivalent classes. An arbitrary choice of one of these classes will be called an orientation of L_α.

Let us choose now an orientation $\mathcal{O}_\alpha$ of L_α for each α and call the family $\mathcal{O} = \{\mathcal{O}_\alpha\}$ an orientation in E. Given $\alpha < \beta$ with $d(\beta) - d(\alpha) = 1$ and $l_\alpha \in \mathcal{O}_\alpha$, there exists $l_\beta \in \mathcal{O}_\beta$ such that $l_\beta(x) = l_\alpha(x)$ for all $x \in L_\alpha$. We let

$$L_\beta^+ = l_\beta^{-1}(R_+^{d(\beta)}) \quad \text{and} \quad L_\beta^- = l_\beta^{-1}(R_-^{d(\beta)}) .$$

Clearly, the definition of L_β^+ and L_β^- depends only on the orientation of L_α and L_β.

Given an object X we let

$$X_\beta^+ = X \cap L_\beta^+ \quad \text{and} \quad X_\beta^- = X \cap L_\beta^- .$$

If $f\colon X \to Y$ is an α-field, then clearly $f(X_\alpha) \subset Y_\alpha$ and hence f determines the map $f_\alpha\colon X_\alpha \to Y_\alpha$ given by $f_\alpha(x) = f(x)$ for $x \in X_\alpha$. We note that if α, β are two elements of $\mathfrak{L}_X$ with $\alpha < \beta$, $d(\beta) - d(\alpha) = 1$, and $f\colon X \to Y$ is an α-field, then $f_\beta\colon X_\beta \to Y_\beta$ maps the triad $(X_\beta, X_\beta^+, X_\beta^-)$ into the triad $(Y_\beta, Y_\beta^+, Y_\beta^-)$.

§2. THE COHOMOLOGY FUNCTOR $\bar{H}^{\infty - n}\colon \mathfrak{L}_0^- \to Ab$.

Let G be an arbitrary but fixed Abelian group and n a positive integer. By H^* (resp. H_*) we shall denote the Cech cohomology (resp. the singular homology) with coefficients in G. By H^0 and H_0 we shall denote the reduced zero-dimensional groups.

A Lemma on the Alexander Pontriagin duality. For a compact subset $X \subset R^k$ we shall denote by D_k the Alexander-Pontriagin isomorphism

$$D_k\colon H_{n-1}(R^k - X) \to H^{k-n}(X)$$

determined by the standard orientation of R^k (see [5], p. 296). Let $X_0 = X \cap R^{k-1}$, $i\colon R^{k-1} - X_0 \to R^k - X$ be the inclusion and denote by Δ^* the Mayer-Vietoris homomorphism of the triad $(X, X \cap R_+^k, X \cap R_-^k)$. It follows from the definition of D_k that the following diagram is sign-commutative:

$$\begin{array}{ccc} H^{k-1-n}(X_0) & \xrightarrow{\Delta^*} & H^{k-n}(X) \\ \Big\downarrow D_{k-1} & & \Big\downarrow D_k \\ H_{n-1}(R^{k-1}-X_0) & \xrightarrow{i*} & H'_{n-1}(R^k - X) \end{array}$$

Thus, $D_k \circ \Delta = a_{k,n} i* \circ D_{k-1}$, where $a_{k,n} = \pm 1$. Let $b_{k,n}$, $k \geq n$, be defined by induction on k:

$$b_{n,n} = 1, \quad b_{k,n} = a_{k,n} b_{k-1,n}$$

Define the isomorphism

$$\mathfrak{D}_k\colon H^{k-n}(X) \to H_{n-1}(R^k - X)$$

by $\mathfrak{D}_k = b_{k,n} D_k'$.

We have the following:

Lemma A. *Let* X *be a compact subset of* R^{k+1}, $X_0 = X \cap R^k$ *and let* Δ^* *be the Mayer-Vietoris homomorphism of the triad* $(X, X \cap R^{k+1}_+, X \cap R^{k+1}_-)$. *Then the following diagram commutes*

$$\begin{array}{ccc} H^{k-n}(X_0) & \xrightarrow{\Delta^*} & H^{k-n+1} \\ \Big\downarrow \mathfrak{D}_k & & \Big\downarrow \mathfrak{D}_{k+1} \\ H_{n-1}(R^k - X_0) & \xrightarrow{i*} & H_{n-1}(R^{k+1} - X) \end{array}$$

(i: $R^k - X_0 \to R^{k+1} - X$ *denotes the inclusion*).

Let n be a positive integer, X be an object and $U = E - X$. Let $\mathcal{O} = \{\mathcal{O}_\alpha\}$ be an orientation in E and $l_\alpha \in \mathcal{O}_\alpha$.

For each $\alpha \in \mathfrak{L}_X$ with $d(\alpha) > n$ define

$$\mathfrak{D}_\alpha\colon H^{d(\alpha)-n}(X_\alpha) \to H_{n-1}(U_\alpha)$$

to be the Alexander-Pontriagin isomorphism "transferred" by l_α^{-1} from $R^{d(\alpha)}$ to L_α.

LEMMA A′. *Let* X *be an object,* $U = E - X$ *and* $\alpha, \beta \in \mathfrak{L}_X$ *with* $d(\beta) - d(\alpha) = 1$. *Let* $i_{\alpha\beta}\colon U_\alpha \to U_\beta$ *be the inclusion and*

$$\Delta_{\alpha\beta}\colon H^{d(\alpha)-n}(X_\alpha) \to H^{d(\beta)-n}(X_\beta)$$

be the Mayer-Vietoris homomorphism of the triad (X, X^+, X^-). *Then the following diagram commutes*

$$\begin{array}{ccc} H^{d(\alpha)-n}(X_\alpha) & \xrightarrow{\Delta_{\alpha\beta}} & H^{d(\beta)-n}(X_\beta) \\ \downarrow{\scriptstyle \mathfrak{D}_\alpha} & & \downarrow{\scriptstyle \mathfrak{D}_\beta} \\ H_{n-1}(U_\alpha) & \xrightarrow{(i_{\alpha\beta})_*} & H_{n-1}(U_\beta) \end{array}$$

This follows from the definition of $\Delta_{\alpha\beta}$ and Lemma A.

The group $H^{\infty-n}(X)$. Let $\alpha, \beta \in \mathfrak{L}_X$, with $\alpha < \beta$, and let $\alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_k = \beta$ be a chain of elements of $\mathfrak{L}_X$ such that $d(\alpha_{i+1}) - d(\alpha_i) = 1$ $(i = 0, 1, \ldots, k-1)$. Define the homomorphism

$$\Delta_{\alpha\beta}\colon H^{d(\alpha)-n}(X_\alpha) \to H^{d(\beta)-n}(X_\beta)$$

to be the composition of the homomorphisms $\Delta_{\alpha\alpha_1}, \Delta_{\alpha_1\alpha_2}, \ldots, \Delta_{\alpha_{k-1}\beta}$. It follows from Lemma A′ that the definition of $\Delta_{\alpha\beta}$ does not depend on the choice of the chain $\alpha_1, \ldots, \alpha_{k-1}$ joining α and β.

For an object X consider the groups $H^{d(\alpha)-n}(X_\alpha)$ together with the homomorphisms $\Delta_{\alpha\beta}$ given for $\alpha \le \beta$. The family $\{H^{d(\alpha)-n}(X_\alpha), \Delta_{\alpha\beta}\}$ indexed by $\alpha \in \mathfrak{L}_X$ is a direct system of Abelian groups called the $(\infty-n)$-th *cohomology system of* X corresponding to the orientation $\mathcal{O}$ in E.

We define the Abelian group

$$H^{\infty-n}(X) = \varinjlim_{\alpha} \{H^{d(\alpha)-n}(X_\alpha), \Delta_{\alpha\beta}\}$$

to be the direct limit of the $(\infty-n)$-th cohomology system of X.

We note, taking into account Lemma A′, that if $\{H^{d(\alpha)-n}(X_\alpha), \Delta_{\alpha\beta}\}$, $\{H^{d(\alpha)-n}(X_\alpha), \bar{\Delta}_{\alpha\beta}\}$ are two $(\infty-n)$-th cohomology systems of X corresponding to different orientations $\mathcal{O} = \{\mathcal{O}_\alpha\}$ and $\bar{\mathcal{O}} = \{\bar{\mathcal{O}}_\alpha\}$ in E, then the above systems are isomorphic. Consequently, the group $H^{\infty-n}(X)$ does not depend on the orientation $\mathcal{O}$ in E.

f^* *for finite dimensional* f. Let $X, Y \in \mathfrak{L}_0^{\sim}$, $a_0 \in \mathfrak{L}_X$ and $f: X \to Y$ be an a_0-field. Then for each $a, \beta \in \mathfrak{L}_X$ with $a_0 \leq a < \beta$ the following diagram commutes:

$$\begin{array}{ccc} H^{d(a)-n}(X_a) & \xleftarrow{f_a^*} & H^{d(a)-n}(Y_a) \\ \downarrow \Delta_{a\beta} & & \downarrow \Delta_{a\beta} \\ H^{d(\beta)-n}(X_\beta) & \xleftarrow{f_\beta^*} & H^{d(\beta)-n}(Y_\beta) \end{array}$$

Consequently, f induces a map $\{f_a^*\}$ from the $(\infty-n)$-th cohomology system of Y into that of X, and therefore determines a map

$$f^* = \operatorname{Lim}_{\vec{a}} \{f_a^*\}$$

from $H^{\infty-n}(Y)$ to $H^{\infty-n}(X)$.

Let us denote now by $\overline{H}^{\infty-n}$ the functor which assigns to an object $X \in \mathfrak{L}_0^{\sim}$ the $(\infty-n)$-th cohomology group $H^{\infty-n}(X)$ and to a finite-dimensional field $f: X \to Y$ the induced homomorphism $f^*: H^{\infty-n}(Y) \to H^{\infty-n}(X)$.

From the properties of the Mayer-Vietoris homomorphism we deduce the following:

THEOREM 1. *The functor* $\overline{H}^{\infty-n}$ *is a contravariant* h-*functor from the category* $\mathfrak{L}_0^{\sim}$ *to the category of Abelian groups Ab.*

The continuity of the functor $\overline{H}^{\infty-n}$. In order to extend the functor $\overline{H}^{\infty-n}: \mathfrak{L}_0^{\sim} \to Ab$ over $\mathfrak{L}^{\sim}$, we shall use the property which is analogous to the property of continuity of the Cech cohomology.

We shall make use of the following algebraic lemma:

LEMMA B. *Let* $\mathfrak{N} = \{k, l, m, \ldots\}$ *and* $\mathfrak{L} = \{a, \beta, \gamma, \ldots\}$ *be two directed sets and* $\{H_a^k, \cdot\}$ *be a double direct system of abelian groups indexed by* $\mathfrak{N} \times \mathfrak{L}$. *Then we have a natural isomorphism between the limit groups*

$$\operatorname*{Lim}_{\vec{a}} \operatorname*{Lim}_{\vec{k}} \{H^k_\alpha, \cdot\} \approx \operatorname*{Lim}_{\vec{k}} \operatorname*{Lim}_{\vec{a}} \{H^k_\alpha, \cdot\}.$$

Let X be an object. For a natural number k let

$$X^{(k)} = \{x \in E;\ \rho(x, X) \leq \frac{1}{k}\}.$$

We shall say that a sequence of objects $\{X_k\}$ is an *approximating sequence* or an *(a)-sequence* for X provided

(i) $X_k \supset X_{k+1}$ for each $k = 1, 2, \ldots,$

(ii) $X = \bigcap_{k=1} X_k$.

We note that

(a) if $\{X_k\}$ is an (a)-sequence for X, then the enlarged sequence $\{\tilde{X}_k\}$, where $\tilde{X}_k = (X_k)^k = \{x \in E;\ \rho(x, X_k) \leq 1/k\}$ is also an (a)-sequence for X.

(b) if $\{X_k\}$ and $\{Y_k\}$ are two (a)-sequences for X and Y, respectively, then $\{X_k \cup Y_k\}$ is an (a)-sequence for $X \cup Y$.

(c) if $\{X_k\}$ is an (a)-sequence for X and $f\colon X_1 \to E$ is a compact field, then $\{f(X_k)\}$ is an (a)-sequence for $f(X)$.

Let Y be an object, $\alpha \in \mathfrak{L}_Y$ and let $\{Y_k\}$ be an (a)-sequence for Y. We let $Y^k_\alpha = Y_k \cap L_\alpha$. Consider the following inclusions, all of them being finite-dimensional fields:

$$i_{kl}\colon Y_l \to Y_k,\ \ i^\alpha_{kl}\ \ Y^l_\alpha \to Y^k_\alpha, \qquad (k \leq l);$$

$$j_k\colon Y \to Y_k,\ \ j_{k\alpha}\colon Y \to Y^k,$$

and the direct system of Abelian groups $\{H^{\infty-n}(Y_k), i^*_{kl}\}$ over $\mathfrak{N}$. We have the commutativity relations $j^*_k = j^*_l \circ i^*_{kl}$ for $k \leq l$ and hence

$$\{j^*_k\}\colon \{H^{\infty-n}(Y_k),\ i^*_{kl}\} \to H^{\infty-n}(Y)$$

is a direct family of homomorphisms.

THEOREM 2. *The map*

$$\varinjlim_k \{j_k^*\}: \varinjlim_k \{H^{\infty-n}(Y_k), i_{kl}^*\} \to H^{\infty-n}(Y)$$

is an isomorphism.

Proof: Consider over $\mathfrak{N} \times \mathfrak{L}$ the following double direct systems of Abelian groups $\kappa = \{H^{d(\alpha)-n}(Y_\alpha^k), \Delta_{\alpha\beta}^{kl}\}$, and $\bar{\kappa} = \{H^{d(\alpha)-n}(Y_\alpha), \Delta_{\alpha\beta}\}$, where $\Delta_{\alpha\beta}^{kl} = (i_{kl}^*) \circ \Delta_{\alpha\beta}$, for $\alpha \leq \beta$, $k \leq l$. Clearly $\{j_{k\alpha}^*\}$ is a map from κ to $\bar{\kappa}$.

In view of the continuity of the Cech cohomology [1], the map

$$\varinjlim_k \{j_{k\alpha}\}$$

is an isomorphism for each α, and therefore so is the map

$$\varinjlim_\alpha \varinjlim_k \{j_{k\alpha}^*\}.$$

Consequently, in view of the naturality of the isomorphism in Lemma B, the map

$$\varinjlim_k \{j_k^*\} = \varinjlim_k \varinjlim_\alpha \{j_{k\alpha}^*\}$$

is also an isomorphism and the proof is completed.

§4. THE FUNCTOR $H^{\infty-n}$: $\mathfrak{L}^- \to Ab$.

Let X, Y be two objects and let $f: X \to Y$ be a compact field. A sequence $\{Y_k, f_k\}$ of objects Y_k and a_k-fields $f_k: X \to Y_k$ is called an *approximating system* or (a) system for f, provided:

(i) $\{Y_k\}$ is an (a)-sequence for Y,

(ii) $f_k \sim j_k f$ in $\mathfrak{L}^-$, where $j_k: Y \to Y_k$ is the inclusion,

(iii) $f_k \sim i_{kl} f_l$ in $\mathfrak{L}_0^-$, where $i_{kl}: Y_l \to Y_k$ is the inclusion $(k \leq l)$.

Every compact field $f: X \to Y$ admits an (a)-system. In fact, let $Y_k = Y^{(k)}$, and let $f_k: X \to Y_k$ be an a_k-field such that

$$\|f(x) - f_k(x)\| \leq \frac{1}{k} \quad \text{for all } x \in X.$$

Clearly, $\{Y_k, f_k\}$ is an (a)-system for f. In what follows any such system $\{Y_k, f_k\}$ will be called a standard (a)-system for f.

Let $f\colon X \to Y$ be a compact field and $\{Y_k, f_k\}$ be an arbitrary (a)-system for f. In view of (iii) the diagram

$$\begin{array}{ccc} H^{\infty-n}(Y_k) & \xrightarrow{\ i^*_{kl}\ } & H^{\infty-n}(Y_l) \\ & {\scriptstyle f_k}\searrow \quad \swarrow{\scriptstyle f_l} & \\ & H^{\infty-n}(X) & \end{array}$$

commutes in *Ab*. Consequently, $\{f^*_k\}$ is a direct family of maps and

$$\operatorname*{Lim}_{\vec{k}} \{f^*_k\}\colon \operatorname*{Lim}_{\vec{k}} \{H^{\infty-n}(Y_k), i^*_{kl}\} \to H^{\infty-n}(X).$$

We define the induced homomorphism

$$f^*\colon H^{\infty-n}(Y) \to H^{\infty-n}(X)$$

by the formula

$$f^* = \operatorname*{Lim}_{\vec{k}} \{f^*_k\} \circ (\operatorname*{Lim}_{\vec{k}} \{j^*_k\})^{-1}.$$

It is easily shown that the definition of f^* does not depend on the choice of $\{Y_k, f_k\}$.

Now define the functor $H^{\infty-n}$ from the main category $\mathfrak{L}^{\sim}$ to the category of Abelian groups *Ab* by putting $H^{\infty-n}(X) = \overline{H}^{\infty-n}(X)$, $H^{\infty-n}(f) = f^*$. If $f\colon X \to Y$ is an a-field, it is easily seen (by taking $\{Y_k, f_k\}$ with $Y_k = Y$, $f_k = f$) that $\overline{H}^{\infty-n}(f) = H^{\infty-n}(f)$, i.e., $H^{\infty-n}$ extends $\overline{H}^{\infty-n}$ over $\mathfrak{L}^{\sim}$.

THEOREM 3. *The induced map* f^* *satisfies the following properties:*

(a) *the homotopy* $f \sim g$ *implies* $f^* = g^*$;

(b) $(gf)^* = f^* \circ g^*$.

In other words, $H^{\infty-n}$ *is an* h*-functor from the main category* $\mathfrak{L}^-$ *to the category of Abelian groups Ab.*

Proof of the property (a). Let f, g: $X \to Y$ be two compact fields and $h_t: X \to Y$ be a compact homotopy such that $f = h_0$, $g = h_1$. Let $h_t^{(k)}: X \to Y^{(k)}$ be an a_k-homotopy satisfying

$$\|h_t^{(k)}(x) - h_t(x)\| \leq \frac{1}{k} \quad \text{for all } (x, t) \in X \times I.$$

Let $f_k = h_0^{(k)}$, $g_k = h_1^{(k)}$. Evidently, $\{Y^{(k)}, f_k\}$ and $\{Y^{(k)}, g_k\}$ are (a)-systems for f and g respectively. Thus, since $f_k \widetilde{a}_k g_k$, we have

$$\operatorname*{Lim}_{\vec{k}} \{f_k^*\} = \operatorname*{Lim}_{\vec{k}} \{g_k^*\}$$

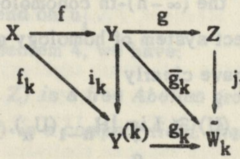
and therefore $f^* = g^*$.

Proof of the property (b). (Special Case). Let f: $X \to Y$ be a compact field, g: $Y \to Z$ be an a_0-field and let $h = gf$. We shall prove that $h^* = f^* \circ g^*$.

Take a standard (a)-system $\{Y^{(k)}, f_k\}$ for f and take an arbitrary finite dimensional extension $\overline{g}$: $Y^{(l)} \to E$ of g over $Y^{(l)}$. The existence of g follows from the Tietze Extension Theorem.

Let us put $W_k = \overline{g}(Y^{(k)}) \cup Z$ for each k and consider the enlarged (a)-sequence $\{\tilde{W}_k\}$ for Z.

In the diagram



define $\tilde{g}_k$ by putting $\tilde{g}_k(y) = \overline{g}(y)$ for all $y \in Y^{(k)}$ and let $\overline{g}_k = \tilde{g}_k i_k$, $h_k = \tilde{g}_k f_k$. It is easily seen that $\{\tilde{W}_k, h_k\}$, $\{\tilde{W}_k, \overline{g}_k\}$ are (a)-systems for h and g, respectively. We have

$$\operatorname*{Lim}_{\vec{k}} \{h_k^*\} = \operatorname*{Lim}_{\vec{k}} \{f_k^*\} \operatorname*{Lim}_{\vec{k}} \{\tilde{g}_k^*\} = \operatorname*{Lim}_{\vec{k}} \{f_k^*\} (\operatorname*{Lim}_{\vec{k}} \{i_k^*\})^{-1} \operatorname*{Lim}_{\vec{k}} \{\overline{g}_k^*\}$$

and thus

$$\lim_{\vec{k}} \{h_k^*\} = f^*(\lim_{\vec{k}} \{\bar{g}_k^*\}).$$

This implies $h^* = f^* \circ g^*$, and the proof is completed.

Proof of the property (b). (General case) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two compact fields and let $h = gf$.

Let $\{Z^{(k)}, h_k\}$ and $\{Z^{(k)}, g_k\}$ be two standard (a)-systems for h and g, respectively.

From the inequalities

$$\|h_k(x) - h(x)\| \leq \frac{1}{k} \quad \text{for all } x \in X,$$

$$\|g_k f(x) - h(x)\| \leq \frac{1}{k} \quad \text{for all } x \in X,$$

it follows that the fields $g_k f$, $h_k\colon X \to Z^{(k)}$ are homotopic in $\mathfrak{L}^\infty$ for each $k = 1, 2, \ldots$. This implies, in view of the property (a), that $h_k^* = (g_k f)^*$. Since each g_k is finite-dimensional, we have $(g_k f)^* = f^* g_k^*$ and thus we obtain

$$\lim_{\vec{k}} \{h_k^*\} = \lim_{\vec{k}} \{f^* g_k^*\} = f^* \lim_{\vec{k}} \{g_k^*\}.$$

This implies $h^* = f^* g^*$ and the proof of Theorem 3 is completed.

§5. THE ALEXANDER-PONTRIAGIN DUALITY IN E.

In view of Lemma A′ the $(\infty - n)$-th cohomology system of an object X is isomorphic to the direct system of homology groups $\{H_{n-1}(U_\alpha), (i_{\alpha\beta})_*\}$, where $U = E - X$. We have clearly

$$H_{n-1}(U) \cong \lim_{\substack{\rightarrow \\ \alpha}} \{H_{n-1}(U_\alpha), (i_{\alpha\beta})_*\}.$$

Consequently, in view of the definition of the group $H^{\infty-n}(X)$, we obtain the following theorem:

THEOREM 4. *For every object* X *we have an isomorphism*

$$H^{\infty-n}(X) \cong H_{n-1}(E - X)$$

between the $(\infty - n)$*-th cohomology group of* X *and the* $(n-1)$*-th singular homology group of the complement of* X *in* E.

Two objects X and Y are called equivalent (resp. *h-equivalent*) in $\mathfrak{L}^\infty$ provided there exist compact fields $f: X \to Y$, $g: Y \to X$ such that $gf = 1_X$, $fg = 1_Y$ (resp. $gf \sim 1_X$, $fg \sim 1_Y$).

It follows from Theorem 3 that if the objects X and Y are equivalent or h-equivalent in $\mathfrak{L}^\infty$. then the $(\infty - n)$-th cohomology groups $H^{\infty - n}(X; G)$ and $H^{\infty - n}(Y; G)$ with the coefficients in G are isomorphic. From this, taking into account Theorem 4, we obtain the following.

THEOREM 5. *If* X *and* Y *are two equivalent or* h*-equivalent objects of the category* $\mathfrak{L}^\infty$, *then for each* $n = 0, 1, 2, \ldots$ *and a coefficient group* G, *the singular homology groups* $H_n(E-X; G)$ *and* $H_n(E-Y; G)$ *are isomorphic.*

§6. THE FUNCTOR $H^{\infty-1}$ AND THE LERAY-SCHAUDER DEGREE

Let $A(r, R) = \{x \in E;\ r \leq \|x\| \leq R\}$, $0 < r \leq R$. By Theorem 4, $H^{\infty-1}(A(r, R); Z) \approx Z$. We will identify groups $H^{\infty-1}(A(r, R); Z)$ with Z by isomorphisms which are compatible with the isomorphisms induced by the inclusions $A(r, R) \subset A(r_1, R_1)$, for $r \geq r_1$, $R \leq R_1$. We denote by 1 the generator of Z.

Let X be an object, $\{U_i\}_{i \in I}$ the family of all bounded components of $E - X$. Let $f_i(x) = x - u_i$, $u_i \in U_i$; f_i: $X \to A(r, R)$ for some $R \geq r > 0$. Evidently, f_i does not depend on u_i.

As a consequence of Theorem 4, we have:

THEOREM 6. $H^{\infty-1}(X; Z)$ *is a free Abelian group with generators* $\alpha_i = f_i^*(1)$. *If* g: $X \to A(r, R)$ *is a compact field, then* $g^*(1) = \Sigma_{q=1}^{p} n_q \alpha_{i_q}$.

The integer $n_q = \gamma(g, U_{i_q})$ is called the *Leray-Schauder index of* g *with respect to* U_{i_q}.

As an immediate consequence we obtain the main theorem of the Leray-Schauder theory of topological degree:

COROLLARY. *Let* U *be a bounded open set with boundary* X. *To every compact field* $f: \bar{U} \to E$ *and every point* $y \in E - f(X)$ *corresponds an integer* $d(y, f, U)$ *with the properties:*

(i) *if* $h_t: (\bar{U}, X) \to (E, E - y)$ *is a compact homotopy, then*

$$d(y, h_1, U) = d(y, h_0, U) \ ;$$

(ii) *if* $d(y, f, U) \neq 0$, *then* $y = f(x)$ *for some* $x \in U$;

(iii) *if* $U = \bigcup U_i$ *is the union of open disjoint sets* U_i *all having their boundary in* X, *then*

$$d(y, f, U) = \sum_i d(y, f, U_i) \ ,$$

(iv) *if* $A \subset \bar{U}$ *is closed and* $y \notin f(A)$, *then*

$$d(y, f, U) = d(y, f, U - A) \ .$$

If U is connected, consider the field $g: X \to A(r, R)$ defined by $g(x) = f(X) - y$ for $x \in X$ and put

$$d(f, U, y) = \gamma(g, U) \ .$$

Now, Theorem 6 permits to define $d(y, f, U)$ in an evident manner for an arbitrary U. The properties (i)-(iv) follow from Theorems 5 and 6.

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INFINITE DIMENSIONAL COHOMOLOGY THEORIES

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INTRODUCTION

The subject-matter of this work belongs to the branch of infinite dimensional topology known as the theory of compact vector fields, and its aim is to establish foundations of algebraic topology in infinite dimensional normed spaces. More specifically, we propose a systematic development of the infinite dimensional cohomology theories; these provide a convenient algebraic tool for the treatment of various infinite dimensional problems and, at the same time, generalize the classical Leray-Schauder theory. The principal topics treated may be listed as follows :

a. Infinite dimensional cohomology theories;

b. Alexander type of duality and Representation Theorems;

c. Applications to some non-linear problems and to the theory of codimension.

Many essential results which we intend to present here were announced in [6] and [7]. Some of our basic ideas and techniques go back to Leray-Schauder [13], A. Granas [8] and K. Gęba [5]; on the other hand, an important part of this work depends on the duality theory due to G. W. Whitehead [21].

In this introduction, we shall not attempt to describe in detail the contents of the present work or state our main results in a precise form. Rather, we shall try to explain the background and some of the ideas, mainly concentrating on (*a*) and (*b*). Before proceeding further, some general remarks are in order. To facilitate matters for the reader, we have tried to make our exposition as self-contained as possible. Thus, a preliminary chapter on general cohomology theories and a summary of basic facts on compact vector fields were included (in Chapter II). Among the essential references, we mention the article of G. W. Whitehead [21] and the Eilenberg-Steenrod book [4], abbreviated here to [GWW] and [ES], respectively. Our notation and terminology largely follow that of [GWW] and [ES] with one notable exception. By a homology (respectively cohomology) theory we shall understand here a theory which satisfies the Eilenberg-Steenrod axioms, except for the dimension axiom. If, in addition, the latter holds, we shall talk about the ordinary homology (respectively cohomology) theories.

We now turn to describing our basic underlying category. Let E be an infinite dimensional normed space. A continuous mapping $f : X \to Y$ between two subsets X and Y of E is called a *compact vector field* provided it can be represented in the form $f(x) = x - F(x)$, where $F : X \to E$ is a compact mapping [i. e., the closure of F (X) is a compact subset of E]. Two such fields $f, g : X \to Y$ are compactly homotopic provided there exists a homotopy $h_t : X \to Y$ joining f and g which is representable in the form $h_t(x) = x - H(x, t)$, where $H : X \times [0, 1] \to E$ is compact. Since compact fields compose well, we have the category $\mathfrak{C}$ with subsets of E as objects and compact fields as morphisms. By the *Leray-Schauder category* $\mathfrak{L}$ we understand the subcategory of $\mathfrak{C}$ generated by closed bounded subsets of E. This category is of primary interest to us and we shall be concerned with such properties of its objects that remain invariant under equivalences, or homotopy equivalences in $\mathfrak{L}$.

Next, some historical remarks. The concept of a compact vector field arose naturally in connection with the question of solvability of the nonlinear equation $x = F(x)$, where F is a compact operator, and was introduced in the early thirties by J. Schauder and J. Leray ([17], [13]). Furthermore, the above authors made the important discovery that many familiar geometrical facts of finite dimensional topology can be carried over to infinitely many dimensions provided attention is restricted to the above category of maps. In particular, for maps of this category, a generalization of Brouwer's theory of degree was established (known presently under the name of the Leray-Schauder theory) and with its aid various applications were obtained. Among these we quote the following theorem, proved by J. Leray [12] : *If* X *and* Y *are two equivalent objects of the category* $\mathfrak{L}$, *then the complements* $E - X$ *and* $E - Y$ *have the same number* *of components; in other words, the singular homology groups* $H_0(E - X; Z)$ *and* $H_0(E - Y; Z)$ *are isomorphic.*

In connection with this theorem, the following problem is of obvious interest in itself :

Problem 1. — If X and Y are two equivalent (or more generally homotopically equivalent) objects of $\mathfrak{L}$, are the homology groups $H_n(E - X)$ and $H_n(E - Y)$ isomorphic for each n ?

One of our aims is to give the affirmative answer to the above problem for a broad class of homology theories. At the moment we remark only that the Leray-Schauder theory is evidently not adequate to deal with Problem 1 and, therefore, a tool of an essentially algebraic character is needed. Thus we are led to the first and main topic of this work.

By an " infinite dimensional " or simply a *cohomology theory* $\mathcal{H}^{\infty-*} = \{\mathcal{H}^{\infty-n}, \delta^{\infty-n}\}$ on $\mathfrak{L}$ we shall understand a sequence of contravariant functors $\mathcal{H}^{\infty-n}(X, A)$ from the pairs in $\mathfrak{L}$ to the category of abelian groups together with a sequence of natural transformations

$$\delta^{\infty-n}_{(X,A)}(A) : \mathcal{H}^{\infty-n}(A) \to \mathcal{H}^{\infty-(n-1)}(X, A)$$

satisfying the Homotopy, Exactness and Strong Excision axioms; the graded group $\{\mathcal{H}^{\infty-n-1}(S)\}$, where S is the unit sphere in E, is called the group of coefficients of the theory. The first basic result of this paper (Theorem V.4.4) states :

To any cohomology theory $\mathcal{H}^*$ *on the category of finite polyhedra corresponds a cohomology theory* $\mathcal{H}^{\infty-*}$ *on* $\mathfrak{L}$ *with the same group of coefficients and satisfying the continuity axiom; moreover, the assignment* $\mathcal{H}^* \mapsto \mathcal{H}^{\infty-*}$ *is natural with respect to maps of the theories.*

Thus, in particular, we have the ordinary cohomology $H^{\infty-*}(\ ;G)$ with coefficients in G, the stable cohomotopy $\Sigma^{\infty-*}$ and the Hopf-Hurewicz map $h^* : \Sigma^{\infty-*} \to H^{\infty-*}(\ ;Z)$.

We shall now try, for the benefit of the reader, to give some general idea about how the " infinite dimensional " cohomologies are defined. In the subsequent more formal treatment of this matter, our presentation will look slightly different for technical reasons.

Assume for simplicity that we start with the ordinary Čech cohomology $H^* = \{H^q, \delta^q\}$ for compacta with coefficients in G and want to define $H^{\infty-n}(X)$ for an object X in $\mathfrak{L}$. We take the directed (by inclusion) set $\mathcal{L}_X = \{L_\alpha, L_\beta, L_\gamma, \dots\}$ of all finite dimensional subspaces of E such that $X_\alpha = X \cap L_\alpha$ for each α is not empty and we fix an orientation of every L_α. Let $d(\alpha) \doteq \dim L_\alpha$ and write $\alpha \leqq \beta$ instead of $L_\alpha \subset L_\beta$. We observe that, given $\alpha \leqq \beta$ with $d(\beta) = d(\alpha) + 1$, the orientations of L_α and L_β determine the triad $(L_\beta; L_\beta^+, L_\beta^-)$ with $L_\alpha = L_\beta^+ \cap L_\beta^-$ and, therefore, the triad $(X_\beta; X_\beta^+, X_\beta^-)$ with $X_\alpha = X_\beta^+ \cap X_\beta^-$ (where $X_\beta^+ = X \cap L_\beta^+$ and $X_\beta^- = X \cap L_\beta^-$). Now, given $\alpha \leqq \beta$, we define

$$\Delta_{\alpha\beta} : \quad H^{d(\alpha)-n}(X_\alpha) \to H^{d(\beta)-n}(X_\beta)$$

as follows : if $d(\beta) = d(\alpha) + 1$, we let $\Delta_{\alpha\beta}$ be the Mayer-Vietoris homomorphism of the triad $(X_\beta; X_\beta^+, X_\beta^-)$; otherwise, we take a chain of consecutive elements of $\mathcal{L}_X$ and define $\Delta_{\alpha\beta}$ to be the composition of the corresponding Mayer-Vietoris homomorphism.

It turns out that if $U = E - X$, $\alpha \leqq \beta$ and $i_{\alpha\beta} : U_\alpha \to U_\beta$ is the inclusion, then we have the commutative diagram :

$$\begin{array}{ccc} H^{d(\alpha)-n}(X_\alpha) & \xrightarrow{\Delta_{\alpha\beta}} & H^{d(\beta)-n}(X_\beta) \\ \wr\downarrow D_\alpha & & \wr\downarrow D_\beta \\ H_{n-1}(U_\alpha) & \xrightarrow{(i_{\alpha\beta})_*} & H_{n-1}(U_\beta) \end{array}$$

in which D_α and D_β stand for the appropriate Alexander-Pontrjagin isomorphisms in L_α and L_β, respectively.

It follows that the groups $\{ H^{d(\alpha)-n}(X_\alpha); \Delta_{\alpha\beta} \}$ form a direct system of Abelian groups and we define

$$H^{\infty-n}(X) = \underset{\overrightarrow{\alpha}}{\operatorname{Lim}} \{H^{\infty-n}(X_\alpha) ; \Delta_{\alpha\beta}\}.$$

It remains to define the induced homomorphism $H^{\infty-n}(f) = f^*$ and to prove its functorial properties; this is done in two steps : first, for a finite dimensional, and then for an arbitrary compact field f. It should be emphasized that in the second and more involved step, the crucial role is played by the continuity property of the functors under consideration.

Next we observe that the passage to the limit in the above commutative diagram indicates how we are led to our next basic theorem : *The group* $H^{\infty-n}(X)$ *is isomorphic to the* $(n-1)$*-th singular homology group* $H_{n-1}(U; G)$. A more general result holds in fact, and our second main theorem (Theorem VI.6.7) stated for a broad class of cohomology theories, expresses also the naturality of the duality isomorphism.

Consider compact fields from an object X in $\mathfrak{L}$ to the open set $E - E_{n-1}$ (where E_{n-1} is a linear subspace of E with $\dim E_{n-1} = n - 1$) and denote by $\pi^{\infty-n}(X)$ the corresponding set of homotopy classes. Our next important result (Representation Theorem for stable cohomotopy) says : *There is* *a natural isomorphism between* $\pi^{*-n}(X)$ *and the stable cohomotopy group* $\Sigma^{*-n}(X)$.

In connection with the duality in E the following problem arises :

Problem 2. — Do there exist two equivalent objects X and Y of $\mathfrak{L}$ such that the fundamental groups $\pi_1(E - X)$ and $\pi_1(E - Y)$ are not isomorphic ?

The answer is " yes " and the corresponding example is given in Chapter II. In fact, we shall see that, from the geometrical point of view, the Leray-Schauder category is in a certain sense as rich as the category of compacta in R^n.

We remark now that from the proved results a number of consequences follow. Some of them (as the Mayer-Vietoris sequence) follow evidently from the axioms alone, while others (as the affirmative answer to Problem 1) do so from both results together. The duality combined with the Hurewicz Theorem in S-theory yields the important Hopf Theorem, relating the ordinary cohomology over Z and the stable cohomotopy on $\mathfrak{L}$.

In Chapter IX we consider some applications to the theory of non-linear equations. Among these we deduce first a number of well-known results (basic properties of the Leray-Schauder characteristic, the Leray-Schauder alternative, Invariance of Domain, etc.) to illustrate the generality of our theory. Then we pass to the case in which the classical Leray-Schauder approach is not applicable. Using the cohomology functors we consider some particular extension problems in $\mathfrak{C}$. Then with the aid of the notion of an essential compact field we translate the corresponding results into the new existence criteria for the equation $x = F(x)$.

It remains to say a few words about the notion of codimension treated in Chapter X. First of all, we have for objects of the Leray-Schauder category the " basic " codimension Codim defined in terms of the extension problem for compact fields with special ranges $E - E^n$, where $\dim E^n = n$. Our definition coincides in the finite dimensional case with a theorem of Alexandroff [1] which characterizes the dimension of a compactum by maps into S^n. Further, using cohomology theories on $\mathfrak{L}$, we define various cohomological codimensions; we have in particular Codim_Z defined in terms of the ordinary cohomology on $\mathfrak{L}$ with integer coefficients Z. *If the space* E *is complete, then* $\mathrm{Codim} = \mathrm{Codim}_Z$. The proof of this theorem uses the above theorem of Hopf, the representability of the stable cohomotopy on $\mathfrak{L}$ and the Homotopy Extension Lemma; the latter is known to be true in needed generality only under the assumption of completeness. We note that the above result corresponds in the finite dimensional case to " the fundamental theorem in the homological dimension theory " in [1]. Among other results we mention the extension of the well-known Phragmen-Brouwer Theorem to the infinite dimensional, case given at the end of Chapter X.

In conclusion, the authors express thanks to Monique Granas for her generous help and assistance during the preparation of this work.

CHAPTER I

Preliminaries

In this chapter we give an account of those definitions and results of the homology and cohomology theories that will be relevant for the purposes of this work. For obvious reasons, we treat the entire matter very sketchily. We do, however, indicate some of the proofs and offer some additional comments on facts of further importance; this should make the general ideas and the logical sequence reasonably clear. For the details, the reader is refered to [GWW] and also (when the necessity arises) to [ES].

1. Remarks on the notation. — The notation adopted in this work should be clear from the preliminaries and the context. Nevertheless, it might be worthwhile to list those symbols which appear frequently in our discussion.

We denote by $E^\infty = (E^\infty, \| \ \|)$ or simply by E an arbitrary but fixed infinite dimensional linear normed space over the field R. We fix a sequence $\{ E^{\infty-n} \oplus E_n \}$ of direct sum decompositions of E^∞ such that :

(i) $E_0 \subset E_1 \subset \ldots \subset \ldots$;

(ii) $E^\infty \supset E^{\infty-1} \supset E^{\infty-2} \supset \ldots$;

(iii) $\operatorname{codim} E^{\infty-n} = \dim E_n = n$.

We let

$$S^{\infty-n} = \{ x \in E^{\infty-n+1} ; \| x \| = 1 \}$$

denote the unit sphere in $E^{\infty-(n-1)}$; $n \geqq 1$; and we reserve the symbol $U^{\infty-n}$ for the open set $E^\infty - E^n$.

Next, we let R^∞ be the normed space consisting of all sequences $x = (x_1, x_2, \ldots)$ of real numbers such that $x_i = 0$ for almost all i with the norm $\|x\| = \left(\sum_{i=1}^{\infty} x_i^2\right)^{1/2}$. The following symbols stand for subsets of R^{∞} :

$$R^k = \{x \in R^{\infty}; x_i = 0 \text{ for } i \geq k+1\},$$
$$R^k_+ = \{x \in R^k; x_k \geq 0\},$$
$$R^k_- = \{x \in R^k; x_k \leq 0\},$$
$$S^k = \{x \in R^{k+1}; \|x\| = 1\},$$
$$S^k_+ = S^k \cap R^{k+1}_+, S^k_- = S^k \cap R^{k+1}_-.$$

There are inclusions $R^k \subset R^{k+1}$, $S^k \subset S^{k+1}$ and we have clearly $S^k = S^k_+ \cup S^k_-$ and $S^{k-1} = S^k_+ \cap S^k_-$. The closed intervals $[-1, 1]$ and $[0, 1]$ of the real line are denoted by J and I, respectively.

Finally, we use the following fixed notation :

$\mathcal{E}ns$ = the category of sets;
$\mathcal{E}ns^*$ = the category of based sets;
$\mathcal{A}b$ = the category of abelian groups;
$\mathcal{A}^*$ = either $\mathcal{A}b$ or $\mathcal{E}ns^*$;
$\mathcal{A}$ = either $\mathcal{E}ns$ or $\mathcal{A}^*$;
$\mathcal{C}(A, B)$ = the set of maps (= morphisms) $f : A \to B$ in a category $\mathcal{C}$.

All standardly used categories are denoted by script letters; the category of compact vector fields and its subcategories will be denoted by German letters.

2. *h*-CATEGORIES. — Let $\mathcal{C}$ be a concrete category and X and A be two objects in $\mathcal{C}$. Then A is called a sub-object of X (written $A \subset X$) provided A is a subset of X and the inclusion map $i : A \to X$ is in $\mathcal{C}$.

Let $f : X \to Y$ be a map in $\mathcal{C}$ such that $f(A) \subset B$, where $A \subset X$ and $B \subset Y$. Consider the function $f' : A \to B$ defined by $f'(a) = f(a)$ for $a \in A$. If f' is in $\mathcal{C}$, then it is called the *contraction* of f to the pair (A, B). A contraction f' of f to the pair (A, Y), being the restriction of f to A, will be denoted by $f \mid A$. In this case, we write also $f' \subset f$ and call f an *extension* of f' over X.

2.1. DEFINITION. — An *h*-category $(\mathcal{C}, \sim)$ is a category $\mathcal{C}$ such that for each pair of objects A and B in $\mathcal{C}$ there is defined in the set $\mathcal{C}(A, B)$ an equivalence relation $\sim$ (called homotopy) satisfying the following (compositive) property :

$$f_1 \sim f_2, \quad g_1 \sim g_2 \quad \Rightarrow \quad g_1 f_1 \sim g_2 f_2.$$

If $f \in \mathcal{C}$ (A, B), then by $[f]$ we denote the equivalence (homotopy) class containing f and we let π (A, B) be the set of such homotopy classes. By $\mathcal{C}/\sim$ we denote the category having the same objects as $(\mathcal{C}, \sim)$ and as maps the homotopy classes between objects.

A subcategory $\mathcal{C}_0$ of $\mathcal{C}$ will be called *dense* provided it has the same objects as $\mathcal{C}$. We say that $(\mathcal{C}_0, \simeq)$ is an *h*-subcategory of $(\mathcal{C}, \sim)$ provided $\mathcal{C}_0 \subset \mathcal{C}$ and the relation $f \simeq g$ implies $f \sim g$ for any f and g in $\mathcal{C}_0$.

Remark. — In what follows $\mathcal{E}ns$ and $\mathcal{A}^*$ will be considered as *h*-categories with the relation of homotopy $\sim$ defined by : $f \sim g \Leftrightarrow f = g$.

A map $f : A \to B$ in $(\mathcal{C}, \sim)$ is called *inversible* (respectively *h-inversible*) provided there is a map $f' : B \to A$ such that $f' \circ f = 1_A$ and $f \circ f' = 1_B$ (respectively $f' \circ f \sim 1_A$ and $f \circ f' \sim 1_B$). In the first case, we write $A \sim B$ and call the objects A and B *equivalent.* In the second case, A and B are said to be *homotopically equivalent* and we write $A \underset{h}{\sim} B$.

A map $r : A \to B$ is called an *r*-map (respectively r_h-map) if there exists a map $s : B \to A$ such that $r \circ s = 1_B$ (respectively $r \circ s \sim 1_B$).

Two objects A and B are called *r-equivalent* (respectively r_h*-equivalent*) provided there exists a pair of *r*-maps (respectively r_h-maps) $r_1 : A \to B$ and $r_2 : B \to A$. We write $A \underset{r}{\sim} B$ in the first case and $A \underset{r_h}{\sim} B$ in the second.

2.2. Proposition. — *The relations* $\sim$, $\underset{h}{\sim}$, $\underset{r}{\sim}$, $\underset{r_h}{\sim}$ *defined in the class of objects of* $(\mathcal{C}, \sim)$ *are equivalence relations and we have*

$$A \underset{h}{\sim} B \text{ in } (\mathcal{C}, \sim) \iff A \sim B \text{ in } \mathcal{C}/\sim,$$

$$A \underset{r_h}{\sim} B \text{ in } (\mathcal{C}, \sim) \iff A \underset{r}{\sim} B \text{ in } \mathcal{C}/\sim.$$

Examples of h-categories :

1º The category $\mathcal{T}$ (respectively $\mathcal{K}$) of all topological (respectively, compact Hausdorff) spaces and all continuous maps with the ordinary relation of homotopy.

2º The category $\mathcal{P}$ of finite polyhedra and the category $\mathcal{W}$ of all CW-complexes.

3º Let E be a linear normed space and denote by $\mathcal{K}_E$ the full subcategory of $\mathcal{K}$ whose objects are compact subsets of E contained in finite dimensional subspaces of E. We say that a polyhedron $K \subset E$ is a *geometric subpolyhedron* of E if K has a triangulation which is a finite union of geometric simplexes. We will denote by $\mathcal{P}_E$ the full subcategory of $\mathcal{K}_E$ whose objects are geometric subpolyhedra of E. We consider $\mathcal{K}_E$ and $\mathcal{P}_E$ as *h*-categories with the ordinary relation of homotopy.

4° For any concrete h-category $\mathcal{D}$ that will appear in this paper, we shall denote by $\mathcal{D}_*$ the corresponding *based* category. For example, the objects of $\mathcal{T}_*$ are based topological spaces (X, x_0), the morphisms are based maps $f : (X, x_0) \to (Y, y_0)$ and the relation $\sim$ in $\mathcal{T}_*$ means the based homotopy.

5° The category $\mathcal{W}^2$ of pairs in $\mathcal{W}$ is defined as follows. The objects of $\mathcal{W}^2$ are pairs (X, A) such that A is subcomplex of X. The maps in $\mathcal{W}^2$ are continuous mappings $f : (X, A) \to (Y, B)$. The relation of homotopy in $\mathcal{W}$ induces in an obvious way the structure of an h-category in $\mathcal{W}^2$. The category $\mathfrak{T}^2$ of pairs in $\mathfrak{T}$ is defined in a similar way.

6° Similarly, for any of h-categories $\mathcal{D}$ introduced in the examples 1° and 3° we will denote by $\mathcal{D}^2$ the category of all pairs in $\mathcal{D}$. The relation of homotopy in $\mathcal{D}$ induces clearly the structure of an h-category in $\mathcal{D}^2$.

7° For any of h-categories $\mathcal{D}$ introduced in the examples 1°, 2° and 3° we will denote by $\mathcal{D}^2_*$ the corresponding h-category whose objects are based pairs $((X, x_0), (A, x_0))$ such that $(X, A) \in \mathcal{D}^2$.

3. h-FUNCTORS.

3.1. DEFINITION. — A functor $\lambda : \mathcal{C}_1 \to \mathcal{C}_2$ between two h-categories is called an *h-functor* provided it sends homotopy commutative diagrams in $\mathcal{C}_1$ into such in $\mathcal{C}_2$.

Example : Let $(\mathcal{L}, \approx)$ be an h-subcategory of $(\mathcal{C}, \sim)$ and U be a fixed object in $\mathcal{C}$. For $f : X \to Y$ in $\mathcal{L}$ we let $f^* : \pi(Y, U) \to \pi(X, U)$ be the map induced by f and defined by $[g] \to [gf]$. The assignments $X \to \pi(X, U)$, $f \to f^*$ define a contravariant h-functor $\pi(U)$ from $(\mathcal{L}, \approx)$ to the category of sets $\mathcal{E}ns$.

3.2. PROPOSITION. — *Let* $\lambda : \mathcal{C}_1 \to \mathcal{C}_2$ *be an h-functor. If* $\approx$ *is any one of the equivalence relations defined above, then*

$$X \approx Y \quad \text{in } \mathcal{C}_1 \quad \textit{implies} \quad \lambda(X) \approx \lambda(Y) \quad \text{in } \mathcal{C}_2.$$

Examples of h-functors :

1° *The functor* $+$: Let $\{\star\}$ be a fixed space consisting of exactly one point $\star$. For X in $\mathcal{T}$, let X^+ be the based space defined by $X^+ = (X \cup \{\star\}, \star)$. If $f : X \to Y$ is a map in $\mathcal{T}$, then $f : X \to Y$ has a unique extension $f^+ : X^+ \to Y^+$ with $f^+(\star) = \star$. The correspondences $X \to X^+$, $f \to f^+$ define an h-functor $+ : \mathcal{T} \to \mathcal{T}^*$ and we may consider $\mathcal{T}$ as a subcategory of $\mathcal{T}^*$.

2° *The unreduced suspension* S : For a topological space X take the product J×X and identify $\{1\}\times X$ to one point and $\{-1\}\times X$ to another. The quotient space SX under this identification is called the (unreduced) suspension of X. Denote by $p : J\times X \to SX$ the corresponding identification map and by $i : X \to J\times X$ an embedding given by $i(x) = (0, x)$. Then the composite $p \circ i : X \to SX$ defines an embedding of X into SX. In what follows we identify X with its image under this embedding. Let us put

$$C_+ X = p(X\times[0, 1]) \quad \text{and} \quad C_- X = p(X\times[-1, 0]);$$

then $(SX; C_+ X, C_- X)$ is an additive triad with $C_+ X \cap C_- X = X$. Now, if $f : X \to Y$ is a continuous map, then $\tilde{f} : J\times X \to J\times Y$, given by $\tilde{f}(t, x) = (t, f(x))$, defines the map $Sf : SX \to SY$, called the *suspension* of f. It is easily seen that S is a covariant h-functor from the category $\mathfrak{T}$ into itself [1].

3° *The reduced suspension* Σ : For a based topological space (X, x_0) the quotient (based) space

$$X = J\times X/\{-1\}\times X \cup \{1\}\times X \cup J\times\{x_0\}$$

is called the *reduced suspension* of X. A based map $f : (X, x_0) \to (Y, y_0)$ induces the based map $\Sigma f : \Sigma X \to \Sigma Y$. It is easily seen that Σ is a functor; since, in addition, Σ preserves based homotopies, it follows that Σ is an h-functor from the category $\mathfrak{T}_*$ into itself [1].

We note that, by putting $S(X, x_0) = (SX, x_0)$, we may view the unreduced suspension S as an h-functor from $\mathfrak{T}_*$ into itself.

We have then the following relation between Σ and S. Denote by $p : SX \to \Sigma X$ the identification map. If $f : (X, x_0) \to (Y, y_0)$ is a based map then the following diagram commutes in $\mathfrak{T}^*$:

$$\begin{array}{ccc} SX & \xrightarrow{Sf} & SY \\ \downarrow p & & \downarrow p \\ \Sigma X & \xrightarrow{\Sigma f} & \Sigma Y \end{array}$$

4° *The reduced join* ∧ : For two based spaces (X, x_0) and (Y, y_0) we let $X \vee Y = X\times\{y_0\} \cup \{x_0\}\times Y$. The *reduced join*

$$X \wedge Y = X\times Y/X \vee Y$$

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[1] Note that our notation differs from that of [GWW] where the unreduced suspension is denoted by Σ and the reduced one by S.

is obtained from $X \times Y$ by collapsing $X \vee Y$ to a point. For two based maps $f : (X, x_0) \to (X', x'_0)$, $g : (Y, y_0) \to (Y', y'_0)$, the product map $f \times g : X \times Y \to X' \times Y'$ given by $f \times g(x, y) = (f(x), g(y))$ induces a map $f \wedge g : X \wedge X' \to Y \wedge Y'$. Thus we obtain the reduced join h-functor $\wedge$ from the category $\mathcal{C}_* \times \mathcal{C}_*$ into $\mathcal{C}_*$.

In the n-dimensional sphere S^n choose $x_0 = (1, 0, \ldots, 0)$ as a base point. Thus we may regard S^n as an object of $\mathcal{C}_*$. For $X \in \mathcal{C}_*$, we let $\Sigma^p X = \Sigma(\Sigma^{p-1} X)$, $p > 0$ and we note that there is a natural identification

$$\Sigma^p X = S^p \wedge X$$

which will be used later on.

4. Homology and cohomology theories. — *Convention.* — In what follows, by a " topological category " we shall understand any of the h-categories introduced in the examples 1°-3° in Section 2.

Let $\mathcal{D}$ be a topological category and $\mathcal{D}^2$ (resp. $\mathcal{D}^2_*$) the corresponding category of pairs (resp. based pairs). In what follows $\mathcal{D}^2_0$ denotes either $\mathcal{D}^2$ or $\mathcal{D}^2_*$. For $A \in \mathcal{D}$ we identify A with the pair $(A, \emptyset) \in \mathcal{D}^2$. If $(A, a_0) \in \mathcal{D}_*$ we identify (A, a_0) with the pair $((A, a_0), (\{a_0\}, a_0)) \in \mathcal{D}^2$. Using this identifications we will regard $\mathcal{D}$ as a subcategory of $\mathcal{D}^2$ and $\mathcal{D}_*$ as a subcategory of $\mathcal{D}^*_2$.

Let $\rho : \mathcal{D}^2_0 \to \mathcal{D}^2_0$ be the covariant functor defined by:

$\rho(X, A) = A$ for any (X, A) in $\mathcal{D}^2_0$;

$\rho(f) = f|_A : A \to B$ for any map $f : (X, A) \to (Y, B)$ in $\mathcal{D}^2_0$.

A *homology theory* $\mathcal{H}_*$ on $\mathcal{D}^2_0$ is a sequence of covariant h-functors

$$\mathcal{H}_n : \mathcal{D}^2_0 \to \mathcal{A}b \qquad (-\infty < n < +\infty)$$

together with a sequence of natural transformations

$$\partial_n : \mathcal{H}_n \to \mathcal{H}_{n-1} \circ \rho \qquad (-\infty < n + \infty)$$

satisfying the following conditions:

(A) (*Excision*) : *If* $(X, A) \in \mathcal{D}^2_0$, U *is an open subset of* X *whose closure is contained in the interior of* A, *and the inclusion map*

$$e : (X - U, A - U) \to (X, A)$$

is in $\mathcal{D}^2_0$, *then the induced homomorphism*

$$\mathcal{H}_n(e) : \mathcal{H}_n(X - U, A - U) \to \mathcal{H}_n(X, A)$$

is an isomorphism for every integer n.

(B) (*Exactness*) : *If* $(X, A) \in \mathcal{O}_0^2$ *and* $i : A \to X$, $j : X \to (X, A)$ *are inclusions then the homology sequence*

$$\ldots \to \mathcal{H}_{n+1}(X, A) \xrightarrow{\partial_{n+1}} \mathcal{H}_n(A) \xrightarrow{\mathcal{H}_n(i)} \mathcal{H}_n(X) \xrightarrow{\mathcal{H}_n(j)} \mathcal{H}_n(X, A) \xrightarrow{\partial_n} \mathcal{H}_{n-1}(A) \to \ldots$$

of (X, A) *is exact.*

The graded group $\{\mathcal{H}_n(p_0)\}$, where p_0 is a point, is called the *group of coefficients* of the theory $\mathcal{H}_*$.

Let $\mathcal{H}_* = \{\mathcal{H}_n, \partial_n\}$ be a homology theory on $\mathcal{O}^2$. If $(A, x_0) \subset (X, x_0)$ is a pair in $\mathcal{O}_*^2$ let

$$\tilde{\mathcal{H}}_n((X, x_0), (A, x_0)) = \mathcal{H}_n(X, A).$$

Note, that according to our convention

$$\tilde{\mathcal{H}}_n(X, x_0) = \mathcal{H}_n(X, \{x_0\}) \quad \text{for} \quad (X, x_0) \in \mathcal{O}_* \subset \mathcal{O}_*^2.$$

We let

$$\tilde{\partial}_n : \tilde{\mathcal{H}}_n((X, x_0), (A, x_0)) \to \tilde{\mathcal{H}}_{n-1}(A, x_0)$$

be the boundary homomorphism of the triple $(X, A, \{x_0\})$,

$$\tilde{\partial}_n : \mathcal{H}_n(X, A) \to \mathcal{H}_{n-1}(A, \{x_0\}).$$

If $f : ((X, x_0), (A, x_0)) \to ((Y, y_0), (B, y_0))$ is a map in $\mathcal{O}_*^2$ we may regard f as a map of (X, A) into (Y, B) in $\mathcal{O}^2$ and let

$$\tilde{\mathcal{H}}_n(f) = \mathcal{H}_n(f).$$

With the above definitions $\tilde{\mathcal{H}}_* = \{\tilde{\mathcal{H}}_n, \tilde{\partial}_n\}$ is a homology theory on $\mathcal{O}_*^2$. It turns out that the assignment $\mathcal{H}_* \to \tilde{\mathcal{H}}_*$ defines one-to-one correspondence between homology theories on $\mathcal{O}^2$ and on $\mathcal{O}_*^2$. In what follows we call $\tilde{\mathcal{H}}_*$ the reduced homology theory corresponding to the homology theory $\mathcal{H}_*$. The converse correspondence may be described as follows : given a homology theory $\tilde{\mathcal{H}}_*$ on $\mathcal{O}_*^2$ we let

$$\mathcal{H}_n(X, Y) = \tilde{\mathcal{H}}_n(X^+, Y^+) \quad \text{for} \quad (X, Y) \in \mathcal{O}^2,$$

$$\mathcal{H}_n(f) \quad = \tilde{\mathcal{H}}_n(f^+) \quad \text{for} \quad f \text{ in } \mathcal{O}^2;$$

finally, if $(X, Y) \in \mathcal{O}^2$, let

$$\partial_n = \tilde{\partial}_n : \mathcal{H}_n(X^+, Y^+) \to \mathcal{H}_{n-1}(Y^+).$$

Similarly, a cohomology theory $\mathcal{H}^*$ on $\mathcal{O}_0^2$ is a sequence of contravariant h-functors

$$\mathcal{H}^n : \mathcal{O}_0^2 \to \mathcal{A}b \quad (-\infty < n < +\infty)$$

together with a sequence of natural transformations

$$\partial_n : \mathcal{H}^{n-1} \circ \rho \to \mathcal{H}^n \qquad (-\infty < n < +\infty)$$

satisfying the analogous excision and exactness axioms; the graded group $\mathcal{H}^n(p_0)$ is the *group of coefficients* of the theory $\mathcal{H}^*$.

As in the case of homology theories there is a one-to-one correspondence between cohomology theories on $\mathcal{D}^2$ and cohomology theories on $\mathcal{D}^2_*$. We will use the same terminology : the cohomology theory $\tilde{\mathcal{H}}^*$ on $\mathcal{D}^2_*$ corresponding to the cohomology theory $\mathcal{H}^*$ on $\mathcal{D}^2$ will be called the *reduced cohomology theory.*

Thus, the homology theory (respectively cohomology theory) satisfies the Eilenberg-Steenrod axioms, except for the dimension axiom. If the dimension axiom is satisfied, then $\mathcal{H}_*$ (respectively $\mathcal{H}^*$) is said to be an *ordinary homology* (respectively *cohomology theory*). Note that the above terminology differs from that of [GWW] where the terms " generalized homology (cohomology) theory " and " homology (cohomology) theory " are used.

Remark. — For notational convenience in all that follows the homology and cohomology theories are denoted by the script letters $\mathcal{H}^*$ and $\mathcal{H}^*$ respectively. The symbols H_* and H^* are used to denote the *ordinary* theories.

Consider the following condition, called the *strong excision axiom :*

(A') (*Strong Excision*) *: If* (X; A, B) *is an additive triad in* $\mathcal{D}$ *with* $A \cap B \in \mathcal{D}$, *and* $k : (A, A \cap B) \to (X, B)$ *is the inclusion, then*

$$\mathcal{H}_n(k) : \mathcal{H}_n(A, A \cap B) \; \mathcal{H}_n(X, B)$$

is an isomorphism for all n.

We consider also the analogous axiom for cohomology theory. Note that on $\mathfrak{T}^2$ or $\mathfrak{T}^2_E$ any homology and cohomology theory satisfies the strong excision axiom.

The axioms for homology (respectively cohomology) may also be introduced in terms of the reduced homology (respectively cohomology) and the suspension isomorphism.

A (*reduced*) *homology theory* $\tilde{\mathcal{H}}_*$ on $\mathcal{D}_*$ is a sequence of h-functors

$$\tilde{\mathcal{H}}_n : \mathcal{D}_* \to \mathcal{A}b \qquad (-\infty < n < +\infty)$$

together with a sequence of natural transformations

$$\sigma_n : \tilde{\mathcal{H}}_n \to \tilde{\mathcal{H}}_{n-1} \circ \Sigma \qquad (-\infty < n < +\infty)$$

satisfying the following conditions :

(A) (*Suspension*) : For any $X \in \mathcal{O}_*$, *the map*

$$\sigma_n(X) : \tilde{\mathcal{H}}_n(X) \to \tilde{\mathcal{H}}_{n-1}(\Sigma X)$$

is an isomorphism.

(B) (*Exactness*) : *If* (X, A) *is a based pair in* $\mathcal{O}_*$ *with* A *closed in* X, $i : A \to X$ *the inclusion map and* $p : X \to X/A$ *the identification map, then the sequence*

$$\tilde{\mathcal{H}}_n(A) \overset{i_*}{\to} \tilde{\mathcal{H}}_n(X) \overset{p_*}{\to} \tilde{\mathcal{H}}^n(X/A)$$

is exact.

The graded group $\{\tilde{\mathcal{H}}_n(S^0)\}$ is the *group of coefficients* of the theory $\tilde{\mathcal{H}}_*$.

A (*reduced*) *cohomology theory* $\tilde{\mathcal{H}}^* = \{\tilde{\mathcal{H}}^n, \sigma^n\}$ on $\mathcal{O}_*$ is defined in a similar way.

Now, let $\tilde{\mathcal{H}}_* = \{\tilde{\mathcal{H}}_n, \sigma_n\}$ be a homology theory on $\mathfrak{T}_*$. If (X, A) is a pair in $\mathfrak{T}_*^2$, we let

(1) $$\tilde{\mathcal{H}}_n(X, A) = \tilde{\mathcal{H}}_n(X/A);$$

if $f : (X, A) \to (Y, B)$ is a map in $\mathfrak{T}_*^2$, we let

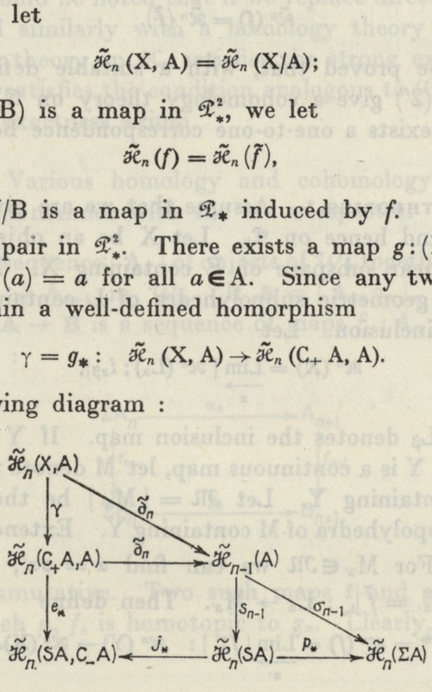
(2) $$\tilde{\mathcal{H}}_n(f) = \tilde{\mathcal{H}}_n(\tilde{f}),$$

where $\tilde{f} : X/A \to Y/B$ is a map in $\mathfrak{T}_*$ induced by f.

Let (X, A) be a pair in $\mathfrak{T}_*^2$. There exists a map $g : (X, A) \to (C_+ A, A)$ in $\mathfrak{T}_*^2$ such that $g(a) = a$ for all $a \in A$. Since any two such maps are homotopic we obtain a well-defined homorphism

$$\gamma = g_* : \tilde{\mathcal{H}}_n(X, A) \to \tilde{\mathcal{H}}_n(C_+ A, A).$$

Consider the following diagram :



in which $p : SA \to A$ is the identification map, $j : SA \to (SA, C_- A)$ is the inclusion and $e : (C_+ A, A) \to (SA, C_- A)$ is the excision map. Since p is a homotopy equivalence, p_* is an isomorphism and we let $s_n = (p_*)^{-1} \circ \sigma_n$. From the definition of induced homomorphism it follows that e_* and j_* are isomorphisms. We let

$$\tilde{\partial}_n = (s_n)^{-1} \circ (j_*)^{-1} \circ e_* \circ \gamma : \tilde{\mathcal{H}}_n (X, A) \to \tilde{\mathcal{H}}_{n-1} (A). \tag{3}$$

It turns out that $\tilde{\mathcal{H}}_* = \{ \tilde{\mathcal{H}}_n, \tilde{\partial}_n \}$ defined by (1), (2) and (3) is a homology theory on $\mathfrak{T}^2_*$. We will call $\tilde{\mathcal{H}}_* = \{ \tilde{\mathcal{H}}_n, \tilde{\partial}_n \}$ an *extension* of $\tilde{\mathcal{H}}_* = \{ \tilde{\mathcal{H}}_n, \sigma_n \}$ over $\mathfrak{T}^2_*$. It can be easily shown that any reduced homology theory on $\mathfrak{T}_*$ has a unique extension over $\mathfrak{T}^2_*$. The extension determines a one-to-one correspondence between reduced homology theories on $\mathfrak{T}_*$ and on $\mathfrak{T}^2_*$.

Let $\tilde{\mathcal{H}}^* = \{ \tilde{\mathcal{H}}^n, \sigma_n \}$ be a cohomology theory on $\mathfrak{T}_*$. If $(X, A) \in \mathfrak{T}^2_*$ we let

$$\tilde{\mathcal{H}}^n (X, A) = \tilde{\mathcal{H}}^n (X/A); \tag{1'}$$

if $f : (X, A) \to (Y, B)$ is in $\mathfrak{T}^2_*$, we let

$$\tilde{\mathcal{H}}^n (f) = \tilde{\mathcal{H}}^n (\tilde{f}). \tag{2'}$$

Again, it can be proved that, with a suitable definition of ∂_n, the formulas (1′) and (2′) give a cohomology theory on $\mathfrak{T}^2_*$. It can be also proved that there exists a one-to-one correspondence between two kinds of theories.

5. Continuous theories. — Assume that we are given a cohomology theory $\mathcal{H}^*$ on $\mathfrak{T}$ and hence on $\mathfrak{T}_E$. Let X be an object in $\mathcal{K}_E$. Let L be the smallest linear subspace of E containing X. Let $\mathcal{L} = \{ L_\alpha \}$ be the system of all geometric subpolyhedra of L containing and directed downward by the inclusion. Let

$$\mathcal{H}^n (X) = \varinjlim_{\alpha} \{ \mathcal{H}^n (L_\alpha) ; i_{\alpha\beta} \},$$

where $i_{\alpha\beta} : L_\alpha \to L_\beta$ denotes the inclusion map. If Y is another object in $\mathcal{K}_E$ and $f : X \to Y$ is a continuous map, let M denote the smallest linear subspace of E containing Y. Let $\mathfrak{M} = \{ M_{\alpha'} \}$ be the directed system of all geometric subpolyhedra of M containing Y. Extend f to a continuous map $\tilde{f} : L \to M$. For $M_{\alpha'} \in \mathfrak{M}$ we can find $\alpha = \varphi\alpha'$, $L_\alpha \in \mathcal{L}$ such that $\tilde{f} (L_\alpha) \subset M_{\alpha'}$. Let $f_{\alpha'} = \tilde{f}|_{L_\alpha} : L_\alpha \to M_{\alpha'}$. Then define

$$f^* = \mathcal{H}^n (f) = \varinjlim_{\alpha'} \{ f^*_{\alpha'} \} : \mathcal{H}^n (Y) \to \mathcal{H}^n (X).$$

Similarly, by the straightforward passage to the limit, one can define the relative groups and the homomorphism δ. In this way we obtain an extension of $\mathcal{H}^*$ over $\mathcal{K}_E$. This extended cohomology theory (which we still denote by $\mathcal{H}^*$) satisfies the strong excision axiom. Moreover, it is continuous, i. e., the following condition is satisfied :

(C) (*Continuity*) : *If* (X_α, A_α) *is a system of objects of* $\mathcal{K}_E$ *directed downward by the inclusion and* $X = \bigcap X_\alpha$, $A = \bigcap A_\alpha$ *then the inclusion maps* $i_\alpha : (X, A) \to (X_\alpha, A_\alpha)$ *induce an isomorphism*

$$\mathcal{H}^n(X, A) \approx \varinjlim_{\alpha} \{ \mathcal{H}^n(X_\alpha, A_\alpha); i^*_{\alpha\beta} \},$$

where $i_{\alpha\beta} : (X_\alpha, A_\alpha) \to (X_\beta, A_\beta)$ *denotes the corresponding inclusion.*

Let us consider now various theories on $\mathfrak{L}$ and their extensions over $\mathcal{K}_E$. Again by the straightforward passage to the limit, one shows that any natural transformation between two theories on $\mathfrak{L}$ extends to the natural transformation between the corresponding extensions on $\mathcal{K}_E$.

Remark. — It should be noted that if we replace direct limits by inverse limits and proceed similarly with a homology theory on $\mathcal{K}_E$ then the resulting homology theory on $\mathcal{K}_E$ satisfies the strong excision axiom and is continuous [i. e., satisfies the condition analogous to (C)] but, in general, fails to satisfy the exactness axiom.

6. Spectra. — Various homology and cohomology theories can be treated in a unified manner with the aid of spectra.

A *spectrum* **A** is a sequence $\{ A_n \}$ of objects of $\mathcal{W}_*$ together with a sequence of maps $\alpha_n : \Sigma A_n \to A_{n+1}$ in $\mathcal{W}_*$. If $\mathbf{A} = \{ A_n, \alpha_n \}$, $\mathbf{B} = \{ B_n, \beta_n \}$ are spectra, a *map* $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ is a sequence of maps $f_n : A_n \to B_n$ in $\mathcal{W}_*$ such that the diagrams :

$$\begin{array}{ccc} \Sigma A_n & \xrightarrow{\alpha_n} & A_{n+1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ \Sigma B_n & \xrightarrow{\beta_n} & B_{n+1} \end{array}$$

are homotopy commutative. Two such maps **f** and **g** are *homotopic* if and only if, for each n, f_n is homotopic to g_n. Clearly, the spectra form an h-category.

The simplest example is provided by the spectrum of *spheres* $\mathbf{S} = \{S^n, \sigma^n\}$ in which $\sigma_n : \Sigma S^n \to S^{n+1}$ is the natural identification. Another important example is the Eilenberg-Mac Lane spectrum $\mathbf{K}(\Pi)$ defined for an abelian group Π.

Suppose that $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are spectra. A *pairing* $\mathbf{f} : (\mathbf{A}, \mathbf{B}) \to \mathbf{C}$ is a double sequence of maps in $\mathcal{W}_*$

$$f_{p,q} : A_p \wedge B_q \to C_{p+q}$$

satisfying suitable commutativity conditions (*see* [GWW], p. 254). Let A be a spectrum. Identifying $\Sigma^p A_q$ with $S^p \wedge A_q$ we obtain the natural pairing $(\mathbf{S}, \mathbf{A}) \to \mathbf{A}$, where $f_{p,q}: S^p \wedge A_q \to A_{p+q}$ is the composite map

$$\Sigma^p A_q \xrightarrow{\Sigma^{p-1}\alpha_q} \Sigma^{p-1} A_{q+1} \xrightarrow{\Sigma^{p-2}\alpha_{q+1}} \dots \longrightarrow \Sigma A_{p+q-1} \xrightarrow{\sigma_{p+q-1}} A_{p+q}.$$

In what follows we shall consider homology and cohomology theories on various categories with coefficients in a spectrum. First, we outline the construction of such a theory on the category $\mathfrak{T}_*$.

We begin by defining the reduced homology functor $\tilde{\mathcal{H}}_n$ on $\mathcal{W}_*$ with coefficients in a spectrum A. Let $X \in \mathfrak{T}_*$. In the following we identify S^1 with $J/\{-1, +1\}$; if $(s, x) \in J \times X$, we let $s \wedge x$ denote the corresponding point in the factor-space $\Sigma X = S^1 \wedge X$.

Let $\varphi : S^{n+k} \to A_k \wedge X$ be a representative of an element γ of the homotopy group $\pi_{n+k}(A_k \wedge X)$. Then the composition

$$S^{n+k+1} = \Sigma S^{n+k} \xrightarrow{\Sigma\varphi} \Sigma A_k \wedge X \xrightarrow{\alpha_k \wedge 1_X} A^{k+1} \wedge X$$

represents an element $\lambda_k(\gamma) \in \pi_{n+k+1}(A_{k+1} \wedge X)$. The assignment $\gamma \to \lambda_k(\gamma)$ defines a homomorphism

$$\lambda_k : \pi_{n+k}(A_k \wedge X) \to \pi_{n+k+1}(A_{k+1} \wedge X).$$

We let

(1) $$\tilde{\mathcal{H}}_n(X; \mathbf{A}) = \varinjlim_k \{\pi_{n+k}(A_k \wedge X); \lambda_k\}.$$

If $f : X \to Y$ is a map in $\mathfrak{T}_*$ then the maps $1_{A_k} \wedge f : A_k \wedge X \to A_k \wedge Y$ induce

(2) $$\tilde{\mathcal{H}}_n(f) : \tilde{\mathcal{H}}_n(X; \mathbf{A}) \to \tilde{\mathcal{H}}_n(Y; \mathbf{A}).$$

Next, the definition of σ_n. We let

$$S_* : \pi_{n+k}(A_k \wedge X) \to \pi_{n+k+1}(S^1 \wedge A_k \wedge X)$$

be given by the assignment

$$\begin{bmatrix} S^{n+k} \\ \big\downarrow f \\ A_k \wedge X \end{bmatrix} \longrightarrow \begin{bmatrix} S^{n+k+1} \\ \big\downarrow \Sigma f \\ S^1 \wedge A_k \wedge X \end{bmatrix}$$

and

$$\omega_* : \pi_{n+k+1} (S^1 \wedge A_k \wedge X) \xrightarrow{\approx} \pi_{n+k+1} (A_k \wedge \Sigma X)$$

be an isomorphism induced by the map

$$\omega : S^1 \wedge A_k \wedge X \to A_k \wedge S^1 \wedge X = A_k \wedge \Sigma X$$

given by $s \wedge a \wedge x \to a \wedge s \wedge x$.

We define

$$\sigma_{n,k} : \pi_{n+k} (A_k \wedge X) \to \pi_{n+k+1} (A_k \wedge \Sigma X)$$

by $\sigma_{n,k} = (-1)^k \omega_* \circ S_*$ and observe the commutativity in the following diagramm :

$$\begin{array}{ccc} \pi_{n+k}(A_k \wedge X) & \xrightarrow{\lambda_k} & \pi_{n+k+1}(A_{k+1} \wedge X) \\ \big\downarrow \sigma_{n,k} & & \big\downarrow \sigma_{n,k+1} \\ \pi_{n+k+1}(A_k \wedge \Sigma X) & \xrightarrow{\lambda_{k+1}} & \pi_{n+k+2}(A_{k+1} \wedge \Sigma X) \end{array}$$

Since the $\sigma_{n,k}$ are isomorphisms for all sufficiently large k we obtain, by passage to the limit with k, a natural isomorphism

(3) $$\sigma_n (X) = \varinjlim_k \{ \sigma_{n,k} \} : \tilde{\mathcal{H}}_n (X) \xrightarrow{\approx} \tilde{\mathcal{H}}_{n+1} (\Sigma X).$$

It turns out that $\tilde{\mathcal{H}}_* (\quad ; \mathbf{A}) = \{ \tilde{\mathcal{H}}_n, \sigma_n \}$ defined by (1), (2)and (3) is a homology theory on $\mathfrak{T}_*$.

The reduced cohomology with coefficients in $\mathbf{A}$ is defined as follows. Take $\varphi : \Sigma^k X \to A_{n+k}$ representing an element $\delta \in \pi (\Sigma^k X, A_{n+k})$ [2]. Then the composition

$$\Sigma^{k+1} X \xrightarrow{\Sigma\varphi} \Sigma A_{n+k} \xrightarrow{\alpha_{n+k}} A_{n+k+1}$$

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[2] $\pi (\Sigma^k X, A_{n+k})$ is an abelian group for $k \geq 2$.

represents an element $\mu_k(\delta) \in \pi(\Sigma^{k+1} X, A_{n+k+1})$. The assignment, $\delta \to \mu_k(\delta)$ defines a homomorphism

$$\mu_k : \pi(\Sigma^k X, A_{n+k}) \to \pi(\Sigma^{k+1} X, A_{n+k+1}),$$

and we let

(1′) $$\tilde{\mathcal{H}}_n(X; \mathbf{A}) = \varinjlim_k \{ \pi(\Sigma^k X, A_{n+k}); \mu_k \}.$$

if $f : X \to Y$ is in $\mathfrak{T}_*$ the homomorphisms

$$(\Sigma^k f)^* : \pi(\Sigma^k Y, A_{n+k}) \to \pi(\Sigma^k X, A_{n+k})$$

give rise to

(2′) $$\tilde{\mathcal{H}}^n(f) : \tilde{\mathcal{H}}^n(Y; \mathbf{A}) \to \tilde{\mathcal{H}}^n(X; \mathbf{A}).$$

To define σ^n, let

$$\sigma_k : \pi(\Sigma^{k+1} X, A_{n+k+1}) \to \pi(S^1 \wedge \Sigma^k X, A_{(n+1)+k})$$

be the identity. Putting

(3′) $$\sigma^n(X) = \varinjlim_k \{ \sigma_k \}$$

we obtain the suspension isomorphism

$$\sigma^n(X) : \tilde{\mathcal{H}}^n(X) \to \tilde{\mathcal{H}}^{n+1}(\Sigma X).$$

It can be proved that $\tilde{\mathcal{H}}^*(\ ;\mathbf{A}) = \{ \tilde{\mathcal{H}}^n, \sigma^n \}$ so defined is a cohomology theory on $\mathfrak{T}_*$.

Now, combining the preceeding remarks with those in Section 4, we get for a spectrum $\mathbf{A}$ the homology $\mathcal{H}_*(\ ;\mathbf{A})$ and the cohomology $\mathcal{H}^*(\ ;\mathbf{A})$ on the category $\mathfrak{T}^2$ with coefficients in $\mathbf{A}$ (3). It is of importance that both $\mathcal{H}_*(\ ;\mathbf{A})$ and $\mathcal{H}^*(\ ;\mathbf{A})$ are also functors of the second variable; thus, given a map $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ of spectra, we have natural transformations

$$f_* : \mathcal{H}_*(\ ;\mathbf{A}) \to \mathcal{H}_*(\ ;\mathbf{B}),$$
$$f^* : \mathcal{H}^*(\ ;\mathbf{A}) \to \mathcal{H}^*(\ ;\mathbf{B})$$

between corresponding theories.

We note that if $\mathbf{A} = \mathbf{K}(\Pi)$, then the corresponding homology and cohomology theories are naturally isomorphic to the ordinary singular homology and cohomology theories with coefficient group Π. The homology

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(3) It can be shown [2] that, under a countability restriction on the coefficients, *any* homology (resp. cohomology) theory on $\mathfrak{T}$ is of the form $\mathcal{H}_*(\ ;\mathbf{A})$ [resp. $\mathcal{H}^*(\ ;\mathbf{A})$] for some spectrum $\mathbf{A}$.

and cohomology theory with coefficients in S are isomorphic to the stable homotopy and cohomotopy theory, respectively.

A part of the preceeding discussion is now summarized in the following.

6.1. Theorem. — *To every spectrum* $\mathbf{A}$ *corresponds a theory* $\mathcal{H}^*(\ \ ;\mathbf{A})$ *on* $\mathcal{K}_E$ *called the cohomology theory with coefficients in* $\mathbf{A}$. *Every such theory is continuous and satisfies the strong excision axiom. Moreover, the assignment* $\mathbf{A} \to \mathcal{H}^*(\ \ ;\mathbf{A})$ *is natural with respect to maps of spectra.*

7. Homology theory $\mathcal{H}_*(\ \ ;\mathbf{A})$ on $\mathcal{C}^2$. — In the treatment of duality in the infinite dimensional case we shall need our homology groups to be defined for open subsets of a normed space. The homology with coefficients in a spectrum A on $\mathfrak{T}^2$ described in Section 6 would not be anymore sufficient for our purposes. We must turn therefore to the larger category $\mathcal{C}$ and define homology theory $\mathcal{H}_*(\ \ ;\mathbf{A})$ on $\mathcal{C}^2$ and the corresponding reduced homology $\tilde{\mathcal{H}}_*(\ \ ;\mathbf{A})$ on $\mathcal{C}^2_*$.

Let $\mathbf{A} = \{A_k, \alpha_k\}$ be a spectrum and let (X, Y) be a pair in $\mathcal{C}^2_*$. Let $D^n = \{x \in R^n; \|x\| \leqq 1\}$. We will identify $(\Sigma D^n, \Sigma S^{n-1})$ with (D^{n+1}, S^n). Then, repeating considerations of the preceding section, we obtain homomorphisms

$$\lambda_k : \pi_{n+k}(A_k \wedge X, A_k \wedge Y) \to \pi_{n+k+1}(A_{k+1} \wedge X, A_{k+1} \wedge Y),$$

$$\sigma_{n,k} : \pi_{n+k}(A_k \wedge X, A_k \wedge Y) \to \pi_{n+k+1}(A_k \wedge \Sigma X, A_k \wedge \Sigma Y);$$

with the following commutative diagram :

$$\begin{array}{ccc}
\pi_{n+k}(A_k \wedge X, A_k \wedge Y) & \xrightarrow{\lambda_k} & \pi_{n+k+1}(A_{k+1} \wedge X, A_{k+1} \wedge Y) \\
\downarrow{\scriptstyle \sigma_{n,k}} & & \downarrow{\scriptstyle \sigma_{n,k+1}} \\
\pi_{n+k+1}(A_k \wedge \Sigma X, A_k \wedge \Sigma Y) & \xrightarrow{\lambda_k} & \pi_{n+k+2}(A_{k+1} \wedge \Sigma X, A_{k+1} \wedge \Sigma Y)
\end{array}$$

Let

$$\tilde{\mathcal{H}}_n(X, Y; \mathbf{A}) = \underset{\overrightarrow{k}}{\mathrm{Lim}}\, \{\pi_{n+k}(A_k \wedge X, A_k \wedge Y); \lambda_k\}.$$

We let

$$\tilde{\partial}_{n,k} = (-1)^{k+1}\, \partial_{n,k},$$

where $\partial_{n,k}$ is the boundary homomorphism of $(A_k \wedge X, A_k \wedge Y)$

$$\partial_{n,k} : \pi_{n+k}(A_k \wedge X, A_k \wedge Y) \to \pi_{n-1+k}(A_k \wedge Y).$$

It is clear from the definition of $\tilde{\partial}_{n,k}$ that the following diagram commutes :

$$\begin{array}{ccc}
\pi_{n+k}(A_k \wedge X, A_k \wedge Y) & \xrightarrow{\tilde{\partial}_{n,k}} & \pi_{n-1+k}(A_k \wedge Y) \\
\downarrow{\scriptstyle \lambda_k} & & \downarrow{\scriptstyle \lambda_k} \\
\pi_{n+k+1}(A_{k+1} \wedge X, A_{k+1} \wedge Y) & \xrightarrow{\tilde{\partial}_{n,k+1}} & \pi_{n+k}(A_{k+1} \wedge Y)
\end{array}$$

We let

$$\tilde{\partial}_n = \underset{\overrightarrow{k}}{\mathrm{Lim}} \{ \tilde{\partial}_{n,k} \} : \quad \tilde{\mathcal{H}}_n (X, Y) \to \tilde{\mathcal{H}}_{n-1} (Y).$$

It is clear from the above definitions that $\tilde{\mathcal{H}}_* = \{ \tilde{\mathcal{H}}_n, \tilde{\partial}_n \}$ satisfies the exactness axiom. It remains to prove that $\tilde{\mathcal{H}}_*$ satisfies the excision axiom. Before doing this we will make several comments and prove some lemmas.

Let Y be an object of $\mathfrak{S}_*$ and

$$j : \; SY \to (SY, C_- Y), \qquad e : \; (C_+ Y, Y) \to (SY, C_- Y)$$

denote corresponding inclusions. Then we have a commutative diagram :

$$\begin{array}{ccccc}
\pi_{n+k}(A_k \wedge C_+ Y, A_k \wedge Y) & & \xrightarrow{\tilde{\partial}_{n,k}} & & \pi_{n-1+k}(A_k \wedge Y) \\
\downarrow{\scriptstyle (\mathrm{id} \wedge e)_*} & & & & \searrow{\scriptstyle \sigma_{n-1,k}} \\
\pi_{n+k}(A_k \wedge SY, A_k \wedge C_- Y) & \xleftarrow{(\mathrm{id} \wedge j)_*} & \pi_{n+k}(A_k \wedge SY) & \xrightarrow{(\mathrm{id} \wedge p)_*} & \pi_{n+k}(A_k \wedge \Sigma Y)
\end{array}$$

Thus, passing to the direct limit, we obtain a commutative diagram

$$\begin{array}{ccccc}
\tilde{\mathcal{H}}_n(C_+ Y, Y; A) & & \xrightarrow{\tilde{\partial}_n} & & \tilde{\mathcal{H}}_{n-1}(Y; A) \\
\downarrow{\scriptstyle e_*} & & & & \searrow{\scriptstyle \sigma_{n-1}} \\
\tilde{\mathcal{H}}_n(SY, C_- Y; A) & \xleftarrow{j_*} & \tilde{\mathcal{H}}_n(SY; A) & \xrightarrow{p_*} & \tilde{\mathcal{H}}_n(\Sigma Y; A)
\end{array}$$

If $Y \in \mathfrak{T}_*$ then e_*, j_* and p_* are isomorphisms. This, in view of the definition of $\tilde{\partial}_n$, proves that the homology theory defined above coincides on $\mathfrak{T}^2_*$ with that defined in the preceding section.

Let $X \in \mathfrak{S}$ and $A = (A, a_0) \in \mathfrak{S}_*$. Then

$$A \wedge X^+ = A \times X^+ / A \times \{ \star \} \cup \{ a_0 \} \times X^+.$$

Since $A \times X^+ = A \times X \cup A \times \{\star\}$ we have a natural identification

$$A \wedge X^+ = A \wedge X/\{a_0\} \times X.$$

We let

$$A \wedge X = A \times X/\{a_0\} \times X.$$

Then $\wedge$ is an h-functor from $\mathfrak{T}_* \times \mathfrak{T}$ into $\mathfrak{T}_*$.

If $(X, Y) \in \mathfrak{T}^2$ then, identifying $(A_k \wedge X^+, A_k \wedge Y)$ with $(A_k \wedge X, A_k \wedge Y)$, we have

$$\mathcal{H}_n(X, Y; \mathbf{A}) = \varinjlim_k \pi_{n+k}(A_k \wedge X, A_k \wedge Y).$$

7.1. Lemma. — *Let* $(A, a_0) \in \mathcal{W}_*$. *Then there exist a neighbourhood* U *of* a_0 *and a continuous map* $d : A \times I \to A$ *(a deformation of* A*) such that :*

(i) $d(a, 0) = a$ *for all* $a \in A$;

(ii) $d(a_0, t) = a_0$ *for all* $t \in I$,

(iii) $d(a, 1) = a_0$ *for all* $a \in U$.

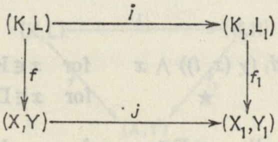
Proof. — Since any CW-complex is locally contractible ([18], p. 420) we can find an open neighbourhood V of a_0 which is contractible in A. Thus there exists a continuous map $h : V \times I \to A$ such that :

$$h(x, 0) = x \quad \text{for all } x \in V,$$
$$h(a_0, t) = a_0 \quad \text{for all } t \in I,$$
$$h(x, 1) = a_0 \quad \text{for all } x \in V.$$

Choose a neighbourhood U of a_0 such that $\overline{U} \subset V$ and let $\theta : A \to I$ be a continuous function equal 1 on $\overline{U}$ and 0 outside V. Then the required deformation is defined by

$$d(a, t) = \begin{cases} a & \text{for } a \in A - V, \\ h(a, \theta(a)\, t) & \text{for } a \in V. \end{cases}$$

7.2. Lemma. — *Let* $j : (X, Y) \to (X_1, Y_1)$ *be an inclusion in* $\mathfrak{T}^2$ *and let* $\alpha \in \mathcal{H}_n(X, Y; \mathbf{A})$. *Suppose* $j_*(\alpha) = 0$. *Then there exist an inclusion* $i : (K, L) \to (K_1, L_1)$ in $\mathfrak{L}_2$, *a commutative diagram of maps in* $\mathfrak{T}^2$:



and $\beta \in \mathcal{H}_n(K, L; \mathbf{A})$ such that $f_*(\beta) = \alpha$, $i_*(\beta) = 0$.

Proof. — Let

$$\varphi: \quad (D^{n+k}, S^{n+k-1}) \to (A_k \wedge X, A_k \wedge Y)$$

represent α. Set $m = n + k$. Since $j_*(\alpha) = 0$ there exists a map (a homotopy in $\mathfrak{C}^2_*$)

$$\chi: \quad (D^m \times I, S^{m-1} \times I) \to (A_k \wedge X_1, A_k \wedge Y_1)$$

such that for all $x \in D^m$, $\chi(x, 0) = \varphi(x)$, $\chi(x, 1) = \star$. Applying Lemma 7.1 with $A = A_k$ we obtain a neighbourhood U of the base-point of A_k and a deformation $d: A_k \times I \to A_k$ satisfying the conditions (i)-(iii) of 7.1. Let V be a neighbourhood of the base-point of A_k such that $\overline{V} \subset U$. There exists a subpolyhedron $K_1 \subset D^m \times I$ such that

$$\chi(K_1) \subset (A_k - V) \times X_1, \qquad \chi^{-1}((A_k - U) \times X_1) \subset K_1.$$

We let

$$L_1 = K_1 \cap (S^{m-1} \times I),$$
$$K = K_1 \cap (D^m \times \{0\}),$$
$$L = K \cap L_1.$$

For $x \in K_1$ we have $\varphi(x) = (\psi_1(x), f_1(x))$, where

$$\psi_1: \quad K_1 \to A_k - V, \qquad f_1: \quad (K_1, L_1) \to (X_1, Y_1)$$

are continuous mappings. Let $f: (K, L) \to (X, Y)$ be given $f(x) = f_1(x)$ for $x \in K$ and define a map

$$\psi: \quad (D^m, S^{m-1}) \to (A_k \wedge K, A_k \wedge L)$$

by setting

$$\psi(x) = \begin{cases} d_1(\psi_1(x)) \wedge x & \text{for } x \in K, \\ \star & \text{for } x \in D^m - K. \end{cases}$$

Let $\beta \in \mathcal{H}_n(K, L; \mathbf{A})$ denote the element represented by ψ. Define a map

$$\eta: \quad (D^m \times I, S^{m-1} \times I) \to (A_k \wedge K_1, A_k \wedge L_1)$$

by setting

$$\eta(x, t) = \begin{cases} d_1(\chi(x, t)) \wedge x & \text{for } x \in K_1, \\ \star & \text{for } x \in D^m \times I - K_1. \end{cases}$$

Since $\chi(x, 1) = \star$ for all $x \in D^m$ we have then $\eta(x, 1) = \star$ for all $x \in D^m$. Hence $\eta_0 = (\mathrm{id} \wedge i) \circ \psi$ represents $i_*(\beta) = 0$.

Finally, consider a homotopy

$$h : (D^m \times I, S^{m-1} \times I) \to (A_k \wedge X, A_k \wedge Y)$$

defined by

$$h(x, t) = \tilde{d}(\varphi(x), t),$$

where $\tilde{d}(a \wedge x, t) = d(a, t) \wedge x$. Since $h_0 = \varphi$, $h_1 = (\mathrm{id} \wedge f) \circ \psi$, $f_*(\beta) = \alpha$ and the proof is completed.

From Lemma 7.2 we obtain at once

7.3 Corollary. — *Let* $(X, Y) \in \mathcal{C}^2$ *and let* $\{(X_\alpha, Y_\alpha)\}$ *be the family of all compact pairs* $(X_\alpha, Y_\alpha) \subset (X, Y)$ *directed inward by inclusion. Then the inclusion maps* $i_\alpha : (X_\alpha, Y_\alpha) \to (X, Y)$ *induce an isomorphism*

$$\varinjlim_{\alpha} \{\mathcal{H}_n(X_\alpha, Y_\alpha; \mathbf{A})\} \approx \mathcal{H}_n(X, Y; \mathbf{A}).$$

7.4. Theorem. — $\mathcal{H}_*(\ \ ,\ \ ;\mathbf{A})$ $\left[resp., \tilde{\mathcal{H}}_*(\ \ ,\ \ ;\mathbf{A})\right]$ *is a homology theory on* $\mathcal{C}^2$ *(resp., on* $\mathcal{C}^2_*$*).*

Proof. — In view of our preceding considerations it suffices to prove that $\mathcal{H}_*$ satisfies the excision axiom. Let (X, Y) be a pair in $\mathcal{C}^2$, let U be an open subset of Y such that $\overline{U} \subset V$, where V denotes the interior of Y. Let $X_0 = X - U$, $Y_0 = Y - U$ and let

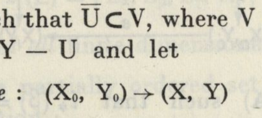
$$e : (X_0, Y_0) \to (X, Y)$$

be the excision map.

To prove that $e_* : \mathcal{H}_n(X_0, Y_0; \mathbf{A}) \to \mathcal{H}_n(X, Y; \mathbf{A})$ is onto, let α be an arbitrary element of $\mathcal{H}_n(X, Y; \mathbf{A})$. By 7.2 there exist a pair $(K, L) \in \mathfrak{T}^2$, an element $\beta \in \mathcal{H}_n(K, L; \mathbf{A})$ and a continuous map $f : (K, L) \to (X, Y)$ such that $f_*(\beta) = \alpha$. There exist subpolyhedra $M, N \subset K$ such that $f(M) \subset X - U$.

$$f^{-1}(X - V) \subset M, \qquad L \subset N, \qquad f(N) \subset Y, \qquad N \cup M = K.$$

Let $i : (K, L) \to (K, N)$ denote the inclusion. There is a commutative diagram :



where $f_1(x) = f(x)$ for all $x \in K$. Let $\beta_1 = i_*(\beta)$.

Hence $(f_1)_*(\beta_1) = \alpha$. Now consider a commutative diagram :

$$\begin{array}{ccc} (M, N \cap M) & \xrightarrow{e_0} & (K, N) \\ \downarrow g & & \downarrow f_1 \\ (X_0, Y_0) & \xrightarrow{e} & (X, Y) \end{array}$$

where e_0 denotes the inclusion and $g(x) = f(x)$ for $x \in M$. Since e_0 induces an isomorphism of homology groups there exists $\beta_2 \in \mathcal{H}_n(M, N \cap M; \mathbf{A})$ such that $(e_0)_*(\beta_2) = \beta_1$. Hence $e_*(g_*(\beta_2)) = \alpha$. Thus we have proved that e_* is an epimorphism.

To prove that e_* is a monomorphism suppose $e_*(\alpha) = 0$ for some $\alpha \in \mathcal{H}_n(X_0, Y_0; \mathbf{A})$. By 7.2 there exists an inclusion in $\mathfrak{T}^2$,

$$i: (K_0, L_0) \to (K, L),$$

a commutative diagram of continuous maps :

$$\begin{array}{ccc} (K_0, L_0) & \xrightarrow{i} & (K, L) \\ \downarrow f_0 & & \downarrow f \\ (X_0, Y_0) & \xrightarrow{e} & (X, Y) \end{array}$$

and $\beta \in \mathcal{H}_n(K_0, L_0; \mathbf{A})$ such that $i_*(\beta) = 0$, $(f_0)_*(\beta) = \alpha$. Using barycentric subdivision we may assume that K has a triangulation such that any simplex of K is mapped by f into one of open subsets $X - U$, V. Then we may assume, without loss of generality, that L consists of all simplexes which are mapped by f into V. Let M be a subcomplex of K consisting of all simplexes mapped by f into $X - U$. Thus $M \cup L = K$ and $(K_0, L_0) \subset (M, M \cap L)$. Consider the following commutative diagram in which j, k denote corresponding inclusions and g is defined by $g(x) = f(x)$, $x \in M$.

$$\begin{array}{ccccc} (K_0, L_0) & & \xrightarrow{i} & & (K, L) \\ & \searrow^{j} & & \nearrow^{k} & \\ \downarrow f_0 & & (M, M \cap L) & & \downarrow f \\ & \swarrow^{g} & & & \\ (X_0, Y_0) & & \xrightarrow{e} & & (X, Y) \end{array}$$

We have $(k \circ j)_*(\beta) = k_*(j_*(\beta)) = i_*(\beta) = 0$. Since k is an excision $j_*(\beta) = 0$. Thus $\alpha = (f_0)_*(\beta) = g_*(j_*(\beta)) = 0$. Therefore e_* is a monomorphism and the proof of the theorem is completed.

CHAPTER II

Basic categories

This chapter is devoted to a summary of basic definitions and results related to the concept of a compact vector field which will enter in our subsequent considerations. We introduce here two categories of primary interest (the category of compact vector fields and the Leray-Schauder category) and two geometrical constructions of further importance (the generalized suspension and the cone functors).

In the last section, with the aid of the generalized suspension, some examples of geometrical interest are given. The cone construction will be used later on (Chapter V) in definition of the relative cohomology functors.

1. The directed set $\mathcal{L}$ (E). — Let E be an infinite dimensional normed space. By

$$\mathcal{L} = \mathcal{L}(\mathrm{E}) = \{ \mathrm{L}_\alpha, \mathrm{L}_\beta, \mathrm{L}_\gamma, \ldots \}$$

we shall denote the family of *all* finite dimensional linear subspaces of E.

We shall consider $\mathcal{L}$ as a partially ordered set with an order relation $\alpha \leqq \beta$ defined by the condition :

$$\mathrm{L}_\alpha \leqq \mathrm{L}_\beta \quad \text{if and only if } \mathrm{L}_\alpha \subset \mathrm{L}_\beta.$$

Evidently, the relation $\leqq$ converts the family $\mathcal{L}$ into a *directed set* ($\mathcal{L}$, $\leqq$) which will be denoted simply by $\mathcal{L}$.

For notational convenience we establish one-to-one correspondence $\alpha \leftrightarrow \mathcal{L}_\alpha$ between the symbols $\alpha, \beta, \gamma, \ldots$ and $\mathrm{L}_\alpha, \mathrm{L}_\beta, \mathrm{L}_\gamma, \ldots$ and in the formulas to occur we replace frequently one sort of symbols by another. Thus, for example, we shall write $\alpha \leqq \beta$ instead of $\mathrm{L}_\alpha \leqq \mathrm{L}_\beta$.

Given an element α of $\mathcal{L}$ we let $d(\alpha)$ denote the dimension of the linear space L_α. A relation $\alpha \leqq \beta$ in $\mathcal{L}$ will be called *elementary* provided $d(\alpha) = d(\beta) - 1$. Given an arbitrary relation $\alpha \leqq \beta$ in $\mathcal{L}$, by a *chain joining* α *and* β we shall understand a finite sequence

$$\alpha = \alpha_0 < \alpha_1 < \ldots < \alpha_k = \beta$$

of elements in $\mathcal{L}$ such that $\alpha_i < \alpha_{i+1}$ is elementary for each $i = 0, 1, \ldots, k-1$.

If X is a subset of E and $\alpha \in \mathcal{L}$, we let $X_\alpha = X \cap L_\alpha$. Evidently, the subset $\mathcal{L}_X$ of $\mathcal{L}$ defined by

$$\mathcal{L}_X = \{ \alpha \in \mathcal{L}_\alpha ; X_\alpha \text{ is non-empty} \}$$

is cofinal in $\mathcal{L}$.

If X and Y are two subsets of E and $f : X \to Y$ is a mapping such that $f(X_\alpha) \subset Y_\alpha$ then by $f_\alpha : X_\alpha \to Y_\alpha$ we denote the contraction of f to the pair (X_α, Y_α).

2. Compact mappings.

2.1. Definition. — A continuous mapping $F : X \to Y$ between topological spaces X and Y is called *compact* provided it maps X into a compact subset of Y. Let α be an element of the directed set $\mathcal{L}$ and $F : X \to E$; we say that F is an α-mapping provided F is compact and $F(X) \subset L_\alpha$. If $F : X \to E$ is an α-mapping for some $\alpha \in \mathcal{L}$ it is called a *finite dimensional mapping*. Compact mappings will be denoted by the capital letters F, G, H.

The following two facts are of basic importance :

2.2 Lemma (*Approximation Lemma*). — *Let* U *be open in* E *and* $F : X \to U$ *be a compact mapping. Then for each* $\varepsilon > 0$ *there exists a finite polyhedron* $P_\varepsilon \subset U$ *and a mapping* $F_\varepsilon : X \to U$ *such that :*

(i) $F_\varepsilon(X) \subset P_\varepsilon$;

(ii) $\| F(x) - F_\varepsilon(x) \| < \varepsilon$ *for each* $x \in X$;

(iii) F *and* F_ε *are homotopic,*

Proof. — Given $\varepsilon > 0$ (which we may assume to be sufficiently small) there exists a finite number of open balls $V(y_i, \varepsilon)$ $(i = 1, 2, \ldots, k)$ such that $F(X) \subset \bigcup V(y_i, \varepsilon) \subset U$.

Putting for each $x \in X$,

$$F_\varepsilon(x) = \frac{\sum_{i=1}^{k} \lambda_i(x) y_i}{\sum_{i=1}^{k} \lambda_i(x)},$$

where

$$\lambda_i(x) = \max \{ 0, \varepsilon - \| F(x) - y_i \| \},$$

we obtain the mapping F_ε satisfying (ii) and (iii); clearly the values of F_ε are in a finite polyhydron $P_\varepsilon \subset U$ with vertices $y_1, y_2, \ldots, y_k$.

2.3. Lemma (*On extension of Compact Mappings*). — *Let* A *be closed in a metric space* X *and* $F : A \to E$ *be a compact mapping. If either* : (i) E *is complete or* (ii) F *is an* α*-mapping, then there is a compact mapping* $\overline{F} : X \to E$ *being an extension of* F *over* X *and such that* $\overline{F}(X) \subset \text{conv}(F(A))$.

Proof. — Since the convex hull of a relatively compact set in a complete E is also relatively compact [14], our assertion follows at once from the Dugundji Extension Theorem [3].

Remark. — If A is a retract of X (for example : X = K the unit ball in E and A = S its boundary, *cf.* J. Dugundji [3]) then, clearly, the assumption of the completeness of E is not needed. In general, however, it is not known to the authors whether a compact mapping $F : A \to E$ admits a compact extension over X, without assuming E to be complete.

3. Compact vector fields. — *Notation.* — Given two subsets X and Y of E and a continuous mapping $f : X \to Y$ we denote by the same but capital letter the mapping $F : X \to E$ defined by

$$F(x) = x - f(x), \quad x \in X.$$

3.1. Definition. — Let X and Y be arbitrary subsets of E. A mapping $f : X \to Y$ is said to be a *compact vector field* (or simply a compact field) provided the map $F : X \to E$ is compact.

The set of all compact vector fields with domain X and range Y will be denoted by $\mathfrak{C}(X, Y)$ and its elements will be denoted by the small letters $f, g, h, \ldots$.

3.2. Proposition. — *Let* $f : X \to E$ *be a compact vector field. Then :* (i) *if* X *is closed* (respectively bounded) *in* E, *then so is the set* $f(X)$; (ii) *if* $C \subset E$ *is relatively compact, then so is* $f^{-1}(C)$.

Proof. — To prove (i), let $\{y_n\}$ be a sequence of points in $f(X)$, i. e. $y_n = x_n - F(x_n)$, $x_n \in X$ and suppose that $\lim_{n\to\infty} y_n = y_0$. Since F is compact we may assume, without loss of generality, that $F(x_n)$ converges to a point y. Then $\lim_{n\to\infty} x_n = y_0 + y$ and hence, by continuity of f, $\lim_{n\to\infty} f(x_n) = f(y_0 + y)$. Since X is closed, we have $y_0 = f(y_0 + y) \in f(X)$ which completes the proof of (i). The proof of the remaining assertions is similar.

3.3. Proposition. — *If* $f : X \to Y$ *is a one-to-one compact vector field of a closed set* X *onto* Y, *then* f *is bicontinuous and* $f^{-1} : Y \to X$ *is a compact field.*

Proof. — This follows from the preceding proposition.

Some other simple but important properties are summarized in the following:

3.5. Proposition. — *The class of compact vector fields has the following properties:*

(i) *If f and g are compact fields* (respectively α-fields), *then so is their composition gf;*

(ii) *If f is a compact field* (respectively an α-field), *then so is every contraction, and in particular every restriction, of f;*

(iii) *The inclusions $i : A \to X$ and in particular the identities $1_X : X \to X$ are α-fields for every $\alpha \in \mathcal{L}$;*

(iv) *If $f : X \to Y$ is a continuous mapping between two subsets of* E *such that* X *is compact, then f is a compact field; if, in addition,* X *and* Y *are contained in L_α, then f is an α-field.*

It follows from Proposition 3.5 that subsets of E as objects and compact vector fields as maps form a category. This category will be denoted by $\mathfrak{C}(E)$ and called the *category of compact vector fields* in E. For each $\alpha \in \mathcal{L}$ we have a dense subcategory $\mathfrak{C}_\alpha(E)$ of $\mathfrak{C}(E)$ whose maps are α-fields between the subsets of E.

Clearly, if $\alpha \leqq \beta$ is a relation in $\mathcal{L}$, then

$$\mathfrak{C}_\alpha(E) \subset \mathfrak{C}_\beta(E).$$

Now we define a category

$$\mathfrak{C}_0(E) = \bigcup \mathfrak{C}_\alpha(E)$$

as the union of all categories $\mathfrak{C}_\alpha(E)$ for $\alpha \in \mathcal{L}$. Evidently, $\mathfrak{C}_0(E)$ is dense in $\mathfrak{C}(E)$. In what follows the maps of $\mathfrak{C}_0(E)$ will be called *finite dimensional fields.*

4. Homotopy of compact vector fields. — *Notation.* — Given two subsets X and Y of E and a homotopy $h_t : X \to Y$ $(0 \leqq t \leqq 1)$ we shall denote by $h : X \times I \to Y$ the mapping defined for $(x, t) \in X \times I$ by $h(x, t) = h_t(x)$. By the capital H we shall denote the mapping $H : X \times I \to E$ defined by

$$H(x, t) = x - h_t(x).$$

4.1. Definition. — Let X and Y be two subsets of E. A family of compact vector fields $h_t : X \to Y$ depending on the parameter t $(0 \leqq t \leqq 1)$ is called a *compact homotopy* provided the mapping $H : X \times I \to E$ is compact. Two compact vector fields $f, g : X \to Y$ are said to be *compactly homotopic*, provided there exists a compact homotopy $h_t : X \to Y$ such that $h_0 = f$, $h_1 = g$.

We shall write $f \sim g$ to mean that the fields f and g are compactly homotopic.

The relation $\sim$ is an equivalence relation in each of the sets $\mathfrak{C}(X, Y)$ and it clearly satisfies the compositive property in the definition of an h-category. Consequently, it converts the category of compact vector fields into an h-category $(\mathfrak{C}, \sim)$. When there is no risk of misunderstanding this category will be denoted simply by $\mathfrak{C}$.

Now an elementary proposition concerning compact homotopies which will be frequently used later on.

4.3. PROPOSITION. — *Let* $h_t : X \to E$ *be a compact homotopy. Then* (i) *if* X *is closed* (respectively *bounded*) *in* E *then so is the set* $h(X \times I)$; (ii) *if* $C \subset E$ *is relatively compact, then so is the set* $h^{-1}(C)$.

Proof. — The proof is similar to that of Proposition 3.2.

4.3. PROPOSITION. — *Let* $X \subset E$, U *be an open set in* $E^{\infty-n}$ *and let* $f, g : X \to U$ *be two compact fields. If the inequality*

$$\| f(x) - g(x) \| \leq \operatorname{dist}(f(x), E^{\infty-n} - U)$$

holds for each $x \in X$, *then the fields* f *and* g *are compactly homotopic.*

Proof. — The above inequality implies that for each $x \in X$ the segment $[f(x), g(x)]$ joining $f(x)$ and $g(x)$ in $E^{\infty-n}$ is entirely contained in U, hence the formula

$$h_t(x) = t\,f(x) + (1-t)\,g(x) = x - [t\,F(x) + (1-t)\,G(x)] \qquad (x \in X,\ t \in I)$$

defines a required compact homotopy between f and g.

4.4. DEFINITION. — Let X and Y be two subsets of E and α be an element of the directed set $\mathcal{L}$. A family of α-fields $h_t : X \to Y$ is called an α-homotopy, provided $H : X \times I \to E$ is an α-mapping. Two α-fields $f, g : X \to Y$ are called α-homotopic if there is an α-homotopy $h_t : X \to Y$ such that $h_0 = f$ and $h_1 = g$.

We shall write $f \underset{\alpha}{\sim} g$ to mean that α-fields f and g are α-homotopic.

The relation of α-homotopy is an equivalence relation in $\mathfrak{C}_\alpha(X, Y)$ and therefore decomposes the above set into disjoint α-homotopy classes. If $f \in \mathfrak{C}_\alpha(X, Y)$, we let $[f]_\alpha$ denote the α-homotopy class which contains f The set of these classes will be denoted by $\pi_\alpha(X, Y)$. We note further that the relation $\underset{\alpha}{\sim}$ satisfies the compositive property in the definition of an h-category and consequently converts $\mathfrak{C}_\alpha$ into an h-subcategory $(\mathfrak{C}_{\alpha}, \widetilde{\alpha})$ of $(\mathfrak{C}, \sim)$.

4.5. Proposition. — *Let* $h_t : X \to E$ *be a compact homotopy. Then for every* $\varepsilon > 0$ *there exists an* α*-homotopy* $h'_t : X \to E$ *such that*

$$\| h_t(x) - h'_t(x) \| < \varepsilon \quad \textit{for all} \quad x \in X \quad \textit{and} \quad 0 \leqq t \leqq 1.$$

Proof. — This clearly follows from Lemma 2.2.

5. The extension problem for compact fields. — Let $h_t : X \to Y$ be a compact homotopy (respectively an α-homotopy) and A be a subset of X. We let $h_t \mid A = h'_t$ denote the partial compact homotopy (respectively α-homotopy); in this case, we shall write also $h'_t \subset h_t$ and say that h_t is an extension (respectively an α-extension) of h'_t over X.

Given a pair $(X, A) \subset E$ with A closed in X and a field (respectively an α-field) $f : A \to U$ we may consider the extension problem for f, i. e., the problem of extending f over X in $\mathfrak{C}$ (respectively in $\mathfrak{C}_\alpha$).

The following important lemma asserts that under some hypotheses this problem depends only on the homotopy (respectively α-homotopy) class of a given field f.

5.1. Lemma (*Homotopy Extension Lemma*). — *Let* (X, A) *be a pair in* E, A *be closed in* X *and* U *an open set in* $E^{\infty-n}$. *Let* $h'_t : A \to U$ $(0 \leqq t \leqq 1)$ *be a compact homotopy such that* $h'_0 \subset h_0 \in \mathfrak{C}(X, U)$. *If either :* (i) E *is complete or* (ii) h_t *is an* α*-homotopy, then there exists a compact homotopy* (α*-homotopy*) $h_t : X \to U$ *such that* $h'_t \subset h_t$.

Proof. — Assume first that $n = 0$, i. e., that U is open in E. Let us put $T = (X \times \{0\}) \cup (A \times I)$. By assumption, there is a compact mapping (α-mapping) $H^*_0 : T \to E$ such that

$$H^*_0(x, t) = \begin{cases} H'(x, t) & \text{for} \quad x \in A, \quad 0 \leqq t \leqq 1, \\ H_0(x) & \text{for} \quad x \in X, \quad t = 0 \end{cases}$$

and

$$x - H^*_0(x, t) \in U \quad \text{for all} \quad (x, t) \in T.$$

Since T is closed in $X \times I$, there is, in view of Lemma 2.2, a compact extension $\overline{H}_0 : X \times I \to E$ of H^*_0 over $X \times I$. Putting

$$B = \{ x \in X ; \; x - \overline{H}_0(x, t) \in E - U \text{ for some } t \in I \},$$

we may suppose that the closed set B is not empty. We note further that A and B are evidently disjoint. Now take a real-valued function

$\lambda : X \to I$ such that $\lambda(B) = 0$ and $\lambda(A) = 1$ and put

$$H(x, t) = \overline{H}_0(x, \lambda(x)\, t) \qquad (x \in X,\ t \in I),$$

$$h_t(x) = x - H(x, t) \qquad (x \in X,\ t \in I).$$

It is easily seen that $h_t : X \to U$ is a required compact homotopy. Next assume that $n > 0$. This case reduces to the proved special case with the aid of the linear projection onto $E^{\infty - n}$.

5.2. Corollary. — *Let* (X, A) *be a pair in* E *with* A *closed in* X *and* $f_0, g_0 : A \to U$ *two* α*-homotopic* α*-fields. If there exists an* α*-extension* $f : X \to U$ *of* f_0 *over* X, *then there exists also an* α*-extension* g *of* g_0 *over* X *and such that* f *and* g *are* α*-homotopic. If the space* E *is complete, the above is true for arbitrary compact fields* f_0 *and* g_0.

Remark. — If $X = K$ is the unit ball in E and $A = S$ stands for its boundary then the Homotopy Extension Lemma and its corollary hold without assuming E to be complete (because in this case $T = A \times I \cup X \times \{0\}$ is a retract of $X \times I$). It is not, however, known to the authors whether the above lemma is true for arbitrary-closed pairs without the above additional hypothesis.

6. The generalized suspension and the cone functors. — *Notation.* — Given a linear (closed) subspace M of E we let S_M denote the unit sph re in M. We assume that we are given a direct sum decomposition $E = M \oplus N$ where M and N are complementary linear subspaces of E.

6.1. Definition. — Given a subset X of N we let

$$S_M(X) = \{ z = tx + (1 - t)\, y \in E;\ x \in X,\ y \in S_M,\ 0 \leq t \leq 1 \}$$

be the union of all segments in E joining points x in X with points y in the unit sphere S_M. Given two subsets X and Y of N and a mapping $f : X \to Y$ we let $S_M(f) : S_M(X) \to S_M(Y)$ be the mapping defined for $x \in X$, $y \in S_M$ and $0 \leq t \leq 1$ by

$$S_M(f)(tx + (1 - t)\, y) = t\, f(x) + (1 - t)\, y.$$

We say that $S_M(X)$ and $S_M(f)$ are the M-*suspensions* of X and f respectively.

For a linear subspace N of E denote by $\mathfrak{E}(N)$ the category whose objects are subsets of N and whose maps are continuous transformations between the objects.

Note that for any two composable mappings f and g in $\mathfrak{E}(N)$ we have

$$S_M(g \circ f) = S_M(g) \circ S_M(f).$$

It follows that the assignments $X \mapsto S_M(X)$, $f \mapsto S_M(f)$ define a covariant functor S_M from $\mathfrak{E}(N)$ to $\mathfrak{E}(E)$ and called the generalized M-suspension functor.

6.2. Proposition. — *The* M-*suspension functor* S_M *has the following properties :*

(i) *If* X *is closed* (respectively bounded) *in* N, *then so is* $S_M(X)$ *in* E;

(ii) *If f is a compact field* (respectively an α-field), *then so is* $S_M(f)$;

(iii) *If the fields* (respectively α-fields) *f and g are compactly homotopic* (respectively α-homotopic), *then so are their* M-*suspensions* $S_M(f)$ *and* $S_M(g)$.

Proof. — (i) is evident. To prove (ii), write $F(x) = x - f(x)$ and take a compact set $C \subset N$ such that $F(X) \subset C$. Note that the set C_1 given by

$$C_1 = \{ z \in N; z = tw, 0 \leqq t \leqq 1, w \in C \}$$

is also a compact subset of N. Since for an arbitrary point

$$z = tx + (1-t)y$$

of $S_M(X)$ we have

$$z - S_M f(z) = t(x - f(x)) \in C_1,$$

it follows that $S_M(f)$ is a compact field. The proof of (ii) is completed. The proof of other assertions is similar.

6.3. Proposition. — *Let* N *be of finite dimension and* X *be a compact subset of* N. *Let us put* $U_0 = N - X$ *and* $U = E - S_M(X)$. *Then the inclusion map* $i : U_0 \to U$ *induces an isomorphism*

$$i_* : \pi_n(U_0) \approx \pi_n(U)$$

of the homotopy groups for all $n < \dim N - 1$.

Proof. — By assumption, $N = L_{\alpha_0}$ for some $\alpha_0 \in \mathcal{L}$. Let us put $\mathcal{L}_0 = \{ \alpha \in \mathcal{L}, \alpha \geqq \alpha_0 \}$. Clearly, $\mathcal{L}_0$ is a cofinal subset of $\mathcal{L}$. Now, for any relation $\alpha \leqq \beta$ in $\mathcal{L}_0$, let

$$i_{\alpha\beta} : U_\alpha \to U_\beta \quad \text{and} \quad i_\alpha : U_\alpha \to U$$

denote the inclusions. Consider the corresponding direct system over $\mathcal{L}_0$ of homotopy groups $\{ \pi_n(U_\alpha); (i_{\alpha\beta})_* \}$ and the direct family $(i_\alpha)_*$ of homomorphisms

$$(i_\alpha)_* : \pi_n(U_\alpha) \to \pi_n(U).$$

It follows from Lemma 2.2 that

$$\varinjlim_\alpha \{ (i_\alpha)_* \} : \varinjlim_\alpha \{ \pi_n(U_\alpha); (i_{\alpha\beta})_* \} \to \pi_n(U)$$

is an isomorphism. On the other hand, if $n < \dim N - 1$, then, by finite dimensional argument, it is clear that

$$(i_{\alpha_0\alpha})_* : \pi_n(U_{\alpha_0}) \to \pi_n(U_\alpha)$$

is an isomorphism and our assertion follows.

6.4. Definition. — Let y be a fixed point S_M. We define the *cone functor* $C : \mathfrak{E}(N) \to \mathfrak{E}(E)$ by putting for $X \subset N$

$$C(X) = \{ z \in E;\ z = tx + (1-t)\, y;\ x \in X,\ 0 \leq t \leq 1 \}$$

and

$$C f(z) = t f(x) + (1-t)\, y$$

for any mapp ng $f : X \to Y$ with $X, Y \subset N$.

6.5. Proposition. — *The cone functor* C *has the following properties :*

(i) *If* X *is closed* (respectively bounded) *in* N, *then so is* C (X) *in* E;

(ii) *If f is a compact field* (respectively α-field), *then so is* C f;

(iii) *If the fields* (respectively α-fields) f *and* g *are compactly homotopic* (respectively α-homotopic), *then so are* C f *and* C g.

7. The Leray Schauder category $\mathfrak{L}$. — Denote by $\mathfrak{L}(E)$ or simply by $\mathfrak{L}$ the h-subcategory of $\mathfrak{E}(E)$ generated by closed bounded subsets of E, $\mathfrak{L}(E)$ will be called the *Leray-Schauder category* corresponding to the linear space E.

In what follows, the category $\mathfrak{L}$ being of primary interest, we shall be concerned with such geometrical properties of its objects that remain invariant under the equivalences or homotopy equivalences in $\mathfrak{L}$.

Remark. — In all that follows the objects of the Leray-Schauder category $\mathfrak{L}(E)$ will be simply called the objects.

7.1. Proposition. — *There exist two equivalent objects* X_1 *and* X_2 *such that* $\pi_1(E - X_1) \approx 0$ *and* $\pi_1(E - X_2) \not\approx 0$.

Proof. — Let $E = M \oplus N$ be the direct sum decomposition of E such that $\dim N = 3$. Let Y_1 be the unit interval in N and $Y_2 \subset N$ be the Artin-Fox example ([4]), i. e., the set homeomorphic to Y_1 with $\pi_1(N - Y_2) \not\approx 0$; let $f : Y_1 \to Y_2$ denote the corresponding homeomorphism.

Now we let

$$X_1 = S_M(Y_1), \qquad X_2 = S_M(Y_2).$$

Clearly, by 6.2,

$$S_M f : X_1 \to X_2$$

([4]) *Cf.* E. Artin and R. Fox, *Some wild cells and spheres in three-dimensional space, Annals of Math.*, 1948, p. 979-990.

is an invertible compact field and thus the objects X_1 and X_2 are equivalent in $\mathfrak{L}$ (E). On the other hand, by 6.3,

$$\pi_1(E - X_2) \approx \pi_1(N - Y_2) \not\approx 0 \quad \text{and} \quad \pi_1(E - X_1) \approx 0.$$

The proof is completed.

Remark. — It can be shown that all the homology groups $H_q(E - X_1)$ and $H_q(E - X_2)$ vanish. By taking, instead of Y_2, the Alexander horned sphere in N and by repeating the above construction, one obtains two equivalent objects X_1 and X_2 in $\mathfrak{L}$ (E) such that

$$\pi_1(E - X_1) \approx 0, \quad \pi_1(E - X_2) \not\approx 0,$$
$$H_0(E - X_1) \approx H_0(E - X_2) \approx Z.$$

We conclude this section by introducing some subcategories of $\mathfrak{L}$. Given $\alpha \in \mathcal{L}$ we let

$$\mathfrak{L}_\alpha = \mathfrak{C}_\alpha \cap \mathfrak{L}$$

and denote by $\mathfrak{L}^*_\alpha$ a subcategory of $\mathfrak{L}_\alpha$ generated by the objects X for which the intersection $X_\alpha = X \cap L_\alpha$ is non-empty; clearly the relation of α-homotopy in $\mathfrak{C}_\alpha$ converts $\mathfrak{L}_\alpha$ and $\mathfrak{L}^*_\alpha$ into *h*-subcategories $(\mathfrak{L}_\alpha, \tilde{\alpha})$ and $(\mathfrak{L}^*_\alpha, \tilde{\alpha})$ of $(\mathfrak{C}_\alpha, \tilde{\alpha})$.

7.2. Definition. — Let $\mathfrak{L}_0 = \mathfrak{C}_0 \cap \mathfrak{L}$. We introduce in $\mathfrak{L}_0$ the relation of *finite dimensional homotopy* $\approx$ as follows : Given two finite dimensional fields $f \in \mathfrak{L}_\alpha(X, Y)$, $g \in \mathfrak{L}_\beta(X, Y)$ we write $f \approx g$ provided there is a γ-homotopy $h_t : X \to Y$ joining f and g such that $\alpha \leqq \gamma$ and $\beta \leqq \gamma$. Clearly $(\mathfrak{L}_0, \approx)$ is an *h*-subcategory of $(\mathfrak{L}, \sim)$.

CHAPTER III

Continuous functors

The basic constructions and results of this paper depend largely on the continuity property of the functors under consideration. This chapter is devoted to the above property and its main result can be briefly stated as follows : every continuous functor defined on the subcategory $\mathfrak{L}_0$ of $\mathfrak{L}$ admits the unique extension over $\mathfrak{L}$. To avoid undesirable repetitions, we shall consider in detail the contravariant case only. The dual results of interest in the covariant case will be merely stated and the proofs will be left to the reader.

Throughout this chapter λ_0 stands for a contravariant and μ_0 for a covariant functor from $\mathfrak{L}_0$ to the category $\mathfrak{A}^*$.

1. Approximating families and the carriers. — *Notation.* — $\mathfrak{L}_0$ being dense in $\mathfrak{L}$, we let for X in

$$\lambda(X) = \lambda_0(X) \quad \text{and} \quad \mu(X) = \mu_0(X);$$

for a field $f: X \to Y$ in $\mathfrak{L}_0$ we denote by

$$\lambda_0(f) = f^*: \ \lambda(Y) \to \lambda(X),$$
$$\mu_0(f) = f_*: \ \mu(X) \to \mu(Y)$$

the maps in $\mathfrak{A}^*$ *induced* by f.

1.1. Definition. — Let Y be an object. A family $\{Y_k\}$ of objects indexed by a directed set $\mathfrak{N}$ is said to be an *approximating family for* Y provided

(i) $Y_n \subset Y_k$ for any relation $k < n$ in $\mathfrak{N}$;

(ii) $Y = \bigcap_{k \in \mathfrak{N}} Y_k$.

In case $\mathfrak{N} = \{1, 2, \ldots\}$ such a family will be refered to as an *approximating sequence* for Y.

We note the following evident proposition :

1.2. Proposition. — *Let* Y *be an object and* α *be an arbitrary element of* $\mathcal{L}_Y$. *If* $\{Y_k\}_{k \in \mathfrak{N}}$ *is an approximating family for* Y, *then so is the family* $\{Y_k \cap L_\alpha\}_{k \in \mathfrak{N}}$ *for* Y_α.

Let Y be an object and $\{Y_k\}_{k \in \mathfrak{N}}$ an approximating family for Y. Denote by

$$i_{kn}: \ Y_n \to Y_k, \quad k < n,$$
$$j_k: \ Y \to Y_k, \quad k \leq \mathfrak{N}$$

the corresponding inclusions, all of them being finite dimensional fields.

We have the following commutative relations in $\mathfrak{L}_0$:

$$(1) \quad \begin{cases} i_{km} = i_{kn}\, i_{nm} & \text{for} \quad k < n < m, \\ j_k \ = i_{kn}\, j_n & \text{for} \quad k < n. \end{cases}$$

Given an object Y and an approximating family $\{Y_k\}_{k \in \mathfrak{N}}$ for Y, we let

$$i^*_{kn} = \lambda_0(i_{kn}), \qquad j^*_k = \lambda_0(j_k),$$
$$i^{kn}_* = \mu_0(i_{kn}), \qquad j^k_* = \mu_0(j_k).$$

In the contravariant case, from the commutativity relations (1) we infer that the objects $\lambda(Y_k)$ together with the maps i^*_{kn} given for every relation $k < n$ in $\mathfrak{N}$ form a direct system $\{\lambda(Y_k); i^*_{kn}\}$ in $\mathfrak{A}^*$ over $\mathfrak{N}$ and the family $\{j^*_k\}$ of maps

$$j^*_k : \lambda(Y_k) \to \lambda(Y)$$

is a direct family of maps in $\mathfrak{A}^*$.

Consequently, we have the direct limit map in $\mathfrak{A}^*$

$$\varinjlim_k \{j^*_k\} : \varinjlim_k \{\lambda(Y_k); i^*_{kn}\} \to \lambda(Y).$$

In the covariant case we have an inverse system $\{\mu(Y_k); i^{kn}_*\}$ of objects in $\mathfrak{A}^*$ over $\mathfrak{N}$, an inverse family $\{j^k_*\}$ of maps

$$j^*_k : \mu(Y) \to \mu(Y_k)$$

and the inverse limit map in $\mathfrak{A}^*$

$$\varprojlim_k \{j^k_*\} : \mu(Y) \to \varprojlim_k \{\mu(Y_k); i^{kn}_*\}.$$

1.3. Definition. — We shall say that a functor $\lambda_0 : \mathfrak{L}_0 \to \mathfrak{A}^*$ (respectively a functor $\mu_0 : \mathfrak{L}_0 \to \mathfrak{A}^*$) is *continuous* provided for every object Y and an approximating family $\{Y_k\}_{k\in\mathfrak{N}}$ for Y, the map $\varinjlim_k \{j^*_k\}$ (respectively the map $\varprojlim_k \{j^k_*\}$) is invertible in the category $\mathfrak{A}^*$.

Given a pair of objects (X, A) we denote by $j_{AX} : A \to X$ the corresponding inclusion and by j^*_{AX} the map

$$\lambda_0(j_{AX}) : \lambda(X) \to \lambda(A).$$

In the following definition we assume that $\mathfrak{A}^*$ is either the category $\mathcal{E}ns^*$ or $\mathfrak{A}$; the *zero element* $\mathcal{O}$ in $(X, \star) \in \mathcal{E}ns^*$ is the distinguished element $\star \in X$.

1.4. Definition. — Let A be an object, x be a point in A and ξ a non-trivial element of $\lambda(A)$. An object $S_x(\xi) = Y$ contained in A and containing x is called a *carrier* of ξ (with respect to x) provided $j^*_{YA}(\xi) \neq 0$. A carrier $S_x(\xi)$ of the element ξ is said to be *essential* (with respect to x) provided for any object $X \subset S_x(\xi)$ containing x we have $j^*_{XA}(\xi) = 0$.

The following lemma expresses an important property of continuous functors:

1.5. Lemma. — *Let* A *be an object and* $x \in A$. *If the functor* $\lambda_0 : \mathfrak{L}_0 \to \mathfrak{A}^*$ *is continuous, then for any non-trivial element* ξ *of* $\lambda(A)$ *there exists at least one essential carrier* $S_x(\xi)$ *of* ξ *with respect to the point* x.

Proof. — For an element $\xi \neq 0$ consider the set $\mathfrak{C}_x$ of all carriers $S_x(\xi) \subset A$ of ξ partially ordered downward by inclusion. If $\{Y_k\}_{k\in\mathfrak{N}}$ is a totally ordered subset of $\mathfrak{E}_x$ then the intersection $Y = \bigcap_{k \in \mathfrak{N}} Y_k$ is a non-empty object and $\{Y_k\}_{k \in \mathfrak{N}}$ is an approximating family for Y. From the continuity of the functor λ_0 we infer that Y is a carrier and thus $\{Y_k\}_{k \in \mathfrak{N}}$ has a lower bound in $\mathfrak{E}_x$. By the Zorn Lemma, the set $\mathfrak{E}_x$ contains a minimal element which is a required essential carrier for ξ.

2. Approximating systems. — *Notation.* — For an object Y and a natural number k we let

$$Y^{(k)} = \left\{ x \in E;\ \rho(x, Y) \leq \frac{1}{k} \right\}.$$

To a sequence $\{Y_k\}$ we assign the enlarged sequence $\{\tilde{Y}_k\}$ by putting

$$\tilde{Y}_k = \left\{ x \in E;\ \rho(x, Y_k) \leq \frac{1}{k} \right\}.$$

We begin with a proposition concerning approximating sequences.

2.1. Proposition. — *Let* $\{X_k\}$ *and* $\{Y_k\}$ *be two approximating sequences for* X *and* Y *respectively and let* $f: X_1 \to E$ *be a compact field. Then :*

(i) $\{X_k \cup Y_k\}$ *is an approximating sequence for* $X \cup Y$;

(ii) $\{\tilde{X}_k\}$ *is an approximating sequence for* X;

(iii) $\{f(X_k)\}$ *is an approximating sequence for* $f(X)$.

Proof. — Properties (i) and (ii) are evident. In order to establish (iii) it is sufficient to prove the inclusion

$$\bigcap_{k=1} f(X_k) \subset f\left(\bigcap_{k=1} X_k \right).$$

Let $y \in \bigcap_{k=1} f(X_k)$; we have $y = f(x_k)$, where $x_k \in X_k$ and thus $y = x_k - F(x_k)$. Since F is compact we may assume without loss of generality that $\lim_{k=\infty} x_k = x$. Consequently, $y = \lim_{k=\infty} f(x_k) = f(x)$. Since $x \in \bigcap_{k=1} X_k$, this completes the proof.

2.2. Definition. — Let X and Y be two objects and let $f: X \to Y$ be a compact field. A sequence $\{Y_k, f_k\}$ of objects Y_k and α_k-fields $f_k: X \to Y_k$ is said to be an *approximating system* for f, provided :

(i) $\{Y_k\}$ is an approximating sequence for Y;

(ii) $f_k \sim j_k f$ in $\mathfrak{L}$ where $j_k: Y \to Y_k$ is the inclusion;

(iii) $f_k \approx i_{kn} f_n$ in $\mathfrak{L}_0$, where $i_{kn}: Y_n \to Y_k$ is the inclusion $(k < n)$.

2.3. *Proposition.* — *Let* $f: X \to Y$ *be a compact field. Then, for each* k, *there is an* α_k-*field* $f_k : X \to Y_k$, *where* $Y_k = Y^{(k)}$, *such that*

$$\| f(x) - f_k(x) \| < \frac{1}{k} \quad \text{for all } x \in X;$$

moreover $\{Y_k, f_k\}$ *is an approximating system for* f.

Proof. — This follows clearly from Propositions II.4.5 and II.4.3. In what follows any system $\{Y_k, f_k\}$ as in Proposition 2.3 will be called a *standard* approximating system for f.

We note the following evident proposition :

2.4. Proposition. — *Let* $\{Y_k, f_k\}$ *be an approximating system for a field* $f: X \to Y$ *and* $\{\overline{Y}_k\}$ *be an approximating sequence for* Y *such that for each* k *we have* $Y_k \subset \overline{Y}_k$. *Denote by* $l_k : Y_k \to \overline{Y}_k$ *the corresponding inclusion and put* $\bar{f}_k = l_k \circ f_k$. *Then* $\{\overline{Y}_k, \bar{f}_k\}$ *is again an approximating system for* f.

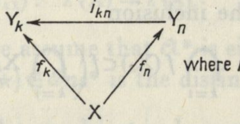
In the rest of this chapter, we assume that

$$\lambda_0 : \mathfrak{L}_0 \to \mathfrak{A}^* \quad \text{and} \quad \mu_0 : \mathfrak{L}_0 \to \mathfrak{A}^*$$

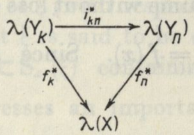
are continuous h-functors from $\mathfrak{L}_0$ to $\mathfrak{A}^*$.

Let X and Y be two objects and $f: X \to Y$ be a compact vector field. Let $\{Y_k, f_k\}$ be an arbitrary approximating system for f.

In view of the definition of an approximating system, to an h-commutative diagram in $\mathfrak{L}_0$:



corresponds the commutative diagram in $\mathfrak{A}^*$:



Consequently $\{f^*_k\}$ is a direct sequence of maps in $\mathfrak{A}^*$ and therefore

$$\varinjlim_k \{f^*_k\} : \varinjlim_k \{\lambda(Y_k); i^*_{kn}\} \to \lambda(X).$$

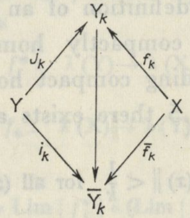
Similarly, $\{f^k_*\}$ is an inverse sequence of maps in $\mathfrak{A}^*$ and

$$\varprojlim_k \{f^k_*\} : \mu(X) \to \varprojlim_k \{\mu(Y_k); i^{kn}_*\}$$

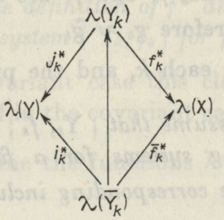
2.5. Proposition. — *Let* $\{ Y_k, f_k \}$ *and* $\{ \overline{Y}_k, \bar{f}_k \}$ *be two approximating systems for f as in Proposition* 2.4 *and let* $j_k : Y \to Y_k$, $i_k : Y \to \overline{Y}_k$ *denote the corresponding inclusions. Then we have*

$$\varinjlim_k \{ f_k^* \} \circ (\varinjlim_k \{ j_k^* \})^{-1} = \varinjlim_k \{ \bar{f}_k^* \} \circ (\varinjlim_k \{ i_k^* \})^{-1}.$$

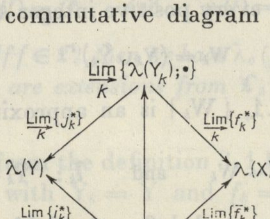
Proof. — Since for each k the diagram :



is commutative in $\mathfrak{L}_0$, it follows that its image under λ_0 in $\mathfrak{A}^*$ is also commutative



By considering the corresponding commutative diagram in the category of direct systems of objects in $\mathfrak{A}^*$ and applying to it the direct limit functor we obtain the following commutative diagram :



This, in view of the continuity of λ_0, implies our assertion.

2.6. Proposition. — *Let* $\{Y_k, f_k\}$, $\{Y_k, \tilde{f}_k\}$ *be two approximating systems for a field* $f: X \to Y$ *with the same sequence* $\{Y_k\}$. *Consider the enlarged sequence* $W_k = \{\tilde{Y}_k\}$, *denote by* $i_k: Y \to W_k$, $l_k: Y_k \to W_k$ *the corresponding inclusions and put* $g_k = l_k \circ f_k$, $\tilde{g}_k = l_k \circ \tilde{f}_k$. *Then* $\{W_k, g_k\}$ *and* $\{W_k, \tilde{g}_k\}$ *are again approximating systems for f and we have*

$$\underset{k}{\underrightarrow{\mathrm{Lim}}}\{g_k^*\} \circ (\underset{k}{\underrightarrow{\mathrm{Lim}}}\{j_k^*\})^{-1} = \underset{k}{\underrightarrow{\mathrm{Lim}}}\{\tilde{g}_k^*\} \circ (\underset{k}{\underrightarrow{\mathrm{Lim}}}\{j_k^*\})^{-1}.$$

Proof. — In view of the definition of an approximating system, the fields f_k, $\tilde{f}_k: X \to Y_k$ are compactly homotopic for every k. Let $h_t^{(k)}: X \to Y_k$ be a corresponding compact homotopy joining f_k and $\tilde{f}_k$. In view of Proposition II.4.5 there exists a α_k-homotopy $\tilde{h}_t^{(k)}: X \to E$ such that

$$\| h_t^k(x) - \tilde{h}_t^k(x) \| < \frac{1}{k} \text{ for all } (x, t) \in X \times I.$$

Clearly for each point $(x, t) \in X \times I$ we have $\tilde{h}_t^{(k)}(x) \in W_k$ and consequently $\tilde{h}_t^{(k)}$ may be viewed as a α_k-homotopy $\tilde{h}_t^{(k)}: X \to W_k$. Assuming without loss of generality that f_k and $\tilde{f}_k$ are α_k-fields, we evidently have $g_k \approx \tilde{h}_0^{(k)}$, $g_k \approx \tilde{h}_1^{(k)}$ and therefore $g_k \approx \tilde{g}_k$.

This implies $g_k^* = \tilde{g}_k^*$ for each k, and the proof is completed.

2.7. Proposition. — *Assume that* $\{Y_k, f_k\}$ *and* $\{\tilde{Y}_k, \tilde{f}_k\}$ *are two arbitrarily given approximating systems for a field* $f: X \to Y$. *Denote by* $j_k: Y \to Y_k$, $i_k: Y \to \tilde{Y}_k$ *the corresponding inclusions. Then we have*

$$\underset{k}{\underrightarrow{\mathrm{Lim}}}\{f_k^*\} \circ (\underset{k}{\underrightarrow{\mathrm{Lim}}}\{j_k^*\})^{-1} = \underset{k}{\underrightarrow{\mathrm{Lim}}}\{\tilde{f}_k^*\} \circ (\underset{k}{\underrightarrow{\mathrm{Lim}}}\{i_k^*\})^{-1}.$$

Proof. — Let us put for every positive integer k

$$W_k = (Y_k \cup \tilde{Y}_k)^{(k)}.$$

In view of Proposition 2.1, $\{W_k\}$ is an approximating sequence for Y. Denote by

$$l_k: \ Y_k \to W_k \quad \text{and} \quad \tilde{l}_k: \ \tilde{Y}_k \to W_k$$

the corresponding inclusions and define the fields

$$g_k, \tilde{g}_k: \ X \to W_k$$

by putting

$$g_k = l_k \circ f_k \quad \text{and} \quad \tilde{g}_k = \tilde{l}_k \circ \tilde{f}_k.$$

Note that $\{W_k, g_k\}$ and $\{W_k, \tilde{g}_k\}$ are both approximating systems for f. It is clear that the pairs $\{W_k, g_k\}$, $\{Y_k, f_k\}$ and $\{W_k, \tilde{g}_k\}$, $\{Y_k, f_k\}$ satisfy the assumptions of Proposition 2.5. Now, our assertion follows from Proposition 2.6.

3. The Extension Theorem for continuous functors.

3.1. Definition. — Given a compact field $f: X \to Y$, let $\{Y_k, f_k\}$ be an approximating system for f and let $j_k: Y \to Y_k$ be the inclusion. We define the induced maps

$$f^*: \quad \lambda(Y) \to \lambda(X)$$

and

$$f_*: \quad \lambda(X) \to \lambda(Y)$$

by the following formulas :

$$f^* = \underset{\overrightarrow{k}}{\mathrm{Lim}}\{f_k^*\} \circ (\underset{\overrightarrow{k}}{\mathrm{Lim}}\{j_k^*\})^{-1}; \tag{2}$$

$$f_* = (\underset{\overleftarrow{k}}{\mathrm{Lim}}\{j_*^k\})^{-1} \circ \underset{\overleftarrow{k}}{\mathrm{Lim}}\{f_*^k\}. \tag{3}$$

3.2. Proposition. — *The definition of f^* and f_* does not depend on the choice of an approximating system $\{Y_k, f_k\}$ for f.*

Proof. — In the contravariant case this clearly is a reformulation of Proposition 2.7. The proof in the covariant case is similar and is omitted.

3.3. Definition. — Define the functions $\lambda: \mathfrak{L} \to \mathcal{A}^*$ and $\mu: \mathfrak{L} \to \mathcal{A}^*$ by putting for X in $\mathfrak{L}$

$$\lambda(X) = \lambda_0(X), \qquad \mu(X) = \mu_0(X)$$

and for a compact field f in $\mathfrak{L}$

$$\lambda(f) = f^* \qquad \text{and} \qquad \mu(f) = f_*.$$

3.4. Proposition. — *If $f \in \mathfrak{L}_0$, then we have $\lambda_0(f) = \lambda(f)$ and $\mu_0(f) = \mu(f)$. In other words, λ and μ are extensions from $\mathfrak{L}_0$ over $\mathfrak{L}$ of the functors λ_0 and μ_0 respectively.*

Proof. — This follows from the definition 3.1 by taking for f an approximating system $\{Y_k, f_k\}$ with $Y_k = Y$ and $f_k = f$ for all $k = 1, 2, \ldots$.

3.5. Proposition. — *λ and μ are h-functors from the Leray-Schauder category $\mathfrak{L}$ to the category $\mathcal{A}^*$. In other words, the induced maps f^* and f_* satisfy the following two properties :*

(*a*) *if the fields f and g are compactly homotopic, then $f^* = g^*$ and $f_* = g_*$;*

(*b*) *for any two composable compact fields f and g we have*

$$(gf)^* = f^* \circ g^* \quad \textit{and} \quad (gf)_* = g_* \circ f_*.$$

Proof of the property (*a*). — Assume that the fields $f, g : X \to Y$ are compactly homotopic and denote by $h_t : X \to Y$ $(0 \leqq t \leqq 1)$ a compact homotopy such that

$$h_0 = f \quad \text{and} \quad h_1 = g.$$

By Proposition II.4.5 there exists an α_k-homotopy $h_t^{(k)} : X \to Y^{(k)}$ $(0 \leqq t \leqq 1)$ satisfying

$$\| h_t^{(k)}(x) - h_t(x) \| < \frac{1}{k} \quad \text{for all } (x, t) \in X \times I.$$

Let us put $f_k = h_0^{(k)}$ and $g_k = h_1^{(k)}$ for every k. It follows from Proposition 2.3 that $\{ Y^{(k)}, f_k \}$ and $\{ Y^{(k)}, g_k \}$ are approximating systems for f and g respectively. Since for every k the α_k-fields $f_k, g_k : X \to Y^{(k)}$ are homotopic in $\mathfrak{L}_0$, it follows that

$$\underset{\overrightarrow{k}}{\text{Lim}} \{ f_k^* \} = \underset{\overrightarrow{k}}{\text{Lim}} \{ g_k^* \}.$$

Consequently, we have $f^* = g^*$ and the proof is completed.

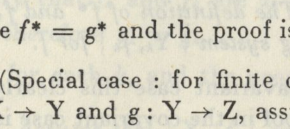
Proof of property (*b*) (Special case : for finite dimensional g). — Given two compact fields $f : X \to Y$ and $g : Y \to Z$, assume that g is an α_0-field and let $h = gf$. We shall prove that $h^* = f^* \circ g^*$.

Take a standard approximating system $\{ Y^{(k)}, f_k \}$ for f; Lemma II.2.3 implies that there exists an α_0-field $\bar{g} : Y^{(1)} \to E$ such that g is the contraction of g to the pair (Y, Z).

Let us put for every $k = 1, 2, \ldots$

$$W_k = g(Y^{(k)}) \cup Z.$$

By Proposition 2.1 both $\{ W_k \}$ and $\{ \tilde{W}_k \}$ are approximating sequences for Z. Now consider the following diagram :



in which i_k and j_k are the inclusions, $\bar{g}_k$ is defined by

$$\bar{g}_k(y) = \bar{g}(y) \quad \text{for} \quad y \in Y^{(k)}$$

and

$$g_k = \bar{g}_k \circ i_k.$$

Since for every k both f_k and g_k are finite dimensional we may apply to the fields

$$h_k = \bar{g}_k \circ f_k \quad \text{and} \quad g_k = \bar{g}_k \circ i_k$$

the functor λ_0 and therefore we have

$$h_k^* = f_k^* \circ \bar{g}_k^* \quad \text{and} \quad g_k^* = i_k^* \circ \bar{g}_k^*.$$

Consequently,

$$\varinjlim_k \{ h_k^* \} = \varinjlim_k \{ f_k^* \} \circ \varinjlim_k \{ \bar{g}_k^* \} = \varinjlim_k \{ f_k^* \} \circ (\varinjlim_k \{ i_k^* \})^{-1} \circ \varinjlim_k \{ g_k^* \}$$

and thus

$$\varinjlim_k \{ h_k^* \} \circ (\varinjlim_k \{ j_k^* \})^{-1} = f^* \circ \varinjlim_k \{ g_k^* \} \circ (\varinjlim_k \{ j_k^* \})^{-1}.$$

Further, it is clear that $\{ \tilde{W}_k, h_k \}$ and $\{ \tilde{W}_k, g_k \}$ are approximating systems for h and g respectively. Therefore, it follows from the last formula and Definition 3.1 that $h^* = f^* \circ g^*$, and the proof is completed.

Proof of the property (*b*) (General case). — Let $f : X \to Y$ and $g : Y \to Z$ be two compact fields and let $h = gf$.

Let $\{ Z^{(k)}, h_k \}$ and $\{ Z^{(k)}, g_k \}$ be two standard approximating systems for h and g, respectively. From the inequalities

$$\| h_k(x) - h(x) \| < \frac{1}{k} \quad \text{for all } x \in X,$$

$$\| g_k f(x) - h(x) \| < \frac{1}{k} \quad \text{for all } x \in X,$$

it follows by Proposition II.4.3 that for every integer k the fields $g_k f, h_k : X \to Z^{(k)}$ are homotopic in $\mathfrak{L}$. This implies, in view of property (*a*), that $h_k^* = (g_k f)^*$. Since each g_k is finite dimensional we have, by the proved special case of property (*b*),

$$(g_k f)^* = f^* \circ g_k^*$$

and thus we obtain

$$\varinjlim_k \{ h_k^* \} = \varinjlim_k \{ f^* \circ g_k^* \} = f^* \circ \varinjlim_k \{ g_k^* \}.$$

This implies $h^* = f^* \circ g^*$ and the proof is completed.

We may now summarize our discussion in the following theorem :

3.6. Theorem. — *Let* $\lambda_0 : \mathfrak{L}_0 \to \mathfrak{A}^*$ *be a continuous (contravariant or covariant) h-functor. Then* λ_0 *can be uniquely extended over* $\mathfrak{L}$ *to an h-functor* $\lambda : \mathfrak{L} \to \mathfrak{A}^*$.

Proof. — In view of Propositions 3.4 and 3.5 it is sufficient to prove only the uniqueness of an extension. But this, in view of the definition of an approximating system, follows clearly from one of the formulas (2) or (3). The proof is completed.

CHAPTER IV

The functor $\mathcal{H}^{\infty-n}$

In this chapter, given a cohomology theory $\mathcal{H}^*$ on the category K_E, we construct for every n a contravariant h-functor $\mathcal{H}^{\infty-n}$ from the Leray-Schauder category $\mathfrak{L}$ to the category of abelian groups. First, we define a functor $\mathcal{H}_0^{\infty-n}$ on $\mathfrak{L}_0$. Then, using the continuity of $\mathcal{H}^*$ on K_E and an algebraic lemma on " interchanging double limits ", we show that the functor $\mathcal{H}_0^{\infty-n}$ is continuous. Finally, we apply the main theorem of the previous chapter, and find a unique extension $\mathcal{H}^{\infty-n}$ of $\mathcal{H}_0^{\infty-n}$ over $\mathfrak{L}$.

1. Preliminaries on the Mayer-Vietoris homomorphism. — *Notation.* — We denote by $\mathcal{O}$ any of the following h-categories :

$\mathfrak{L}$, the Leray-Schauder category on E;

$\mathcal{K}_E$, the full subcategory of $\mathfrak{L}$ generated by the compact subsets of finite dimensional subspaces of E.

In what follows, by a *triad* in $\mathcal{O}$ we shall understand an ordered triple $T = (X; X_1, X_2)$ of objects in $\mathcal{O}$ such that $X = X_1 \cup X_2$ and, by a *map* $f : (X; X_1, X_2) \to (Y; Y_1, Y_2)$ between the triads, a map $f : X \to Y$ in $\mathcal{O}$ which carries X_i into Y_i for $i = 1, 2$.

Let $\mathcal{O}^2$ be the h-category of pairs in $\mathcal{O}$ and let $\rho : \mathcal{O}^2 \to \mathcal{O}^2$ be the covariant functor defined by

$\rho(X, A) = A$ for any $(X, A) \in \mathcal{O}^2$;

$\rho(f) = f|_A : A \to B$ for any map $f : (X, A) \to (Y, B)$ in $\mathcal{O}^2$.

1.1. Definition. — A *cohomology theory* $\mathcal{H}^* = \{\mathcal{H}^n, \delta^n\}$ on $\mathcal{O}$ is a sequence of contravariant h-functors

$$\mathcal{H}^n : \mathcal{O}^2 \to \mathfrak{A}b \quad (-\infty < n < +\infty)$$

together with a sequence of natural transformations

$$\delta^n : \mathcal{H}^n \circ \rho \to \mathcal{H}^{n+1} \qquad (-\infty < n < +\infty)$$

satisfying the following conditions :

(*a*) (*Strong Excision*). — If (X; A, B) is a triad in $\mathcal{O}$ and

$$k \ : (A, A \cap B) \to (X, B)$$

is the inclusion, then $\mathcal{H}^n(k) : \mathcal{H}^n(X, B) \approx \mathcal{H}^n(A, A \cap B)$ for all n.

(*b*) (*Exactness*). — If (X, A) is a pair in $\mathcal{O}$ and $i : A \to X$, $j : X \to (X, A)$ are the inclusion maps, then the cohomology sequence

$$\ldots \to \mathcal{H}^{n-1}(A) \xrightarrow{\delta^{n-1}} \mathcal{H}^n(X, A) \xrightarrow{j^*} \mathcal{H}^n(X) \xrightarrow{i^*} \mathcal{H}^n(A) \xrightarrow{\delta^n} \mathcal{H}^{n+1}(X, A) \to \ldots$$

of (X, A) is exact.

(*c*) (*Continuity*). — Given an approximating family $\{Y_k\}_{k \in N}$ of objects in $\mathcal{O}$ for $Y = \bigcap_{k \in N} Y_k$ we have an isomorphism for each n :

$$\mathcal{H}^n(Y) \approx \underset{\overrightarrow{k}}{\mathrm{Lim}} \{ \mathcal{H}^n(Y^k); i^*_{kl} \},$$

where $i_{kl} : Y_l \to Y_k$, $(k < l)$ stands for the inclusion.

Convention. — Cohomology theories on $\mathfrak{L}$ are denoted by

$$\mathcal{H}^{\infty-*} = \{ \mathcal{H}^{\infty-n}, \delta^{\infty-n} \},$$

where $\mathcal{H}^{\infty-n}$ and $\delta^{\infty-n}$ play the role of $\mathcal{H}^{-n}$ and δ^{-n} in Definition 1.1.

If $\mathcal{H}^{\infty-*} = \{ \mathcal{H}^{\infty-n}, \delta^{\infty-n} \}$ is a cohomology theory on $\mathfrak{L}$, then the graded group $\{ \mathcal{H}^{\infty-n}(S) \}$, where S is the unit sphere in E, is called the *group of coefficients of* the theory $\mathcal{H}^{\infty-*}$.

The aim of this and the next chapter is to show that any cohomology theory on $\mathcal{K}_E$ gives rise to a geometrically meaningful cohomology theory on $\mathfrak{L}$. The rest of this section is devoted to general remarks on the Mayer-Vietoris homomorphism which are applicable both for cohomology theories on $\mathcal{K}_E$ and on $\mathfrak{L}$.

Let $\mathcal{H}^*$ be a cohomology theory on $\mathcal{O}$. Given a triad (X; X_1, X_2) in $\mathcal{O}$ with $A = X_1 \cap X_2$, denote by

$$\begin{aligned} j_{01} &: A \to X_1, & j_{02} &: A \to X_2, \\ i_1 &: X_1 \to X, & i_2 &: X_2 \to X, \\ j_1 &: X \to (X, X_1), & j_2 &: X \to (X, X_2), \end{aligned}$$

the corresponding inclusions.

1.2. Definition. — The *Mayer-Vietoris cohomology sequence* of a triad $(X; X_1, X_2)$ with $A = X_1 \cap X_2$ is the sequence of abelian groups

$$\ldots \to \mathcal{H}^{n-1}(A) \xrightarrow{\Delta^{n-1}} \mathcal{H}^n(X) \xrightarrow{\varphi} \mathcal{H}^n(X_1) \oplus \mathcal{H}^n(X_2) \xrightarrow{\psi} \mathcal{H}^n(A) \to \ldots$$

in which φ and ψ are given by

$$\varphi(\gamma) = (i_1^*(\gamma), i_2^*(\gamma)) \quad \text{for} \quad \gamma \in \mathcal{H}^n(X),$$
$$\psi(\gamma_1 + \gamma_2) = j_{01}^*(\gamma_1) - j_{02}^*(\gamma_2) \quad \text{for} \quad \gamma_i \in \mathcal{H}^n(X_i) \quad (i = 1, 2)$$

and the *Mayer-Vietoris homomorphism* Δ^{n-1} is defined by

$$\Delta^{n-1} = j_1^* \circ (k^*)^{-1} \circ \delta^{n-1},$$

where k^* is the isomorphism induced by the excision $k : (X_2, A) \to (X, X_1)$.

We shall often drop the superscript n on Δ^n, when there is no danger of confusion.

1.3. Definition. — The *cohomology sequence of a triple* $B \subset A \subset X$ with inclusions

$$A \xrightarrow{k} (A, B) \xrightarrow{i} (X, B) \xrightarrow{j} (X, A)$$

is the sequence of abelian groups

$$\ldots \to \mathcal{H}^{n-1}(A, B) \xrightarrow{\delta} \mathcal{H}^n(X, A) \xrightarrow{j^*} \mathcal{H}^n(X, B) \xrightarrow{i^*} \mathcal{H}^n(A, B) \to \ldots$$

in which the coboundary homomorphism δ is defined as the composite

$$\mathcal{H}^{n-1}(A, B) \xrightarrow{k^*} \mathcal{H}^{n-1}(A) \xrightarrow{\delta^{n-1}} \mathcal{H}^n(X, A).$$

As in ordinary cohomology theory, we deduce from the axioms by purely formal argument the following propositions :

1.4. Proposition. — *The Mayer-Vietoris sequence of a triad* $(X; X_1, X_2)$ *is exact. If* $f : (X; X_1, X_2) \to (Y; Y_1, Y_2)$ *is a map of one triad into another, then f induces a homomorphism of the Mayer-Vietoris sequence of the second triad into that of the first.*

1.5. Proposition. — *The cohomology sequence of a triple is exact. If* $f : (X, A, B) \to (X', A', B')$ *is a map between two triples in* $\mathcal{O}$, *then f induces a homomorphism of the cohomology sequence of the second triple into that of the first.*

Let $T_0 = (A; A_1, A_2)$ and $T_1 = (X; X_1, X_2)$ be two triads. Then, T_0 is a *subtriad* of T, written $T_0 \subset T$, provided $A \subset X$ and $A_i \subset X_i$ for $i = 1, 2$; T_0 is said to be a *proper subtriad* of T, written $T_0 \Subset T$, provided $A_i = A \cap X_i$ for $i = 1, 2$.

If $T_0 \Subset T$, then clearly $T_0 \subset T$ and $A_0 = A_1 \cap A_2 = A \cap X_0$, where $X_0 = X_1 \cap X_2$; moreover, the inclusions

$$\begin{aligned} q &: (X_0, A_0) \to (X_0 \cup A_2, A_2), \\ k &: (X_2, X_0 \cup A_2) \to (X, X_1 \cup A) \end{aligned}$$

are excisions.

1.6. Definition. — Given $(A; A_1, A_2) \subset (X; X_1, X_2)$ we define the *relative Mayer-Vietoris homomorphism*

$$\Delta : \mathcal{H}^{n-1}(X_0, A_0) \to \mathcal{H}^n(X, A)$$

by

$$\Delta = j^* \circ (k^*)^{-1} \circ \delta \circ (q^*)^{-1},$$

where

$$\delta : \mathcal{H}^{n-1}(X_0 \cup A_2, A_2) \to \mathcal{H}^n(X_2, X_0 \cup A_2)$$

is the coboundary homomorphism of the triple $(X_2, X_0 \cup A_2, A_2)$ and

$$j : (X, A) \to (X, X_1 \cup A)$$

is the inclusion.

The following proposition is an immediate consequence of the definitions involved :

1.7. Proposition. — *To a commutative diagram of triads :*

$$\begin{array}{ccc} (B; B_1, B_2) & \Subset & (Y; Y_1, Y_2) \\ \uparrow g & & \uparrow f \\ (A; A_1, A_2) & \Subset & (X; X_1, X_2) \end{array}$$

corresponds the following commutative diagram of abelian groups :

$$\begin{array}{ccc} \mathcal{H}^{n-1}(Y_0, B_0) & \xrightarrow{\Delta} & \mathcal{H}^n(Y, B) \\ \downarrow g^* & & \downarrow f^* \\ \mathcal{H}^{n-1}(X_0, A_0) & \xrightarrow{\Delta} & \mathcal{H}^n(X, A) \end{array}$$

2. Orientation in E. — We begin by defining an orientation in L_α. To this end, consider the set of all linear isomorphisms from L_α to the euclidean space $R^{d(\alpha)}$ of dimension $d(\alpha)$ (we recall that $d(\alpha) = \dim L_\alpha$). Call two linear isomorphisms $l_1, l_2 : L_\alpha \to R^{d(\alpha)}$ *equivalent*, $l_1 \sim l_2$, provided $l_1 \circ l_2^{-1} \in GL_+(d(\alpha))$, i. e., the determinant of the corresponding

matrix is positive. With respect to this equivalence relation, the set of linear isomorphisms from L_α to $R^{d(\alpha)}$ decomposes into exactly two equivalence classes. An arbitrary choice $\mathcal{O}_\alpha$ of one of these classes will be called an orientation in L_α.

Let us chose now for each $\alpha \in \mathcal{L}$ an orientation $\mathcal{O}_\alpha$ in L_α and call the familly $\mathcal{O} = \{\mathcal{O}_\alpha\}$ to be an *orientation* in E.

Given an elementary relation $\alpha < \beta$ in $\mathcal{L}$ and $l_\alpha \in \mathcal{O}_\alpha$, there exists $l_\beta \in \mathcal{O}_\beta$ such that $l_\beta(x) = l_\alpha(x)$ for all $x \in L_\alpha$.

We let

$$L_\beta^+ = l_\beta^{-1}(R_+^{d(\beta)}) \quad \text{and} \quad L_\beta^- = l_\beta^{-1}(R_-^{d(\beta)}).$$

Clearly, the definition of L_β^+ and L_β^- depends only on the orientations of L_β and L_α.

As a consequence, given an object X and an elementary relation $\alpha < \beta$ in $\mathcal{L}_X$, the orientations of L_α and L_β determine the triad $(X_\beta; X_\beta^+, X_\beta^-)$, where

$$X_\beta^+ = X \cap L_\beta^+ \quad \text{and} \quad X_\beta^- = X \cap L_\beta^-$$

and such that

$$X_\alpha = X_\beta^+ \cap X_\beta^-.$$

2.1 Proposition. — *Let* X *and* Y *be two objects*, $f : X \to Y$ *be an* α_0-*field and let* $\alpha < \beta$ *be an elementary relation in* $\mathcal{L}_X$ *such that* $\alpha_0 \leqq \alpha < \beta$. *Then* $f(X_\beta) \subset Y_\beta$ *and* $f_\beta : X_\beta \to Y_\beta$ *induces a map, also denoted by* f_β, *of the triad* $(X_\beta; X_\beta^+, X_\beta^-)$, *into the triad* $(Y_\beta; Y_\beta^+, Y_\beta^-)$.

3. Definition of the group $\mathcal{H}^{\infty - n}(X)$. — Let $\mathcal{H}^* = \{\mathcal{H}^q, \partial^q\}$ be a cohomology theory on $\mathcal{K}_E$ and X be an arbitrarily-given object of the Leray-Schauder category $\mathfrak{L}$. In this section, starting with $\mathcal{H}^*$, we shall define for an integer n the group $\mathcal{H}^{\infty - n}(X)$.

First, we fix an orientation $\mathcal{O} = \{\mathcal{O}_\alpha\}$ in the space E. Next, for for any relation $\alpha \leqq \beta$ in $\mathcal{L}_X$ we define a homomorphism

$$\Delta_{\alpha\beta}^{(n)} : \mathcal{H}^{d(\alpha)-n}(X_\alpha) \to \mathcal{H}^{d(\beta)-n}(X_\beta)$$

as follows : If $\alpha = \beta$ we let $\Delta_{\alpha\beta}^{(n)}$ be the identity. If $\alpha < \beta$ is elementary, we let $\Delta_{\alpha\beta}^{(n)}$ be the Mayer-Vietoris homomorphism of the triad $(X_\beta; X_\beta^-, X_\beta^-)$ with $X_\beta^+ \cap X_\beta^- = X_\alpha$.

In order to extend this definition to an arbitrary relation $\alpha < \beta$ in $\mathcal{L}_X$ we shall need the following lemma :

3.1. Lemma. — *Let* X *be an object and* $\alpha < \beta$ *be a relation in* $\mathcal{L}_X$ *such that* $d(\beta) = d(\alpha) + 2$. *Assume that* $\alpha < \gamma < \beta$ *and* $\alpha < \tilde{\gamma} < \beta$ *are two* *different chains in* $\mathcal{L}_X$ *joining* α *and* β. *Then*

$$\Delta_{\gamma\beta} \circ \Delta_{\alpha\gamma} = \Delta_{\tilde{\gamma}\beta} \circ \Delta_{\alpha\tilde{\gamma}}.$$

The proof of Lemma 3.1 is given in section 7.

3.2. Definition. — Let $\alpha < \beta$ be an arbitrary relation in $\mathcal{L}_X$ and let $\alpha = \alpha_0 < \alpha_1 < \ldots < \alpha_{k+1} = \beta$ be a chain of elementary relations in $\mathcal{L}_X$ joining α and β. We define

$$\Delta_{\alpha\beta}^{(n)} = \Delta_{\alpha_k\beta}^{(n)} \circ \ldots \circ \Delta_{\alpha_1\alpha_2}^{(n)} \circ \Delta_{\alpha\alpha_1}^{(n)}$$

as the composition of the corresponding Mayer-Vietoris homomorphisms.

It follows from Lemma 3.1 that the definition of $\Delta_{\alpha\beta}^{(n)}$ does not depend on the choice of the chain $\alpha_1, \ldots, \alpha_k$ joining α and β.

Consider now the abelian groups $\mathcal{H}^{d(\alpha)-n}(X_\alpha)$ together with the homomorphisms $\Delta_{\alpha\beta}^{(n)}$ given for each relation $\alpha < \beta$ in $\mathcal{L}_X$. The family $\{\mathcal{H}^{d(\alpha)-n}(X_\alpha); \Delta_{\alpha\beta}^{(n)}\}$ indexed by $\alpha \in \mathcal{L}_X$ will be called the $(\infty - n)$-th *cohomology system of* X corresponding to the theory $\mathcal{H}^*$ and the orientation $\mathcal{O}$ in E.

3.3. Proposition. — $\{\mathcal{H}^{d(\alpha)-n}(X_\alpha); \Delta_{\alpha\beta}^{(n)}\}$ *is a direct system of abelian groups over* $\mathcal{L}_X$.

Proof. — This follows clearly from Lemma 3.1.

3.4. Definition. — For an object X we define an abelian group

$$\mathcal{H}^{\infty-n}(X) = \varinjlim_{\alpha} \{\mathcal{H}^{d(\alpha)-n}(X_\alpha); \Delta_{\alpha\beta}^{(n)}\}$$

to be the direct limit over $\mathcal{L}_X$ of the $(\infty - n)$-th cohomology system of X.

Remark. — We note that the group $\mathcal{H}^{\infty-n}(X)$ depends only up to an isomorphism, on the orientation in E used for its definition. In fact, suppose that $\{\mathcal{O}_\alpha\}$ and $\{\bar{\mathcal{O}}_\alpha\}$ are two orientations in E. These determine two direct systems of abelian groups $\{\mathcal{H}^{d(\alpha)-n}(X_\alpha), \Delta_{\alpha\beta}^{(n)}\}$ and $\{\mathcal{H}^{d(\alpha)-n}(X_\alpha), \bar{\Delta}_{\alpha\beta}^{(n)}\}$ respectively. For each $\alpha \in \mathcal{L}_X$ define

$$\Phi_\alpha : \mathcal{H}^{d(\alpha)-n}(X_\alpha) \to \mathcal{H}^{d(\alpha)-n}(X_\alpha)$$

by

$$\Phi_\alpha = \begin{cases} 1 & \text{if } \mathcal{O}_\alpha = \bar{\mathcal{O}}_\alpha, \\ -1 & \text{if } \mathcal{O}_\alpha = -\bar{\mathcal{O}}_\alpha. \end{cases}$$

Since $\{\Phi_\alpha\}$ is clearly an isomorphism of the above direct systems, it follows that the corresponding limit groups are isomorphic.

4. Definition of f^* for a finite dimensional field f. — We begin with the following :

4.1. Proposition. — *Let* X *and* Y *be two objects and let* $f : X \to Y$ *be an* α_0*-field, where* $\alpha_0 \in \mathcal{L}_X$. *Then, for every relation* $\alpha < \beta$ *in* $\mathcal{L}_X$ *such that* $\alpha_0 \leq \alpha < \beta$, *the following diagram commutes :*

$$\begin{array}{ccc} \mathcal{H}^{d(\alpha)-n}(Y_\alpha) & \xrightarrow{f_\alpha^*} & \mathcal{H}^{d(\alpha)-n}(X_\alpha) \\ \Big\downarrow {\scriptstyle \Delta_{\alpha\beta}^{(n)}} & & \Big\downarrow {\scriptstyle \Delta_{\alpha\beta}^{(n)}} \\ \mathcal{H}^{d(\beta)-n}(Y_\beta) & \xrightarrow{f_\beta^*} & \mathcal{H}^{d(\beta)-n}(X_\beta) \end{array}$$

Proof. — If $\alpha < \beta$ is elementary, this follows from Proposition 1.3. Our assertion for an arbitrary relation $\alpha < \beta$ follows then from the definition of the homomorphism $\Delta_{\alpha\beta}^{(n)}$.

Proposition 4.1 implies that $f : X \to Y$ induces a map

$$\{ f_\alpha^* \} : \ \{ \mathcal{H}^{d(\alpha)-n}(Y_\alpha); \Delta_{\alpha\beta}^{(n)} \} \to \{ \mathcal{H}^{d(\alpha)-n}(X_\alpha); \Delta_{\alpha\beta}^{n} \}$$

from the $(\infty - n)$-th cohomology system of Y into that of the object X.

4.2. Definition. — Given a finite dimensional field $f : X \to Y$ we define the induced homomorphism

$$f^* = \underset{\overrightarrow{\alpha}}{\mathrm{Lim}} \{ f_\alpha^* \} : \ \mathcal{H}^{\infty - n}(Y) \to \mathcal{H}^{\infty - n}(X)$$

to be the direct limit over $\mathcal{L}_X$ of the family $\{ f_\alpha^* \}$.

4.3. Proposition. — *The induced homomorphism* f^* *satisfies the following properties :*

(*a*) *if* 1 *is the identity on* X, *then* 1^* *is the identity on* $\mathcal{H}^{\infty - n}(X)$;

(*b*) *for any two composable finite dimensional fields f and g we have*

$$(gf)^* = f^* \circ g^*;$$

(*c*) *if the finite dimensional fields f and g are homotopic in* $\mathcal{L}_0$ *then* $f^* = g^*$.

Proof. — This clearly follows from Proposition 1.4 and the definition of $\Delta_{\alpha\beta}^{(n)}$.

We summarize the preceeding discussion in the following :

4.4. Theorem. — *The assignments* $X \to \mathcal{H}^{\infty-n}(X)$ *for* $X \in \mathfrak{L}_0$ *and* $f \to f^*$ *for* $f \in \mathfrak{L}_0$ *define a contravariant h-functor* $\mathcal{H}_0^{\infty-n}$ *from the h-category* $(\mathfrak{L}_0, \approx)$ *to the category of abelian groups.*

5. Two algebraic lemmas. — Given a directed set $\mathcal{L} = \{\alpha, \beta, \gamma, \ldots\}$, denote by the same letter $\mathcal{L}$ the category having as objects the elements of and as maps the relations $\alpha \leqq \beta$ in $\mathcal{L}$. For a small category $\mathcal{L}$, denote by $(\mathcal{L}, \mathcal{D})$ the category of covariant functors from $\mathcal{L}$ to $\mathcal{D}$, i. e., the category of direct systems of objects of $\mathcal{D}$ over $\mathcal{L}$.

By $\underset{\alpha}{\underrightarrow{\operatorname{Lim}}} : (\mathcal{L}, \mathcal{D}) \to \mathcal{D}$ we shall denote the " direct limit " functor, i. e., the left-adjoint to the constant functor from $\mathcal{D}$ to $(\mathcal{L}, \mathcal{D})$.

Let $\mathfrak{N} = \{k, l, m, \ldots\}$ and $\mathcal{L} = \{\alpha, \beta, \gamma, \ldots\}$ be two directed sets. Denote by $\mathfrak{N} \times \mathcal{L}$ the corresponding product category.

Given a direct system of Abelian groups

$$\varkappa \in (\mathfrak{N} \times \mathcal{L}, \mathcal{A}b)$$

let us put

$$\varkappa(\alpha, k) = H_\alpha^k,$$
$$\varkappa(\alpha \leqq \beta, k \leqq k) = \Delta_{\alpha\beta}^k,$$
$$\varkappa(\alpha \leqq \alpha, k \leqq l) = i_{k,l}.$$

For any relations $k \leqq l$ and $\alpha \leqq \beta$ we have the commutative diagram :

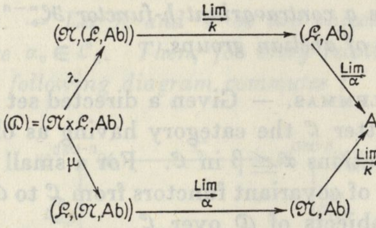
$$\begin{array}{ccc} H_\alpha^k & \xrightarrow{\;i_{kl}^\alpha\;} & H_\alpha^l \\ \Big\downarrow{\scriptstyle \Delta_{\alpha\beta}^k} & & \Big\downarrow{\scriptstyle \Delta_{\alpha\beta}^l} \\ H_\beta^k & \xrightarrow{\;i_{kl}^\beta\;} & H_\beta^l \end{array}$$

Clearly, each double system of Abelian groups $\{H_\alpha^k;\ \}$ indexed by $\mathfrak{N} \times \mathcal{L}$ together with the maps $\{i_{kl}\}$, $\{\Delta_{\alpha\beta}^k\}$, (satisfying the natural functorial properties), may be identified with a functor $\varkappa \in (\mathfrak{N} \times \mathcal{L}, \mathcal{A}b)$. We shall write simply $\varkappa = \{H_\alpha^k\}$.

5.1. Lemma. — *Let* $\mathcal{L} = \{\alpha, \beta, \gamma, \ldots\}$ *and* $\mathfrak{N} = \{k, l, m, \ldots\}$ *be two directed sets,* $\mathfrak{N} \times \mathcal{L}$ *be the product category, and let*

$$\lambda : \ (\mathfrak{N} \times \mathcal{L}, \mathcal{A}b) \to (\mathfrak{N}, (\mathcal{L}, \mathcal{A}b)),$$
$$\mu : \ (\mathfrak{N} \times \mathcal{L}, \mathcal{A}b) \to (\mathcal{L}, (\mathfrak{N}, \mathcal{A}b)),$$

be the natural isomorphisms between the corresponding categories. Then, the following diagram commutes :



The word " commutativity " stands for the natural equivalence of functors.

The commutativity of the diagram $(\mathfrak{D})$ follows from the fact that the left-adjoint functor commutes with direct limits [11].

We shall restate now a part of Lemma 5.1 in equivalent but more convenient form.

Let us denote by

$$\tau : \ \underset{k}{\underrightarrow{\mathrm{Lim}}}\,\underset{\alpha}{\underrightarrow{\mathrm{Lim}}} \circ \mu \to \underset{\alpha}{\underrightarrow{\mathrm{Lim}}} \circ \underset{k}{\underrightarrow{\mathrm{Lim}}} \circ \lambda$$

the natural equivalence between the corresponding functors. In order to simplify the notation, given $\varkappa \in (\mathfrak{N} \times \mathfrak{L}, \mathfrak{Ab})$, let us denote by the same letter $\varkappa$ the direct systems $\lambda(\varkappa)$ and $\mu(\varkappa)$.

5.2. Lemma. — *For any double direct system of Abelian groups* $\varkappa = \{ H^k_\alpha ; \ \}$ *indexed by* $\mathfrak{N} \times \mathfrak{L}$, *we have a natural isomorphism*

$$\tau_\varkappa : \ \underset{\alpha}{\underrightarrow{\mathrm{Lim}}}\,\underset{k}{\underrightarrow{\mathrm{Lim}}} \{ H^k_\alpha ; \ \} \to \underset{k}{\underrightarrow{\mathrm{Lim}}}\,\underset{\alpha}{\underrightarrow{\mathrm{Lim}}} \{ H^k_\alpha ; \ \}$$

between the limit groups; more precisely, if

$$\{ f_{k\alpha} \} : \ \{ H^k_\alpha ; \ \} \to \{ \overline{H}^k_\alpha ; \ \}$$

is a map between two double direct systems of Abelian groups, then the following diagram commutes :

$$\begin{array}{ccc}
\underset{\alpha}{\underrightarrow{\mathrm{Lim}}}\,\underset{k}{\underrightarrow{\mathrm{Lim}}}\{H^k_\alpha ; \ \} & \xrightarrow{\ \tau_\varkappa\ } & \underset{k}{\underrightarrow{\mathrm{Lim}}}\,\underset{\alpha}{\underrightarrow{\mathrm{Lim}}}\{H^k_\alpha ; \ \} \\
\Big\downarrow {\scriptstyle \underset{\alpha}{\underrightarrow{\mathrm{Lim}}}\,\underset{k}{\underrightarrow{\mathrm{Lim}}}\{f_{k\alpha}\}} & & \Big\downarrow {\scriptstyle \underset{k}{\underrightarrow{\mathrm{Lim}}}\,\underset{\alpha}{\underrightarrow{\mathrm{Lim}}}\{f_{k\alpha}\}} \\
\underset{\alpha}{\underrightarrow{\mathrm{Lim}}}\,\underset{k}{\underrightarrow{\mathrm{Lim}}}\{\overline{H}^k_\alpha ; \ \} & \xrightarrow{\ \tau_{\overline{\varkappa}}\ } & \underset{k}{\underrightarrow{\mathrm{Lim}}}\,\underset{\alpha}{\underrightarrow{\mathrm{Lim}}}\{\overline{H}^k_\alpha ; \ \}
\end{array}$$

In the contravariant case we have the following lemma on double inverse limits :

5.3. — Lemma. *Let* $\mathcal{L} = \{ \alpha, \beta, \gamma, \ldots \}$ *and* $\mathcal{N} = \{ k, l, m, \ldots \}$ *be two directed sets and* $\varkappa = \{ \mathrm{H}^\alpha_k; \quad \}$ *be a double inverse system of Abelian groups indexed by* $(\alpha, k) \in \mathcal{L} \times \mathcal{N}$. *Then we have a natural isomorphism*

$$\tau_\varkappa : \underset{\overleftarrow{k}}{\mathrm{Lim}}\ \underset{\overleftarrow{\alpha}}{\mathrm{Lim}} \{ \mathrm{H}^\alpha_k; \quad \} \rightarrow \underset{\overleftarrow{\alpha}}{\mathrm{Lim}}\ \underset{\overleftarrow{k}}{\mathrm{Lim}} \{ \mathrm{H}^\alpha_k; \quad \}$$

between the limit; the meaning of naturality is similar to that in Lemma 5.2.

6. Continuity of the functor $\mathcal{H}^{\infty-n}_0$. — In this section, we show that the functor $\mathcal{H}^{\infty-n}_0$ is continuous.

Let Y be an object and $\{ \mathrm{Y}_k \}_{k \in \mathcal{N}}$ be an approximating family for Y. Denote by

$$j_k : \mathrm{Y} \rightarrow \mathrm{Y}_k, \qquad i_{kl} : \mathrm{Y}_l \rightarrow \mathrm{Y}_k \qquad (k < l)$$

the corresponding inclusions and consider the direct system of abelian groups $\{ \mathcal{H}^{\infty-n}(\mathrm{Y}_k), i^*_{kl} \}$ over $\mathcal{N}$, together with the direct family $\{ j^*_k \}$ of homomorphisms

$$j^*_k : \mathcal{H}^{\infty-n}(\mathrm{Y}_k) \rightarrow \mathcal{H}^{\infty-n}(\mathrm{Y}).$$

6.1. Theorem. — *The map*

$$\underset{\overrightarrow{k}}{\mathrm{Lim}} \{ j^*_k \} : \underset{\overrightarrow{k}}{\mathrm{Lim}} \{ \mathcal{H}^{\infty-n}(\mathrm{Y}_k); i^*_{kl} \} \rightarrow \mathcal{H}^{\infty-n}(\mathrm{Y})$$

is an isomorphism. In other words, the functor $\mathcal{H}^{\infty-n}_0 : \mathfrak{L}_0 \rightarrow \mathcal{A}b$ *is continuous.*

Proof. — For an arbitrary element α in $\mathcal{L}_\mathrm{Y}$ and $k \in \mathcal{N}$, let us put

$$\mathrm{Y}^k_\alpha = \mathrm{Y}_k \cap \mathrm{L}_\alpha$$

and denote by

$$j_{k\alpha} : \mathrm{Y}_\alpha \rightarrow \mathrm{Y}^k_\alpha, \qquad i^\alpha_{kl} : \mathrm{Y}^l_\alpha \rightarrow \mathrm{Y}^k_\alpha \qquad (k \leq l)$$

the corresponding inclusions.

Now, for any relations $k \leq l$ and $\alpha \leq \beta$ in $\mathcal{N}$ and $\mathcal{L}_\mathrm{Y}$ respectively, consider the following diagram :

$$\begin{array}{ccc} \mathcal{H}^{d(\alpha)-n}(\mathrm{Y}^k_\alpha) & \xrightarrow{(i^\alpha_{kl})^*} & \mathcal{H}^{d(\alpha)-n}(\mathrm{Y}^l_\alpha) \\ \Big\downarrow {\scriptstyle \Delta^k_{\alpha\beta}} & & \Big\downarrow {\scriptstyle \Delta^l_{\alpha\beta}} \\ \mathcal{H}^{d(\beta)-n}(\mathrm{Y}^k_\beta) & \xrightarrow{(i^\beta_{kl})^*} & \mathcal{H}^{d(\beta)-n}(\mathrm{Y}^l_\beta) \end{array}$$

It follows from Definition 3.2 and Proposition 1.3 that the above diagram is commutative. Consequently, the groups $\{ \mathcal{H}^{d(\alpha)-n}(\mathrm{Y}^k_\alpha) \}$, together with the homomorphisms $(i^{\alpha}_{kl})^*$ and $\Delta^k_{\alpha\beta}$, determine a double direct system of abelian groups over $\mathfrak{N} \times \mathfrak{L}_Y$ which we denote simply $\varkappa = \{ \mathcal{H}^{d(\alpha)-n}(Y^k_\alpha); \}$.

Let $\bar{\varkappa} = \{ \mathcal{H}^{d(\alpha)-n}(Y^{\cdot}_\alpha); \Delta_{\alpha\beta} \}$ be the $(\infty - n)$-cohomology system of Y. We shall treat $\varkappa$ as a double direct system over $\mathfrak{N} \times \mathfrak{L}_Y$.

Now let us consider the double family of homomorphisms $\{ j^*_{k\alpha} \}$. Taking into account the various commutativity relations between the inclusions, it follows from Definition 3.2 and Proposition 1.3 that $\{ j^*_{k\alpha} \}$ is a map from $\varkappa$ to $\bar{\varkappa}$.

In view of the continuity of the cohomology theory $\{ \mathcal{H}^q, \delta^q \}$, the map $\varinjlim_k \{ j^*_{k\alpha} \}$ is an isomorphism for each $\alpha \in \mathfrak{L}_Y$, and therefore so is the map $\varinjlim_\alpha \varinjlim_k \{ j^*_{k\alpha} \}$. Consequently, in view of Lemma 5.2 the map

$$\varinjlim_k \{ j^*_k \} = \varinjlim_k \varinjlim_\alpha \{ j^*_{k\alpha} \}$$

is also an isomorphism and the proof of the theorem is completed.

Now, Theorem 6.1, in view of Theorem III.3.6, gives us the final result of this chapter :

6.2. Theorem. — *The functor $\mathcal{H}^{\infty-n}_0$ extends uniquely from $\mathfrak{L}_0$ over $\mathfrak{L}$ to an h-functor $\mathcal{H}^{\infty-n} : \mathfrak{L} \to \mathfrak{Ab}$.*

7. Consecutive pairs of triads and Proof of Lemma 3.1. — *Notation.* — The following symbols denote the subsets of R^{k+2} :

$$Q^{k+1} = \{ x \in R^{k+2};\ x_{k+1} = 0 \},$$
$$Q^{k+1}_+ = \{ x \in Q^{k+1};\ x_{k+2} \geqq 0 \},$$
$$Q^{k+1}_- = \{ x \in Q^{k+1};\ x_{k+2} \leqq 0 \},$$
$$P^{k+2}_+ = \{ x \in R^{k+2};\ x_{k+1} \geqq 0 \},$$
$$P^{k+2}_- = \{ x \in R^{k+2};\ x_{k+1} \leqq 0 \}.$$

The proof of Lemma 3.1 will be preceded by a preliminary discussion about the triads. We assume in this section that $\mathcal{H}^* = \{ \mathcal{H}^q, \delta^q \}$ is a cohomology theory on the category $\mathcal{K}_E$. By a triad we understand an additive triad in $\mathcal{K}_E$.

For a triad $T = (X; X_1, X_2)$ we let $-T = (X; X_2, X_1)$ and denote by $\Delta^n(T)$, or simply by $\Delta(T)$, the Mayer-Vietoris homomorphism :

$$\Delta(T) : \mathcal{H}^n(X_1 \cap X_2) \to \mathcal{H}^{n+1}(X)$$

of the triad T. We note that

(1) $$\Delta(T) = -\Delta(-T).$$

Let $T_0 = (Y; Y_1, Y_2)$ and $T = (X; X_1, X_2)$ be two triads. A pair (T_0, T) is a *consecutive pair of triads*, written $T_0 \Rightarrow T$, provided $Y_1 \cup Y_2 = Y = X_1 \cap X_2$; we say in this case that (T_0, T) *starts* at $Y_1 \cap Y_2$ and *ends* at $X_1 \cup X_2$.

We observe, that if (T_0, T) is a consecutive pair of triads, then we may form the composite

$$\Delta(T) \circ \Delta(T_0) = \Delta^{n+1}(T) \circ \Delta^n(T_0) : \mathcal{H}^n(Y_1 \cap Y_2) \to \mathcal{H}^{n+2}(X)$$

of the corresponding Mayer-Vietoris homomorphisms.

7.1. Lemma. — *Let us assume that in the following diagram of triads:*

$$\begin{array}{ccc} T_0 & \Longrightarrow & T \\ \uparrow \cup & & \uparrow \cup \\ T'_0 & \Longrightarrow & T' \end{array}$$

the consecutive pairs

$$(T_0, T) = ((Y; Y_1, Y_2), \quad (X; X_1, X_2)),$$
$$(T'_0, T') = ((Y'; Y'_1, Y'_2), \quad (X'; X'_1, X'_2))$$

both start at $Y_1 \cap Y_2 = Y'_1 \cap Y'_2$ *and both end at* $X = X'$. *Then, for the composites of the corresponding Mayer-Vietoris homomorphisms we have*

$$\Delta(T) \circ \Delta(T_0) = \Delta(T') \circ \Delta(T'_0).$$

Proof. — This is an immediate consequence of Proposition 1.4.

7.2 Lemma. — *Let us assume that*

$$((Y; Y_1, Y_2), \quad (X; X_1, X_2))$$

and

$$((Z; Z_1, Z_2), \quad (X; W_1, W_2))$$

are two consecutive pairs of triads both starting at

$$Y \cap Z = Y_1 \cap Y_2 = Z_1 \cap Z_2$$

and both ending at X. *Assume further that*

$$Z_i = Z \cap X_i \quad \textit{and} \quad Y_i = Y \cap W_i \quad \textit{for} \quad i = 1, 2.$$

Then, we have

$$\Delta(X; X_1, X_2) \circ \Delta(Y; Y_1, Y_2) = -\Delta(X; W_1, W_2) \circ \Delta(Z; Z_1, Z_2).$$

Proof. — Let us consider the following triads :

$$
\begin{aligned}
T_1 &= (Y; Y_1, Y_2), & T_2 &= (X; X_1, X_2),\\
T_3 &= ((W_1 \cap X_2) \cup Y_2); W_1 \cap X_2, Y_2), & T_4 &= (X; X_1 \cup W_1, X_2),\\
T_5 &= (Y_2 \cup Z_2; Z_2, Y_2), & T_6 &= (X; X_1 \cup W_1, X_2 \cap W_2),\\
T_7 &= ((X_1 \cap W_2) \cup Z_2; X_1 \cap W_2, Z_2), & T_8 &= (X; X_1 \cup W_1, W_2),\\
T_9 &= (Z; Z_1, Z_2), & T_{10} &= (X; W_1, W_2).
\end{aligned}
$$

We claim that every pair (T_{2i-1}, T_{2i}) for $i = 1, 2, 3, 4, 5$ is a consecutive pair of triad starting at $Y \cap Z$ and ending at X.

For $i = 1$ and $i = 5$, this is true by assumption.

Assume now that $i = 2$. Taking into account the inclusions

$$Y_2 = Y \cap W_2 \subset X_2 \quad \text{and} \quad Y_1 \subset W_1$$

we have respectively

$$(W_1 \cap X_2) \cap Y_2 = W_1 \cap Y_2 = W_1 \cap W_2 \cap Y = Z \cap Y,$$
$$(X_1 \cup W_1) \cap X_2 = (X_1 \cap X_2) \cup (W_1 \cap X_2) = Y \cup (W_1 \cap X_2) = Y_2 \cup (W_1 \cap X_2).$$

and thus the statement holds for $i = 2$.

Next, we suppose that $i = 4$. In this case, the proof is strictly analogous to that for $i = 2$.

Assuming finally that $i = 3$, we have

$$(Z_2 \cap Y_2) = (Z \cap X_2) \cap (Y \cap W_2) = (Z \cap W_2) \cap (Y \cap X_2) = Z \cap Y,$$

and

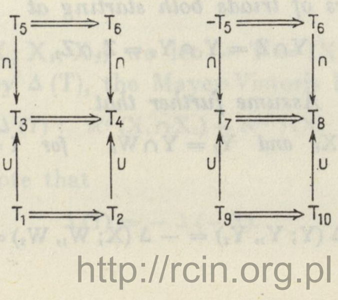
$$(X_1 \cup W_1) \cap (X_2 \cap W_2) = (X_1 \cap X_2 \cap W_2) \cup (W_1 \cap X_2 \cap W_2) = (Y \cap W_2) \cup (Z \cap X_2) = Y_2 \cup Z_2.$$

Thus, the proof of our statement is completed.

Further, we note the following inclusions between the triads

$$T_1, T_5 \subset T_3, \quad T_2, T_6 \subset T_4,$$
$$-T_5, T_9 \subset T_7, \quad T_6, T_{10} \subset T_8,$$

The various established interrelations between the triads may be displayed as follows :



Now, we let

$$\Delta_i = \Delta(T_i) \quad \text{for} \quad i = 1, 2, \ldots, 10$$

and apply Lemma (7.1) and property (1) to our situation. We obtain

$$\Delta_2 \Delta_1 = \Delta_4 \Delta_3 = \Delta_6 \Delta_5 = -\Delta_8 \Delta_7 = -\Delta_{10} \Delta_9$$

and the proof of the Lemma is completed.

Proof of Lemma 3.1. — We shall use the notation given at the beginning of this section. Letting $k = d(\alpha)$ we have $d(\gamma) = d(\tilde{\gamma}) = k + 1$ and $d(\beta) = k + 2$. Define a linear isomorphism $\Psi : R^{k+2} \to R^{k+2}$ by putting

$$\Psi(x_1, \ldots, x_k, x_{k+1}, x_{k+2}) = (x_1, \ldots, x_k, x_{k+2}, x_{k+1}).$$

Now take linear isomorphisms

$$l_\alpha : L_\alpha \to R^k, \qquad l_\gamma : L_\gamma \to R^{k+1}, \qquad l_{\tilde{\gamma}} : L_{\tilde{\gamma}} \to R^{k+1}$$

such that

$$l_\alpha \in \mathcal{O}_\alpha, \qquad l_\gamma \in \mathcal{O}_\gamma, \qquad l_{\tilde{\gamma}} \in \mathcal{O}_{\tilde{\gamma}}$$

and

$$l_\gamma(x) = l_{\tilde{\gamma}}(x) = l_\alpha(x) \quad \text{for all } x \in L_\alpha.$$

There is a unique isomorphism $l : L_\beta \to R^{k+2}$ such that

$$l(x) = l_\gamma(x) \quad \text{for all } x \in L_\gamma,$$
$$l(x) = \Psi \circ l_{\tilde{\gamma}}(x) \quad \text{for all } x \in L_{\tilde{\gamma}}.$$

Consider the following triads :

$$T_1 = X \cap l^{-1}(R^{k+1}; R_+^{k+1}, R_-^{k+1}),$$
$$T_2 = X \cap l^{-1}(R^{k+2}; R_+^{k+2}, R_-^{k+2}),$$
$$T_3 = X \cap l^{-1}(Q^{k+1}; Q_+^{k+1}, Q_-^{k+1}),$$
$$T_4 = X \cap l^{-1}(R^{k+2}; P_+^{k+2}, P_-^{k+2}).$$

By straightforward computation, one easily verifies that (T_1, T_2) and (T_3, T_4) are consecutive pairs of triads satisfying the assumption of Lemma 7.2.

We have therefore

(2) $$\Delta(T_2)\,\Delta(T_1) = -\Delta(T_4)\,\Delta(T_3).$$

There are two possibilities : either $l \in \mathcal{O}_\beta$ or $l \in -\mathcal{O}_\beta$. Now, we shall show that, in any of the above cases, we obtain the desired conclusion

(★) $$\Delta_{\gamma\beta} \circ \Delta_{\alpha\gamma} = \Delta_{\tilde{\gamma}\beta} \circ \Delta_{\alpha\tilde{\gamma}}.$$

If $l \in \mathcal{O}_\beta$, then $\Psi \circ l \in -\mathcal{O}_\beta$ and we have

$$\Delta_{\alpha\gamma} = \Delta(T_1), \quad \Delta_{\gamma\beta} = \Delta(T_2),$$
$$\Delta_{\alpha\tilde{\gamma}} = \Delta(T_3), \quad \Delta_{\tilde{\gamma}\beta} = -\Delta(T_4).$$

Thus, in view of (2), we obtain (★).

If $l \in -\mathcal{O}_\beta$, then $\Psi \circ l \in \mathcal{O}_\beta$ and we have

$$\Delta_{\alpha\gamma} = \Delta(T_1), \quad \Delta_{\gamma\beta} = -\Delta(T_2),$$
$$\Delta_{\alpha\tilde{\gamma}} = \Delta(T_3), \quad \Delta_{\tilde{\gamma}\beta} = \Delta(T_4).$$

Consequently, again by (2), we get the desired formula (★). The proof of Lemma 3.1 is completed.

CHAPTER V

Cohomology theories on $\mathfrak{L}$

Having defined the absolute cohomology functors $\mathcal{H}^{\infty-n}$ we turn in this chapter to the relative case and show that to any cohomology theory $\mathcal{H}^*$ on $\mathcal{K}_E$ corresponds certain "infinite dimensional" cohomology theory $\mathcal{H}^{\infty-*}$ on the Leray-Schauder category $\mathfrak{L}$. More specifically, for every n, we construct the relative cohomology functor $(X, A) \to \mathcal{H}^{\infty-n}(X, A)$, the coboundary transformation $\delta^{\infty-n} : \mathcal{H}^{\infty-n}(A) \to \mathcal{H}^{\infty-n+1}(X, A)$, and then we prove that $\mathcal{H}^{\infty-*} = \{\mathcal{H}^{\infty-n}, \delta^{\infty-n}\}$ is a cohomology theory on $\mathfrak{L}$ in the sense of Definition IV.1.1.

1. The relative cohomology functor $\mathcal{H}^{\infty-n}$. — *Notations.* — Throughout this chapter, $F = E \oplus R$ stands for the direct product of E and the real line R; we consider E as a 1-codimensional linear subspace of F. We fix a point in F not lying in E by putting $y^+ = (0, 1)$, where $0 \in E$, $1 \in R$.

We begin by fixing an orientation $\{\mathcal{O}_\alpha\}$ in the space E. For technical reasons, we shall consider also an orientation in the space F; this will be defined in a specified way as follows. Let $\mathcal{L}_E$ and $\mathcal{L}_F$ be the directed sets of finite dimensional linear subspaces of E and F, respectively, and $\mathcal{L}_0$ be a subset of $\mathcal{L}_F$ consisting of those linear subspaces $\alpha \in \mathcal{L}_F$ which contain the point y; clearly, $\mathcal{L}_0$ is cofinal in $\mathcal{L}_F$. For $\alpha \in \mathcal{L}_E$ denote by α' the element of $\mathcal{L}_0$ given by $L_{\alpha'} = L_\alpha + R\,y^+$.

To the orientation $\{\mathcal{O}_\alpha\}$ in E we assign an orientation $\{\overline{\mathcal{O}}_\alpha\}$ in F by the following rule : If $\alpha \in \mathcal{L}_E$, we let $\overline{\mathcal{O}}_\alpha = \mathcal{O}_\alpha$. If $\alpha \notin \mathcal{L}_E$ and $\alpha \notin \mathcal{L}_0$, we define $\overline{\mathcal{O}}_\alpha$ arbitrarily. Assuming that $\alpha \in \mathcal{L}_0$, there is a $\beta \in \mathcal{L}_E$ such that

$\beta' = \alpha$. We take a representative $l : L_\beta \to R^k$ in $\mathcal{O}_\beta$, where $k = d(\beta)$ and put

$$l(x) = (l_1(x), \ldots, l_k(x)) \quad \text{for} \quad x \in L_\beta.$$

Now let $\bar{l} : L_\alpha \to R^{k+1}$ be a linear map such that

$$\bar{l}(x) = (0, l_1(x), \ldots, l_k(x)) \quad \text{for} \quad x \in L_\beta,$$
$$\bar{l}(y^+) = (1, 0, \ldots, 0)$$

and let $\overline{\mathcal{O}}_\alpha$ be the orientation of L_α determined by $\bar{l}$. Thus, we have defined an orientation $\{\overline{\mathcal{O}}_\alpha\}$ in F; we call $\{\overline{\mathcal{O}}_\alpha\}$ an *extension* of $\{\mathcal{O}_\alpha\}$ from E over F.

From now on, we assume that such an orientation $\{\overline{\mathcal{O}}_\alpha\}$ in F is fixed.

Next, consider the categories

$$\mathfrak{L}_E = \mathfrak{L}(E), \qquad \mathfrak{L}_F = \mathfrak{L}(F)$$

and observe that $\mathfrak{L}_E$ and $\mathfrak{L}_0(E)$ are h-subcategories of $\mathfrak{L}_F$ and $\mathfrak{L}_0(F)$, respectively. We will denote by $\mathfrak{L}_E^2$ and $\mathfrak{L}_F^2$ the corresponding h-categories of pairs.

In what follows, it will be convenient, for technical reasons, to reduce certains facts in the relative case to those in the absolute case. This will be done with the aid of a functor ρ from $\mathfrak{L}_E^2$ to $\mathfrak{L}_F$ which will now be defined in terms of the cone functor as follows. Let $C : \mathfrak{L}_E \to \mathfrak{L}_F$ be the cone functor corresponding to the point y^+. We recall that for $A \subset E$

$$C(A) = \{ x \in F ;\ x = ta + (1-t)\, y^+,\ a \in A,\ 0 \leqq t \leqq 1 \},$$

and for $f : A \to B$ in $\mathfrak{L}_E$ the field $C(f)$ is given by

$$C(f)(x) = t f(a) + (1-t)\, y^+ \quad \text{for all } x \in C(A).$$

Now, given a pair (X, A) in $\mathfrak{L}_E$, let us put

$$\rho(X, A) = \begin{cases} X \cup CA & \text{if } A \neq \emptyset, \\ X & \text{if } A = \emptyset, \end{cases}$$

and for a map $f : (X, A) \to (Y, B)$ in $\mathfrak{L}_E$ define

$$\rho(f) = \tilde{f} : X \cup CA \to Y \cup CB$$

by

$$\tilde{f}(x) = \begin{cases} C f(x) & \text{for all } x \in CA, \\ f(x) & \text{for all } x \in X. \end{cases}$$

As a consequence of Proposition II.6.5 we obtain the following :

1.1. Proposition. — *The assignments* $(X, A) \to (X \cup CA)$ *and* $f \to \tilde{f}$ *define a covariant h-functor* ρ *from the category* $\mathfrak{L}^2_E$ *to the category* $\mathfrak{L}_F$. *Moreover, we have* $\rho(\mathfrak{L}^2_0(E)) \subset \mathfrak{L}_0(F)$ *and* $\rho(\mathcal{K}^2_E) \subset \mathcal{K}_F$.

Now let $\mathcal{H}^*$ be a fixed cohomology theory on $\mathcal{K}_E$ and (X, A) be a pair in $\mathfrak{L}_E$. We turn to defining the relative groups $\mathcal{H}^{*-n}(X, A)$. To this end, for an $\alpha \in \mathcal{L}_E$ such that $X_\alpha \neq \emptyset$, let

$$e_\alpha : (X_\alpha, A_\alpha) \to (X_\alpha \cup CA_\alpha, CA_\alpha),$$
$$j_\alpha : (X_\alpha \cup CA_\alpha) \to (X_\alpha \cup CA_\alpha, CA_\alpha),$$
$$i_\alpha : CA_\alpha \to X_\alpha \cup CA_\alpha$$

denote the corresponding inclusions. Since e_α in an excision the induced map

$$e^*_\alpha : \mathcal{H}^n(X_\alpha \cup CA_\alpha, CA_\alpha) \to \mathcal{H}^n(X_\alpha, A_\alpha)$$

is an isomorphism. We have the exact sequence of the pair $(X_\alpha \cup CA_\alpha, CA_\alpha)$

$$\ldots \to \mathcal{H}^n(X_\alpha \cup CA_\alpha, CA_\alpha) \xrightarrow{j^*_\alpha} \mathcal{H}^n(X_\alpha \cup CA_\alpha) \xrightarrow{i^*_\alpha} \mathcal{H}^n(CA_\alpha) \to \ldots$$

Since CA_α is homotopy equivalent to a point, $\operatorname{Ker} i^*_\alpha = \operatorname{Im} j^*_\alpha$ is a direct summand of $\mathcal{H}^n(X_\alpha \cup CA_\alpha)$ and i^*_α is an epimorphism. Hence j^*_α is a monomorphism. Define a monomorphism

$$\eta_\alpha : \mathcal{H}^n(X_\alpha, A_\alpha) \to \mathcal{H}^n(X_\alpha \cup CA_\alpha)$$

by

$$\eta_\alpha = j^*_\alpha (e^*_\alpha)^{-1}.$$

Note that we have an exact sequence

$$\mathcal{H}^n(X_\alpha, A_\alpha) \xrightarrow{\eta_\alpha} \mathcal{H}^n(X_\alpha \cup CA_\alpha) \xrightarrow{i^*_\alpha} \mathcal{H}^n(CA_\alpha).$$

Let $\alpha < \beta$ be an elementary relation in $\mathcal{L}_E$ and suppose that X_α is non-empty. Then $(A_\beta; A^+_{\bar\beta}, A^-_{\bar\beta})$ is a proper subtriad of $(X_\beta; X^+_{\bar\beta}, X^-_{\bar\beta})$ and we denote the corresponding relative Mayer-Vietoris homomorphism by

$$\Delta^{(n)}_{\alpha\beta} : \mathcal{H}^{d(\alpha)-n}(X_\alpha, A_\alpha) \to \mathcal{H}^{d(\beta)-n}(X_\beta, A_\beta).$$

Note that $(X \cup CA)_{\alpha'} = X_\alpha \cup CA_\alpha$. Let $\Delta_{\alpha'\beta'}$ denote the Mayer-Vietoris homomorphism of the triad $((X \cup CA)_{\beta'}, (X \cup CA)^+_{\bar\beta'}, (X \cup CA)^-_{\bar\beta'})$ and $((CA)_{\beta'}, (CA)^+_{\bar\beta'}, (CA)^-_{\bar\beta'})$. Clearly we have the following :

1.2. Lemma. — *The following diagram commutes* :

$$\begin{array}{ccccc}
\mathcal{H}^{d(\alpha)-n}(X_\alpha, A_\alpha) & \xrightarrow{\eta_\alpha} & \mathcal{H}^{d(\alpha)-n}(X_\alpha \cup CA_\alpha) & \xrightarrow{i^*_\alpha} & \mathcal{H}^{d(\alpha)-n}(CA_\alpha) \\
\downarrow \Delta_{\alpha\beta} & & \downarrow \Delta_{\alpha'\beta'} & & \downarrow \Delta_{\alpha'\beta'} \\
\mathcal{H}^{d(\beta)-n}(X_\beta, A_\beta) & \xrightarrow{\eta_\beta} & \mathcal{H}^{d(\beta)-n}(X_\beta \cup CA_\beta) & \xrightarrow{i^*_\beta} & \mathcal{H}^{d(\beta)-n}(CA_\beta)
\end{array}$$

Now let $\alpha < \beta$ be an arbitrary relation in $\mathcal{L}_E$ and

$$\alpha = \alpha_0 < \alpha_1, <\ldots< \alpha_{k+1} = \beta.$$

We define

$$\Delta_{\alpha\beta}^{(n)} = \Delta_{\alpha_k\beta}^{(n)} \circ \ldots \circ \Delta_{\alpha\alpha_1}^{(n)}$$

as the composition of the corresponding Mayer-Vietoris homomorphisms. In view of Lemmas IV.3.1, 1.2 and since η_α are monomorphisms, the above definition does not depend on the choice of a chain. Furthermore, the groups $\mathcal{H}^{d(\alpha)-n}(X_\alpha, A_\alpha)$ together with the homomorphisms $\Delta_{\alpha\beta}^{(n)}$ form a direct system of abelian groups over $\mathcal{L}_X$ which we will call the $(\infty - n)$-th *cohomology system* of the pair (X, A) (corresponding to the theory $\mathcal{H}^*$ and the orientation $\{\mathcal{O}_\alpha\}$ in E).

1.3. Definition. — For an integer n we define the relative cohomology group

$$\mathcal{H}^{\infty-n}(X, A) = \varinjlim_{\alpha} \{\mathcal{H}^{d(\alpha)-n}(X_\alpha, A_\alpha); \Delta_{\alpha\beta}^{(n)}\}$$

as the direct limit of the $(\infty - n)$-th cohomology system of the pair (X, A). Evidently, this definition extends that of the absolute group $\mathcal{H}^{\infty-n}(X)$ given in Chapter IV.

Now, for the orientation $\{\overline{\mathcal{O}}_\alpha\}$ in F, apply the construction of the previous chapter to the space F and denote by $\overline{\mathcal{H}}^{\infty-n} : \mathfrak{L} \to \mathcal{A}b$ the functor corresponding to $\mathcal{H}^*$ and the orientation $\{\overline{\mathcal{O}}_\alpha\}$ in F.

Observe that, by Lemma 1.2, the family $\{\eta_\alpha\}$ is a direct family of maps between $\{\mathcal{H}^{d(\alpha)-n}(X_\alpha, A_\alpha); \Delta_{\alpha\beta}^{(n)}\}$ and $\{\mathcal{H}^{d(\alpha')-n-1}((X \cup CA)_{\alpha'}); \Delta_{\alpha'\beta'}^{(n+1)}\}$. Using Lemma 1.2, we conclude that the direct limit map

$$\eta = \varinjlim_{\alpha} \{\eta_\alpha\} : \mathcal{H}^{\infty-n}(X, A) \to \overline{\mathcal{H}}^{\infty-(n+1)}(X \cup CA)$$

is a monomorphism. Moreover, the sequence

$$\mathcal{H}^{\infty-n}(X, A) \to \overline{\mathcal{H}}^{\infty-(n+1)}(X \cup CA) \xrightarrow{\overline{\mathcal{H}}(i)} \overline{\mathcal{H}}^{\infty-(n+1)}(CA)$$

in which $i : CA \to X \cup CA$ denotes the inclusion, is exact.

1.4. Definition. — For $f : (X, A) \to (Y, B)$ in $\mathfrak{L}_E^2$ we define the induced map $f^* = \mathcal{H}^{\infty-n}(f)$ by imposing commutativity on the following diagram:

$$\begin{array}{ccccc}
\mathcal{H}^{\infty-n}(Y,B) & \xrightarrow{\ \eta\ } & \overline{\mathcal{H}}^{\infty-n-1}(Y \cup CB) & \xrightarrow{\ i^*\ } & \overline{\mathcal{H}}^{\infty-n-1}(CB) \\
\downarrow{\scriptstyle \mathcal{H}^{\infty-n}(f)} & & \downarrow{\scriptstyle \overline{\mathcal{H}}^{\infty-n-1}(\tilde{f})} & & \downarrow{\scriptstyle \overline{\mathcal{H}}^{\infty-n-1}(\tilde{f}|CA)} \\
\mathcal{H}^{\infty-n}(X,A) & \xrightarrow{\ \eta\ } & \overline{\mathcal{H}}^{\infty-n-1}(X \cup CA) & \xrightarrow{\ i^*\ } & \overline{\mathcal{H}}^{\infty-n-1}(CA)
\end{array}$$

Since γ are monomorphisms and the rows are exact this definition makes sense.

1.5. Proposition. — *The assignments* $(X, A) \to \mathcal{H}^{*-n}(X, A)$ *and* $f \to f^*$ *define a contravariant h-functor* $\mathcal{H}^{*-n}$ *from the category* $\mathfrak{L}^2_E$ *to the category of abelian groups. Moreover,* γ *is a natural transformation from the functor* $\mathcal{H}^{*-n}$ *to* $\overline{\mathcal{H}}^{*-(n+1)} \circ \rho$.

Proof. — Proposition 1.5 follows clearly from the definitions involved and Theorem IV.6.2.

2. The homomorphism δ^n_α. — In this section, we give some definitions and prove lemmas which will be used in defining the coboundary transformation δ^{*-n}. For an object A in $\mathfrak{L}_E$ we let

$$\mathcal{L}_A = \{ \alpha \in \mathcal{L}_E;\ A_\alpha \neq \emptyset \}.$$

2.1. Definition. — For a pair (X, A) in $\mathfrak{L}^2_E$ and $\alpha \in \mathcal{L}_A$ define the homomorphism

$$\delta^n_\alpha : \mathcal{H}^{d(\alpha)-n-1}(A_\alpha) \to \mathcal{H}^{d(\alpha)-n}(X_\alpha, A_\alpha)$$

by putting

$$\delta^n_\alpha = (-1)^{d(\alpha)} \circ \delta_{(X_\alpha, A_\alpha)},$$

where $\delta_{(X_\alpha, A_\alpha)}$ is the coboundary homomorphism of the pair (X_α, A_α).

2.2. Lemma. — *Let* (X, A) *be a pair in* $\mathfrak{L}^2_E$. *Then, for every relation* $\alpha < \beta$ *in* $\mathcal{L}_A$ *the following diagram commutes :*

$$\begin{array}{ccc}
\mathcal{H}^{d(\alpha)-n-1}(A_\alpha) & \xrightarrow{\ \delta^n_\alpha\ } & \mathcal{H}^{d(\alpha)-n}(X_\alpha, A_\alpha) \\
\downarrow \Delta^{(n+1)}_{\alpha\beta} & & \downarrow \Delta^{(n)}_{\alpha\beta} \\
\mathcal{H}^{d(\beta)-n-1}(A_\beta) & \xrightarrow{\ \delta^n_\beta\ } & \mathcal{H}^{d(\beta)-n}(X_\beta, A_\beta)
\end{array}$$

Proof. — Assume first that $\alpha < \beta$ is elementary. Let

$$\tilde{\delta}^n_\alpha : \mathcal{H}^{d(\alpha)-n-1}(A_\alpha) \to \mathcal{H}^{d(\alpha)-n}(X_\alpha \cup CA_\alpha) = \mathcal{H}^{d(\alpha')-n-1}((X \cup CA)_{\alpha'})$$

denote the Mayer-Vietoris homomorphism of the triad $(X \cup CA_\alpha, CA_\alpha, X_\alpha)$. Evidently, we have

$$\tilde{\delta}^n_\alpha = (-1)^{d(\alpha)} \circ \gamma_{1\alpha} \circ \delta^n_\alpha.$$

Next, we observe that the consecutive pairs of triads

$$(X_\alpha \cup CA_\alpha;\ CA_\alpha, X_\alpha),\quad ((X \cup CA)_{\beta'};\ (X \cup CA)^+_{\beta'},\ (X \cup CA)^-_{\beta'}),$$
$$(A_\beta;\ A^+_\beta, A^-_\beta),\quad (X_\beta \cup CA_\beta;\ CA_\beta, X_\beta)$$

satisfy the assumptions of Lemma IV.7.2. Consequently,

$$\Delta_{\alpha'\beta'} \circ \tilde{\delta}_\alpha^n = - \tilde{\delta}_\beta^n \circ \Delta_{\alpha\beta}^{n+1}.$$

Now we consider the diagram

$$\begin{array}{ccccc}
\mathcal{H}^{d(\alpha)-n-1}(A_\alpha) & \xrightarrow{\delta_\alpha^n} & \mathcal{H}^{d(\alpha)-n}(X_\alpha, A_\alpha) & \xrightarrow{\eta_\alpha} & \mathcal{H}^{d(\alpha)-n}(X_\alpha \cup CA_\alpha) \\
\downarrow{\scriptstyle \Delta_{\alpha\beta}^{n+1}} & & \downarrow{\scriptstyle \Delta_{\alpha\beta}^{n}} & & \downarrow{\scriptstyle \Delta_{\alpha'\beta'}^{n}} \\
\mathcal{H}^{d(\beta)-n-1}(A_\beta) & \xrightarrow{\delta_\beta^n} & \mathcal{H}^{d(\beta)-n}(X_\beta, A_\beta) & \xrightarrow{\eta_\beta} & \mathcal{H}^{d(\beta)-n}(X_\beta \cup CA_\beta)
\end{array}$$

The composition of the top row homomorphisms equals $(-1)^{d(\alpha)} \tilde{\delta}_\alpha^n$ and the composition of the botton row homomorphisms equals $(-1)^{d(\beta)} \tilde{\delta}_\beta^n$. Since the right-hand square is, by Lemma 1.2, commutative we have

$$\eta_\beta \circ \Delta_{\alpha\beta}^n \, \delta_\alpha^n = \Delta_{\alpha'\beta'}^n \circ \eta_\alpha \circ \delta_\alpha^n = (-1)^{d(\alpha)} \Delta_{\alpha'\beta'}^n \circ \tilde{\delta}_\alpha^n = (-1)^{d(\beta)} \tilde{\delta}_\beta^n \circ \Delta_{\alpha\beta}^{n+1} = \eta_\beta \, \delta_\beta^n \, \Delta_{\alpha\beta}^{n+1}.$$

Since η_β is a monomorphism $\Delta_{\alpha\beta}^n \, \delta_\alpha^n = \delta_\beta^n \, \Delta_{\alpha\beta}^{n+1}$ and the proof is completed.

The following two propositions are immediate consequences of the definition of δ_α^n.

2.3. Proposition. — *Let* (X, A) *be a pair in* $\mathfrak{L}_E$ *and let* $i : A \to X$, $j : X \to (X, A)$ *denote the inclusion maps. Then, for every* α *in* $\mathcal{L}_A$, *the following sequence is exact :*

$$\ldots \xrightarrow{j_\alpha^*} \mathcal{H}^{d(\alpha)-n}(X_\alpha) \xrightarrow{i_\alpha^*} \mathcal{H}^{d(\alpha)-n}(A_\alpha) \xrightarrow{\delta_\alpha^{n+1}} \mathcal{H}^{d(\alpha)-n-1}(X_\alpha, A_\alpha) \xrightarrow{j_\alpha^*} \ldots$$

2.4. Proposition. — *Let* (X, A) *and* (Y, B) *be two pairs in* $\mathfrak{L}_E^2$ *and let* $f : (X, A) \to (Y, B)$ *be an* α_0*-field. Then, for any* α *in* $\mathcal{L}_A$ *such that* $\alpha_0 \leqq \alpha$, *the following diagram commutes :*

$$\begin{array}{ccc}
\mathcal{H}^{d(\alpha)-n-1}(A_\alpha) & \xrightarrow{\delta_\alpha^n} & \mathcal{H}^{d(\alpha)-n}(X_\alpha, A_\alpha) \\
\uparrow{\scriptstyle (f_\alpha|A_\alpha)^*} & & \uparrow{\scriptstyle (f_\alpha)^*} \\
\mathcal{H}^{d(\alpha)-n-1}(B_\alpha) & \xrightarrow{\delta_\alpha^n} & \mathcal{H}^{d(\alpha)-n}(Y_\alpha, B_\alpha)
\end{array}$$

3. Definition of the coboundary transformation $\delta^{\infty-n}$. — Now we are prepared to define the coboundary transformation $\delta^{\infty-n}$. Let (X, A) be a pair in $\mathfrak{L}_E^2$. It follows from Lemma 2.2 that the family $\{ \delta_\alpha^n \}$ is a direct family of homomorphisms. Taking into account definitions 1.3 and 2.1 and Lemma 2.2, the coboundary homomorphism

$$\delta^{\infty-n}(X, A) : \mathcal{H}^{\infty-n-1}(A) \to \mathcal{H}^{\infty-n}(X, A)$$

is defined by

$$\delta^{\infty-n}_{(X,A)} = \varinjlim_{\alpha} \{ \delta^n_\alpha \}.$$

Similarly, we let

$$\tilde{\delta}^{\infty-n}_{(X,A)} = \varinjlim_{\alpha} \{ (-1)^{d(\alpha)} \delta^n_\alpha \} : \quad \mathcal{H}^{\infty-n-1}(A) \to \mathcal{H}^{\infty-n-1}(X \cup CA).$$

3.1. Proposition. — *The following diagram commutes :*

$$\begin{array}{ccc} \mathcal{H}^{\infty-n-1}(A) & \xrightarrow{\delta^{\infty-n}} & \mathcal{H}^{\infty-n}(X,A) \\ & \searrow^{\tilde{\delta}^{\infty-n}} & \downarrow{\scriptstyle\eta} \\ & & \mathcal{H}^{\infty-n-1}(X \cup CA) \end{array}$$

Proof. — This clearly follows from the definitions involved.

3.2. Proposition. — *The family $\delta^{\infty-n} = \{ \delta^{\infty-n}_{(X,A)} \}$ indexed by the pairs (X, A) in $\mathcal{L}_E$ is a natural transformation from $\mathcal{H}^{\infty-n-1} \circ \rho$ to $\mathcal{H}^{\infty-n}$.*

Proof. — Consider the diagram

$$\begin{array}{ccccc} \mathcal{H}^{\infty-n-1}(B) & \xrightarrow{\delta^{\infty-n}} & \mathcal{H}^{\infty-n}(Y,B) & \xrightarrow{\eta} & \mathcal{H}^{\infty-n-1}(Y \cup CB) \\ \downarrow{\scriptstyle (f|_A)^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \tilde{f}^*} \\ \mathcal{H}^{\infty-n-1}(A) & \xrightarrow{\delta^{\infty-n}} & \mathcal{H}^{\infty-n}(X,A) & \xrightarrow{\eta} & \mathcal{H}^{\infty-n-1}(X \cup CA) \end{array}$$

Assume we have proved $\tilde{f}^* \circ \tilde{\delta}^{\infty-n} = \tilde{\delta}^{\infty-n} (f|_A)^*$.

Since the right-hand square is commutative, we have

$$\eta \circ f^* \circ \delta^{\infty-n} = \tilde{f}^* \circ \eta \circ \delta^{\infty-n} = \tilde{f}^* \circ \tilde{\delta}^{\infty-n} = \tilde{\delta}^{\infty-n} \circ (f|_A)^* = \eta \circ \delta^{\infty-n} (f|_A)^*,$$

Since η is a monomorphism this implies

$$f^* \circ \delta^{\infty-n} = \delta^{\infty-n} \circ (f|_A)^*.$$

Therefore it suffices to prove that, for $f : (X, A) \to (Y, B)$ in $\mathcal{L}^2_E$, the following diagram commutes :

$$\begin{array}{ccc} \mathcal{H}^{\infty-n-1}(B) & \xrightarrow{\tilde{\delta}^{\infty-n}} & \mathcal{H}^{\infty-n-1}(Y \cup CB) \\ \downarrow{\scriptstyle (f|_A)^*} & & \downarrow{\scriptstyle \tilde{f}^*} \\ \mathcal{H}^{\infty-n-1}(A) & \xrightarrow{\tilde{\delta}^{\infty-n}} & \mathcal{H}^{\infty-n-1}(X \cup CA) \end{array}$$

Assume first that the field f is finite dimensional. In this special case, we apply a straightforward passage to the limit in the commutative diagram of Proposition 2.2 and 2.4 and the desired conclusion follows by Proposition 3.1.

Consider now the general case and take an approximating system

$$f^{(k)} : (X, A) \to (Y_k, B_k)$$

for f. The definition and the proof of the existence of such a system is similar to that in the absolute case. It follows from (1.1) that the sequence

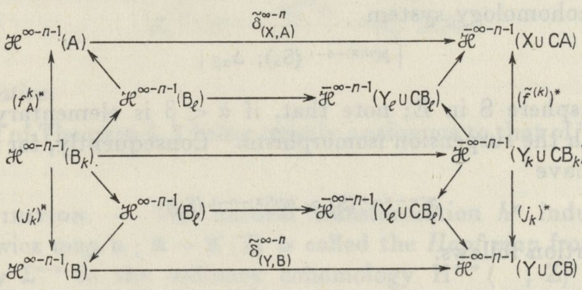
$$\tilde{f}^{(k)} : X \cup CA \to Y_k \cup CB_k$$

forms an approximating system for $\tilde{f} : X \cup CA \to Y \cup CB$.

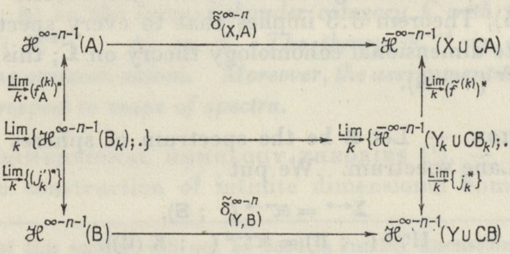
Consider the inclusions

$$j_k : Y \cup CB \to Y_k \cup CB_k, \qquad j'_k : B \to B_k,$$
$$i_{kl} : Y_l \cup CB_l \to Y_k \cup CB_k, \qquad i'_{kl} : B_l \to B_k \qquad (k \leq l).$$

By the special case of our assertion, the following diagram commutes for each pair $k \leq l$:



Applying the direct limit functor to the corresponding commutative diagram in the category of direct systems of abelian groups, we obtain the following commutative diagram :



By Theorem IV.6.1. the homomorphisms $\underset{k}{\underrightarrow{\mathrm{Lim}}}\{(j'_k)^*\}$ and $\underset{k}{\underrightarrow{\mathrm{Lim}}}\{j^*_k\}$ are invertible. This, in view of Proposition 3.1 and the definition of the induced map, implies our assertion and thus the proof is completed.

3.3. Theorem. — $\mathcal{H}^{\infty-*} = \{\mathcal{H}^{\infty-n}, \delta^{\infty-n}\}$ *is a cohomology theory on* $\mathfrak{L}$. *Moreover for each n, we have*

$$\mathcal{H}^{\infty-n-1}(S) \approx \mathcal{H}^n \text{ (point)},$$

i. e., the coefficients of the theory $\mathcal{H}^{\infty-*}$ *coincide with those of the theory* $\mathcal{H}^*$.

Proof. — The Exactness Axiom follows from 2.3 and the definition of $\mathcal{H}^{\infty-n}$ by passing to the limit with α.

To show that the Excision Axiom is fullfilled, let (X; A, B) be a triad in $\mathfrak{L}$ with $A \cup B = X$; if $k : (A, A \cap B) \to (X, B)$ is the inclusion then so is $\tilde{k} : A \cup C(A \cap B) \to X \cup CB$. Since $(\tilde{k}_\alpha)^*$ is an isomorphism for each α and $(\tilde{k})^* = \underset{k}{\underrightarrow{\mathrm{Lim}}}\{\tilde{k}^*_\alpha\}$, it follows that $\mathcal{H}^{\infty-n}(k)$ is an isomorphism.

To show that the last assertion of Theorem 3.3 is satisfied, take the $(\infty - n - 1)$-cohomology system

$$\{\mathcal{H}^{d(\alpha)-n-1}(S_\alpha);\ \Delta_{\alpha\beta}\}$$

of the unit sphere S in E; note that, if $\alpha < \beta$ is elementary, then $\Delta_{\alpha\beta}$ coincides with the suspension isomorphism. Consequently, for sufficiently large α, we have

$$\mathcal{H}^{\infty-n-1}(S) \approx \mathcal{H}^{d(\alpha)-n-1}(S_\alpha)$$

and our assertion follows.

3.4. Corollary. — *If* (X, A) *and* (Y, B) *are two equivalent (or more generally h-equivalent) pairs in* $\mathfrak{L}$, *then for every n we have an isomorphism*

$$\mathcal{H}^{\infty-n}(X, A) \approx \mathcal{H}^{\infty-n}(Y, B).$$

4. Naturality. — Taking into account the results of Chapter I (Theorem I.6.4), Theorem 3.3 implies that to every spectrum **A** corresponds an infinite dimensional cohomology theory on $\mathfrak{L}$; this theory will be denoted by $\mathcal{H}^{\infty-*}(\ ;\mathbf{A})$.

4.1. Definition. — Let **S** be the spectrum of spheres and **K** (Π) the Eilenberg-Mac Lane spectrum. We put

$$\Sigma^{\infty-*} = \mathcal{H}^{\infty-*}(\ ;\mathbf{S}),$$
$$H^{\infty-*}(\ ;\Pi) = \mathcal{H}^{\infty-*}(\ ;\mathbf{K}(\Pi)),$$

$\Sigma^{\infty-*}$ is called the *stable cohomology* and $H^{\infty-*}(\ ;\Pi)$ the *ordinary cohomology* with coefficients in Π on the category $\mathfrak{L}$.

Let $\mathbf{A}_i$ $(i=1, 2)$ be two spectra $\mathcal{H}_i^{\infty-*} = \mathcal{H}^{\infty-*}(\ ;\mathbf{A}_i)$ $(i=1, 2)$ the corresponding cohomology theories on $\mathfrak{L}$ and let $\mathbf{h}:\mathbf{A}_1 \to \mathbf{A}_2$ be an arbitrarily given map between spectra.

4.2. Theorem. — *The map* $\mathbf{h}$ *induces a natural transformation* h^* *from* $\mathcal{H}_1^{\infty-*}$ *to* $\mathcal{H}_2^{\infty-*}$. *More precisely, for any compact field* $f:(X, A) \to (Y, B)$ *in* $\mathfrak{L}^2$ *the following diagrams :*

$$\begin{array}{ccc} \mathcal{H}_1^{\infty-n-1}(A) & \xrightarrow{\ \delta\ } & \mathcal{H}_1^{\infty-n}(X,A) \\ {\scriptstyle h^*}\downarrow & & \downarrow{\scriptstyle h^*} \\ \mathcal{H}_2^{\infty-n-1}(A) & \xrightarrow{\ \delta\ } & \mathcal{H}_2^{\infty-n}(X,A) \end{array}$$

$$\begin{array}{ccc} \mathcal{H}_1^{\infty-n}(Y,B) & \xrightarrow{\ \mathcal{H}_1^{\infty-n}(f)\ } & \mathcal{H}_1^{\infty-n}(X,A) \\ {\scriptstyle h^*}\downarrow & & \downarrow{\scriptstyle h^*} \\ \mathcal{H}_2^{\infty-n}(Y,B) & \xrightarrow{\ \mathcal{H}_2^{\infty-n}(f)\ } & \mathcal{H}_2^{\infty-n}(X,A) \end{array}$$

are commutative.

The proof of Theorem 4.1 being strictly analogous to that of Theorem 3.3 is omitted.

4.3. Definition. — The natural transformation h^* induced by the Hopf-Hurewicz map $\mathbf{h}:\mathbf{S} \to \mathbf{K}(Z)$ is called the *Hopf map* from the stable cohomotopy $\Sigma^{\infty-*}$ to the ordinary cohomology $H^{\infty-*}(\ ; Z)$ with integer coefficients on $\mathfrak{L}$.

We summarize now an essential part of the preceding discussion in the following :

4.4. Theorem. — *To every spectrum* $\mathbf{A}$ *there corresponds a cohomology theory* $\mathcal{H}^{\infty-*}(\ ;\mathbf{A})$ *on the Leray-Schauder category* $\mathfrak{L}$ *with the same group of coefficients as* $\mathcal{H}^*(\ ;\mathbf{A})$ *on* $\mathfrak{P}$. *The theory* $\mathcal{H}^{\infty-*}$ *is continuous and satisfies the strong excision axiom. Moreover, the assignment* $\mathbf{A} \to \mathcal{H}^{\infty-*}(\ ;\mathbf{A})$ *is natural with respect to maps of spectra.*

5. Infinite dimensional homology theories [5]. — In this section we indicate the construction of intinite dimensional homology theories.

[5] The results of this section will not be used in further discussion.

The proofs, being analogous to those in the contravariant case, are omitted and, to simplify the exposition, we treat only the " ordinary " theories. The symbol $\mathcal{A}b$ stands either : (i) for the category of abelian groups or (ii) for the category of compact topological abelian groups. Whenever several functors into $\mathcal{A}b$ occur these are to be interpreted in a fixed manner; in particular, in case (ii) the " homomorphism " will mean always continuous homomorphism.

Let $H_* = \{H_n, \partial_n\}$ be the ordinary Čech homology for compacta over a group of coefficients G. First, we sketch the construction of the functor $H_{\infty-n}(\ ;G)$ from $\mathfrak{L}$ to the category of abelian groups $\mathcal{A}b$.

We begin by fixing an orientation $\{\mathcal{O}_\alpha\}$ in E.

Let X be an object in $\mathfrak{L}$. Given an elementary relation $\alpha < \beta$ in $\mathcal{L}_X$ denote by

$$\Delta_n^{\alpha\beta} : \ H_{d(\beta)-n}(X_\beta) \to H_{d(\alpha)-n}(X_\alpha)$$

the Mayer-Vietoris homomorphism of the triad $(X_\beta; X_\beta^+, X_\beta^-)$. Given an arbitrary relation $\alpha < \beta$ in $\mathcal{L}_X$ take a chain $\alpha = \alpha_0 < \alpha_1 < \ldots < \alpha_K = \beta$ of elementary relations in $\mathcal{L}_X$ joining α and β and define $\Delta^{\alpha\beta}$ to be the composition of the corresponding Mayer-Vietoris homomorphisms.

The definition of $\Delta_{(n)}^{\alpha\beta}$ does not depend on the choice of chain of elementary relations joining α and β. Consequently, the groups $H_{d(\alpha)-n}(X_\alpha)$ together with the homomorphisms $\Delta_{(n)}^{\alpha\beta}$ given for each relation $\alpha < \beta$ in $\mathcal{L}_X$ form an inverse system of abelian groups over $\mathcal{L}_X$.

5.1. Definition. — For an object X in $\mathfrak{L}$ (E) we define an abelian group

$$H_{\infty-n}(X) = \varprojlim_{\alpha} \{H_{d(\alpha)-n}(X_\alpha); \Delta_{(n)}^{\alpha\beta}\},$$

as the inverse limit over $\mathcal{L}_X$ of the system $\{H_{d(\alpha)-n}(X_\alpha); \Delta_{(n)}^{\alpha\beta}\}$.

Let $f : X \to Y$ be in $\mathfrak{L}_0$. Then $x - f(x)$ has values in L_{α_0} for some α_0 and for each $\alpha > \alpha_0$ we have $f(X_\alpha) \subset Y_\alpha$. Consequently, f determines the map $f^\alpha : X_\alpha \to Y_\alpha$. Moreover, for every elementary relation $\alpha < \beta$, f^β maps the triad $(X_\beta; X_\beta^+, X_\beta^-)$ into $(Y_\beta; Y_\beta^+, Y_\beta^-)$. It follows that f induces a map $\{f_*^\alpha\}$ from the inverse system $\{H_{d(\alpha)-n}(X_\alpha); \Delta_{(n)}^{\alpha\beta}\}$ to $\{H_{d(\alpha)-n}(Y_\alpha); \Delta_{(n)}^{\alpha\beta}\}$.

5.2. Definition. — We define the *induced map*

$$f_* = \varprojlim_{\alpha} \{f_*^\alpha\} : \ H_{\infty-n}(X) \to H_{\infty-n}(Y)$$

as the inverse limit of the system of homomorphisms $\{f_*^\alpha\}$.

Let us put now

$$H_{\infty-n}^0(X) = H_{\infty-n}(X) \quad \text{for X in } \mathfrak{L}_0,$$
$$H_{\infty-n}^0(f) = f_* \quad \text{for } f \text{ in } \mathfrak{L}_0.$$

5.3. Proposition. — *The function* $H^0_{\infty-n} : \mathfrak{L}_0 \to \mathfrak{Ab}$ *is a covariant h-functor from the category* $\mathfrak{L}_0$ *to the category of abelian groups* $\mathfrak{Ab}$.

Let Y be an object in $\mathfrak{L}$ and $\{Y_k\}$ be an approximative sequence for Y. Consider the inclusions $j^k : Y \to Y_k$, $i^{kl} : Y_l \to Y_k$ $(k \leq l)$ (all being in $\mathfrak{L}_0$), the corresponding inverse system of abelian groups $\{H_{\infty-n}(Y_k);\ i^{kl}_*\}$ and the family $\{j^k_*\}$ of the induced maps

$$j^k_* :\ H_{\infty-n}(Y) \to H_{\infty-n}(Y_k).$$

5.4. Lemma. — *The family* $\{j^k_*\}$ *is an inverse family of homomorphisms and the map*

$$\underset{\underset{k}{\longleftarrow}}{\mathrm{Lim}}\{j^k_*\} :\ H_{\infty-n}(Y) \to \underset{\underset{k}{\longleftarrow}}{\mathrm{Lim}}\{H_{\infty-n}(Y_k);\ i^{kl}_*\};$$

is invertible; in other words, the functor $H^0_{\infty-n}$ *is continuous.*

The proof uses the continuity of the Čech homology and the lemma on interchanging the double inverse limits.

Now from Lemma 5.4, in view of Theorem III.3.6, we get the following:

5.5. Theorem. — $H^0_{\infty-n}$ *extends uniquely from* $\mathfrak{L}_0$ *over* $\mathfrak{L}$ *to a covariant h-functor* $H_{\infty-n} : \mathfrak{L} \to \mathfrak{Ab}$.

By proceeding as in the case of cohomology one gets a sequence of covariant functors $(X, A) \mapsto H_{\infty-n}(X, A)$ from the pairs in $\mathfrak{L}$ to $\mathfrak{Ab}$ together with a sequence of natural transformations $\partial_{\infty-n} : H_{\infty-n+1}(X, A) \to H_{\infty-n}(A)$.

5.6. Definition. — Call an object X in $\mathfrak{L}$ a *polyhedron* provided it is equivalent in $\mathfrak{L}$ to an object Y such that for every α the intersection $Y_\alpha = Y \cap L_\alpha$ is a polyhedron.

Now we may state the main result of this section :

5.7. Theorem. — *The sequence of homology functors* $\{H_{\infty-n}\}$ *together with the sequence of boundary transformations* $\{\delta_{\infty-n}\}$ *satisfies the following properties :*

(*Semi-exactness*) *: For the inclusions*

$$i :\ A \to X, \qquad j :\ X \to (X, A)$$

the homology sequence

$$\ldots \longrightarrow H_{\infty-n}(A) \xrightarrow{i_*} H_{\infty-n}(X) \xrightarrow{j_*} H_{\infty-n}(X, A) \xrightarrow{\partial_{n-1}} H_{\infty-n-1}(A) \longrightarrow \ldots$$

is semi-exact. The above sequence is exact, if either G *is compact, or* (X, A) *is a polyhedral pair and* G *is a vector space over a field.*

(*Excision*) : *Let* $(X; A, B)$ *be a triad in* $\mathfrak{L}$ *with* $X = A \cup B$. *Then the inclusion*

$$k : (A, A \cap B) \to (X, B)$$

induces for every n an isomorphism

$$k_* : H_{\infty - n}(A, A \cap B) \to H_{\infty - n}(X, B).$$

(*Coefficients*) : *For the unit sphere* S *in* E *we have* $H_{\infty - 1}(S) = G$, $H_{\infty - n}(S) = 0$ *for* $n > 1$.

CHAPTER VI

Duality theorems

We know from the previous chapters that, given a spectrum **A**, we have corresponding "infinite dimensional" cohomology theory $\{\mathcal{H}^{\infty - n}(\ ; \mathbf{A})\}$, on the Leray-Schauder category $\mathfrak{L}$ and the homology theory $\{\mathcal{H}_n(\ ; \mathbf{A})\}$ on the category $\mathfrak{C}$. In this chapter we are concerned with the Alexander type of duality in the infinite dimensional normed space E. We show that for an arbitrary spectrum **A** we have an isomorphism

$$D : \mathcal{H}^{\infty - n}(X, Y; \mathbf{A}) \to \mathcal{H}_n(E - Y, E - X; \mathbf{A})$$

and establish some of its properties of further importance. For the convenience of the reader, we first treat the case when **A** is either $\mathbf{K}(\Pi)$ or **S** and then pass to the case of an arbitrary spectrum.

1. Cap product. — In this section we assume that we are given spectra **A**, **B**, **C** together with a pairing $(\mathbf{A}, \mathbf{B}) \to \mathbf{C}$. The pairing gives rise to a pairing

$$\diagdown : \tilde{\mathcal{H}}_n(X \wedge Y; \mathbf{A}) \otimes \tilde{\mathcal{H}}^q(X; \mathbf{B}) \to \tilde{\mathcal{H}}_{n-q}(Y; \mathbf{C})$$

called the slant product ([GWW], p. 258).

Let $(X, x_0) \in \mathfrak{T}_*$. Assume that (X, A_i), $(X, B_i) \in \mathfrak{T}^2_*$, $i = 1, 2$, are such that :

(*a*) $B_i \subset A_i$, $i = 1, 2$;

(*b*) $(A_1 \cap A_2) \cup B_1 \cup B_2 = X$.

Denote by $\pi_i : A_i \to A_i/B_i$, $i = 1, 2$, the projection map. Define the *reduced diagonal map* $\mu : X \to A_1/B_1 \wedge A_2/B_2$ by

$$\mu(x) = \begin{cases} \pi_1(x) \wedge \pi_2(x) & \text{for } x \in A_1 \cap A_2, \\ \star & \text{for } x \in B_1 \cup B_2. \end{cases}$$

The conditions (a) and (b) imply that μ is well defined and continuous. Define a pairing

$$\cap: \tilde{\mathcal{H}}_n(X; \mathbf{A}) \otimes \tilde{\mathcal{H}}^q(A_1/B_1; \mathbf{B}) \to \tilde{\mathcal{H}}_{n-q}(A_2/B_2; \mathbf{C})$$

called the *cap product*, by

$$\alpha \cap \beta = \mu^*(\alpha) \diagdown \beta, \qquad \alpha \in \tilde{\mathcal{H}}_n(X; \mathbf{A}), \qquad \beta \in \tilde{\mathcal{H}}^q(A_1/B_1; \mathbf{B}).$$

Let X' be another object in $\mathfrak{T}_*$ and assume (X', A_i'), $(X', B_i') \in \mathfrak{T}_*^2$ satisfy conditions (a) and (b). Then the diagonal map

$$\mu': \quad X' \to A_1'/B_1' \wedge A_2'/B_2'$$

gives rise to the cap product pairing

$$\cap': \quad \tilde{\mathcal{H}}_n(X'; \mathbf{A}) \otimes \tilde{\mathcal{H}}^q(A_1'/B_1'; \mathbf{B}) \to \tilde{\mathcal{H}}_{n-q}(A_2'/B_2'; \mathbf{C}).$$

Assume further that $f: X \to X'$ is a map in $\mathfrak{T}_*$ such that $f(A_i) \subset A_i'$, $f(B_i) \subset B_i'$ for $i = 1, 2$. Then f induces maps

$$\varphi: \quad A_1/B_1 \to A_1'/B_1', \qquad \psi: \quad A_2/B_2 \to A_2'/B_2'.$$

The following proposition describes the behavior of the cap product under the induced homomorphisms.

1.1. Proposition. — *If* $\alpha \in \tilde{\mathcal{H}}_n(X; \mathbf{A})$, $\beta \in \tilde{\mathcal{H}}^q(A_1'/B_1'; \mathbf{B})$, *then*

$$\psi_*(\alpha \cap \varphi^*(\beta)) = f_*(\alpha) \cap' \beta.$$

Proof. — Clearly $(\varphi \wedge \psi) \circ \mu = \mu' \circ f$. Therefore, using the definition of the cap product and 6.10 of [GWW] we have

$$\psi_*(\alpha \cap \varphi^*(\beta)) = \psi_*(\mu_*(\alpha) \diagdown \varphi^*(\beta)) = (\varphi \wedge \psi)_* \mu_*(\alpha) \diagdown \beta = \mu'_* f_*(\alpha) \diagdown \beta = f_*(\alpha) \cap' \beta.$$

Let X, A_i, B_i be as before and assume that $(Y; Y_1, Y_2)$ is a triad in $\mathfrak{T}_*$ such that $X = Y_1 \cap Y_2$. Assume further that (Y, C_i), $(Y, D_i) \in \mathfrak{T}_*^2$, $i = 1, 2$, satisfy conditions (a) and (b) and the following condition :

(c) $C_i \cap X = A_i$, $D_i \cap X = B_i$, $i = 1, 2$.

Hence there is the reduced diagonal map

$$\nu: \quad Y \to C_1/D_1 \wedge C_2/D_2$$

and corresponding cap product pairing

$$\cap: \quad \tilde{\mathcal{H}}_n(Y; \mathbf{A}) \otimes \tilde{\mathcal{H}}^q(C_1/D_1; \mathbf{B}) \to \tilde{\mathcal{H}}^n{}_{-q}(C_2/D_2; \mathbf{C}).$$

The inclusions $A_2 \subset C_2$, $B_2 \subset D_2$ induce an inclusion $i: A_2/B_2 \to C_2/D_2$. From (c) we obtain

$$(C_1 \cap Y_1)/(D_1 \cap Y_1) \cap (C_1 \cap Y_2)/(D_1 \cap Y_2) = A_1/B_1.$$

Let $\Delta^* : \tilde{\mathcal{H}}^q(A_1/B_1 ; \mathbf{B}) \to \tilde{\mathcal{H}}^{q+1}(C_1/D_1 ; \mathbf{B})$ denote the Mayer-Vietoris homomorphism of the triad

$$(C_1/D_1), \quad (C_1 \cap Y_1)/(D_1 \cap Y_1), \quad (C_1 \cap Y_2)/(D_1 \cap Y_2)$$

and

$$\Delta_* : \quad \tilde{\mathcal{H}}_{n+1}(Y ; \mathbf{A}) \to \tilde{\mathcal{H}}_n(X ; \mathbf{A})$$

denote the Mayer-Vietoris homomorphism of the triad $(Y ; Y_1, Y_2)$.

Now we are in a position to formulate the theorem which will be needed in the next section.

1.2. Theorem. — *Let* $\alpha \in \tilde{\mathcal{H}}_{n+1}(Y ; \mathbf{A})$, $\beta \in \tilde{\mathcal{H}}^q(A_1/B_1 ; \mathbf{B})$. *If* A_1 *and* B_1 *are strong deformation retracts of* C_1 *and* D_1, *respectively, then*

$$\alpha \cap \Delta^* \beta = (-1)^n i_* (\Delta_* \alpha \cap \beta) \quad \text{in} \quad \tilde{\mathcal{H}}_{n-q}(C_2/D_2 ; \mathbf{C}).$$

The proof of this theorem will be given separately in the last section of this chapter.

2. Duality in S^n for polyhedra. — In this section, with the aid of the cap product, we define (following G. W. Whitehead) the Alexander duality map D_n for polyhedra. By selecting for each n an appropriate orientation of S^n we specify D_n and then exhibit an important relation between D_{n-1} and D_n in terms of the Mayer-Vietoris homomorphism (Theorem 2.4). First a few preliminary definitions.

We assume that we are given a spectrum $\mathbf{A}$ and let $(\mathbf{S}, \mathbf{A}) \to \mathbf{A}$ be the natural pairing. We regard S^n as an object of $\mathfrak{T}_*$ and identify SS^{n-1} with S^n.

Choose a generator $z_1 \in \mathcal{H}_1(S^1 ; \mathbf{S}) = H_1(S^1 ; Z)$, where Z denotes the group of integers and define inductively

$$z_n \in \mathcal{H}_n(S^n ; \mathbf{S}) = H_n(S^n ; Z) \quad \text{such that} \quad z_{n-1} = (-1)^{n-1} \Delta(z_n).$$

Assume that we are given a triangulation of S^n and let K denote the corresponding simplicial complex. Let K′, K″ denote the first and second barycentric subdivisions, respectively. If M is a subcomplex of K′ we let N (M) denote the smallest subcomplex of K″ containing all simplexes which have non-empty intersections with M; N (M) is called the *normal neighbourhood* of M.

Note that N (M) is a closed subset of K. If L is a subcomplex of K we denote by L* the subcomplex of K′ consisting of all simplexes none of whose vertices are in L′; L* is called the *supplement* of L.

Now let L, M, $L \subset M$, be subcomplexes of K. Let $N(L) = N(L')$, $N(M) = N(M')$. Then the inclusions

$$(L, M) \subset (N(L), N(M)),$$
$$(M^*, L^*) \subset (N(M^*), N(L^*))$$

are homotopy equivalences. Therefore we have isomorphisms

$$\mathcal{H}^q(L, M; \mathbf{A}) \approx \mathcal{H}^q(N(L), N(M); \mathbf{A}) = \tilde{\mathcal{H}}^q(N(L)/N(M); \mathbf{A}),$$

$$\mathcal{H}_{n-q}(M^*, L^*; \mathbf{A}) \approx \mathcal{H}_{n-q}(N(M^*), N(L^*); \mathbf{A}) = \tilde{\mathcal{H}}_{n-q}(N(M^*)/N(L^*); \mathbf{A}).$$

In what follows we identify the above groups. Define the *Alexander duality*

$$D_n : \mathcal{H}^q(L, M; \mathbf{A}) \to \mathcal{H}_{n-q}(M^*, L^*; \mathbf{A})$$

by putting for $w \in \mathcal{H}^q(L, M; \mathbf{A}) = \tilde{\mathcal{H}}^q(N(L)/N(M); \mathbf{A})$,

$$D_n(w) = z_n \cap w.$$

This definition makes sense since $A_1 = N(L)$, $B_1 = N(M)$, $A_2 = N(M)$, $B_2 = N(L)$ satisfy the conditions (*a*) and (*b*) of the preceding section.

The following basic fact is an immediate consequence of the Theorem 7.4 of [GWW].

2.1. Proposition. — *The duality map* D_n *is an isomorphism.*

From the Theorem 6.31 of [GWW] we deduce the following important property :

2.2. Proposition. — *Let* $N \subset M \subset L$ *be subcomplexes of* K. *Then the diagram :*

$$\begin{array}{ccccccccc}
\cdots \to & \mathcal{H}^q(L,M;A) & \to & \mathcal{H}^q(L,N;A) & \to & \mathcal{H}^q(M,N;A) & \to & \mathcal{H}^{q+1}(L,M;A) & \to \cdots \\
& \downarrow D_n & & \downarrow D_n & & \downarrow D_n & & \downarrow D_n & \\
\cdots \to & \mathcal{H}_{n-q}(M^*,L^*,A) & \to & \mathcal{H}_{n-q}(N^*,L^*,A) & \to & \mathcal{H}_{n-q}(N^*,M^*;A) & \to & \mathcal{H}_{n-q-1}(M^*,L^*;A) & \to \cdots
\end{array}$$

in which the upper row is the cohomology sequence of the triple (L, M, N) *and the lower row is the homology sequence of the triple* (N^*, M^*, L^*) *has two left-hand squares commutative and the third square commutative up to the sign* $(-1)^{n+1}$

Let $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ be a map of spectra. Then $\mathbf{f}$ induces homomorphisms

$$\mathcal{H}^q(L, M; \mathbf{f}) : \mathcal{H}^q(L, M; \mathbf{A}) \to \mathcal{H}^q(L, M; \mathbf{B}),$$
$$\mathcal{H}_q(L, M; \mathbf{f}) : \mathcal{H}_q(L, M; \mathbf{A}) \to \mathcal{H}_q(L, M; \mathbf{B}).$$

We leave to the reader the proof of the next property.

2.3. Proposition. — *The following diagram is commutative :*

$$\begin{array}{ccc} \mathcal{H}^q(L,M;A) & \xrightarrow{\mathcal{H}^q(L,M;f)} & \mathcal{H}^q(L,M;B) \\ \downarrow{\scriptstyle D_n} & & \downarrow{\scriptstyle D_n} \\ \mathcal{H}_{n-q}(M^*,L^*;A) & \xrightarrow{\mathcal{H}_{n-q}(M^*,L^*;f)} & \mathcal{H}_{n-q}(M^*,L^*;B) \end{array}$$

Now assume that K is a triangulation of S^n such that S^{n-1} is a subcomplex of K. Let $M \subset L$ be subcomplexes of K and let (M^*, L^*) denote the corresponding dual pair in S^n. Let $M_0 = M \cap S^{n-1}$, $L_0 = L \cap S^{n-1}$ and let (M_0^*, L_0^*) denote the dual pair in S^{n-1}. Let $i : (M_0^*, L_0^*) \to (M^*, L^*)$ denote the inclusion map. Denote by Δ the relative Mayer-Vietoris homomorphism corresponding to the proper inclusion of triads

$$(M, M \cap C_+ S^{n-1}, M \cap C_- S^{n-1}) \subset (L, L \cap C_+ S^{n-1}, L \cap C_- S^{n-1}).$$

Now we state a theorem which will be the main tool in extending the Alexander type of duality to the infinite dimensional case.

2.4. Theorem. — *The following diagram commutes :*

$$\begin{array}{ccc} \mathcal{H}^{q-1}(L_0,M_0;A) & \xrightarrow{\Delta} & \mathcal{H}^q(L,M;A) \\ \downarrow{\scriptstyle D_{n-1}} & & \downarrow{\scriptstyle D_n} \\ \mathcal{H}_{n-q}(M_0^*,L_0^*;A) & \xrightarrow{i_*} & \mathcal{H}_{n-q}(M^*,L^*;A) \end{array}.$$

Proof. — If K_0 is a subcomplex of S^{n-1} we let $N_0(K_0)$ denote the normal neighbourhood of K_0 in S^{n-1}. In what follows we identify the following groups :

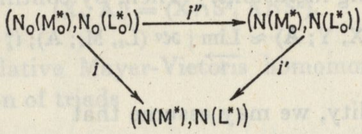
$$\mathcal{H}^{q-1}(L_0, M_0; A) = \mathcal{H}^{q-1}(N_0(L_0); N_0(M_0); A),$$
$$\mathcal{H}_{n-q}(M_0^*, L_0^*; A) = \mathcal{H}_{n-q}(N_0(M_0^*); N_0(L_0^*); A),$$
$$\mathcal{H}_{n-q}(M^*, L^*; A) = \mathcal{H}_{n-q}(N(M^*), N(L^*); A).$$

Using these identifications we let

$$i : (N_0(M_0^*), N_0(L_0^*)) \to (N(M^*), N(L^*))$$

denote the inclusion map. Let L_1 (resp. M_1) denote the subcomplex of K' consisting of all simplexes none of whose vertices are in L_0^* (resp. M_0^*).

Consider the following diagram of inclusions :



Let $j : (N(L), N(M)) \to (N(L_1), N(M_1))$ denote the inclusion map and let

$$\Delta' : \mathcal{H}^{q-1}(N_0(L_0), N_0(M_0); \mathbf{A}) \to \mathcal{H}^q(N(L_1), N(M_1); \mathbf{A})$$

denote the Mayer-Vietoris homomorphism corresponding to the proper inclusion of triads

$$(N(M_1), N(M_1) \cap C_+ S^{n-1}, N(M_1) \cap C_- S^{n-1}) \subset (N(L_1), N(L_1) \cap C_+ S^{n-1}, N(L_1) \cap C_- S^{n-1}).$$

Clearly

$$(1) \qquad \Delta = j^* \circ \Delta'.$$

Now take an arbitrary $\beta \in \mathcal{H}^q(N_0(L_0), N_0(M_0); \mathbf{A})$. The inclusion

$$(S^n; N(L), N(M); N(M^*), N(L^*)) \subset (S^n; N(L_1), N(M_1); N(M^*), N(L^*))$$

implies, by the Proposition 1.1,

$$(2) \qquad z_n \cap j^* \Delta' \beta = z_n \cap \Delta' \beta.$$

The inclusion

$$(S^n; N(L_1), N(M_1); N(M_0^*), N(L_0^*)) \subset (S^n; N(L_1), N(M_1); N(M^*), N(L^*))$$

implies (again by 1.1)

$$(3) \qquad z_n \cap \Delta' \beta = (i')_* (z_n \cap \Delta' \beta).$$

Applying Theorem 1.2 to $(S^n; N(L_1), N(M_1); N(M_0^*), N(L_0^*))$, we have

$$(4) \qquad z_n \cap \Delta' \beta = (i'')_* (z_{n-1} \cap \beta).$$

Therefore, by (1), (2), (3) and (4),

$$D_n \circ \Delta(\beta) = z_n \cap \Delta\beta = z_n \cap \Delta' \beta = (i')_* (z_n \cap \Delta' \beta)$$
$$= (i')_* (i'')_* (z_{n-1} \cap \beta) = i_* (z_{n-1} \cap \beta) = i_* \circ D_{n-1}(\beta).$$

The proof is completed.

3. Duality in S^n for compacta. — In this section $\mathcal{H}^* = \{\mathcal{H}^n(\ ; \mathbf{A}), \}$ stands for the continuous cohomology theory with coefficients in a spectrum $\mathbf{A}$ on the category $\mathcal{K}_{R^\infty}$ and $\mathcal{H}_* = \{\mathcal{H}_n(\ ; \mathbf{A}), \}$ for the homology theory with coefficients in $\mathbf{A}$ on the category $\mathcal{C}^2$.

Now let $Y \subset X$ be a pair of compact subsets of S^n. Let $\{M_k\}$, $\{L_k\}$, $M_k \subset L_k$ be approximative sequences of Y and X, respectively, consisting of subcomplexes of triangulations of S^n. Let $i_k : (L_{k+1}, M_{k+1}) \to (L_k, M_k)$ denote the corresponding inclusions. Then, by continuity of $\mathcal{H}^*$, we have

$$\mathcal{H}^q(X, Y; \mathbf{A}) \approx \underset{k}{\underrightarrow{\operatorname{Lim}}} \{ \mathcal{H}^q(L_k, M_k; \mathbf{A}); i_k^* \}.$$

Without loss of generality, we may assume that

$$M_k^* \subset M_{k+1}^*, \qquad L_k^* \subset L_{k+1}^*, \qquad \bigcup_{k=1}^{\infty} M_k^* = S^n - Y, \qquad \bigcup_{k=1}^{\infty} L_k^* = S^n - X.$$

Let $j_k : (M_k^*, L_k^*) \to (M_{k+1}^*, L_{k+1}^*)$ be the inclusion. By 1.7.4 we have

$$\mathcal{H}_m(S^n - Y, S^n - X; \mathbf{A}) \approx \underset{k}{\underrightarrow{\operatorname{Lim}}} \{ \mathcal{H}_m(M_k, L_k; \mathbf{A}); (j_k)_* \}.$$

It is evident that, by straightforward passage to the limit, D_n extends uniquely to an isomorphism (which we continue to denote by D_n)

$$D_n : \mathcal{H}^q(X, Y; \mathbf{A}) \to \mathcal{H}_{n-q}(S^n - Y, S^n - X; \mathbf{A})$$

defined for all compact subsets of $Y \subset X$ of S^n.

Now, as an immediate consequence of Propositions 2.2 and 2.3, we obtain, by passing to the limit :

3.1. Proposition. — *For a triple* (X, Y, Z) *of compact subsets of* S^n *let* $U = S^n - X$, $V = S^n - Y$, $W = S^n - Z$. *Then the diagram :*

$$\begin{array}{ccccccc}
\mathcal{H}^q(X,Y;A) & \longrightarrow & \mathcal{H}^q(X,Z;A) & \longrightarrow & \mathcal{H}^q(Y,Z;A) & \longrightarrow & \mathcal{H}^{q+1}(X,Y;A) \\
\downarrow D_n & & \downarrow D_n & & \downarrow D_n & & \downarrow D_n \\
\mathcal{H}_{n-q}(V,U;A) & \longrightarrow & \mathcal{H}_{n-q}(W,U;A) & \longrightarrow & \mathcal{H}_{n-q}(W,V;A) & \longrightarrow & \mathcal{H}_{n-q+1}(V,U;A)
\end{array}$$

in which the upper row is the cohomology sequence of the triple (X, Y, Z) *and the lower row is the homology sequence of the triple* (W, V, U), *has two left-hand squares commutative and the third square commutative up to the sign* $(-1)^{n+1}$.

3.2. Proposition. — *For a map* $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ *of spectra we have the following commutative diagram :*

$$\begin{array}{ccc}
\mathcal{H}^q(X,Y;A) & \longrightarrow & \mathcal{H}^q(X,Y;B) \\
\downarrow D_n & & \downarrow D_n \\
\mathcal{H}_{n-q}(V,U;A) & \longrightarrow & \mathcal{H}_{n-q}(V,U;B)
\end{array}$$

Let (X, A) be a pair of compacta in S^{n+1} and let

$$(X_0, A_0) = (X \cap S^n, A \cap S^n).$$

Denote by Δ the relative Mayer-Vietoris homomorphism corresponding to the proper inclusion of triads

$$(A, A \cap E_+^{n+1}, A \cap E_-^{n+1}) \subset (X, X \cap E_+^{n+1}, X \cap E_-^{n+1})$$

and by $i : (S^n - A_0, S^n \to X) - (S^{n+1} - A, S^{n+1} - X)$ the inclusion. Then from 2.4, we obtain

3.3. Proposition. — *The following diagram commutes :*

$$\begin{array}{ccc} \mathcal{H}^q(X_0, A_0; \mathbf{A}) & \xrightarrow{\ \Delta\ } & \mathcal{H}^{q+1}(X, A; \mathbf{A}) \\ \downarrow D_n & & \downarrow D_{n+1} \\ \mathcal{H}_{n-q}(S^n - A_0, S^n - X_0; \mathbf{A}) & \xrightarrow{\ i_*\ } & \mathcal{H}_{n-q}(S^{n+1} - A, S^{n+1} - X; \mathbf{A}) \end{array}$$

4. Duality in R^n. — *Notation.*— We choose a sequence $\{\omega_n\}$ of continuous maps $\omega_n : R^n \to S^n$ with the following properties :

(i) $\omega_{n+1}(x) = \omega_n(x)$ for all $x \in R^n$;

(ii) ω_n maps R^n homeomorphically onto $S^n - \{q\}$, where

$$q = (-1, 0, \ldots, 0) \in S^n \subset R^{n+1};$$

(iii) $\omega_n(R_+^n) \subset C_+ S^{n-1}$, $\omega_n(R_-^n) \subset C_- S^{n-1}$.

If X is a subset of R^n we let $\omega_X : X \to \omega(X)$ denote the map defined by $\omega_X(x) = \omega_n(x)$.

In this section we assume that $\mathbf{A}$ is either the Eilenberg-Mac Lane spectrum K (Π) or the spectrum $\mathbf{S}$ of spheres. Note that, in either case, the homomorphism

$$(\omega_{R^n - X})_* : \mathcal{H}_q(R^n - X; \mathbf{A}) \to \mathcal{H}_q(S^n - \omega(X); \mathbf{A})$$

is an isomorphism for all $q \leq n - 2$.

4.1. Definition. — Assume that (X, Y) is a pair of compacta in R^n and $q \leq n - 2$. Define the duality map

$$D_n : \mathcal{H}^{n-q}(X, Y; \mathbf{A}) \to \mathcal{H}_q(R^n - Y, R^n - X; \mathbf{A})$$

by imposing commutativity on the following diagram :

$$\begin{array}{ccc} \mathcal{H}^{n-q}(X,Y;A) & \xrightarrow{\omega_X^*} & \mathcal{H}^{n-q}(\omega(X),\omega(Y);A) \\ \downarrow D_n & & \downarrow D_n \\ \mathcal{H}_q(R^n-Y,R^n-X;A) & \xrightarrow{(\omega_{R^n-X})_*} & \mathcal{H}_q(S^n-\omega(Y),S^n-\omega(X);A) \end{array}$$

Thus,

$$D_n = (\omega_{R^n-X})_*^{-1} \circ D_n \circ (\omega_X^*)^{-1}.$$

From the definition of D_n and Propositions 3.1, 3.2 and 3.3, we obtain the following properties :

4.2. Proposition. — *For any compact pair* (X, Y) *in* R^n, *the duality map*

$$D_n : \mathcal{H}^{n-q}(X, Y; A) \to \mathcal{H}_q(R^n - Y, R^n - X; A)$$

is an isomorphism.

4.3. Proposition. — *Let* (X, Y, Z) *be a compact triple in* R^n *and* $q \leq n-2$. *Then the diagram :*

$$\begin{array}{ccccccc} \mathcal{H}^{n-q}(X,Y;A) & \to & \mathcal{H}^{n-q}(X,Z;A) & \to & \mathcal{H}^{n-q}(Y,Z;A) & \to & \mathcal{H}^{n-q+1}(X,Y;A) \\ \downarrow D_n & & \downarrow D_n & & \downarrow D_n & & \downarrow D_n \\ \mathcal{H}_q(R^n-Y,R^n-X;A) & \to & \mathcal{H}_q(R^n-Z,R^n-X;A) & \to & \mathcal{H}_q(R^n-Z,R^n-Y;A) & \to & \mathcal{H}_{q-1}(R^n-Y,R^n-X;A) \end{array}$$

in which the upper row is the cohomology sequence of the triple (X, Y, Z) *and the lower row is the homology sequence of the triple* $(R^n - Z, R^n - Y, R^n - X)$ *has two left-hand squares commutative and the third square commutative up to the sign* $(-1)^{n+1}$.

Let (X, Y) be a compact pair in R^{n+1} and $(X_0, Y_0) = (X \cap R^n, Y \cap R^n)$. Denote by Δ the relative Mayer-Vietoris homomorphism corresponding to the proper inclusion of triads

$$(Y, Y\cap R_+^{n+1}, Y\cap R_-^{n+1}) \subset (X, X\cap R_+^{n+1}, X\cap R_-^{n+1})$$

and by

$$i : (R^n - Y_0, R^n - X_0) \to (R^{n+1} - Y, R^{n+1} - X)$$

the inclusion map.

4.4. Proposition. — *If $q \leq n-2$, then the following diagram commutes :*

$$\begin{array}{ccc} \mathcal{H}^{n-q}(X_0,Y_0;A) & \xrightarrow{\ \Delta\ } & \mathcal{H}^{n+1-q}(X,Y;A) \\ \downarrow{\scriptstyle D_n} & & \downarrow{\scriptstyle D_{n+1}} \\ \mathcal{H}_q(R^n-Y_0,R^n-X_0;A) & \xrightarrow{\ i_*\ } & \mathcal{H}_q(R^{n+1}-Y,R^{n+1}-X;A) \end{array}$$

Proof. — Follows at once from the Proposition 3.3.

Let Z denote the group of integers. Let us put

$$\begin{aligned} \Sigma^*(\ ,\ ) &= \mathcal{H}^*(\ ,\ ;\mathbf{S}), \\ \Sigma_*(\ ,\ ) &= \mathcal{H}_*(\ ,\ ;\mathbf{S}), \\ H^*(\ ,\ ;Z) &= \mathcal{H}^*(\ ,\ ;\mathbf{K}(Z)), \\ H_*(\ ,\ ;Z) &= \mathcal{H}_*(\ ,\ ;\mathbf{K}(Z)). \end{aligned}$$

The Hopf-Hurewicz map $\mathbf{h} : \mathbf{S} \to \mathbf{K}(Z)$ induces the following transformations of homology and cohomology theories : the *Hurewicz homomorphism*

$$h_* : \Sigma_*(\ ,\ ) \to H_*(\ ,\ ;Z),$$

the *Hopf homomorphism*

$$h^* : \Sigma^*(\ ,\ ) \to H^*(\ ,\ ;Z).$$

4.5. Proposition. — *If* (X, Y) *is a compact pair in* R^n *and* $q \leq n-2$, *then the following diagram commutes :*

$$\begin{array}{ccc} \Sigma^{n-q}(X,Y) & \xrightarrow{\ h^*\ } & H^{n-q}(X,Y;Z) \\ \downarrow{\scriptstyle D_n} & & \downarrow{\scriptstyle D_n} \\ \Sigma_q(R^n-Y,R^n-X) & \xrightarrow{\ h_*\ } & H_q(R^n-Y,R^n-X;Z) \end{array}$$

Proof. — Follows at once from the Proposition 3.2.

5. Duality in E (special case). — In this section we assume that $\mathbf{A}$ is either the Eilenberg-Mac Lane spectrum $\mathbf{K}(\Pi)$ or the spectrum $\mathbf{S}$ of spheres. Let $\{\mathcal{O}_\alpha\}$ be an orientation of E. For $\alpha \in \mathcal{L}$ let $l_\alpha : L_\alpha \to R^{d(\alpha)}$ be a representative of $\mathcal{O}_\alpha$. If X is a subset of E let $X'_\alpha = l_\alpha(X) \subset R^{d(\alpha)}$. We will denote by the same letter l_α the homeomorphism of X_α onto X'_α defined by the assignment $x \to l_\alpha(x)$. Thus $l_\alpha : X_\alpha - X'_\alpha$.

Throughout the rest of this section (X, Y, Z) will denote a triple in $\mathfrak{L}$. We let $U = E - X$, $V = E - Y$, $W = E - Z$. If α, $\beta \in \mathcal{L}$, $\alpha \leq \beta$, then we let

$$i_\alpha : \ (V_\alpha, U_\alpha) \to (V, U), \qquad i_{\alpha\beta} : \ (V_\alpha, U_\alpha) \to (V_\beta, U_\beta)$$

denote the corresponding inclusion maps.

5.1. Definition. — Assume that $\alpha \in \mathcal{L}$ and $d(\alpha) \geqq n+2$. Define the *duality isomorphism*

$$D_\alpha : \ \mathcal{H}^{d(\alpha)-n}(X_\alpha, Y_\alpha; A) \to \mathcal{H}_n(V_\alpha, U_\alpha; A)$$

by imposing commutativity on the diagram :

$$\begin{array}{ccc}
\mathcal{H}^{d(\alpha)-n}(X_\alpha, Y_\alpha; A) & \xleftarrow{(\ell_\alpha)^*} & \mathcal{H}^{d(\alpha)-n}(X'_\alpha, Y'_\alpha; A) \\
\downarrow D_\alpha & & \downarrow D_{d(\alpha)} \\
\mathcal{H}_n(V_\alpha, U_\alpha; A) & \xrightarrow{(\ell_\alpha)_*} & \mathcal{H}_n(V'_\alpha, U'_\alpha; A)
\end{array}$$

Clearly D_α depends only on the orientation of L_α.

5.2. Proposition. — *Let* $\alpha \in \mathcal{L}$ *and* $d(\alpha) \geqq n+2$. *Then the diagram :*

$$\begin{array}{ccccccc}
\mathcal{H}^{d(\alpha)-n}(X_\alpha, Y_\alpha; A) & \to & \mathcal{H}^{d(\alpha)-n}(X_\alpha, Z_\alpha; A) & \to & \mathcal{H}^{d(\alpha)-n}(Y_\alpha, Z_\alpha; A) & \to & \mathcal{H}^{d(\alpha)-n+1}(X_\alpha, Y_\alpha; A) \\
\downarrow D_\alpha & & \downarrow D_\alpha & & \downarrow D_\alpha & & \downarrow D_\alpha \\
\mathcal{H}_n(V_\alpha, U_\alpha; A) & \to & \mathcal{H}_n(W_\alpha, V_\alpha; A) & \to & \mathcal{H}_n(W_\alpha, V_\alpha; A) & \to & \mathcal{H}_{n-1}(V_\alpha, U_\alpha; A)
\end{array}$$

in which the upper row is the cohomology sequence of the triple $(X_\alpha, Y_\alpha, Z_\alpha)$ *and the lower row is the homology sequence of the triple* $(W_\alpha, V_\alpha, U_\alpha)$ *has two left-hand squares commutative and the third square commutative up to the sign* $(-1)^{d(\alpha)+1}$.

Proof. — Follows at once from the definition of D_α and Proposition 4.3.

5.3. Proposition. — *If* $\alpha, \beta \in \mathcal{L}$, $\alpha \leqq \beta$, $d(\alpha) \geqq n+2$, *then the following diagram commutes :*

$$\begin{array}{ccc}
\mathcal{H}^{d(\alpha)-n}(X_\alpha, Y_\alpha; A) & \xrightarrow{\Delta_{\alpha\beta}} & \mathcal{H}^{d(\beta)-n}(X_\beta, Y_\beta; A) \\
\downarrow D_\alpha & & \downarrow D_\beta \\
\mathcal{H}_n(V_\alpha, U_\alpha; A) & \xrightarrow{(i_{\alpha\beta})_*} & \mathcal{H}_n(V_\beta, U_\beta; A)
\end{array}$$

Proof. — If the relation $\alpha < \beta$ is elementary this follows from Proposition 4.4. Then the assertion in the general case follows from the definition of $\Delta_{\alpha\beta}$.

5.4. Proposition. — *If $\alpha \in \mathcal{L}$ and $d(\alpha) \geqq n+2$, then the following diagram commutes :*

$$\begin{array}{ccc}
\Sigma^{d(\alpha)-n}(X_\alpha, Y_\alpha) & \xrightarrow{h^*} & H^{d(\alpha)-n}(X_\alpha, Y_\alpha; Z) \\
\downarrow {\scriptstyle D_\alpha} & & \downarrow {\scriptstyle D_\alpha} \\
\Sigma_n(V_\alpha, U_\alpha) & \xrightarrow{h_*} & H_n(V_\alpha, U_\alpha; Z)
\end{array}$$

Proof. — Follows at once from the Proposition 4.5.

Now let (K, L) be a compact pair contained in (V, U) and let k : (K, L) — (V, U) denote the inclusion map. It follows from II.2.2 that k is homotopic to a map $k_1 : (K, L) \to (V, U)$ such that $k_1(K) \subset V_\alpha$ for some $\alpha \in \mathcal{L}$. Hence we obtained

5.5. Proposition. — *The map*

$$\varinjlim_\alpha \{(i_\alpha)_*\} : \ \varinjlim_\alpha \{\mathcal{H}_n(V_\alpha, U_\alpha; \mathbf{A}), (i_{\alpha\beta})_*\} \to \mathcal{H}_n(V, U; \mathbf{A})$$

is an isomorphism.

Note that the above proposition is valid for an arbitrary spectrum **A**. Recall that

$$\mathcal{H}^{\infty-n}(X, Y; \mathbf{A}) = \varinjlim_\alpha \{\mathcal{H}^{d(\alpha)-n}(X_\alpha, Y_\alpha; \mathbf{A}); \Delta_{\alpha\beta}\}.$$

5.6. Definition. — We define the *duality isomorphism*

$$D : \ \mathcal{H}^{\infty-n}(X, Y; \mathbf{A}) \to \mathcal{H}_n(V, U; \mathbf{A})$$

by the formula

$$D = \varinjlim_\alpha \{(i_\alpha)_*\} \circ \varinjlim_\alpha \{D_\alpha\}.$$

5.7. Theorem. — *The diagram*

$$\begin{array}{ccccccc}
\mathcal{H}^{\infty-n}(X,Y;A) & \to & \mathcal{H}^{\infty-n}(X,Z;A) & \to & \mathcal{H}^{\infty-n}(Y,Z;A) & \to & \mathcal{H}^{\infty-n+1}(X,Y;A) \\
\downarrow {\scriptstyle D} & & \downarrow {\scriptstyle D} & & \downarrow {\scriptstyle D} & & \downarrow {\scriptstyle D} \\
\mathcal{H}_n(V,U;A) & \to & \mathcal{H}_n(W,U;A) & \to & \mathcal{H}_n(W,V;A) & \to & \mathcal{H}_{n-1}(U,V;A)
\end{array}$$

in which the upper row is the $\mathcal{H}^{\infty-*}$-cohomology sequence of the triple (X, Y, Z) *and the lower row is the $\mathcal{H}_*(\ ;\mathbf{A})$-homology sequence of the triple* (W, V, U) *is commutative.*

Proof. — We recall that the coboundary homomorphism

$$\delta : \ \mathcal{H}^{\infty-n}(Y, Z) \to \mathcal{H}^{\infty-n+1}(X, Y)$$

is, by definition, the composition of the following homomorphisms :

$$\mathcal{H}^{\infty-n}(Y, Z; A) \xrightarrow{j^*} \mathcal{H}^{\infty-n}(Y, A) \to \mathcal{H}^{\infty-n+1}(X, Y; A),$$

where $j : (Y, \emptyset) \to (Y, Z)$ denotes the inclusion map and the second homomorphism is the coboundary homomorphism of the pair (X, Y). According to Definition V.2.1

$$\delta^{n+1} = \varinjlim_{\alpha} \{ \delta_\alpha^{n+1} \}, \qquad \delta_\alpha^{n+1} = (-1)^{d(\alpha)+1} \delta_{(X_\alpha, Y_\alpha)},$$

where $\delta_{(X_\alpha, Y_\alpha)} : \mathcal{H}^{d(\alpha)-n}(Y_\alpha; A) \to \mathcal{H}^{d(\alpha)-n+1}(X_\alpha, Y_\alpha; A)$ is the coboundary homomorphism of the pair (X_α, Y_α).

Hence, in view of Proposition 5.2, the diagram

$$\begin{array}{ccccc}
\mathcal{H}^{d(\alpha)-n}(Y_\alpha, Z_\alpha; A) & \xrightarrow{(j_\alpha)^*} & \mathcal{H}^{d(\alpha)-n}(Y_\alpha; A) & \xrightarrow{\delta_\alpha^{n+1}} & \mathcal{H}^{d(\alpha)-n}(X_\alpha, Y_\alpha; A) \\
\searrow^{D_\alpha} & & & & \swarrow_{D_\alpha} \\
\mathcal{H}_n(W_\alpha, V_\alpha; A) & & \xrightarrow{\quad\partial\quad} & & \mathcal{H}_{n-1}(V_\alpha, U_\alpha; A)
\end{array}$$

is commutative for each $\alpha \in \mathfrak{C}$. This proves that the third square is commutative. Since the commutativity of the first two squares is an immediate consequence of 5.2 the theorem is proved.

5.8. Theorem (*Hopf Theorem*). — *If* (X, Y) *is a pair in* $\mathfrak{L}^2$, *then*

(*a*) $\Sigma^{\infty-q}(X, Y) \approx 0$ for $0 \leq q < n$ *if and only if*

$$H^{\infty-q}(X, Y; Z) \approx 0 \quad \text{for} \quad 0 \leq q < n;$$

(*b*) *if* $\Sigma^{\infty-q}(X, Y) \approx 0$ *for* $0 \leq q < n$, *then the Hopf homomorphism* h^* : $\Sigma^{\infty-n}(X, Y) \to H^{\infty-n}(X, Y, Z)$ *is an isomorphism.*

Proof. — By 5.4, we have a commutative diagram :

$$\begin{array}{ccc}
\Sigma^{\infty-q}(X, Y) & \xrightarrow{h^*} & H^{\infty-n}(X, Y; Z) \\
\downarrow D & & \downarrow D \\
\Sigma_q(V, U) & \xrightarrow{h_*} & H_q(V, U; Z)
\end{array}$$

and the theorem follows from the Hurewicz Isomorphism Theorem in S-theory ([10], p. 57).

6. Duality in E (general case). — Let X be a closed and bounded subset of E and let $U = E - X$. For $\alpha \in \mathcal{L}$ denote by $\hat{U}_\alpha$ the space U_α augumented with " the point at infinity ". More precisely, we let $\hat{U}_\alpha = U_\alpha \cup \{ e \}$. $e \notin U_\alpha$, with the sets $\{ x \in U_\alpha,\ x > r \} \cup \{ e \}$ forming a fundamental system of neighbourhoods of e.

Let $\{ \mathcal{O}_\alpha \}$ be an orientation of E. For each $\alpha \in \mathcal{L}$ choose $l_\alpha : L_\alpha \to R^{d(\alpha)}$ which represents $\mathcal{O}_\alpha$. Let

$$X'_\alpha = (\omega_{d(\alpha)} \circ l_\alpha)(X_\alpha)$$

and let $\lambda_\alpha : X_\alpha \to X'_\alpha$ be the homeomorphism defined by

$$\lambda_\alpha(x) = (\omega_{d(\alpha)} \circ l_\alpha)(x), \qquad x \in X_\alpha.$$

Let $\varkappa_\alpha : \hat{U}_\alpha \to S^{d(\alpha)} - X'_\alpha$ be the homeomorphism which is the unique extension of the map of U_α into $S^{d(\alpha)} - X'_\alpha$ defined by the assignment $x \to (\omega_{d(\alpha)} \circ l_\alpha)(x)$.

Let Y be a closed bounded subset of X and let $V = E - Y$. Then for each $\alpha \in \mathcal{L}$ we have homeomorphisms

$$\lambda_\alpha : \ (X_\alpha, Y_\alpha) \to (X'_\alpha, Y'_\alpha), \qquad \varkappa_\alpha : \ (\hat{V}_\alpha, \hat{U}_\alpha) \to (S^{d(\alpha)} - Y', S^{d(\alpha)} - X').$$

Putting

$$D_\alpha = (\varkappa_\alpha)_*^{-1} \circ D_{d(\alpha)} \circ (\lambda_\alpha^*)^{-1}$$

we obtain an isomorphism

$$D_\alpha : \ \mathcal{H}^{d(\alpha)-q}(X_\alpha, Y_\alpha; \mathbf{A}) \to \mathcal{H}_q(\hat{V}_\alpha, \hat{U}_\alpha; \mathbf{A});$$

clearly D_α depends only on the orientation of $\mathcal{L}_\alpha$.

Thus we have a commutative diagram :

$$\begin{array}{ccc}
\mathcal{H}^{d(\alpha)-q}(X_\alpha, Y_\alpha; A) & \xrightarrow{\lambda_\alpha^*} & \mathcal{H}^{d(\alpha)-q}(X'_\alpha, Y'_\alpha; A) \\
\downarrow D_\alpha & & \downarrow D_{d(\alpha)} \\
\mathcal{H}_q(\hat{V}_\alpha, \hat{U}_\alpha; A) & \xrightarrow{(\varkappa_\alpha)_*} & \mathcal{H}_q(S^{d(\alpha)} - X'_\alpha, S^{d(\alpha)} - Y'_\alpha; A)
\end{array}$$

From the definition of D_α and Proposition 3.1, 3.2 we obtain the following two propositions :

Proposition 6.1. — *Assume that* $Z \subset Y \subset X$ *are closed and bounded subsets of* E *and let* $U = E - X$, $V = E - Y$, $W = E - Z$. *Then for* *each $\alpha \in \mathcal{L}$ the diagram :*

$$\begin{array}{ccc}
\cdots \to \mathcal{H}^{d(\alpha)-q}(X_\alpha, Y_\alpha; A) & \longrightarrow & \mathcal{H}^{d(\alpha)-q}(X_\alpha, Z_\alpha; A) \to \\
\downarrow D_\alpha & & \downarrow D_\alpha \\
\cdots \to \mathcal{H}_q(\hat{V}_\alpha, \hat{U}_\alpha; A) & \longrightarrow & \mathcal{H}_q(\hat{W}_\alpha, \hat{U}_\alpha; A) \to
\end{array}$$

$$\begin{array}{ccc}
\to \mathcal{H}^{d(\alpha)-q}(Y_\alpha, Z_\alpha; A) & \longrightarrow & \mathcal{H}^{d(\alpha)-q+1}(X_\alpha, Y_\alpha; A) \to \cdots \\
\downarrow D_\alpha & & \downarrow D_\alpha \\
\to \mathcal{H}_q(\hat{W}_\alpha, \hat{V}_\alpha; A) & \longrightarrow & \mathcal{H}_{q-1}(\hat{V}_\alpha, \hat{U}_\alpha; A) \to \cdots
\end{array}$$

in which the upper row is the cohomology sequence of the triple $(X_\alpha, Y_\alpha, Z_\alpha)$ and the lower row is the homology sequence of the triple $(\hat{W}_\alpha, \hat{V}_\alpha, \hat{U}_\alpha)$ has two left hand squares commutative and the third square commutative up to the sign $(-1)^{d(\alpha)+1}$.

Proposition 6.2. — *Let* $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ *be a map of spectra. Then the following diagram commutes for each $\alpha \in \mathcal{L}$:*

$$\begin{array}{ccc}
\mathcal{H}^{d(\alpha)-q}(X_\alpha, Y_\alpha; A) & \xrightarrow{\mathcal{H}^{d(\alpha)-q}(X_\alpha, Y_\alpha; f)} & \mathcal{H}^{d(\alpha)-q}(X_\alpha, Y_\alpha; B) \\
\downarrow D_\alpha & & \downarrow D_\alpha \\
\mathcal{H}_q(\hat{V}_\alpha, \hat{U}_\alpha; A) & \xrightarrow{\mathcal{H}_q(\hat{V}_\alpha, \hat{U}_\alpha; f)} & \mathcal{H}_q(\hat{V}_\alpha, \hat{U}_\alpha; B)
\end{array}$$

If $\alpha < \beta$, we let

$$i_{\alpha\beta} : \ (V_\alpha, U_\alpha) \to (V_\beta, U_\beta), \qquad \hat{i}_{\alpha\beta} : \ (\hat{V}_\alpha, \hat{U}_\alpha) \to (\hat{V}_\beta, \hat{U}_\beta)$$

denote the corresponding inclusions.

Proposition 6.3. — *If $\alpha < \beta$, then the following diagram commutes :*

$$\begin{array}{ccc}
\mathcal{H}^{d(\alpha)-q}(X_\alpha, Y_\alpha; A) & \xrightarrow{\Delta_{\alpha\beta}} & \mathcal{H}^{d(\alpha)-q}(X_\alpha, Y_\alpha; A) \\
\downarrow D_\alpha & & \downarrow D_\beta \\
\mathcal{H}_q(\hat{V}_\alpha, \hat{U}_\alpha; A) & \xrightarrow{(\hat{i}_{\alpha\beta})_*} & \mathcal{H}_q(\hat{V}_\alpha, \hat{U}_\alpha; A)
\end{array}$$

Proof. — If $d(\beta) - d(\alpha) = 1$, then we may assume, without a loss of generality, that $l_\alpha(x) = l_\beta(x)$ for all $x \in L_\alpha$. In this case the conclusion follows from the definition of $\Delta_{\alpha\beta}$ and Proposition 3.3. Now the conclusion in the general case follows at once from the definition of $\Delta_{\alpha\beta}$.

From the above proposition we obtain that $\{D_\alpha\}$ is an isomorphism between directed sets

$$\{\mathcal{H}^{d(\alpha)-q}(X_\alpha, Y_\alpha; \mathbf{A}); \Delta_{\alpha\beta}\} \quad \text{and} \quad \{\mathcal{H}_q(\hat{V}_\alpha, \hat{U}_\alpha; \mathbf{A}); (\hat{\imath}_{\alpha\beta})_*\}.$$

In view of the definition of $\mathcal{H}^{\infty-q}(X, Y; \mathbf{A})$ we have

COROLLARY 6.4. — *The map*

$$\varinjlim_\alpha \{D_\alpha\} : \mathcal{H}^{\infty-q}(X, Y; \mathbf{A}) \to \varinjlim_\alpha \{\mathcal{H}_q(\hat{V}, \hat{U}; \mathbf{A})(\hat{\imath}_{\alpha\beta})_*\}$$

is an isomorphism.

For $\alpha < \beta$ consider the following commutative diagram of inclusions:

$$\begin{array}{ccc}
(V_\alpha, U_\alpha) & \xrightarrow{\ i_{\alpha\beta}\ } & (V_\beta, U_\beta) \\
\downarrow \eta_\alpha & & \downarrow \eta_\beta \\
(\hat{V}_\alpha, \hat{U}_\alpha) & \xrightarrow{\ \hat{\imath}_{\alpha\beta}\ } & (\hat{V}_\beta, \hat{U}_\beta)
\end{array}$$

PROPOSITION 6.5. — *The map*

$$\varinjlim_\alpha \{(\eta_\alpha)_*\} : \varinjlim_\alpha \{\mathcal{H}_q(V_\alpha, U_\alpha; \mathbf{A}); (i_{\alpha\beta})_*\} \to \varinjlim_\alpha \{\mathcal{H}_q(\hat{V}_\alpha, \hat{U}_\alpha; \mathbf{A}); (\hat{\imath}_{\alpha\beta})_*\}$$

is an isomorphism.

Proof. — For any $\alpha \in \mathcal{C}$ we have a commutative diagram:

$$\begin{array}{ccccccccc}
\cdots \to & \mathcal{H}_q(U_\alpha; A) & \to & \mathcal{H}_q(V_\alpha; A) & \to & \mathcal{H}_q(V_\alpha, U_\alpha; A) & \to & \mathcal{H}_{q-1}(U_\alpha; A) & \to \cdots \\
& \downarrow (\eta_\alpha)_* & & \downarrow (\eta_\alpha)_* & & \downarrow (\eta_\alpha)_* & & \downarrow (\eta_\alpha)_* & \\
\cdots \to & \mathcal{H}_q(\hat{U}_\alpha; A) & \to & \mathcal{H}_q(\hat{V}_\alpha; A) & \to & \mathcal{H}_q(\hat{V}_\alpha, \hat{U}_\alpha; A) & \to & \mathcal{H}_{q-1}(\hat{U}_\alpha; A) & \to \cdots
\end{array}$$

Since by applying the direct limit functor we obtain the corresponding commutative diagram, then by the Five Isomorphism Lemma, it suffices to prove that

$$\varinjlim_\alpha \{(\eta_\alpha)_*\} : \varinjlim_\alpha \{\mathcal{H}_q(U_\alpha; \mathbf{A}); (i_{\alpha\beta})_*\} \to \varinjlim_\alpha \{\mathcal{H}_q(\hat{U}_\alpha; \mathbf{A}); (\hat{\imath}_{\alpha\beta})_*\}$$

is an isomorphism.

For any relations $k \leq l$ in $\mathfrak{N}$ = the set of positive integers and $\alpha \leq \beta$ in $\mathfrak{T}$ we have the commutative diagrams :

$$\begin{array}{ccc}
\pi_{q+k}(A_k \wedge U_\alpha) & \xrightarrow{(\mathrm{id}_k \wedge i_{\alpha\beta})_*} & \pi_{q+k}(A_k \wedge U_\beta) \\
\downarrow \lambda_{k,l} & & \downarrow \lambda_{k,l} \\
\pi_{q+l}(A_l \wedge U_\alpha) & \xrightarrow{(\mathrm{id}_l \wedge i_{\alpha\beta})_*} & \pi_{q+l}(A_l \wedge U_\beta)
\end{array}$$

$$\begin{array}{ccc}
\pi_{q+k}(A_k \wedge \hat{U}_\alpha) & \xrightarrow{(\mathrm{id}_k \wedge \hat{i}_{\alpha\beta})_*} & \pi_{q+k}(A_k \wedge \hat{U}_\beta) \\
\downarrow \lambda_{k,l} & & \downarrow \lambda_{k,l} \\
\pi_{q+l}(A_l \wedge \hat{U}_\alpha) & \xrightarrow{(\mathrm{id}_l \wedge \hat{i}_{\alpha\beta})_*} & \pi_{q+l}(A_l \wedge \hat{U}_\beta)
\end{array}$$

where id_k denotes the identity map of A_k into itself and

$$\lambda_{k,l} = \lambda_{l-1} \circ \ldots \circ \lambda_{k+1} \circ \lambda_k .$$

Thus we have two double directed systems of abelian groups

$$\{\pi_{q+k}(A_k \wedge U_\alpha), \quad\}, \quad \{\pi_{q+k}(A_k \wedge \hat{U}_\alpha), \quad\}$$

together with a map

$$\{(\mathrm{id}_k \wedge \eta_\alpha)_*\} : \{\pi_{q+k}(A_k \wedge U_\alpha), \quad\} \to \{\pi_{q+k}(A_k \wedge \hat{U}_\alpha), \quad\}.$$

Since

$$\varinjlim_\alpha \mathcal{H}_q(U_\alpha; A) = \varinjlim_\alpha \varinjlim_k \{\pi_{q+k}(A_k \wedge U_\alpha; A), \quad\},$$

$$\varinjlim_\alpha \mathcal{H}_q(\hat{U}_\alpha; A) = \varinjlim_\alpha \varinjlim_k \{\pi_{q+k}(A_k \wedge \hat{U}_\alpha; A), \quad\},$$

we have, according to IV.5.2, the following commutative diagram in which the horizontal maps are isomorphisms :

$$\begin{array}{ccc}
\varinjlim_\alpha \varinjlim_k \{\pi_{q+k}(A_k \wedge U_\alpha)\} & \longrightarrow & \varinjlim_k \varinjlim_\alpha \{\pi_{q+k}(A_k \wedge U_\alpha)\} \\
\downarrow \varinjlim_\alpha \varinjlim_k (\mathrm{id}_k \wedge \eta_\alpha)_* & & \downarrow \varinjlim_k \varinjlim_\alpha (\mathrm{id}_k \wedge \eta_\alpha)_* \\
\varinjlim_\alpha \varinjlim_k \{\pi_{q+k}(A_k \wedge \hat{U}_\alpha)\} & \longrightarrow & \varinjlim_k \varinjlim_\alpha \{\pi_{q+k}(A_k \wedge \hat{U}_\alpha)\}
\end{array}$$

Thus it suffices to prove that, for each k,

$$\varinjlim_\alpha \{(\mathrm{id}_k \wedge \eta_\alpha)_*\} : \varinjlim_\alpha \{\pi_{q+k}(A_k \wedge U_\alpha)\} \to \varinjlim_\alpha \{\pi_{q+k}(A_k \wedge \hat{U}_\alpha)\}$$

is an isomorphism. We will prove that if $d(\alpha) \geqq q + k + 2$, then $(\mathrm{id}_k \wedge \eta_\alpha)_*$ is an isomorphism.

To prove that $(\mathrm{id}_k \wedge \eta_\alpha)_*$ is an epimorphism, let γ be any element of $\pi_{q+k}(\mathrm{A}_k \wedge \hat{\mathrm{U}}_\alpha)$. By 1.7.2 there exists a polyhedron K with $\dim \mathrm{K} \leqq q + k$ and a continuous map $f : \mathrm{K} \to \mathrm{U}_\alpha$ such that $\gamma \in \mathrm{Im}\,(\mathrm{id}_k \wedge f)_*$. Since $\dim \mathrm{K} < \dim \mathrm{U}_\alpha = d(\alpha)$, f is homotopic to a map $g : \mathrm{K} \to \hat{\mathrm{U}}_\alpha$ such that $g(\mathrm{K}) \subset \mathrm{U}_\alpha$. Hence $\gamma \in \mathrm{Im}\,(\mathrm{id}_k \wedge \eta_\alpha)_*$ and $(\mathrm{id}_k \wedge \eta_\alpha)_*$ is an epimorphism.

To prove that $(\mathrm{id}_k \wedge \eta_\alpha)_*$ is a monomorphism, let γ be an element of $\pi_{q+k}(\mathrm{A}_k \wedge \mathrm{U}_\alpha)$ with $(\mathrm{id}_k \wedge \eta_\alpha)_*(\gamma) = 0$. Again by 1.7.2, there exists a polyhedral pair $(\mathrm{K}_1, \mathrm{K})$ with $\dim \mathrm{K}_1 \leqq q + k + 1$, commutative diagram of continuous maps

$$\begin{array}{ccc} \mathrm{K} & \xrightarrow{\;i\;} & \mathrm{K}_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_1} \\ \mathrm{U}_\alpha & \longrightarrow & \mathrm{U}_\alpha \end{array}$$

and $\gamma_1 \subset \pi_{q+k}(\mathrm{A}_k \wedge \mathrm{K})$ such that

$$\gamma = (\mathrm{id}_k \wedge f)_*(\gamma_1) \qquad (\mathrm{id}_k \wedge i)_*(\gamma_1) = 0.$$

Since $\dim \mathrm{K}_1 < d(\alpha)$, f_1 is homotopic to a map $g_1 : \mathrm{K}_1 \to \hat{\mathrm{U}}_\alpha$ with $g_1(\mathrm{K}_1) \subset \mathrm{U}_\alpha$. Define $g : \mathrm{K}_1 \to \mathrm{U}_\alpha$ by $g(x) = g_1(x)$ for $x \in \mathrm{K}_1$. Then $\gamma = g_*(i_*(\gamma_1)) = 0$. Hence $\gamma = 0$ and $(\mathrm{id}_k \wedge \eta_\alpha)_*$ is a monomorphism.

Recall that, by 1.7.4, the map

$$\varinjlim_\alpha \{(i_\alpha)_* : \varinjlim_\alpha \mathcal{H}_q(\mathrm{V}_\alpha, \mathrm{U}_\alpha; \mathbf{A})\,(i_{\alpha\beta})_*\} \to \mathcal{H}_q(\mathrm{V}, \mathrm{U}; \mathbf{A})$$

is an isomorphism.

6.6 Definition. — *We define the duality isomorphism*

$$\mathrm{D} : \mathcal{H}^{\infty - q}(\mathrm{X}, \mathrm{Y}; \mathbf{A}) \to \mathcal{H}_q(\mathrm{V}, \mathrm{U}; \mathbf{A})$$

by the formula

$$\mathrm{D} = \varinjlim_\alpha \{(i_\alpha)_*\} \circ \left(\varinjlim_\alpha \{(\eta_\alpha)_*\}\right)^{-1} \circ \varinjlim_\alpha \{\mathrm{D}_\alpha\}.$$

Note, that if $\mathbf{A} = \mathbf{S}$ or $\mathbf{K}(\Pi)$ the above definition coincides with that given in Section 5.

6.7. Theorem. — *Assume that* $Z \subset Y \subset X$ *are closed and bounded subsets of* E *and let* $U = E - X$, $V = E - Y$, $W = E - Z$. *Then the diagram:*

$$\begin{array}{ccccccc}
\mathcal{H}^{\infty-q}(X,Y) & \longrightarrow & \mathcal{H}^{\infty-q}(X,Z) & \longrightarrow & \mathcal{H}^{\infty-q}(Y,Z) & \longrightarrow & \mathcal{H}^{\infty-q+1}(X,Y) \\
\downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D} \\
\mathcal{H}_q(V,U) & \longrightarrow & \mathcal{H}_q(W,U) & \longrightarrow & \mathcal{H}_q(W,V) & \longrightarrow & \mathcal{H}_{q-1}(V,U)
\end{array}$$

in which the upper row is the $\mathcal{H}^{\infty-*}$(; **A**)*-cohomology sequence of the triple* (X, Y, Z) *and the lower row is the* $\mathcal{H}_*$(; **A**)*-homology sequence of the triple* (W, V, U) *is commutative. Moreover, if* $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ *is a map of spectra then the following diagram commutes:*

$$\begin{array}{ccc}
\mathcal{H}^{\infty-q}(X,Y;A) & \xrightarrow{\mathcal{H}^{\infty-q}(X,Y;f)} & \mathcal{H}^{\infty-q}(X,Y;B) \\
\downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D} \\
\mathcal{H}_q(V,U;A) & \xrightarrow{\mathcal{H}_q(V,U;f)} & \mathcal{H}_q(V,U;B)
\end{array}$$

Proof. — The theorem follows from Propositions 6.1, 6.2, 6.3 and the definitions involved.

7. Proof of the Theorem 1.3. — Before proving Theorem 1.2 in full generality we shall consider first a special case.

Assume that $B_i \subset A_i \subset X$, $i = 1, 2$, are the same as in Section 1 and let $Y' = \Sigma X$,

$$C'_1 = \Sigma A_1 \cup \left\{ s \wedge x \in \Sigma X ;\ |s| \geq \frac{1}{2} \right\},$$

$$D'_1 = \Sigma B_1 \cup \left\{ s \wedge x \in \Sigma X ;\ |s| \geq \frac{1}{2} \right\},$$

$$C'_2 = \left\{ s \wedge x \in \Sigma A_2 ;\ |s| \leq \frac{1}{2} \right\},$$

$$D'_2 = \left\{ s \wedge x \in \Sigma B_2 ;\ |s| \leq \frac{1}{2} \right\}.$$

If Y, Z are objects in $\mathcal{T}_*$ we let

$$q : \ \Sigma (Y \wedge Z) \to \Sigma Y \wedge Z$$

denote the map defined by the assignment

$$s \wedge (y \wedge z) \to (s \wedge y) \wedge z.$$

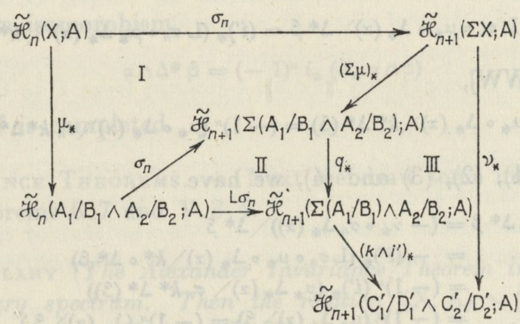
Define

$$L_{\sigma_n} : \mathcal{H}_n(Y \wedge Z; A) \to \mathcal{H}_{n+1}(\Sigma Y \wedge Z; A)$$

by $L\sigma_n = q_* \circ \sigma_n$ (*see* [GWW], p. 260, where $L\sigma_n$ is denote by σ_L).

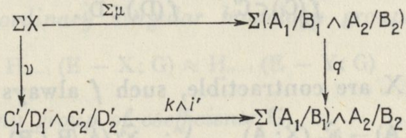
Let $\rho_i : C_i' \to C_i'/D_i'$, $i = 1, 2$, denote the identification map. Let $k : \Sigma(A_1/B_1) \to C_1'/D_1'$ denote the map defined by the assignment $s \wedge \rho_1(x) \to \rho_1(s \wedge x)$.

7.1. Lemma. — *The following diagram commutes :*



(ν_*'' is induced by the reduced diagonal map).

Proof — The regions I and II are commutative by definition. To prove the commutativity of III consider the diagram :



If $x \in A_1 \cap A_2$ and $|s| \leq \frac{1}{2}$, then $s \wedge x \in C_1' \cap C_2'$. Hence

$$\nu(s \wedge x) = \rho_1(s \wedge x) \wedge \rho_2(s \wedge x),$$
$$(k \wedge i') \circ q \circ \Sigma\mu(x) = \rho_1(s \wedge x) \wedge \rho_2(0 \wedge x).$$

Define a homotopy $h : \Sigma X \times I \to C_1/D_1 \wedge C_2/D_2$ by

$$h(s \wedge x, t) = \begin{cases} \rho_1(s \wedge x) \wedge \rho_2(ts \wedge x) & \text{for } s \wedge x \in C_1' \cap C_2', \\ \star & \text{for } s \wedge x \in B_1' \cup B_2'. \end{cases}$$

Since the inclusion $i' : A_2/B_2 \to C_2'/D_2'$ is defined by $i'(\pi_2(x)) = \rho_2(0 \wedge x)$ the last diagram is homotopy commutative and the lemma follows.

Proof of Theorem 1.2 (*special case*). — Let $\alpha \in \tilde{\mathcal{H}}_{n+1}(\Sigma X; \mathbf{A})$, $\beta \in \tilde{\mathcal{H}}^q(A_1/B_1; \mathbf{B})$. In view of the definition of Δ_* and Δ^* (*see* Section 1.4) we have

(1) $$\alpha = -\sigma_n \circ \Delta_*(\alpha), \qquad \beta = -\sigma^q \circ k^* \circ \Delta^*(\beta).$$

By Lemma 7.1,

(2) $$\nu_* \circ \sigma_n \circ \Delta_*(\alpha) = (k \wedge i')_* \circ L\,\sigma_n \circ \mu_* \circ \Delta_*(\alpha).$$

By 6.10 of [GWW],

(3) $$(k \wedge i')_* \circ L\,\sigma_n \circ \mu_* \circ \Delta_*(\alpha) \backslash \Delta^* \beta = (i')_*(L\,\sigma_n \circ \mu_* \Delta_*(\alpha) \backslash k^* \Delta^*(\beta)).$$

By 6.13 of [GWW],

(4) $$L\,\sigma_n \circ \mu_* \circ \Delta_*(\alpha) \backslash k^* \Delta^*(\beta) = (-1)^n (\mu_* \circ \Delta_*(\alpha) \backslash \sigma_n k^* \Delta^* \beta).$$

Therefore, by (1), (2), (3) and (4), we have

$$\begin{aligned} \alpha \cap \Delta^* \beta &= (-\nu_* \circ \sigma_n \Delta_*(\alpha)) \backslash \Delta^* \beta \\ &= -(i')_*(L\,\sigma_n \circ \mu_* \circ \Delta_*(\alpha) \backslash k^* \circ \Delta^* \beta) \\ &= (-1)^n (i')_*(\mu_* \Delta_*(\alpha) \backslash \sigma_n k^* \Delta^*(\beta)) \\ &= (-1)^n (\mu_* \Delta_*(\alpha) \backslash \beta) = (-1)^n (\Delta_*(\alpha) \backslash \beta), \end{aligned}$$

and the special case is proved.

Proof of Theorem 1.2 (*general case*). — Let $f : Y \to \Sigma X$ be an extension of the inclusion $X \subset \Sigma X$ such that $f(Y_1) \subset \Gamma_+ X$, $f(Y_2) \subset \Gamma_- X$,

$$f(C_i) \subset C'_i, \qquad f(D_i) \subset D'_i.$$

Since $\Gamma''_+ X$ and $\Gamma''_- X$ are contractible, such f always exists. Let

$$\Delta_1 : \tilde{\mathcal{H}}_{n+1}(\Sigma X; \mathbf{A}) \to \tilde{\mathcal{H}}_n(X; \mathbf{A}), \qquad \Delta' : \tilde{\mathcal{H}}^q(A_1/B_1; \mathbf{B}) \to \tilde{\mathcal{H}}^{q+1}(C'_1/D'_1; \mathbf{B})$$

denote the corresponding boundary and coboundary homomorphism. Let $i' : A_1/B_1 \to C'_1/D'_1$ denote the inclusion map. Denote by $\varphi : C_1/D_1 \to C'_1/D'_1$, $\psi : C_2/D_2 \to C'_2/D'_2$ the maps induced by f. Note, that the assumptions of Theorem 1.2 imply that ψ is a homotopy equivalence.

Let $\alpha \in \tilde{\mathcal{H}}_{n+1}(Y; \mathbf{A})$, $\beta \in \tilde{\mathcal{H}}^q(A_1/B_1; \mathbf{B})$. Hence

(5) $$\Delta^* \beta = \varphi^* \Delta' \beta, \qquad \Delta_* \alpha = \Delta_1 f_*(\alpha).$$

Therefore $\psi_*(\alpha \cap \Delta^* \beta) = \psi_*(\alpha \cap \varphi^* \Delta' \beta)$.

By Lemma 1.1,

$$\psi_*(\alpha \cap \varphi^* \Delta' \beta) = f_*(\alpha) \cap \Delta' \beta.$$

By the special case of our theorem,

$$f_*(\alpha)\cap\Delta'\beta = (-1)^n (i')_* (\Delta_1 f_*(\alpha)\cap\beta).$$

Hence, by (5),

$$\psi_*(\alpha\cap\Delta^*\beta) = (-1)^n (i')_* (\Delta_* \alpha\cap\beta).$$

Because $i' = \psi\circ i$, we have shown that

$$\psi_*(\alpha\cap\Delta^*\beta) = (-1)^n \psi_*(i)_* (\Delta_* \alpha\cap\beta).$$

Since ψ_* is an isomorphism

$$\alpha\cap\Delta^*\beta = (-1)^n i_* (\Delta_* \alpha\cap\beta)$$

and the proof is completed.

8. Invariance Theorems. — Next we draw some immediate corrollaries of Theorems 6.7 and V.3.4.

8.1. Corollary (*The Alexander Invariance Theorem in* E). — *Let* **A** *be an arbitrary spectrum. Then the relation* $(X, A) \sim (Y, B)$ [*or more generally the relation* $(X, A) \underset{h}{\sim} (Y, B)$] *in* $\mathfrak{L}^2$ *implies the isomorphism*

$$\mathcal{H}_n(E - A, E - X; \mathbf{A}) \approx \mathcal{H}_n(E - B, E - Y; \mathbf{A}).$$

8.2. Corollary (*The Alexander-Pontrjagin Invariance in* E). — *The relation* $X \sim Y$ (*or more generally the relation* $X \underset{h}{\sim} Y$) *in* $\mathfrak{L}$ *implies an isomorphism of the ordinary singular homology groups*

$$H_{n-1}(E - X; G) \approx H_{n-1}(E - Y; G)$$

for any $n \geqq 1$ *and any group of coefficients* G.

8.3. Corollary (*The Leray Invariance Theorem in* E). — *Any of the relations*

$$X \sim Y, \quad X \underset{h}{\sim} Y, \quad X \underset{r}{\sim} Y, \quad X \underset{rh}{\sim} Y$$

in $\mathfrak{L}$ *implies that the complements* $E - X$ *and* $E - Y$ *have the same number of components.*

8.4. Corollary (*The Spanier-Whitehead Invariance in* E). — *The relation* $X \sim Y$ (*or more generally the relation* $X \underset{h}{\sim} Y$) *implies that for any* $n \geqq 1$ *we have the isomorphism of the stable homotopy groups*

$$\Sigma_{n-1}(E - X) \approx \Sigma_{n-1}(E - Y).$$

CHAPTER VII

Group-like objects in $\mathfrak{C}/\sim$

In this chapter we establish some results needed for the proof of the representability of the stable cohomotopy and the ordinary cohomology and which will be given in the next chapter. More specifically, we are concerned with a contravariant h-functor $X \to \pi(X, U)$ from the category $\mathfrak{L}$ to the category of sets and which is associated with an object U in $\mathfrak{C}/\sim$. We show that for some objects U (called algebraically admissible) this functor can be converted into an h-functor from $\mathfrak{L}$ to the category of abelian groups.

1. Compact fields with admissible range U. — Call an object $U \in \mathfrak{C}$ *admissible*, provided U is open in E and its complement is contained in some finite dimensional subspace of E. In the rest of this chapter U will stand for an arbitrary but fixed admissible set and $W = E - U$ for its complement.

For notational convenience, we shall use the following abbreviations :

$$\mathfrak{C}(X) = \mathfrak{C}(X, U); \qquad \pi(X) = \pi(X, U);$$
$$\mathfrak{C}_\alpha(X) = \mathfrak{C}_\alpha(X, U); \qquad \pi_\alpha(X) = \pi_\alpha(X, U);$$
$$\mathfrak{C}(X_\alpha) = \mathfrak{C}(X_\alpha, U_\alpha); \qquad \pi(X_\alpha) = \pi(X_\alpha, U_\alpha).$$

The objects of the Leray-Schauder category $\mathfrak{L}$ are denoted by X, Y, A. By $\mathcal{L}_U$ we denote the cofinal subset of $\mathcal{L}$ defined by the condition :

$$\alpha \in \mathcal{L}_U \iff \begin{cases} \text{(i)} & W \subset L_\alpha, \\ \text{(ii)} & U_\alpha = L_\alpha - W \quad \text{is connected} \end{cases}$$

and we let $\mathcal{L}^U_X = \mathcal{L}_U \cap \mathcal{L}_X$. The elements of $\mathcal{L}^U_X$ are said to be *admissible* with respect to X. We assume that the elements of $\mathcal{L}$ which appear in the sequel are admissible with respect to the objects under consideration.

1.1. Proposition. — *Let* $f' : X_\alpha \to U_\alpha$, *where* $U_\alpha = L_\alpha - W$, *be a continuous mapping. There exists an* α*-field* $f : X \to U$ *such that* $f_\alpha = f'$.

Proof. — Let $F : X \to L_\alpha$ be an α-extension of $F' : X_\alpha \to L_\alpha$ over X. We claim that f defined on X by $f(x) = x - F(x)$ has values in U. For, suppose to the contrary that $f(x) \in W$ for some $x \in X$. Since $W \subset L_\alpha$ and $F(X) \subset L_\alpha$, we conclude that $x \in L_\alpha \cap X = X_\alpha$. But then, $f(x) = f'(x) \in W$. This contradiction shows that $f : X \to U$. We have clearly $f_\alpha = f'$, and therefore f is a required α-field.

1.2. Proposition. — *For each* $\alpha \in \mathcal{L}_X^U$, *the sets* $\mathfrak{C}_\alpha(X)$ *and consequently* $\pi_\alpha(X)$ *and* $\pi(X)$ *are non-empty.*

1.3. Proposition. — *Let* $f, g : X \to U$ *be two* α*-fields and* $h'_t : X_\alpha \to U_\alpha$ *be a homotopy joining* f_α *and* g_α. *Then there exists an* α*-homotopy* $h_t : X \to U$ *joining* f *and* g *and such that* $h_t(x) = h'_t(x)$ *for all points* $x \in X$ *and* $t \in I$.

Proof. — Define on $T = (X \times \{0\}) \cup (X_\alpha \times I) \cup (X \times \{1\})$ a mapping H^* by

$$H^*(x, t) = \begin{cases} F(x) & \text{if } x \in X \text{ and } t = 0; \\ H(x, t) & \text{if } (x, t) \in X_\alpha \times I; \\ G(x) & \text{if } x \in X \text{ and } t = 1. \end{cases}$$

Evidently, $H^* : T \to L_\alpha$ is an α-mapping. Let $H : X \times I \to L_\alpha$ be an arbitrary compact extension of H from T over $X \times I$ and let $h_t(x) = x - H(x, t)$ for $x \in X$, $t \in I$. We claim that the values of h_t are in U. For, suppose the contrary that $h_t(x) \in W$ for some $t \in I$ and $x \in X$. Since $W \subset L_\alpha$ and $H(X \times I) \subset L_\alpha$ we conclude that $x \in L_\alpha$. But then $h_t(x) = h'_t(x) \in W$. This contradiction shows that $h_t : X \to U$. It follows from the construction that h_t is a required α-homotopy.

1.4. Definition. — For a given object X and $\alpha \in \mathcal{L}_X^U$ let

$$\tau_\alpha : \pi_\alpha(X) \to \pi(X_\alpha)$$

be a map defined by the correspondance

$$[f]_\alpha \to [f_\alpha].$$

1.5. Proposition. — *The map* τ_α *is bijective.*

Proof. — This clearly is a consequence of Propositions 1.1 and 1.3.

1.6. Proposition. — *Let* X *be an object and let* $h_t : X \to U$ *be a compact homotopy. Then the number* $\operatorname{dist}(h(X \times I), W)$ *is positive and for each* $\varepsilon > 0$ *satisfying*

$$0 < \varepsilon < \operatorname{dist}(h(X \times I), W)$$

there exists an α*-homotopy* $h'_t : X \to U$ *such that*

$$\| h'_t(x) - h_t(x) \| < \varepsilon \quad \textit{for all } x \in X,\ t \in I.$$

Proof. — The first assertion being obvious, let $H' : X \times I \to L_\alpha$ be an α-approximation of the compact mapping $H : X \times I \to E$ (comp. Lemma II.2.2). Assuming without loss of generality that $W \subset L_\alpha$ and putting

$$h'_t(x) = x - H'(x, t) \qquad (x \in X,\ t \in I)$$

we obtain clearly a required α-homotopy.

As a consequence, we obtain the following two propositions :

1.7. Proposition. — *Let f, $g \in \mathfrak{C}(X)$ be two homotopic compact fields. Then there exist two α-fields f^*, $g^* \in \mathfrak{C}_\alpha(X)$ such that :*

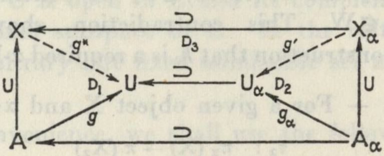
(i) $f^* \underset{\alpha}{\simeq} g^*$;

(ii) $f^* \sim f$ *and* $g^* \sim g$.

1.8. Proposition. — *Let $f \in \mathfrak{C}_\alpha(X)$ be an α-field and $g \in \mathfrak{C}_\beta(X)$ be a β-field. If the fields f and g are homotopic, then there exists γ with $\alpha \leqq \gamma$, $\beta \leqq \gamma$ and such that $f \underset{\gamma}{\simeq} g$.*

2. Inessential fields. — First we shall establish two elementary facts concerning the extension problem for finite dimensional fields.

Let (X, A) be a closed bounded pair in E and $g : A \to U$ be an α-field. Consider the following diagram :



and its sub-diagrams D_1, D_2, D_3.

The following proposition states that the extension problem for g reduces to the extension problem for the partial mapping g_α.

2.1. Proposition. — *If an α-field g^* completes the diagram D_1, then the mapping g' defined by $g' = g^*_\alpha$ completes the diagram D_2. Conversely, if a mapping g' completes D_2 then there exists an α-field g^*, which completes the whole diagram D, i. e., $g^* \mid A = g$ and $g^*_\alpha = g'$.*

Proof. — Let $T = A \cup U_\alpha$ and define on T a mapping $\overline{G}$ by

$$\overline{G}(x) = \begin{cases} G'(x) & \text{for } x \in X_\alpha, \\ G(x) & \text{for } x \in A. \end{cases}$$

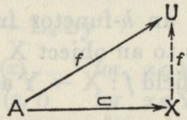
Evidently, $\overline{G} : T \to L_\alpha$ and if $\bar{g}(x) = x - \overline{G}(x)$ for $x \in T$, then $\bar{g}(x) \in U$. Since T is closed in X, in view of the Extension of Compact Mappings Lemma, there exists a compact extension $G^* : X \to L_\alpha$ of $\overline{G}$ over X. Putting $g^*(x) = x - G^*(x)$ for $x \in X$, we obtain the required α-field $g^* : X \to U$. The proof is completed.

Using the same argument as in the proof of the previous theorem we obtain the following result :

2.2. Proposition. — *Let g'_1, $g'_2 : X \to U_\alpha$ be two homotopic mappings. If each of them completes the diagram* D_2, *then there exist two α-homotopic α-fields g_1 and g_2 such that each of them completes the whole diagram* D.

Now we introduce the following :

2.3. Definition. — A compact vector field (respectively an α-field) $f : A \to U$ is called *inessential* provided each diagram :



can be completed by a compact field (respectively an α-field) $f^* : X \to U$.

2.4. Proposition. — *The set* $\mathfrak{C}(X, U)$ [*respectively the set* $\mathfrak{C}_\alpha(X, U)$] *contains an inessential compact field* (*respectively an α-field*).

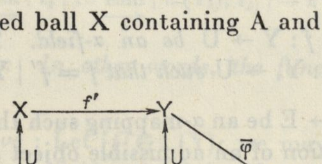
Proof. — Clearly, it is sufficient to prove the existence of an inessential α-field in $\mathfrak{C}_\alpha(X, U)$. Let $g' : X_\alpha \to U_\alpha$ be a constant map. In view of Proposition 1.1, there is an α-field $g : X \to U$ such that $g_\alpha = g'$. By the Homotopy Extension Lemma and Proposition 2.2, we conclude that the field g is inessential.

2.5. Proposition. — *Any two inessential compact fields* (*respectively α-fields*) f', f'' : $X \to U$ *are homotopic* (*respectively α-homotopic*).

Proof. — It is clearly sufficient to prove the second part of our proposition. We may assume without loss of generality that X is a closed ball. Then $f'_\alpha \sim f''_\alpha$ since X_α is contractible and $L_\alpha - W = U_\alpha$ is connected. Applying Proposition 1.3, we conclude that $f' \underset{\alpha}{\simeq} f''$. The proof is completed.

2.6. Proposition. — *Let* A *and* B *be two objects*, $f : A \to B$ *be a compact field. Assume that either :* (*a*) f *is finite dimensional or* (*b*) E *is complete. Then, if a field* $\varphi : B \to U$ *is inessential, so is the composite* $\varphi f : A \to U$.

Proof. — Take a closed ball X containing A and consider the following diagram :



By the Extension of Compact Mappings Lemma, in either of these cases (*a*) and (*b*), there exist a closed ball Y containing B and a field $f' : X \to Y$ such that $f(a) = f'(a)$ for all $a \in A$. On the other hand, by assumption, there is an extension $\bar{\varphi} : Y \to U$ of φ over Y. Putting $\bar{f} = \bar{\varphi} f'$, we obtain a field $\bar{f} : X \to U$ which extends $\varphi f : A \to U$ over X. Thus φf is inessential and the proof is completed.

3. Continuity of the functor π_0. — We know that, given a fixed admissible object U we have an *h*-functor from the category $\mathfrak{L}$ to the category of sets, which assigns to an object X the set of homotopy classes $\pi(X) = \pi(X, U)$ and to each field $f : X \to Y$ assigns the induced map

$$f^* : \pi(Y) \to \pi(X).$$

The purpose of this section is to show that the above functor may be converted into an *h*-functor from $\mathfrak{L}$ to the category of based sets.

First we note that by the results of the previous section, each homotopy set $\pi(X)$ contains a distinguished element, namely, the class which contains all inessential fields and called the *zero homotopy class*. Moreover, if *f* is a finite dimensional field, then as a consequence of Proposition 2.6, we observe that f^* is the based map, i. e., $f^*(0) = 0$.

Let us denote by π_0 the function defined on the dense subcategory $\mathfrak{L}_0 \subset \mathfrak{L}$ by the assignments $X \to \pi(X)$ and $f \to f^*$. Then the above remarks may be expressed equivalently as follows :

3.1. Proposition. — *The function* $\pi_0 : \mathfrak{L}_0 \to \mathcal{E}ns$ *is a contravariant h-functor from the category* $\mathfrak{L}_0$ *into the category of sets with distinguished elements.*

In order to prove that f^* is the based map for an arbitrary compact field *f*, we shall establish the continuity of the functor π_0. This fact will also be of importance in our further discussion.

We begin with two lemmas :

Let Y be an object and $\{Y_k\}$ be an approximating sequence for Y.

3.2. Lemma. — *Let* $f : Y \to U$ *be an* α*-field. There exists an integer* $k \geq 1$ *and an* α*-field* $f' : Y_k \to U$ *such that* $f = f' \mid Y$.

Proof. — Let $\bar{F} : Y_1 \to E$ be an α-mapping such that $\bar{F} \mid Y = F$. Taking into account the definition of an admissible object and Proposition II.3.2 we infer that the set

$$C = \{ x \in Y_1 ; f(x) = x - F(x) \in W \}$$

is compact; therefore, for some $k \geqq 1$, Y_k does not intersect C. Now putting $f' = f \mid Y_k$, we obtain a required α-field f'.

3.3. Lemma. — *Let $f, g : Y_k \to U$ be two α-fields such that $f \mid Y \underset{\alpha}{\simeq} g \mid Y$. There exists an integer $m \geqq k$ such that $f \mid Y_m \underset{\alpha}{\simeq} g \mid Y_m$.*

Proof. — Let $h_t : Y \to U$ be an α-homotopy such that $h_0 = f$ and $h_1 = g$. On the set

$$T = (Y_k \times \{0\}) \cup (Y \times I) \cup (Y_k \times \{1\})$$

define an α-mapping $H' : T \to L_\alpha$ by

$$H'(x, t) = \begin{cases} F(x) & \text{for } x \in Y_k, \quad t = 0; \\ H(x, t) & \text{for } x \in Y, \quad 0 \leqq t \leqq 1; \\ G(x) & \text{for } x \in Y_k, \quad t = 1. \end{cases}$$

Clearly, we have

$$x - H'(x, t) \in U \quad \text{for all } (x, t) \in T.$$

By extending H' from T over $Y_k \times I$, we obtain an α-mapping $\overline{H} : Y_k \times I \to L_\alpha$. Clearly, the set

$$C = \{(x, t) \in Y_k \times I; \bar{h}_t(x) \in W\}$$

is compact; therefore, for some $m \geqq k$ the intersection $C \cap (Y_m \times I)$ is empty. Now, putting $\tilde{h}_t = \bar{h}_t \mid Y_m$, we obtain an α-homotopy $\tilde{h}_t : Y_m \to U$ such that $\tilde{h}_0 = f \mid Y_m$ and $\tilde{h}_1 = g \mid Y_m$. The proof is completed.

3.4. Definition. — For each relation $k \leqq l$ denote by

$$i^*_{kl} : \pi(Y_k) \to \pi(Y_l)$$

the based maps induced by the corresponding inclusions. Clearly, $\{\pi(Y_k); i^*_{kl}\}$ is a direct system of based sets and the family $\{i^*_k\}$ is a direct family of based maps.

Now we may state the main result of this section :

3.5. Theorem. — *The map*

$$i^* = \underset{\overrightarrow{k}}{\text{Lim}} \{i^*_k\} : \underset{\overrightarrow{k}}{\text{Lim}} \{\pi(Y_k); i^*_{kl}\} \to \pi(Y)$$

is a bijective based map. In other words, the functor $\pi_0 : \mathfrak{L}_0 \to \mathcal{E}ns^*$ is continuous.

Proof. — i^* is surjective : Let $[f] \in \pi(Y)$; we may assume without loss of generality that f is an α-field. Applying Lemma 3.2, there exist k and $[f'] \in \pi(Y_k)$ such that $i^*_k[f'] = [f]$.

i^* is injective : Suppose that for two elements $[f], [g] \in \pi(Y_k)$ we have

$$i^*_k([f]) = i^*_k([g]).$$

Assuming, without loss of generality, that both f and g are α_0-fields, this implies $f \mid Y \simeq_\alpha g \mid Y$ for some $\alpha \geqq \alpha_0$; consequently by 3.3 we have

$$i^*_{kl}([f]) = i^*_{kl}([g])$$

for some $l \geqq k$ and the proof is completed.

i^* is a based map : Evident.

3.6. Corollary. — *For an arbitrary compact field f, we have $f^*(0) = 0$.*

4. $\pi(X)$ is a direct limit of the homotopy system $\{\pi_\alpha(X); i_{\alpha\beta}\}$. — In this section, X denotes an arbitrary but fixed object and the indices α, β, ... stand for the elements of the directed set $\mathcal{L}^U_X$.

Definition. — For each α consider the set $\pi_\alpha(X)$ as a set with the 0-homotopy class as the distinguished element. For each relation $\alpha \leqq \beta$, let

$$i_{\alpha\beta} : \pi_\alpha(X) \to \pi_\beta(X)$$

be the map defined by the assignment

$$[f]_\alpha \to [f]_\beta.$$

The family

$$\mu(X) = \{\pi_\alpha(X); i_{\alpha\beta}\}$$

indexed by $\mathcal{L}^U_X$ will be called the *homotopy system* of X.

4.2. Proposition. — *The homotopy system $\mu(X)$ of X is a directed system of based sets over the directed set $\mathcal{L}^U_X$.*

Proof. — This follows immediately from the definitions.

Now we turn to the result indicated in the title of this section.

For each α let

$$i_\alpha : \pi_\alpha(X) \to \pi(X)$$

be defined by the assigment

$$[f]_\alpha \to [f].$$

Since for every relation $\alpha \leqq \beta$ in $\mathcal{L}^U_X$ we have

$$i_\beta \, i_{\alpha\beta} = i_\alpha,$$

the family $\{i_\alpha\}$ is a direct family of maps in $\mathcal{E}ns^*$.

4.5. Theorem. — *The direct limit map*

$$i^* = \underset{\overrightarrow{\alpha}}{\text{Lim}} \{ i_\alpha \} : \ \underset{\longrightarrow}{\text{Lim}} \{ \pi_\alpha (X); i_{\alpha\beta} \} \to \pi (X)$$

is invertible in the category of based sets.

Proof. — *i^* is surjective :* Let $[f] \in \pi (X)$; by Proposition 1.7, there exists an α and an α-homotopy class $[g]_\alpha \in \pi_\alpha (X)$ such that $i_\alpha ([g]_\alpha) = [f]$.

i^* is injective : Assume that $i_\alpha ([g]_\alpha) = i_\beta ([f]_\beta)$, where $[g]_\alpha \in \pi_\alpha (X)$ and $[f]_\beta \in \pi_\beta (X)$. Then, by Proposition 1.8, there exists an γ such that $i_{\alpha\gamma} ([g]_\alpha) = i_{\beta\gamma} ([f]_\beta)$.

i^* is a based map : Evident.

5. The functor $\tilde{\pi}$ and π_0. — For every relation $\alpha \leq \beta$ in $\mathcal{L}^U_X$, consider the based map

$$\tau^\beta_\alpha : \ \pi_\alpha (X_\beta) \to \pi (X_\alpha)$$

defined by the correspondence

$$[f]_\alpha \to [f_\alpha]$$

and the based map

$$i^\beta_\alpha : \ \pi_\alpha (X_\beta) \to \pi (X_\beta)$$

given by the assignment

$$[f]_\alpha \to [f].$$

It follows from Proposition 1.5 that τ^β_α is bijective.

5.1. Definition. — For each relation $\alpha \leq \beta$ define

$$j_{\alpha\beta} : \ \pi (X_\alpha) \to \pi (X_\beta)$$

by putting

$$j_{\alpha\beta} = i^\beta_\alpha \circ (\tau^\beta_\alpha)^{-1}.$$

The family

$$\mu (X) = \{ \pi (X_\alpha); j_{\alpha\beta} \}$$

will be called the *restricted homotopy system* of X.

The following two propositions follow clearly from the definitions involved

5.2. Proposition. — *The restricted homotopy system of* X *is a directed system of based sets over the directed set* $\mathcal{L}^U_X$.

5.3. Proposition. — *Let* X *and* Y *be two objects and* $f: X \to Y$ *an* α_0*-field. Then, for any* $\alpha, \beta \in \mathcal{L}_X^U$ *with* $\alpha_0 \leqq \alpha \leqq \beta$ *we have the following commutative diagram of based maps :*

$$\begin{array}{ccc} \pi(X_\alpha) & \xrightarrow{j_{\alpha\beta}} & \pi(X_\beta) \\ \downarrow{\scriptstyle f_\alpha^*} & & \downarrow{\scriptstyle f_\beta^*} \\ \pi(Y_\alpha) & \xrightarrow{j_{\alpha\beta}} & \pi(Y_\beta) \end{array}$$

Now let us put for an object X

(i) $$\tilde{\pi}(X) = \varinjlim_{\alpha} \{ \pi(X_\alpha); j_{\alpha\beta} \}.$$

If $f: X \to Y$ is a finite dimensional field, then, by Proposition 5.3, $\{ f_\alpha^* \}$ is a direct family of based maps and therefore the formula

(ii) $$\tilde{\pi}(f) = \tilde{f} = \varinjlim_{\alpha} \{ f_\alpha^* \}$$

defines the based map $\tilde{f}: \tilde{\pi}(Y) \to \tilde{\pi}(X)$.

5.4. Proposition. — *The formulas* (i) *and* (ii) *define an h-functor* $\tilde{\pi}$ *from* $\mathcal{L}_0$ *to the category of based sets.*

Our aim now is to prove that $\tilde{\pi}$ is naturally equivalent with the functor π_0.

For an object X and $\alpha \in \mathcal{L}_X^U$, denote by

$$\tau_{\alpha X}: \pi_\alpha(X) \to \pi(X_\alpha)$$

the *restriction map* given by the correspondence

$$[f]_\alpha \to [f_\alpha].$$

5.5. Proposition. — *For every relation* $\alpha \leqq \beta$ *we have the following commutative diagram of based maps :*

$$\begin{array}{ccc} \pi_\alpha(X) & \xrightarrow{i_{\alpha\beta}} & \pi_\beta(X) \\ \downarrow{\scriptstyle \tau_{\alpha X}} & & \downarrow{\scriptstyle \tau_{\beta X}} \\ \pi(X_\alpha) & \xrightarrow{j_{\alpha\beta}} & \pi(X_\beta) \end{array}$$

5.6. Proposition. — *Let* X *and* Y *be two objects and* $f: X \to Y$ *be an* α_0*-field. Then, for any* α *with* $\alpha_0 \leqq \alpha$ *we have the following commutative* *diagram of based maps :*

$$\begin{array}{ccc} \pi_\alpha(X) & \xleftarrow{\ f^*\ } & \pi_\alpha(Y) \\ \downarrow{\scriptstyle \tau_{\alpha X}} & & \downarrow{\scriptstyle \tau_\alpha} \\ \pi(X_\alpha) & \xleftarrow{\ f^*_\alpha\ } & \pi(Y_\alpha) \end{array}$$

By passing to the limit in the diagram of Proposition 5.5 and putting for every object X

$$\tau_X = \underset{\overrightarrow{\alpha}}{\mathrm{Lim}} \{ \tau_{\alpha X} \} \qquad (\alpha \in \mathcal{L}^U_X)$$

we obtain a family $\tau = \{ \tau_X \}$ of based maps

$$\tau_X : \pi_0(X) \to \tilde{\pi}(X).$$

Now we are prepared to state the main result of this section :

5.7. Theorem. — *τ is a natural equivalence between the functors π_0 and $\tilde{\pi}$. More precisely (a) τ_X is a bijective based map for each X; (b) given a finite dimensional field $f : X \to Y$ we have the following commutative diagram of based maps :*

$$\begin{array}{ccc} \pi_0(X) & \xrightarrow{\ \tau_X\ } & \tilde{\pi}(X) \\ \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f} \\ \pi_0(Y) & \xrightarrow{\ \tau_Y\ } & \tilde{\pi}(Y) \end{array}$$

Proof. — This follows clearly from Propositions 1.5 and 5.6.

6. Natural group strucure in π (X). — In this section, for an admissible object U, we denote by $\bar{\pi} : \mathfrak{L} \to \mathcal{E}ns^*$ the functor which assigns to an object X the based set π (X) and to a compact field $f : X \to Y$ the based map $f^* : \pi(Y) \to \pi(X)$. We shall establish that for some U (called algebraically admissible) the functor $\bar{\pi}$ may be converted to a functor into the category of abelian groups.

Let us denote by $\bar{\pi}_0$ the restriction of $\bar{\pi}$ to the subcategory $\mathfrak{L}_0$ and denote by

$$\iota : \mathcal{A}b \to \mathcal{E}ns^*$$

the forgetful functor from the category of abelian groups to the category of based sets.

Now consider the problems $(\mathfrak{D})$ and $(\mathfrak{D}_0)$ of completing (by functors) the diagrams

$$D = \begin{array}{ccc} & & Ab \\ & \overset{\pi}{\nearrow} & \cap\,\iota \\ \mathfrak{L} & \xrightarrow{\bar{\pi}} & \mathcal{E}ns^* \end{array} \qquad \text{and } D_0 = \begin{array}{ccc} & & Ab \\ & \overset{\pi_0}{\nearrow} & \cap\,\iota \\ \mathfrak{L} & \xrightarrow{\bar{\pi}_0} & \mathcal{E}ns^* \end{array}$$

respectively.

6.1. Lemma. — *The problems* $(\mathfrak{D})$ *and* $(\mathfrak{D}_0)$ *are equivalent.*

Proof. — The implication in one direction is evident. To prove the implication in the opposite direction, let us assume that $\pi_0 : \mathfrak{L}_0 \to \mathfrak{A}b$ completes D_0. Since $\mathfrak{L}_0$ is dense in $\mathfrak{L}$, every based set $\pi(X)$ carries a structure of abelian group. It will be, therefore, sufficient to show that every based map

$$f^* : \ \pi(Y) \to \pi(X)$$

induced by a compact field $f : X \to Y$ is a homomorphism. To prove this, take an approximating sequence $\{Y_k, f_k\}$ for f. Since by Theorem 3.5 the functor π_0 is continuous we have the following formula in the category $\mathcal{E}ns^*$:

$$(\star) \qquad f = \varinjlim_k \{ f_k^* \} \circ (\varinjlim_k \{ j_k^* \})^{-1}.$$

By assumption, all the maps $\pi_0(f_k) = f_k^*$ and $\pi_0(j_k) = j_k^*$ are homomorphisms. Consequently, in view of the formula $(\star)$, f^* is also a homomorphism and the proof. of the lemma is completed.

6.2. Definition. — An admissible object U is said to be *algebraically admissible* provided there exists a cofinal subset $\mathcal{L}'_U$ of $\mathcal{L}_U$ such that :

(i) for each $\alpha \in \mathcal{L}'_U$ the assignment $X \to \pi(X_\alpha)$ is an h-functor from $(\mathfrak{L}^*_{\alpha,\tilde{\alpha}})$ to the category of abelian groups [6];

(ii) for every relation $\alpha \leq \beta$ in $\mathcal{L}'_U$ and every X such that X_β is non-empty, the map $j_{\alpha\beta} : \pi(X_\alpha) \to \pi(X_\beta)$ is a homomorphism.

Now we may state the final result of this chapter :

6.3. Theorem. — *If* U *is algebraically admissible, then the diagram* $\mathfrak{D}$ *can be completed by an h-functor* $\pi : \mathfrak{L} \to \mathfrak{A}b$.

---

(6) We recall that $(\mathfrak{L}_{\alpha,\tilde{\alpha}}^*)$ is an h-subcategory of $(\mathfrak{L}_{\alpha,\tilde{\alpha}}^*)$ whose objects have non-empty intersection with L_α.

Proof. — By the definition of an algebraically admissible object and taking into account Theorem 5.7, we infer that the diagram $\mathcal{D}_0$ can be completed by π_0. This, in view of Lemma 6.1, implies our assertion and the proof is completed.

Remark. — One could prove that every admissible object U is algebraically admissible. Since this fact will not be used, we do not give any details.

CHAPTER VIII

Representation theorems

The main purpose of this chapter is to show that the infinite dimensional stable cohomotopy theory is representable. In the second part of the chapter, we shall prove that (under some restrictions on the group of coefficients) the ordinary cohomology theory is also representable.

1. The groups $\pi^{\infty-n}(X)$ (7). — Let $\{E^{\infty-n} \oplus E_n\}$ be a fixed sequence of direct sum decompositions of E as in the section I.1. For $n \geq 1$, we let

$$U^{(n)} = U^{\infty-n} = E - E_{n-1}.$$

Clearly, $U^{\infty-n}$ is admissible and we denote by $\pi^{\infty-n}$ the corresponding functor from $\mathfrak{L}$ to $\mathcal{E}ns^*$.

Next, we let

$$\mathcal{L}_n = \{\alpha \in \mathcal{L}\,;\, E_{n-1} \subset L_\alpha;\, d(\alpha) \geq 2n + 2\},$$
$$\mathcal{L}_{n,X} = \{\alpha \in \mathcal{L}_n;\, X_\alpha \neq \emptyset\}.$$

Clearly, $\mathcal{L}_{n,X}$ is cofinal in $\mathcal{L}_n$ and $\mathcal{L}_n$ is cofinal in $\mathcal{L}$.

For $k > n > 0$, the map from S^{k-n} to $R^k - R^{n+1}$ given by the assignment

$$(x_1, \ldots, x_{k-n+1}) \to (0, \ldots, 0, x_1, \ldots, x_{k-n+1})$$

is a homotopy equivalence and we denote by

$$\gamma_{k,n}: \quad R^k - R^{n+1} \to S^{k-n}$$

a homotopy inverse of this map.

Let $\mathcal{O} = \{\mathcal{O}_\alpha\}$ be a fixed orientation in E. For each $n \geq 1$ choose $l_n : E_n \to R^n$ which represents the orientation of E_n and $l_n(x) = l_{n+1}(x)$ for $x \in E_n$.

For $\alpha \in \mathcal{L}_{n,X}$ choose $l_\alpha \in \mathcal{O}_\alpha$ such that $l_\alpha(x) = l_{n-1}(x)$ for all $x \in E_{n-1}$ and define a map

$$\gamma_{\alpha,n}: \quad U_\alpha^{\infty-n} \to S^{d(\alpha)-n}$$

(7) These groups were introduced in K. Gęba [5].

by
$$\gamma_{\alpha,n}(x) = \gamma_{d(\alpha),n} \circ l_\alpha(x) \quad \text{for} \quad x \in U_\alpha^{\infty-n};$$
$\gamma_{\alpha,n}$ is a homotopy equivalence and therefore :
$$(\gamma_{\alpha,n})_* : \ \pi(X_\alpha, U_\alpha^{\infty-n}) \to \pi(X_\alpha, S^{d(\alpha)-n})$$
is bijective. Moreover, since
$$\dim X_\alpha \leqq d(\alpha) \leqq d(\alpha) + d(\alpha) - 2n - 2 = 2(d(\alpha) - n) - 2,$$
the set of homotopy classes $\pi(X_\alpha, S^{d(\alpha)-n})$ may be identified with the $(d(\alpha)-n)$-th cohomotopy group of X_α [and also with the $(d(\alpha)-n)$-th stable cohomotopy of X], i. e.,
$$\pi(X_\alpha, S^{d(\alpha)-n}) = \pi^{d(\alpha)-n}(X_\alpha) = \Sigma^{d(\alpha)-n}(X_\alpha).$$
Consequently, $\pi(X_\alpha, U_\alpha^{\infty-n})$ admits a unique abelian group structure such that $(\gamma_{\alpha,n})_*$ is an isomorphism; this structure is determined only by the orientation $\mathcal{O}_\alpha$ in L_α.

1.1. Proposition. — *Every* $U^{\infty-n}$ *is algebraically admissible* $(n \geqq 1)$. *Moreover, for each* $X \in \mathfrak{L}$ *the family* $(\gamma_{\alpha,n})_*$ *where* $\alpha \in \mathcal{L}_{n,X}$ *defines an isomorphism from* $\{\pi(X_\alpha, U_\alpha^{\infty-n}); j_{\alpha\beta}\}$ *to* $\{\Sigma^{d(\alpha)-n}(X_\alpha); \Delta_{\alpha\beta}\}$.

Proof. — If X, $Y \in \mathfrak{L}$ and $f : X \to Y$ is an α-field, $\alpha \in \mathcal{L}_{n,X}$, then we have a commutative diagram :
$$\begin{array}{ccc} \pi(Y_\alpha, U_\alpha^{(n)}) & \xrightarrow{(\gamma_{\alpha,n})_*} & \Sigma^{d(\alpha)-n}(Y_\alpha) \\ \downarrow f^* & & \downarrow f^* \\ \pi(X_\alpha, U_\alpha^{(n)}) & \xrightarrow{(\gamma_{\alpha,n})_*} & \Sigma^{d(\alpha)-n}(X_\alpha) \end{array}$$
Therefore, the assignment $X \to \pi(X_\alpha, U_\alpha^{(n)})$ is an h-functor from $\mathcal{L}_\alpha^*$ to the category of abelian groups. If $\alpha \leqq \beta$ is a relation in $\mathcal{L}_{n,X}$, then the following diagram commutes :
$$\begin{array}{ccc} \pi(X_\alpha, U_\alpha^{(n)}) & \xrightarrow{j_{\alpha\beta}} & \pi(X_\beta, U_\beta^{(n)}) \\ \downarrow (\gamma_{\alpha,n})_* & & \downarrow (\gamma_{\beta,n})_* \\ \Sigma^{d(\alpha)-n}(X_\alpha) & \xrightarrow{\Delta_{\alpha\beta}} & \Sigma^{d(\beta)-n}(X_\beta) \end{array}$$
Hence $j_{\alpha\beta}$ is a homomorphism and the proof is completed.

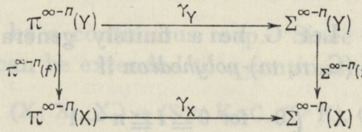
1.2. Corollary. — $\pi^{\infty-n}$ *is an h-functor from* $\mathfrak{L}$ *to the category of abelian groups.*

2. Representability of the stable cohomotopy. — We shall prove now that the functors $\pi^{\infty-n}$ and $\Sigma^{\infty-n}$ are naturally equivalent. To this end, recall that, in view of the results of Chapter VII, we may identify $\pi^{\infty-n}(X)$ with $\varinjlim_{\alpha} \{ \pi(X_\alpha, U_\alpha^{(n)}); j_{\alpha\beta} \}$. Using this identification we let

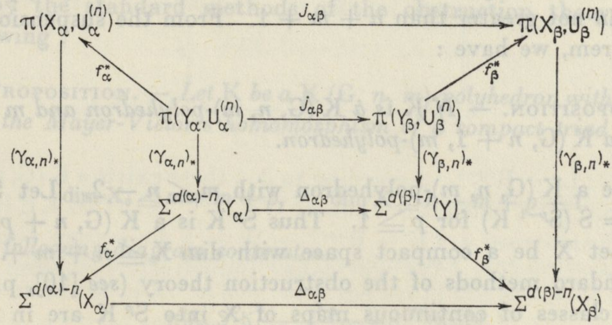
$$\gamma_X = \varinjlim_{\alpha} \{ (\gamma_{\alpha,n})_* \} : \quad \pi^{\infty-n}(X) \to \Sigma^{\infty-n}(X).$$

2.1. Theorem. — *The family $\gamma = \{ \gamma_X \}$ is a natural equivalence between the functors $\pi^{\infty-n}$ and $\Sigma^{\infty-n}$.*

Proof. — By definition γ_X is an isomorphism of abelian groups for each object X in $\mathfrak{L}$. It remains to prove that if $f: X \to Y$ is a map in $\mathfrak{L}$ then the following diagram commutes:



In view of the continuity of $\pi^{\infty-n}$ and $\Sigma^{\infty-n}$ it suffices to prove it for $f \in \mathfrak{L}_0$. Suppose that f is an α_0-field. If $\alpha, \beta \in \mathcal{E}_{n,X}$, $\alpha_0 \leq \alpha \leq \beta$, then the following diagram commutes:



Since we have identified $\varinjlim_{\alpha} \{ \pi(X_\alpha, U_\alpha^{(n)}); j_{\alpha\beta} \}$ with $\pi^{\infty-n}(X)$ and under this identification $\varinjlim_{\alpha} \{ f_\alpha^* \} = \pi^{\infty-n}(f)$, the desired conclusion follows. This completes the proof.

Remark. — The entire argument can be repeated in the relative case. Namely, letting $V^{\infty-n} = E - (l_{n+1})^{-1}(R_+^{n+1})$ we obtain a pair $(U^{\infty-n}, V^{\infty-n})$

of open subsets of E. One can prove that for a pair (X, A) in $\mathfrak{L}$ there exists a natural isomorphism

$$\gamma : \pi(X, A; U^{\varkappa-n}, V^{\varkappa-n}) \to \Sigma^{\varkappa-n}(X, A).$$

Thus Theorem 2.1 remains valid in the relative case.

An immediate consequence of 2.1 is the following :

2.2. Corollary. — *For an object* X *in* $\mathfrak{L}$, E — X *is connected if and only if any two compact fields* $f, g : X \to E - \{0\}$ *are compactly homotopic.*

3. K (G, *n*, *m*)-polyhedra. — In this section [8] we prove some lemmas which will be used in the proof of the representability of the ordinary cohomology.

3.1. Definition. — Let G be a finitely generated abelian group. A polyhedron K is a K (G, *n*, *m*)*-polyhedron* if

$$\pi_i(K) \approx \begin{cases} 0 & \text{for } 0 \leqq i \leqq n-1 \\ G & \text{for } i = n, \\ 0 & \text{for } n+1 \leqq i \leqq n+m. \end{cases}$$

It is a standard fact from the homotopy theory that, given a finitely generated G and integers *n* and *m*, there exists a K (G, *n*, *m*)-polyhedron of dimension not greater than $n + m + 1$. From the suspension isomorphism theorem, we have :

3.2. Proposition. — *If* K *is a* K (G, *n*, *m*)*-polyhedron and* $m \leqq n - 2$, *then* SK *is a* K (G, $n + 1$, *m*)*-polyhedron.*

Let K be a K (G, *n*, *m*)-polyhedron with $m \leqq n - 2$. Let $S^0 K = K$ and $S^p K = S(S^{p-1} K)$ for $p \geqq 1$. Thus $S^p K$ is a K (G, $n + p$, *m*)-polyhedron. Let X be a compact space with $\dim X \leqq n + m + p$. Then by the standard methods of the obstruction theory (*see* [10], p. 194) the homotopy classes of continuous maps of X into $S^p K$ are in a natural one-to-one correspondence with the elements of the Čech cohomology group $H^{n+p}(X; G)$. We will denote the corresponding map by

$$\lambda_p : \pi(X, S^p K) \to H^{n+p}(X; G).$$

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[8] In connection with the results of this section the authors thank P. Hilton for critical remarks and several helpful comments.

It turns out that λ_p is a transformation of functors : if Y is another compact space with $\dim Y \leqq n + p + m$ and $\varphi : X \to Y$ is a continuous map then the following diagram commutes :

$$\begin{array}{ccc} \pi(Y, S^p K) & \xrightarrow{\lambda_p} & H^{n+p}(Y; G) \\ \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \varphi^*} \\ \pi(X, S^p K) & \xrightarrow{\lambda_p} & H^{n+p}(X; G) \end{array}$$

Now let $(X; X_1, X_2)$ be a triad of compact spaces and let $X_0 = X_1 \cap X_2$. Assume

$$\dim X_0 \leqq n + m + p, \qquad \dim X \leqq n + m + p + 1.$$

Let $\varphi : X_0 \to S^p K$ be a continuous map. Since $C_+(S^p K)$ and $C_-(S^p K)$ are contractible φ can be extended to a continuous map

$$\tilde{\varphi} : \ (X; X_1, X_2) \to (S^{p+1} K; C_+(S^p K), C_-(S^p K)).$$

Clearly, the homotopy class of $\tilde{\varphi}$ depends only on the homotopy class of φ. Thus, the assignment $\varphi \mapsto \tilde{\varphi}$ induces a map

$$\sigma : \ \pi(X_0, S^p K) \to \pi(X, S^{p+1} K).$$

Again, by the standard methods of the obstruction theory we obtain the following :

3.3. Proposition. — *Let* K *be a* K (G, *n*, *m*)-*polyhedron with* $m \leqq n - 2$. *Let* Δ *be the Mayer-Vietoris homomorphism of a compact triad* $(X; X_1, X_2)$ *such that*

$$\dim X_0 \leqq n + m + p, \qquad \dim X \leqq n + m + p + 1.$$

Then the following diagram commutes :

$$\begin{array}{ccc} \pi(X_0, S^p K) & \xrightarrow{\lambda_p} & H^{n+p}(X_0; G) \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \Delta} \\ \pi(X, S^{p+1} K) & \xrightarrow{\lambda_p} & H^{n+p+1}(X; G) \end{array}$$

3.4. Lemma. — *Let* G *be a finitely generated abelian group. Let* $n > 0$ *be a fixed integer and let* $p = 4n + 7$. *Then there exists a closed subset* $W \subset R^p$ *such that* $R^p - W$ *has the homotopy type of a* K (G, $n + 2$, *n*)-*polyhedron.*

Proof. — There exists a $K(G, n+2, n)$-polyhedron K with

$$\dim K \leq n + 2 + n + 1 = 2n + 3.$$

Since $p = 2(2n+3) + 1$, K can be realized as a subcomplex of a triangulation of S^p. Let K^* be the dual subcomplex. Then $S^p - K^*$ has the homotopy type of K. Choose $x_0 \in K^*$ and let $\eta : R^p \to S^p$ be a continuous map which maps R^p homeomorphically onto $S^p - \{x_0\}$. Let $W = \eta^{-1}(K)$. Then η maps $R^p - W$ homeomorphically onto $S_n - K^*$ and hence W has the required property.

4. Representability of the ordinary cohomology $H^{\infty-*}(\ ; G)$. — Throughout this section we assume that we are given a finitely generated abelian group G. Let $n \geq 0$ be a fixed integer. Let $F = E \oplus R^{2n+5}$. Choose $\alpha_0 \in \mathcal{L}(F)$ such that $R^{2n+5} \subset L_{\alpha_0}$, $d(\alpha_0) = 4n + 7$. Let $L_0 = L_{\alpha_0}$. Let $\mathcal{L}_0 = \{\alpha \in \mathcal{L}(F); \alpha_0 \leq \alpha\}$. Clearly, $\mathcal{L}_0$ is a cofinal subset of $\mathcal{L}(F)$. For $\alpha \in \mathcal{L}_0$ we let $L_{\alpha'} = L_\alpha \cap E$. The assignment $\alpha \to \alpha'$ defines a bijective correspondence between $\mathcal{L}_0$ and a cofinal subset of $\mathcal{L}(E)$. Let $\{\mathcal{O}_\alpha\}$ be an orientation in E. Extend $\{\mathcal{O}_\alpha\}$ to an orientation in F which satisfies the following condition : if $\alpha \in \mathcal{L}_0$, $l_{\alpha'} \in \mathcal{O}_{\alpha'}$, $l_\alpha \in \mathcal{O}_\alpha$ then the following diagram, in which i denotes inclusion and $q(x) = (0, \ldots, 0, x_1, \ldots, x_{d(\alpha')})$, is commutative :

$$\begin{array}{ccc} L_{\alpha'} & \xrightarrow{\ l_{\alpha'}\ } & R^{d(\alpha')} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle q} \\ L_\alpha & \xrightarrow{\ l_\alpha\ } & R^{d(\alpha')+2n+5} \end{array}$$

Fix $l_0 \in \mathcal{O}_{\alpha_0}$, $l_0 : L_0 \to R^{4n+7}$. According to 3.4 there exists a $K(G, n+2, n)$-polyhedron $K \subset R^{4n+7}$ and a closed subset $W \subset R^{4n+7}$ such that $K \subset R^{4n+7} - W$ and the inclusion map $i_0 : K \to R^{4n+7} - W$ is a homotopy equivalence. Let $U = F - (l_0)^{-1}(W)$. We will define inductively inclusions

$$i_k : \ S^k K \to R^{k+4n+7} - W.$$

Assuming $i_{k-1} : S^{k-1} K \to R^{k-1+4n+7} - W$ is already defined, we let i_k be the map of $S^k K = S(S^{k-1} K)$ which maps the north pole onto $(0, \ldots, 0, 1)$, the south pole onto $(0, \ldots, 0, -1)$ and is linearly extended onto $C_+(S^{k-1} K)$ and $C_-(S^{k-1} K)$. Evidently, each i_k is a homotopy equivalence. Let $\varkappa_k : R^{k+4n+7} - W \to S^k K$ be a homotopy inverse to i_k.

Now let X be an object of $\mathfrak{L}(E)$ and let $\alpha \in \mathcal{L}_0$ be such that $X_\alpha \neq 0$. Choose a representative $l_\alpha : L_\alpha \to R^{d(\alpha)}$ such that $l_\alpha(x) = l_0(x)$ for all $x \in L_0$.

Let $\varkappa_\alpha : U \to S^{k(\alpha)} K$, where $k(\alpha) = d(\alpha) - 4n - 7$, be the map defined by $\varkappa_\alpha(x) = (\varkappa_{k(\alpha)} \circ l_\alpha)(x)$. Then $\varkappa_\alpha$ induces a bijective map

$$(\varkappa_\alpha)_* : \ \pi(X_\alpha, U_\alpha) \to \pi(X_\alpha, S^{k(\alpha)} K).$$

4.1. Lemma. — *If* $\alpha, \beta \in \mathfrak{L}_0$, $\alpha \leqq \beta$, $X_\alpha \neq 0$, *then the following diagram commutes :*

$$\begin{array}{ccccc}
\pi(X_\alpha, U_\alpha) & \xrightarrow{(\varkappa_\alpha)_*} & \pi(X_\alpha, S^{k(\alpha)}K) & \xrightarrow{\lambda_{k(\alpha)}} & H^{d(\alpha')-n}(X_\alpha; G) \\
\downarrow j_{\alpha\beta} & & & & \downarrow \Delta_{\alpha'\beta'} \\
\pi(X_\beta, U_\beta) & \xrightarrow{(\varkappa_\beta)_*} & \pi(X_\beta, S^{k(\beta)}K) & \xrightarrow{\lambda_{k(\beta)}} & H^{d(\beta')-n}(X_\beta; G)
\end{array}$$

Proof. — The lemma follows from 3.3 and the definitions involved.

In view of the Theorem VII.4.5 we may identify $\pi(X, U)$ with the limit of the direct system $\{\pi(X_\alpha, U_\alpha); j_{\alpha\beta}\}$. Thus

$$\lambda = \varinjlim_{\alpha} \{\lambda_{k(\alpha)} \circ (\varkappa_\alpha)_*\} : \ \pi(X, U) \to H^{\infty - n}(X; G).$$

Hence, we obtain the following :

4.2. Theorem. — *Let* $n \geqq 0$ *be a given integer and let* G *be a finitely generated abelian group. Then there exists an open subset* U *of* $E \oplus R^{2n+5}$ *such that for any* $X \in \mathfrak{C}(E)$ *there is a natural bijection*

$$\lambda : \ \pi(X, U) \to H^{\infty - n}(X; G).$$

CHAPTER IX

Some applications to non-linear problems

In this chapter, using cohomology functors, we consider some particular extension problems in $\mathfrak{C}$. Then, with the aid of the notion of the essential vector field, we treat some aspects of the theory of the equation $x = F(x)$ in cases when the Leray-Schauder degree theory is not applicable.

1. Essential fields from S into $E^{\infty - n} - \{0\}$. — *Notation.* — By an annular ring T in E we understand a set of the form

$$T = \{x \in E;\ r_1 \leqq \|x\| \leqq r_2\},$$

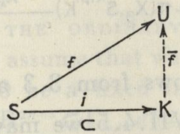
where $0 < r_1 \leqq r_2$. By K we denote a closed ball in E with the center 0 and by S its boundary.

We let

$$T^{\infty-n} = T \cap E^{\infty-n},$$
$$K^{\infty-n} = K \cap E^{\infty-n},$$
$$S^{\infty-n-1} = S \cap E^{\infty-n}$$

and we reserve the symbol $U^{\infty-n}$ for the open set $E - E_{n-1}$

1.1. Definition (comp. [8]). — Let U be open in $E^{\infty-n}$ and $f: S \to U$ be a compact field; f is called *essential* provided the diagram



cannot be completed in the category $\mathfrak{C}$, i. e., there is no compact field $\bar{f}: K \to U$ such that $f(x) = \bar{f}(x)$ for $x \in S$; otherwise, f is said to be *inessential.*

From the Homotopy Extension Lemma (*cf* remark after corollary II.5.2) it follows that the property of a field f to be essential depends only on the homotopy class of f. More precisely, we have the following:

1.2. Proposition. — *Let f, $g: S \to U$ be two fields which are compactly homotopic. If f is essential then so is g.*

Remarks. — Let $f: S \to E^{\infty-n} - \{0\}$ be an arbitrary compact field and

$$j: \; E^{\infty-n} - \{0\} \to U^{\infty-n},$$
$$i: \; E^{\infty-n} - \{0\} \to E - \{0\}$$

stand for the inclusions. Then:

(i) f is essential if and only if so is the composite $j \circ f$;

(ii) the composite $i \circ f$ is inessential;

(iii) the restriction

$$f \mid S^{\infty-n}: \; S^{\infty-n} \to E^{\infty-n} - \{0\}$$

is inessential.

Let $f: S \to E^{\infty-n} - \{0\}$ be a compact field. We may treat f as a field into $T^{\infty-n}$ for some annular ring $T^{\infty-n}$, i. e., we may write

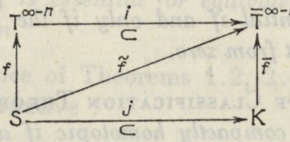
$$f: \; S \to T^{\infty-n}.$$

Take the stable cohomotopy functor $\Sigma^{\infty-n-1}$ and consider the induced homomorphism

$$f^*: \Sigma^{\infty-n-1}(T^{\infty-n}) \to \Sigma^{\infty-n-1}(S).$$

1.3. Theorem. — *If the induced homomorphism f^* is non-trivial, then the field f is essential.*

Proof. — Assume to the contrary that f is inessential. Then for some annular ring $\overline{T}^{\infty-n} \supset T^{\infty-n}$ there is a field $\bar{f}: K \to \overline{T}^{\infty-n}$ such that the diagram in $\mathfrak{L}$



in which f and $\tilde{f}$ stand for the obvious contractions of $\bar{f}$ is commutative. Since $\Sigma^{\infty-n-1}(K) = 0$ it follows that

$$\tilde{f}^* = j^* \circ \bar{f}^* = f^* \circ i^* = 0.$$

This in turn (because i^* is clearly an isomorphism) implies that $f^* = 0$. The obtained contradiction completes the proof.

Remarks. — 1. It can be shown that the groups

$$\Sigma^{\infty-n}(T^{\infty-m}) \approx \Sigma^{\infty-m}(S^{\infty-n-1})$$

are isomorphic to the stable homotopy groups of spheres (*cf.* [5]). In the case considered above

$$^{\infty-n-1}(T^{\infty-n}) \approx \Sigma_m(S^m) \approx Z,$$

$$\Sigma^{\infty-n-1}(S) \approx \Sigma_{m+n}(S^m).$$

2. In view of remark 1, the algebraic criterion given by Theorem 1.3 can only be of interest in cases when the group $\Sigma_{m+n}(S^m)$ is non-trivial. (This is known, for example, to be the case for $n = 1, 2, 3, 6, 7, 8$ but not for $n = 4, 5$.)

2. The Leray-Schauder characteristic. — We consider now the case $n = 0$ and deduce some known facts about the Leray-Schauder characteristic. We use the notation of the previous section.

2.1. Definition. — Let $f: S \to E - \{0\}$ be a compact field. Then we have

$$Z = \Sigma^{\infty-1}(T) \xrightarrow{f^*} \Sigma^{\infty-1}(S) = Z$$

and hence $f^* (1) = \gamma (f) \circ 1$; the integer $\gamma (f)$ is called the *Leray-Schauder characteristic* of f; clearly, f^* is non trivial if and only if $\gamma (f) \neq 0$.

Remark. — It follows from the Hopf theorem that, in the definition of $\gamma (f)$ one could use the cohomology functor $H^{\infty-1}(\ \ ; Z)$ instead of $\Sigma^{\infty-1}$. For $n > 0$, the ordinary cohomology could not provide any information because always $f^* = 0$.

Now, by taking into account the Representation Theorem for the stable cohomotopy, we draw the following consequences of Theorem 1.3 :

2.2. The Leray-Schauder Theorem. — *A compact vector field $f : S \to E - \{0\}$ is essential if and only if the Leray-Schauder characteristic $\gamma (f)$ of f is different from zero.*

2.3. The Rothe-Hopf Classification Theorem. — *Two vector fields $f, g : S \to E - \{0\}$ are compactly homotopic if and only if $\gamma (f) = \gamma (g)$.*

The only known general criterion for a field to be essential (in both cases $n = 0$ and $n > 0$) is given by the following result :

2.4. Borsuk's Antipodal Theorem. — *If $f : S \to E - \{0\}$ is odd [i. e., $f(x) = -f(-x)$ for every $x \in S$], then f is essential; in particular, the inclusion $j : S \to E - \{0\}$ is essential. More precisely, $\gamma (f)$ is odd and $\gamma (j) = 1$.*

Proof. — It can be easily shown (comp. [8]) that f is compactly homotopic to an odd finite dimensional field $g : S \to E - \{0\}$ and thus $\gamma (f) = \gamma (g)$. Then the fact that $\gamma (g)$ is odd, follows readily from the finite dimensional statement of Borsuk's Theorem.

3. The equation $x = F(x)$ (*The Leray-Schauder case*). — Each result concerning the essentiality of a compact field $f : S \to E^{\infty-n} - \{0\}$ can be " translated " into the existence criterion for the equation $x = F(x)$. In this and the next section we consider (for the sake of completeness) the known case $n = 0$. Our exposition follows essentially that of [8] (except we do not assume the space E to be complete).

Suppose that

$$F : K \to E$$

is a compact map and we are interested in solving the equation

$$F(x) = x, \tag{1}$$

or, equivalently, the equation

$$f(x) = x - F(x) = 0. \tag{2}$$

To treat (2) we may make an assumption that

(3) $$f(x) \neq 0 \quad \text{for} \quad x \in S.$$

The assumption (3) permits to consider f on S as a compact field

$$\bar{f}: \ S \to E - \{0\}$$

with the *range* equal to $E - \{0\}$.

3.1. Theorem. — *If $\bar{f}$ is essential* [*or equivalently* $\gamma(\bar{f}) \neq 0$], *then the equation* (2) *has a solution.*

As a simple consequence of Theorems 1.2, 2.4 and 3.1 one gets the following non-linear alternative :

3.2. Theorem ([9]). — *Let* $F : K \to E$ *be a compact mapping. Then either : (a) the equation* $x = F(x)$ *has a solution for some* $x \in K$ *or (b)* $x = \lambda F(x)$ *for some* $0 < \lambda < 1$ *and* $x \in S$.

Proof. — For suppose to the contrary that

(i) $$x \neq F(x) \quad \text{for all } x \in K$$

and at the same time,

(ii) $$x \neq \lambda F(x) \quad \text{for all } x \in S \quad \text{and} \quad 0 < \lambda < 1.$$

Then putting

(iii) $$h_t(x) = x - t F(x) \quad \text{for} \quad (x, t) \in S \times I$$

we have, in view of (i) and (ii),

$$h_t(x) \neq 0 \quad \text{for all } (x, t) \in S \times I$$

and hence the formula (iii) defines a compact homotopy $h_t : S \to E - \{0\}$ joining the inclusion $h_0 = j : S \to E - \{0\}$ and the field $h_1 = \bar{f} : S \to E - \{0\}$. Since j is essential, so is $\bar{f}$. Consequently, by Theorem 3 1, there exists a fixed point for F, which contradicts (i). The proof is completed.

3.3. Corollary (*The Leray-Schauder Alternative* [13]). — *Let* $F : E \to E$ *be a completely continuous operator and let the set* $\mathcal{E}_F \subset E$ *be given by the*

([9]) In the finite dimensional case Theorem 3.2 goes back to Bohl (*cf. J. für Reine Angew. Math.*, vol. 127, 1904, p. 179-276); *see* also A. Granas, *Introduction à la topologie des espaces de Banach*, Institut Henri Poincaré, 1966.

condition

$$x \in \mathcal{E}_F \iff \{ x = \lambda F(x) \text{ for some } 0 < \lambda < 1 \}.$$

Then either : (a) $\mathcal{E}_F$ is not bounded or (b) the equation $x = F(x)$ has at least one solution.

3.4. Corollary. — *Let* $F: K \to E$ *be a compact mapping and assume that any of the following conditions is satisfied :*

(i) $\| F(x) \| \leq \| x \|$ *for* $x \in S$ (E. Rothe);

(ii) $\| F(x) \| \leq \| x - F(x) \|$ *for* $x \in S$;

(iii) $\| F(x) \| \leq \sqrt{\| x - F(x) \|^2 + \| x \|^2}$ *for* $x \in S$ (M. Altman);

(iv) $\| F(x) \| \leq \max \{ \| x \|, \| x - F(x) \| \}$ *for* $x \in S$.

Then the equation $x = F(x)$ has at least one solution in K.

Proof. — Let us prove, for example, that (ii) implies the existence of a fixed point for F. For suppose to the contrary that $x \neq F(x)$ for all $x \in K$. Then, by Theorem 3.2, there is a point $x_0 \in S$ and $0 < \lambda < 1$ such that $x_0 = \lambda F(x_0)$. Then, in view of the inequality (ii),

$$\| F(x_0) \| \leq (1 - \lambda) \| F(x_0) \|$$

and since $\| F(x_0) \| \neq 0$, we get $1 \leq 1 - \lambda$. The obtained contradiction completes the proof.

4. Invariance of Domain. — Let $f: X \to Y$ be a continuous mapping between two metric spaces and $\varepsilon > 0$. Call f an ε-*mapping* provided

$$[f(x_1) = f(x_2)] \Rightarrow \rho(x_1, x_1) < \varepsilon \quad \text{for all } x_1, x_2, \in X;$$

f is said to be an ε-*mapping in the narrow sense* provided for some $\delta > 0$

$$\rho(f(x_1), f(x_2)) < \delta \Rightarrow \rho(x_1, x_2) < \varepsilon \quad \text{for all } x_1, x_2 \in X.$$

Denote by $K(x_0, \varepsilon)$ a ball in E with center x_0 and radius ε.

4.1. Lemma (*on ε-mappings*). — *Let* $f: K(x_0, \varepsilon) \to E$ *be a compact field. If in addition f is an ε-mapping, then $f(x_0)$ is an interior point of $f(K(x_0, \varepsilon))$.*

Proof. — We may assume without loss of generality that $\varepsilon = 1$ and $x_0 = 0$ is the origin in E. Let us put for $(x, t) \in S \times I$

$$h_t(x) = f\left(\frac{x}{1+t}\right) - f\left(-\frac{tx}{1+t}\right)$$

$$= x - \left[F\left(\frac{x}{1+t}\right) - F\left(-\frac{tx}{t+1}\right) \right].$$

Since f is an ε-mapping with $\varepsilon = 1$ and $\| x \| = 1$, it follows that $h_t(x) \neq 0$ for all $(x, t) \in S \times I$; thus $h_t : S \to E - \{0\}$ is a compact homotopy joining h_0 and h_1, where

$$h_0(x) = f(x) - f(0) \qquad (x \in S),$$

$$h_1(x) = f\left(\frac{x}{2}\right) - f\left(-\frac{x}{2}\right) \qquad (x \in S).$$

Since h_1 is odd on S and $h_0 \sim h_1$ it follows, by Borsuk's Antipodal Theorem, that h_0 is essential. This implies easily that $f(0) \in \mathrm{Int}\,(f(K))$.

A number of consequences of Lemma 4.1 follow :

4.2. Corollary (*Schauder's Theorem on Invariance of Domain* [17]). — *Let* U *be open in* E *and* $f : U \to E$ *be an injective completely continuous field* [10]. *Then* $f(U)$ *is open in* E.

Proof. — Since an injective map is clearly an ε-mapping for all ε, it follows that every point $y \in f(U)$ is an interior point in $f(U)$ and hence $f(U)$ is open.

4.3. Corollary (*Theorem on* ε-*mappings* [8]). — *Let* $f : E \to E$ *be a completely continuous field. If, in addition,* f *is an* ε-*mapping, then* $f(E)$ *is open in* E. *If, moreover,* f *is an* ε-*mapping in the narrow sense then* f *is surjective.*

4.4. Corollary. — *Let* $f : E \to E$ *be a completely continuous field satisfying for some* $C > 0$ *the condition*

$$(\star) \qquad \| f(x_1) - f(x_2) \| \geq C \| x_1 - x_2 \| \qquad (x_1, x_2 \in E).$$

Then f *is invertible.*

Proof. — Condition $(\star)$ implies that f is an ε-mapping in the narrow sense and hence, by 4.3, f is surjective. Clearly f is also injective. Hence, by the Invariance of Domain Theorem, f maps open sets onto open sets and therefore f is bicontinuous.

4.5. Corollary (*The Fredholm Alternative*). — *Let* $F : E \to E$ *be a completely continuous linear operator. Then either : (a) the equation* $x = F(x)$ *has a non-trivial solution or (b) the equation* $y = x - F(x)$ *has for each* y *exactly one solution.*

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(10) $f : X \to E$ is called a completely continuous field provided $F : X \to E$ is a completely continuous operator.

Proof. — Let us put $f(x) = x - F(x)$ for $x \in E$; clearly either : (i) f is not injective or (ii) f is injective. In the first case the condition (a) holds. In the second case, by the Invariance of Domain Theorem, f is surjective and bicontinuous and hence the condition (b) is satisfied. The proof is completed.

5. The equation $x = F(x)$ (The Hopf case). — If the Leray-Schauder characteristic $\gamma(\bar{f})$ of $\bar{f}$ is zero, then, in general, there is no solution of the equation (1). Nevertheless, one could still assert the existence of solutions for (1) provided *some additional information about f is available.*

To explain this consider first the finite dimensional case. Let us make an additional assumption that

$$f(K^{n+1}, S^n) \subset (R^{m+1}, R^{m+1} - \{0\}),$$

where $R^{m+1} \subset R^{n+1}$; we have the obvious contraction $\tilde{f}$ of f

$$\tilde{f}: \ S^n \to R^{m+1} - \{0\}.$$

Putting

$$\varphi(x) = \frac{\tilde{f}(x)}{\|\tilde{f}(x)\|} \quad \text{for} \quad x \in S^n$$

we get a map $\varphi : S^n \to S^m$.

Clearly, the following conditions are equivalent :

(a) $\tilde{f}: S^n \to R^{m+1} - \{0\}$ is essential;

(b) $\varphi : S^n \to S^m$ is not homotopic to a constant map;

(c) the homotopy class $[\varphi]$ is a non-trivial element of the homotopy group $\pi_n(S^m)$.

If any of the above conditions is satisfied, then the equation (2) has at least one solution. Thus, in brief, in finite dimensional case the types of the equation (1) can be classified according to the homotopy classification of spheres.

Passing to the infinite dimensional case let us keep the notation of section 3 and assume that for some $n > 0$,

(4) $$f(K) \subset E^{\infty - n}.$$

The assumptions (3) and (4) permit to consider f on S as a compact field

$$\tilde{f}: \ S \to E^{\infty - n} - \{0\}$$

with the range equal to $E^{\infty - n} - \{0\}$.

The following statement contains Theorem 3.1 and links the notion of the essential field and the theory of the equation (1).

4.1. Theorem. — *If the condition* (4) *is satisfied and* $\tilde{f} : S - E^{\infty - n} - \{0\}$ *is essential then the equation* (2) *and hence* (1) *has at least one solution.*

Remark. — We know (*cf.* Remark following 1.2) that $f : S \to E^{\infty-n} - \{0\}$ is essential if and only if so is the composite $jf : S \to U^{\infty-n}$, where $j : E^{\infty-n} - \{0\} \to U^{\infty-n}$ stands for the inclusion. But jf is essential if and only if the homotopy class $[jf]$ is a non-trivial element of the group

$$\pi^{\infty-n-1}(S) \approx \Sigma^{\infty-n-1}(S) \approx \Sigma_{m+n}(S^m).$$

It follows that in infinite dimensional case the various types of the equation (1) can be classified according to the *stable homotopy* classification of spheres.

CHAPTER X

Codimension

In this chapter we are concerned with the problem of finding a satisfactory notion of codimension for objects of the Leray-Schauder category. First for the objects in $\mathfrak{L}$ we define the "basic" codimension Codim in terms of the extension problem for compact fields with special ranges $E - E_n$. Further, we have various cohomological codimensions; in particular, Codim_Z defined in terms of the ordinary cohomology over Z. The main result of the chapter says that if E is a Banach space, then $\mathrm{Codim} = \mathrm{Codim}_Z$.

1. Extension objects and the function Codim. — We recall that for a fixed increasing sequence of linear subspaces of E :

$$E_0 \subset E_1 \subset \ldots \subset E_n \subset E_{n+1} \subset \ldots \qquad (\dim E_n = n).$$

we let

$$U^{\infty-n} = E^{\infty} - E_{n-1} \quad \text{for} \quad n = 1, 2, \ldots.$$

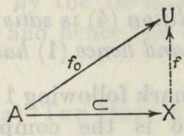
To each $U^{\infty-n}$ corresponds a functor $\pi^{\infty-n} : \mathfrak{L} \to \mathfrak{A}b$ which is naturally equivalent to the stable cohomotopy functor $\Sigma^{\infty-n} : \mathfrak{L} \to \mathfrak{A}b$.

We recall that on objects

$$\pi^{\infty-n}(X) = \pi(X, U^{\infty-n}).$$

1.1. Definition. — Let X be an object in $\mathfrak{L}$ and $U \in \mathfrak{C}$. We say that U is an *extension object* for X provided that, given an object $A \subset X$

and a compact field $f_0 : A \mapsto U$, there exists a field $f : X \to U$ being an extension of f_0 over X. In other words, every diagram



with A in $\mathfrak{L}$ can be completed in $\mathfrak{C}$.

We denote by $\mathfrak{E}$ (U) the set of objects defined by the condition :

$$X \in \mathfrak{E}(U) \iff U \text{ is an extension object for } X$$

and we let

$$\mathfrak{E}^{\infty-k} = \mathfrak{E}(U^{\infty-k}).$$

Example. — Given an object X in $\mathfrak{L}$ it is natural to assume that

$$\text{Codim } X = 0 \iff \text{the interior Int}(X) \text{ is non-empty.}$$

If E is a Banach space one could prove then that

$$\text{Codim } X = 0 \iff U^{\infty-1} = E - \{0\} \text{ is not an extension object for X.}$$

The above example suggests the following :

1.2. Definition. — For an object $X \in \mathfrak{L}$ we define the *codimension* Codim X of X with respect to E to be the smallest integer n for which $U^{\infty-(n+1)}$ is not an extension object for X.

Thus,

$$\text{Codim } X = n \iff \begin{cases} \text{(i)} & X \in \mathfrak{E}^{\infty-(k+1)} \quad \text{for} \quad k = 0, 1, \ldots, n-1; \\ \text{(ii)} & X \notin \mathfrak{E}^{\infty-(k+1)} \quad \text{for} \quad k = n. \end{cases}$$

The following is an immediate consequence of the definitions :

1.3. Proposition. — *For any two equivalent objects* X *and* Y *in* $\mathfrak{L}$, *we have* Codim X = Codim Y.

2. Cohomological codimension Codim_G. — Let G be an abelian group and $H^{\infty-*}(\ \ ; G)$ the corresponding cohomology theory on $\mathfrak{L}$.

2.1. Definition. — We define the *cohomological codimension* $\text{Codim}_G(X)$ of an object X with respect to E as the smallest number n, such that $H^{\infty-n}(X, A) \neq 0$ for some object $A \subset X$.

Now our aim is to prove that Codim = $\mathrm{Codim}_Z(X)$. This result will be established with the aid of the Hopf Theorem, and the Representation Theorem after some preliminary lemmas.

First, as a consequence of the Homotopy Extension Lemma, we have the following :

2.2. Lemma. — *If the space* E *is complete, then for an object* $X \in \mathfrak{L}$ *the following two conditions are equivalent :*

(i) $X \in \mathfrak{C}^{\infty-(n+1)}$;

(ii) *For any pair of objects* $A \subset B \subset X$ *the restriction map*

$$j^*_{AB} : \pi^{\infty-n-1}(B) \to \pi^{\infty-n-1}(A)$$

is an epimorphism.

Next, two lemmas based on the continuity of the functors under consideration :

2.3. Lemma. — *Let* X *be a given object. Assume that for any pair of objects* $A \subset B \subset X$ *the map*

(i) $$j^*_{AB} : H^{\infty-n-1}(B) \to H^{\infty-n-1}(A)$$

is an epimorphism. Then for any object $A \subset X$ *the group* $H^{\infty-n}(A)$ *is trivial.*

Proof. — Assuming that our assertion is not true, take a nontrivial element ξ of the group $H^{\infty-n}(A)$.

For a point x in A, let $Y = S(\xi)$ be an essential carrier of ξ with respect to x. Now, take an additive triad $(Y; Y_1, Y_2)$ in which both Y_1, Y_2 are *proper* subsets of Y such that $x \in Y_0 = Y_1 \cap Y_2$ and consider the corresponding Mayer-Vietoris exact sequence :

$$\ldots \to H^{\infty-n-1}(Y_1) \oplus H^{\infty-n-1}(Y_2) \xrightarrow{\Psi} H^{\infty-n-1}(Y_0) \xrightarrow{\Delta} H^{\infty-n}(Y) \xrightarrow{0} H^{\infty-n}(Y_1) \oplus H^{\infty-n}(Y_2) \to \ldots$$

In view of the definition of the triad $(Y; Y_1, Y_2)$ we have :

(ii) $$j^*_{AY}(\xi) \neq 0, \qquad 0\, j^*_{AY}(\xi) = 0$$

and therefore by exactness

(iii) $$j^*_{AY}(\xi) \in \mathrm{Im}\,\Delta.$$

Further, by the assumption (i), the map Ψ is an epimorphism. From here, in view of (iii), we infer that for some ξ'

$$j^*_{AY}(\xi) = \Delta\Psi(\xi').$$

Consequently, again by exactness, $j^*_{AY}(\xi) = 0$, contrary to (ii).

2.4. Lemma. — *For an object* X *the following two conditions are equivalent :*

1° $H^{\infty-n}(X, A) = 0$ *for all objects* $A \subset X$;

2° *For any pair of objects* $A \subset B \subset X$ *the map*

$$j^*_{AB} : H^{\infty-n-1}(B) \to H^{\infty-n-1}(A)$$

induced by the inclusion $j_{AB} : A \to B$ *is an epimorphism.*

Proof. — 1° ⟹ 2° : Assuming 1° we infer, in view of the exactness of the cohomology sequence of a pair, that both j^*_{AX} and j^*_{BX} are epimorphisms. Since $j^*_{AB} \circ j^*_{BX} = j^*_{AX}$, it follows that j^*_{AB} is also an epimorphism.

2° ⟹ 1° : Consider the cohomology sequence of the pair (X, A). Assuming 2°, we infer by Lemma 2.3 that $H^{\infty-n}(X) = 0$ and, consequently, by exactness $H^{\infty-n}(X, A) = 0$. The proof is completed.

Remark. — In the proof of Lemma 2.3 we used only the continuity of the ordinary cohomology theory $H^{\infty-*}(\ ; G)$; the same proof is valid therefore for any cohomology theory $\mathcal{H}^{\infty-*}$.

Thus, in particular, Lemma 2.3 and hence Lemma 2.4 is valid for the stable cohomotopy theory $\Sigma^{\infty-*}$.

3. Theorems of Alexandroff and Phragmen-Brouwer in E. — We turn now to the main result of this chapter, which is analogous to the " Fundamental Theorem in Dimension Theory " due to P. Alexandroff [1]. In this section we denote by $H^{\infty-*}$ the ordinary cohomology with integer coefficients Z.

3.1. Theorem (*The Alexandroff Theorem in* E). — *If the space* E *is complete, then for every object* X *in* $\mathfrak{L}$ *we have* $\mathrm{Codim}(X) = \mathrm{Codim}_Z(X)$.

Proof. — Assume that $\mathrm{Codim}_Z X = n$. Thus we have :

(1) $H^{\infty-k}(X, A) = 0$ for all $A \subset X$ and $k = 0, 1, 2, \ldots, n-1$;

(2) $H^{\infty-n}(X, B) \neq 0$ for some $B \subset X$.

Consequently, by the Hopf Theorem, we get from (1),

(3) $\Sigma^{\infty-k}(X, A) = 0$ for all $A \subset X$ and $k = 0, 1, 2, \ldots, n-1$.

From here, by Lemma 2.2, the Hopf Theorem and the Representation Theorem, we infer that

(4) The restriction map $j^*_{AX} : \pi^{\infty-(k+1)}(X) \to \pi^{\infty-(k+1)}(A)$ is an epimorphism for any $A \subset X$ and $k = 0, 1, 2, \ldots, n-1$.

Let A be an object contained in X, k be an integer satisfying $0 \leq k \leq n-1$ and $g : A \to U^{\infty-(k+1)}$ be an arbitrarily given compact field. In view of (4), $[g] = j^*_{AX}[f] = [f | A]$ for some field $f : X \to U^{\infty-(k+1)}$; thus, g is compactly homotopic to the restriction $f | A$ of f to A.

It follows now from the Homotopy Extension Lemma (which we can use in full generality because E is complete) that g can be extended over X; hence (because of the arbitrariness of A) we have

(5) $$X \in \mathfrak{E}^{\infty-(k+1)} \quad \text{for } k = 0, 1, \ldots, n-1.$$

We now claim that

(6) $$X \notin \mathfrak{E}^{\infty-(n+1)}.$$

For, suppose to the contrary that $X \in \mathfrak{E}^{\infty-(n+1)}$. Let $A \subset B$ be a pair of objects contained in X. Then, by Lemma 2.2 and the Representation Theorem, the restriction map

$$j^*_{AB} : \Sigma^{\infty-(n+1)}(B) \to \Sigma^{\infty-(n+1)}(A)$$

is an epimorphism. This implies, by Lemma 2.4, for stable homotopy (*cf.* Remark after proof of Lemma 2.4) that

(7) $$\Sigma^{\infty-n}(X, A) = 0 \quad \text{for all } A \subset X.$$

From (7), in view of (3) and the Hopf Theorem, we infer that

(8) $$H^{\infty-n}(X, A) = 0 \quad \text{for all } A \subset X,$$

which is in contradiction with (2).

By comparing (6) and (5) we get Codim X $= n$ and the proof is completed.

Now, with the aid of the Mayer-Vietoris sequence, we shall deduce the following :

3.2. THEOREM (*The Phragmen-Brouwer Theorem in* E). — *For an object* X, *denote by* b_0 (E — X) *the number of bounded components of* E — X. *Let* $(Y; Y_1, Y_2)$ *be an additive triad in* $\mathfrak{L}$ *such that* $\text{Codim}_Y (Y_1 \cap Y_2) > 2$. *Then*

$$b_0(E - Y) = b_0(E - Y_1) + b_0(E - Y_2).$$

Proof. — Take the ordinary cohomology $H^{\infty-*}(\quad ; Z)$ and consider the following part of the Mayer-Vietoris sequence of the triad $(Y; Y_1, Y_2)$

$$\ldots \to H^{\infty-2}(Y_1 \cap Y_2) \to H^{\infty-1}(Y_1 \cup Y_2) \xrightarrow{\theta} H^{\infty-1}(Y_1) \oplus H^{\infty-1}(Y_2) \to H^{\infty-1}(Y_1 \cap Y_2) \to \ldots$$

It follows from the assumption that

$$H^{\infty-2}(Y_1 \cap Y_2) \approx H^{\infty-1}(Y_1 \cap Y_2) \approx 0$$

and therefore, by the exactness, θ is an isomorphism. Consequently, because

$$\operatorname{rank} H^{\infty-1}(X) = b_0(E - X),$$

the conclusion of the theorem follows.

Among other facts which follow easily from the proved theorems, we mention also :

3.3. Proposition. — *Let* X *be an object such that* $\operatorname{Codim}_Z(X) \geqq 2$. *Then* X *does not disconnect the space* E.

Proof. — It follows from the assumption that $H^{\infty-1}(X) = 0$ and hence $E - X$ is connected.

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On some generalizations of the Leray–Schauder theory

by

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Let E be an infinite dimensional normed space. A continuous mapping $f\colon X \to Y$ between two subsets X and Y of E is called a *compact vector field* provided it is of the form $f(x) = x - F(x)$, where $F\colon X \to E$ is a compact operator. This notion arose naturally in connection with the question of solvability of the non-linear equation $x = F(x)$ and was introduced in the early thirties by J. Schauder and J. Leray. Furthermore, the above authors made the important discovery that many familiar geometrical facts of finite dimensional topology can be carried over to infinitely many dimensions provided attention is restricted to the class of compact vector fields. In particular, a generalization of Brouwer's degree (or of the equivalent notion of a fixed point index) was established, known presently under the name of the Leray–Schauder theory, and with its aid various applications were obtained.

In this report we present a survey of the work of K. Gęba and the author on infinite dimensional cohomology theories(*). These theories generalize the Leray–Schauder degree theory and provide a convenient algebraic tool for the treatment of various infinite dimensional problems. Among these, we shall discuss briefly the questions of duality in E and also some aspects of the theory of the non-linear equation $x = F(x)$ in cases when the Leray–Schauder theory is not applicable. Some applications of infinite dimensional cohomology to bifurcation theory are presented in the report of K. Gęba.

1. The Leray–Schauder category.

Notation. We denote by E^{α} or simply by E an infinite dimensional

(*) The extension to the infinite dimensional case of topological invariants other than the Brouwer degree (as Betti numbers etc.) was initiated by J. Leray (cf. the report of J. Schauder 117] at the Topological Conference in Moscow in 1935); the corresponding results however never appeard in print.

normed space. We fix a sequence $E^{\infty-n}\oplus E_n$ of direct sum decompositions of E such that

(i) $E_0 \subset E_1 \subset \ldots \subset \ldots,$
(ii) $E^{\infty} \supset E^{\infty-1} \supset \ldots \supset \ldots,$
(iii) $\operatorname{codim} E^{\infty-n} = \dim E_n = n.$

By an annular ring T in E we shall understand a set of the form $T = \{x \in E\colon r_1 \leqslant \|x\| \leqslant r_2\}$ where $0 < r_1 \leqslant r_2$. By K we shall denote a closed ball in E with the center 0 and by S its boundary.

We let

$$T^{\infty-n} = T \cap E^{\infty-n},$$

$$K^{\infty-n} = K \cap E^{\infty-n},$$

$$S^{\infty-n} = S \cap E^{\infty-n}$$

and we reserve the symbol $U^{\infty-n}$ for the open $E-E_n$.

Compact mappings. By $L = \{L_\alpha, L_\beta, L_\gamma, \ldots\} = \{\alpha, \beta, \gamma, \ldots\}$ we shall denote the directed set of all finite dimensional linear subspaces of E with the natural order relation given by $\alpha \leqslant \beta \Leftrightarrow L_\alpha \subset L_\beta$.

A mapping $F\colon X \to Y$ between two topological spaces X and Y is *compact* provided $F(X)$ is contained in a compact subset of Y; a continuous $F\colon X \to E$ is said to be an *α-mapping* provided (i) F is compact and (ii) $F(X) \subset L_\alpha$. The following theorem due to J. Schauder is of importance.

APPROXIMATION THEOREM. *If $F\colon X \to E$ is compact, then for each $\varepsilon > 0$ there is an α-mapping $F_\varepsilon\colon X \to E$ such that*

$$\|F(x) - F_\varepsilon(x)\| < \varepsilon \quad \textit{for all } x \in X.$$

Compact vector fields. Given a map $f\colon X \to Y$ between two subsets X and Y of E we shall denote by the same but capital letter the map $F\colon X \to E$ defined by $F(x) = x - f(x)$ for $x \in X$.

A continuous map $f\colon X \to Y$ is said to be a *compact vector field* (resp. an *α-field*) provided F is compact (resp. an α-mapping).

The set of all compact fields (resp. α-fields) with the domain X and the range Y will be denoted by $\mathscr{K}(X, Y)$ (resp. by $\mathscr{K}_\alpha(X, Y)$). Since compact fields compose well we have the category $\mathscr{K}$ of *compact vector fields* with subsets of E as objects and the compact fields as morphisms. Similarly, for each $\alpha \in L$, we have the subcategory $\mathscr{K}_\alpha$ of $\mathscr{K}$. The union of all subcategories $\mathscr{K}_a$, $a \in L$, will be denoted by $\mathscr{K}_0$. The morphisms of $\mathscr{K}_0$ are called *finite dimensional fields.*

Homotopy of compact vector fields. Given X and Y in E and a homotopy $h_t\colon X \to Y$ $(0 \leqslant t \leqslant 1)$ we shall denote by H the mapping from $X \times I$ to E given by $H(x, t) = x - h_t(x)$ for $(x, t) \in X \times I$.

A continuous family of compact fields (resp. α-fields) $h_t\colon X \to Y$ is called a *compact homotopy* (resp. *α-homotopy*) provided $H\colon X\times I\to E$ is a compact mapping (resp. α-mapping).

Two compact fields (resp. α-fields) $f, g\colon X\to Y$ are said to be *compactly homotopic*, $f\sim g$ (resp. *α-homotopic*, $f\sim_\alpha g$) provided there is a compact homotopy (resp. α-homotopy) $h_t\colon X\to Y$ such that $h_0 = f$, $h_1 = g$.

The sets of the corresponding homotopy (resp. α-homotopy) classes will be denoted by $\pi(X, Y)$ and $\pi_\alpha(X, Y)$ respectively.

Example 1. Let $f\colon S\to E-\{0\}$ be a compact field and denote by $j\colon S\to E-\{0\}$ the inclusion. Let us put

$$(*)\qquad h_t(x) = x-tF(x)\quad \text{for}\quad (x,t)\in S\times I.$$

It follows from (∗) that *either* (i) $x = tF(x)$ *for some* $0 < t < 1$ *and* $x\in S$ *or* (ii) *the fields f and j are compactly homotopic.*

Example 2. Let U be open in $E^{\infty-n}$ and $f, g\colon S\to U$ be two compact fields such that

$$\|f(x)-g(x)\| \leqslant \operatorname{dist}\big(f(x), E-U\big)\quad \text{for all } x\in S.$$

Then for all $(x, t)\in S\times I$ we have

$$h_t(x) = tf(x)+(1-t)g(x)$$
$$= x-[tF(x)+(1-t)G(x)] \neq 0$$

and hence $h_t\colon S\to E^{\infty-n}-\{0\}$ is a compact homotopy joining f and g.

Example 3. Let $f, g\colon S\to E^{\infty-n}-\{0\}$ be two compact fields such that for all $x\in S$ $\|f(x)-g(x)\| \leqslant \|f(x)\|$. Then f and g are compactly homotopic.

The extension problem for compact fields. Given a pair (X, A), E with A closed in X, and a field $f\colon A\to U$, we may consider the extension problem for f, i.e., the problem of extending f over X in $\mathscr{L}$. The following theorem asserts that under some hypotheses this problem depends only on the homotopy class of a given field f.

Homotopy Extension Theorem. *Let U be an open subset of* $E^{\infty-n}$ *and* $h_t\colon S\to U$ $(0\leqslant t\leqslant 1)$ *a compact homotopy. If* h_0 *can be extended over K to a compact field* $h_0\colon K\to U$, *then there is a compact homotopy* $\bar{h}_t\colon K\to U$ *which is an extension of* h_1 *over K, i.e.,* $h_t = \bar{h}_t|S$.

Remark. The above theorem is valid for an arbitrary pair (X, A) with A closed in X.

The Leray–Schauder category. By $\mathscr{L}(E)$ or simply $\mathscr{L}$ we denote the subcategory of $\mathscr{K}$ generated by closed bounded subsets of E; $\mathscr{L}$ will be called the *Leray–Schauder category* corresponding to the linear space E.

A compact field $f\colon X\to Y$ in $\mathscr{L}$ is *invertible* (resp. *homotopically invertible*) provided there is a field $f'\colon Y\to X$ such that $f\circ f' = 1_Y$ and $f'\circ f = 1_X$ (resp. $f\circ f'\sim 1_Y$ and $f'\circ f\sim 1_X$). In the first case, we write $X\sim Y$ and call X and Y *equivalent*; in the second case X and Y are said to be *homotopically equivalent*, and we write $X \sim_h Y$.

Remark. One could show that there exist two equivalent objects X and Y in $\mathscr{L}$ such that $\pi_1(E-X) = 0$ and $\pi_1(E-Y) \neq 0$.

2. Infinite dimensional cohomology theories. We begin by recalling the following theorem proved by J. Leray [13] (with the aid of the degree): If X and Y are two equivalent objects of the Leray–Schauder category $\mathscr{L}$, then the complements $E-X$ and $E-Y$ have the same number of components. In connection with this theorem, the following problem arises: *If X and Y are two equivalent (or more generally homotopically equivalent) objects of $\mathscr{L}$, are the homology groups $H_n(E-X)$ and $H_n(E-Y)$ isomorphic for each n?*

The Leray–Shauder degree theory,is not adequate to treat this problem and therefore a tool of an essentially algebraic character is needed. Thus we are led to infinite dimensional cohomology theories.

Definition. An infinite dimensional or simply a cohomology theory $H^{\infty-n}$ on $\mathscr{L}$ is a sequence of contravariant functors $\{H^{\infty-n}(X, A)\}$ from the pairs in $\mathscr{L}$ to the category of abelian groups together with a sequence of natural transformations $\delta^{\infty-n}\colon H^{\infty-n}(A) \to H^{\infty-n+1}(X, A)$ satisfying the Homotopy, Exactness, and Strong Excision axioms; the graded group $\{H^{\infty-n-1}(S)\}$, where S in the unit sphere in E, is the *group of coefficients* of the theory.

The question arises whether such theories exists. The answer is "yes" and more precisely we have the following:

Theorem. *To any (generalized) cohomology theory on the category of finite polyhedra corresponds a cohomology theory on $\mathscr{L}$ with the same group of coefficients; moreover, the assignment $H^* \to H^{\infty-*}$ is natural with respect to maps of the theories.*

Thus, in particular, for any spectrum A in the sense of G. Whitehead corresponds $\{H^{\infty-*}(, A)\}$ called the cohomology theory on $\mathscr{L}$ with coefficients in A. If $A = K(\Pi)$ is the Eilenberg–MacLane spectrum corresponding to an abelian group Π then we get the "*ordinary*" *cohomology* theory on $\mathscr{L}$ with coefficients in Π. If $A = S$ is the sphere spectrum, then the corresponding theory denoted by $\Sigma^{\infty-*}$ is called the *stable cohomotopy* on $\mathscr{L}$. The Hopf–Hurewicz map $h\colon S \to K(Z)$ induces a natural transformation h^* from $\Sigma^{\infty-*}$ to $\Pi^{\infty-*}(; Z)$.

The construction of $H^{\infty-n}$. We shall now try to give some general idea about how the infinite dimensional cohomologies are defined. Assume for simplicity that we start with the ordinary Čech cohomology $H^* = \{H^q, \delta^q\}$ for compacta with coefficients in G and want to define $H^{\infty-n}(X)$ for an object X in $\mathscr{L}$. We take the directed (by inclusion) set $L_x = \{L_\alpha, L_\beta, L_\gamma, \ldots\}$ of all finite dimensional subspaces of E such that $X_\alpha = X \cap L_\alpha$ for each α is not empty and we fix an orientation of every L_α. Let $d(\alpha) = \dim L_\alpha$ and write $\alpha \leqslant \beta$ instead of $L_\alpha \subset L_\beta$. We observe that, given $\alpha \leqslant \beta$ with $d(\beta) = d(\alpha)+1$, the orientations of L_α and L_β determine the triad $(L_\beta, L_\beta^+, L_\beta^-)$ with $L_\alpha = L_\beta^+ \cap L_\beta^-$ and, therefore, the triad $(X_\beta, X_\beta^+, X_\beta^-)$ with $X_\alpha = X_\beta^+ \cap X_\beta^-$ (where $X_\beta^+ = X \cap L_\beta^+$ and $X_\beta^- = X \cap L_\beta^-$). Now, given $\alpha \leqslant \beta$, we define $\Delta_{\alpha\beta}$: $H^{d(\alpha)-n}(X_\alpha) \to H^{d(\beta)-n}(X_\beta)$ as follows: if $d(\beta) = d(\alpha)+1$, we let $\Delta_{\alpha\beta}$ be the Mayer–Vietoris homomorphism of the triad $(X_\beta, X_\beta^+, X_\beta^-)$; otherwise, we take a chain of consecutive elements of L_x and define $\Delta_{\alpha\beta}$ to be the composition of the corresponding Mayer–Vietoris homomorphisms.

It turns out that if $U = E-X$, $\alpha \leqslant \beta$ and $i_{\alpha\beta}$: $U_\alpha \to U_\beta$ is the inclusion, then we have the commutative diagram

$$\begin{array}{ccc} H^{d(\alpha)-n}(X_\alpha) & \xrightarrow{\Delta_{\alpha\beta}} & H^{d(\alpha)-n}(X_\beta) \\ \downarrow \mathscr{D}_\alpha & & \downarrow \mathscr{D}_\beta \\ H_{n-1}(U_\alpha) & \xrightarrow{(i_{\alpha\beta})_*} & H_{n-1}(U_\beta) \end{array}$$

in which $\mathscr{D}_\alpha$ and $\mathscr{D}_\beta$ stand for the appropriate Alexander–Pontriagin isomorphisms in L_α and L_β respectively.

It follows that the groups $\{H^{d(\alpha)-n}(X_\alpha); \Delta_{\alpha\beta}\}$ form a direct system of abelian groups and we define

$$H^{\infty-n}(X) = \varinjlim_{\alpha} \{H^{\infty-n}(X_\alpha); \Delta_{\alpha\beta}\}.$$

It remains to define the induced homomorphism $H^{\infty-n}(f) = f^*$ and to prove its functorial properties; this is done in two steps: first, for a finite dimensional, and then for an arbitrary compact field f. We remark that in the second and more involved step, the crucial role is played by the continuity property of the functors under consideration.

Duality theorems. We note that the passage to the limit in the commutative diagram of the previous section indicates how we are led to the next main result: *The group $H^{\infty-n}(X)$ is isomorphic to the $(n-1)$-th singular homology group $H_{n-1}(U; G)$.* A more general theorem holds in fact and may be viewed as an extension to the infinite dimensional case of the duality theory due to G. Whitehead [18].

Denote by (X, Y, Z) a triple in $\mathscr{L}$ and let $U = E-X$, $V = E-Y$, $W = E-Z$.

THEOREM (*The Alexander duality in E*). *Let A be either the spectrum of spheres S or the Eilenberg–MacLane spectrum $K(\Pi)$. Then*

(i) *for each pair (X, Y) there exists the duality map*

$$D: H^{\infty-n}(X, Y; A) \to H_n(V, U; A).$$

(ii) *D maps the cohomology sequence of a triple (X, Y, Z) into the homology* *sequence of the complementary triple (W, V, U), i.e., the following diagram commutes:*

$$\begin{array}{ccc} \dots \to H^{\infty-n}(X, Y; A) & \to & H^{\infty-n}(X, Z; A) \to \\ \downarrow D & & \downarrow D \\ \dots \to H_n(V, U; A) & \to & H_n(W, U; A) \to \end{array}$$

$$\begin{array}{ccc} \to H^{\infty-n}(Y, Z; A) & \to & H^{\infty-n+1}(X, Y; A) \to \dots \\ \downarrow D & & \downarrow D \\ \to H_n(W, V; A) & \to & H_{n-1}(V, U; A) \to \dots \end{array}$$

(iii) *D is natural with respect to the Hopf–Hurewicz map of spectra $h: S \to K(Z)$, i.e., the following diagram commutes:*

$$\begin{array}{ccc} \Sigma^{\infty-n}(X, Y) & \xrightarrow{h_*} & H^{\infty-n}(X, Y; Z) \\ D \downarrow & & \downarrow D \\ \Sigma_n(V, U) & \to & H_n(V, U; Z) \end{array}$$

Corollaries (*The Alexander–Pontrjagin Invariance in E*). *The relation $X \sim_h Y$ in $\mathscr{L}$ implies that*

$$H_{n-1}(E-X; \Pi) \approx H_{n-1}(E-Y; \Pi),$$

for any $n \geqslant 1$ and any group of coefficients Π.

(*The Spanier–Whitehead Invariance in E*). *The relation $X \sim_h Y$ in $\mathscr{L}$ implies that for any $b \geqslant 1$,*

$$\Sigma_{n-1}(E-X) \approx \Sigma_{n-1}(E-Y).$$

The duality combined with the Hurewicz Theorem in S-theory yields the following:

(*The Hopf Theorem*). *For any pair in $\mathscr{L}$ the first non-vanishing stable cohomotopy group is isomorphic to the first non-vanishing cohomology group over Z. More precisely, we have*

(i) $\Sigma^{\infty-q}(X, Y) = 0 \Leftrightarrow H^{\infty-q}(X, Y; Z) = 0$ *for any* $0 \leqslant q < n$.

(ii) *if* $\Sigma^{\infty-n}(X, Y) = 0$ *for* $0 \leqslant q < n$; *then the Hopf map* h^*: $\Sigma^{\infty-n}(X, Y) \to H^{\infty-n}(X, Y; Z)$ *is an isomorphism.*

Representability of the stable cohomotopy and codimension. Consider compact fields from an object X in $\mathscr{L}$ to the open set $E - E_{n-1}$ and denote by $\pi^{\infty-n}(X)$ the corresponding set of homotopy classes.

Representation Theorem. *There exists a natural isomorphism between $\pi^{\infty-n}(X)$ and the stable cohomotopy group $\Sigma^{\infty-n}(X)$.*

For the objects in $\mathscr{L}$ one defines the "basic" codimension Codim in terms of the extension problem for compact fields with special ranges $E - E_n$. This definition coincides in the finite dimensional case with a theorem of P.

Alexandroff which characterizes the dimension of compacta by maps into S^n. Further, one defines various cohomological codimensions; we have, in particular, Codim_Z defined in terms of the ordinary cohomology on $\mathscr{L}$ over Z.

THEOREM. $\mathrm{Codim} = \mathrm{Codim}_Z$.

The proof of this result uses, among others, the representability of the stable cohomotopy and the Homotopy Extension Theorem.

3. Applications to non-linear problems. Using the cohomology functors we may treat various extension problems in $\mathscr{K}$. Then, with the aid of the notion of the essential vector field (stated here in a more general form than in [18]) we translate the corresponding results into the new existence criteria for the equation $x = F(x)$.

Essential and inessential compact fields. Let U be open in $E^{\infty-n}$ and $f\colon S \to U$ be a compact field; f is called *inessential* provided the diagram

$$\begin{array}{ccc} & & U \\ & \overset{f}{\nearrow} & \uparrow \bar{f} \\ S & \xrightarrow[\subset]{i} & K \end{array}$$

can be completed in the category $\mathscr{K}$, i.e., there is a compact field $\bar{f}\colon K \to U$ with $f = \bar{f}|S$; otherwise, f is said to be *essential*.

From the Homotopy Extension Theorem it follows that the property of a field f of being essential depends only on the homotopy class of f. More precisely we have:

HOMOTOPY INVARIANCE, *Let $f, g\colon S \to U$ be two fields which are compactly homotopic. If f is essential then so is g.*

Remarks. Let $f\colon S \to E^{\infty-n}-\{0\}$ be an arbitrary compact field and

$$j\colon E^{\infty-n}-\{0\} \to U^{\infty-n},$$

$$i\colon E^{\infty-n}-\{0\} \to E^{\infty}-\{0\}$$

stand for the inclusions. Then

(i) f is essential if and only if so is the composite $j \circ f$;

(ii) the composite $i \circ f$ is inessential;

(iii) the restriction

$$f|S^{\infty-n}\colon S^{\infty-n} \to E^{-n}-\{0\}$$

is inessential.

The only known general criterion for a field to be essential is given by

BORSUK'S ANTIPODAL THEOREM. *If $f\colon S \to E-\{0\}$ is odd (i.e., $f(x) = -f(-x)$ for every $x \in S$), then f is essential; in particular the inclusion $j\colon S \to E-\{0\}$ is essential.*

An algebraic criterion. Let $f: S \to E^{\infty-n} - 0$ be a compact field. We may treat f as field into $T^{\infty-n}$ for some annular ring $T^{\infty-n}$, i.e., we may write

$$f: S \to T^{\infty-n}.$$

Consider the induced homomorphism

$$f^*: \Sigma^{\infty-n-1}(T^{\infty-n}) \to \Sigma^{\infty-n-1}(S).$$

THEOREM. *If f^* the induced homomorphism is non-trivial, then the field f is essential.*

Assume that $n = 0$. Then

$$Z = \Sigma^{\infty-1}(T) \xrightarrow{f^*} \Sigma^{\infty-1}(S) = Z$$

and $f^*(1) = \gamma(f) \cdot 1$; clearly $f^* \neq 0$ if and only if the integer $\gamma(f)$ (called the Leray–Schauder characteristic of f) is different from zero). Consequently, we obtain as a special case the following:

THE LERAY–SCHAUDER THEOREM. *A field $f: S \to E - \{0\}$ is essential if and only if $\gamma(f)$ is different from zero.*

Moreover, using the Representation Theorem, we get:

THE ROTHE–HOPF CLASSIFICATION THEOREM. *Two fields $f, g: S \to E - \{0\}$ are compactly homotopic if and only if $\gamma(f) = \gamma(g)$.*

Remarks. 1° In case $n = 0$ one could use the cohomology functor $H^{\infty-1}(, Z)$ instead of $\Sigma^{\infty-1}$. For $n > 0$ the ordinary cohomology cannot provide any information because always $f^* = 0$.

2° The groups $\Sigma^{\infty-m}(T^{\infty-n}) \approx \Sigma^{\infty-m}(S^{\infty-n})$ are isomorphic to the stable homotopy groups of spheres (cf. [5]). In the case considered above

$$\Sigma^{\infty-n-1}(T^{\infty-n}) \approx \Sigma_m(S^m) \approx Z,$$

$$\Sigma^{\infty-n-1}(S) \approx \Sigma_{m+n}(S^m).$$

3° In view of remark 2°, the above algebraic criterion can only be of interest in cases when the group $\Sigma_{m+n}(S^m)$ is non-trivial. (This is known to be the case for $n = 1, 2, 3, 6, 7, 8$ but not for $n = 4, 5$.)

The equation $x = F(x)$ (The Leray–Schauder case). Suppose that $F: K \to E$ is a compact map and we are interested in solving the equation

(1) $$F(X) = x$$

or, equivalently, the equation

(2) $$f(x) = x - F(x) = 0.$$

To treat (2) we may make an assumpton that

(3) $$f(x) \neq 0 \quad \text{for} \quad x \in S.$$

The assumption (3) permits to consider f on S as a compact field

$$\bar{f}\colon S \to E - \{0\}$$

with the range equal to $E - \{0\}$.

If $\bar{f}$ is essential, then the quation (2) *has a solution.*

The equation $x = F(x)$ (The Hopf case). If the Leray–Schauder characteristic $\gamma(\bar{f})$ of $\bar{f}$ is zero, then, in general, there is no solution of the equation (1). Nevertheless, one could still assert the existence of solutions for (1) provided *some additional information about f is available.*

To explain this consider first the finite dimensional case. Let us make an additional assumption that

$$f(K^{n+1}, S^n) \subset (R^{m+1}, R^{m+1} - \{0\}), \quad \text{where} \quad R^{m+1} \subset R^{n+1};$$

letting $\bar{f}\colon S^n \to R^{m+1} - \{0\}$ be defined by f and putting

$$\varphi(x) = \frac{\bar{f}(x)}{\|\bar{f}(x)\|} \quad \text{for} \quad x \in S^n$$

we get a map $\varphi\colon S^n \to S^m$.

Clearly, the following conditions are equivalent:

(a) $\bar{f}\colon S^n \to R^{m+1} - \{0\}$ is essential;

(b) $\varphi\colon S^n \to S^m$ is not homotopic to a constant map;

(c) the homotopy class $[\varphi]$ is a non-trivial element of the homotopy group $\pi_n(S^m)$.

If any of the above conditions is satisfied, then the equation (2) has at least one solution. Thus, in brief, in finite dimensional case the types of the equation (1) can be classified according to the homotopy classification of spheres.

Passing to the infinite dimensional case, assume that for some $n > 0$,

$$f(K) \in E^{\infty - n}. \tag{4}$$

The assumptions (3) and (4) permit to consider f on S as a compact field $\bar{f}\colon S \to E^{\infty - n} - \{0\}$.

THEOREM. *If the condition* (4) *is satisfied and $\bar{f}\colon S - E^{\infty - n} - \{0\}$ is essential then the equation* (2) *and hence* (1) *has at least one solution.*

Remark. From the remarks above we know that $f\colon S \to E^{\infty - n} - \{0\}$ is essential if and only if so is the composite $jf\colon S \to U^{\infty - n}$, where $j\colon E^{\infty - n} - \{0\} \to U^{\infty - n}$ stands for the inclusion. But jf is essential if and only if the homotopy class $[if]$ is a non-trivial element of the group

$$\pi^{\infty - n - 1}(S) \approx \Sigma^{\infty - n - 1}(S) \approx \Sigma_{m+n}(S^m).$$

It follows that in infinite dimensional case the various types of the equation (1) can be classified according to the *stable homotopy* classification of spheres.

Bibliographical comments

This work was in progress over a number of years and we give here some additional comments concerning its background. In the 1930's K. Borsuk discovered that a number of important facts concerning the topology of R^n can be developed with the aid of notions of homotopy theory. To this circle of results belonged an observation that the Brouwer fixed point theorem can be equivalently expressed in terms of retraction, the homotopy extension theorem with various applications and the set-theoretical approach to the disconnection theory in R^n based on describing the structure of the cohomotopy group $\pi^n(X)$ of a compactum $X \subset R^{n+1}$ (cf. Borsuk [1]–[3]). In 1959, the author had shown that the above ideas of Borsuk can be conveniently extended to the framework of the theory of compact fields in Banach spaces (cf. Granas [9]–[11]). Some of the techniques developed in those papers (based on the homotopy extension theorem for compact fields) provided the starting point for the work on infinite dimensional cohomology theories. In 1962 K. Gęba in his thesis (cf. Gęba [4], [5]) constructed (in the spirit of the Eilenberg–Steenrod) the first full-fledged infinite-dimensional cohomology theory; he extended to Banach spaces the theory of cohomotopy groups of Borsuk and also the Spanier–Whitehead duality. Extensions to infinite dimensional case of the ordinary cohomology theory were further studied in Gęba and Granas [6] and [7]. A systematic and unified treatment of general infinite cohomology theories (taking into account the work of G. Whitehead [18] appeared in Gęba and Granas [8]. Some applications of infinite-dimensional cohomology to non-linear problems will be found in Ize [12] and Nirenberg [15], [16].

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B. Theorie des Points Fixes

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MATHÉMATIQUE

Sur la notion du degré topologique pour une certaine classe de transformations multivalentes dans les espaces de Banach

par

A. GRANAS

Presenté par K. BORSUK le 7 Fevrier 1959

Notations. Désignons respectivement par E_a un espace quelconque fixé de Banach pour $a = \infty$ et son sous-espace n-dimensionnel pour $a = n < +\infty$. Soient $x_0 \in E_a$, $A \subset E_a$ et ϱ un nombre positif; posons

$$V_a(x_0, \varrho) = \underset{x \in E_a}{E} (\|x - x_0\| < \varrho), \quad V_a(A, \varrho) = \underset{x \in E_a}{E} \underset{a \in A}{\exists} (\|x - a\| < \varrho),$$

$$P_a = E_a \setminus \{0\}, \quad S_{a-1}(x_0, \varrho) = \begin{cases} \mathrm{Fr}(V_\infty(x_0, \varrho)) & \text{pour} \quad a = \infty \\ \mathrm{Fr}(V_n(x_0, \varrho)) & \text{pour} \quad a = n, \end{cases}$$

$$V_a^0 = V_a(0, 1), \quad S_{a-1}^0 = S_{a-1}(0, 1).$$

Si E_n est un sous-espace de E_{n+1} contenant le centre de la sphère $S_n \subset E_{n+1}$, E_{n+1}^+, E_{n+1}^- désignent les demi-espaces fermés de E_{n+1} tels que $E_n = E_{n+1}^+ \cap E_{n+1}^-$. Posons $S_n^+ = S_n \cap E_{n+1}^+$, $S_n^- = S_n \cap E_{n+1}^-$.

Par minuscules latines $f, g, \ldots, r, s, t$ nous désignons les transformations continues univalentes et par majuscules F, G — les transformations univalentes complètement continues.

Transformations multivalentes. Une transformation définie sur $A \subset E_a$ qui à chaque point $x \in A$ attache un ensemble non vide $\varphi(x) \subset E_a$ est dite *supérieurement semi-continue* si les conditions $x_n \to x$, $(x_n, x \in A)$, $y_n \to y$, $y_n \in \varphi(x_n)$ impliquent $y \in \varphi(x)$. Nous ne considérons dans cette note que les fonctions supérieurement semi-continues dont les valeurs sont des ensembles *convexes* dans E_a. Nous écrivons $\varphi: A \to E_a$ pour dire que φ est une transformation supérieurement semi-continue de A dans E_a; $\varphi: (A, B) \to (E_a, C)$ (où $B \subset A$, $C \subset E_a$) signifie de plus que $\varphi(B) \subset C$ où $\varphi(B) = \underset{y \in E_a}{E} \underset{x \in B}{\exists} \{y \in \varphi(x)\}$.

Nous dirons qu'une fonction multivalente $\Phi\colon A\to E_\alpha$ est une *transformation multivalente complètement continue* *) (t. m. c. c.) lorsque l'image $\Phi(A')$ de chaque ensemble $A'\subset A$ borné dans E_α est relativement compact dans E_α.

LEMME 1. *Soit $A\subset E_\infty$ un ensemble borné et $\Phi\colon A\to E_\infty$ une t. m. c. c. de A. Pour chaque $\varepsilon>0$ il existe un sous-espace $E_n\subset E_\infty$ et une transformation $\Phi_\varepsilon\colon A\to E_n$ tels que les inclusions*:

$$\Phi_\varepsilon(x)\subset V_\infty\big(\Phi(x),\varepsilon\big),\quad \Phi(x)\subset V_\infty\big(\Phi_\varepsilon(x),\varepsilon\big). \tag{1}$$

ont lieu pour tous $x\in A$.

Champs vectoriels multivalents complètement continus dans E_α. Nous dirons qu'une transformation $\varphi\colon A\to C\subset E_\alpha$ est un *champ vectoriel multivalent complètement continu* (c. v. m. c. c.) sur A dans E_α, (ou *déplacement multivalent complètement continu* (d. m. c. c.) de l'ensemble A dans E_α) si elle peut être representée sous la forme **)

$$\varphi(x)=x-\Phi(x), \tag{2}$$

où $\Phi\colon A\to E_\alpha$ est une t. m. c. c.

Deux c. v. m. c. c. $\varphi_1,\varphi_2\colon A\to C$ (A — borné, $C\subset E_\alpha$, $\varphi_1(x)=x-\Phi_1(x)$, $\varphi_2(x)=x-\Phi_2(x)$) sont dits *homotopes*, $\varphi_1\simeq\varphi_2$, lorsqu'il existe une famille des c. v. m. c. c. $\psi\colon A\times I\to C$ $\psi(x,t)=x-\Psi(x,t)$ (I — intervalle $[0,1]$) telle, que pour chaque $x\in A$ on a $\Phi_1(x)=\Psi(x,0)$, $\Phi_2(x)=\Psi(x,1)$ et l'ensemble $\Psi(A,I)$ est relativement compact dans E_α.

LEMME 2. *Soit $\varphi\colon A\to E_\alpha$ un d. m. c. c.; si A est un ensemble fermé et borné, $\varphi(A)$ est fermé.*

Notion de la caractèristique. Soit $S_{\alpha-1}=S_{\alpha-1}(x_0,\varrho)$, $V_\alpha=V_\alpha(x_0,\varrho)$ et $f,\varphi\colon S_{\alpha-1}\to P_\alpha$ champs complètement continus (respectivement unis et multivalents) sur $S_{\alpha-1}$. Par $\gamma(f,S_{\alpha-1})$ nous désignons la caractèristique [2] du champ f sur $S_{\alpha-1}$. Un champ univalent $f\colon S_{\alpha-1}\to P_\alpha$ est dit *sélécteur* du champ multivalent $\varphi\colon S_{\alpha-1}\to P_\alpha$, (en symboles: $f\in\varphi$), si $f(x)\in\varphi(x)$ pour chaque $x\in S_{\alpha-1}$.

THÉORÈME 1a. *A chaque c. v. m. c. c. $\varphi\colon S_{\alpha-1}\to P_\alpha$ on peut associer un nombre entier $\gamma(\varphi,S_{\alpha-1})$ d'une telle façon, que les conditions suivantes soient satisfaites*

I_α *si $f\in\varphi$, alors $\gamma(\varphi,S_{\alpha-1})=\gamma(f,S_{\alpha-1})$;*

II_α *si les c. v. m. c. c. $\varphi_1,\varphi_2\colon S_{\alpha-1}\to P_\alpha$ sont homotopes $\varphi_1\simeq\varphi_2$, alors $\gamma(\varphi_1,S_{\alpha-1})=\gamma(\varphi_2,S_{\alpha-1})$. On appelle le nombre $\gamma(\varphi,S_{\alpha-1})$ la caractèristique du c. v. m. c. c. φ sur $S_{\alpha-1}$.*

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*) En désignant par minuscules grecques φ,ψ,η les transformations multivalentes, nous réservons les majuscules Φ, Ψ pour les transformations multivalentes complètement continues.

**) $\varphi(x)$, $\psi(x)$ étant deux fonctions multivalentes, on désigne par $\eta(x)=t_1\varphi(x)+t_2\psi(x)$ (t_1,t_2 — nombres réels) la fonction multivalente qui à chaque point x fait correspondre l'ensemble des points de la forme $t_1z_1+t_2z_2$ où $z_1\in\varphi(x)$, $z_2\in\psi(x)$; $\varphi(x)$, $\psi(x)$ étant convexes, $\eta(x)$ l'est aussi.

Démonstration. Les cas $a = n$. Dans ce cas nous allons utiliser la théorie d'homologie de Vietoris pour les espaces métriques compacts avec les coefficients entiers. Si X, Y sont les espaces compacts et $f: X \to Y$, nous désignons par $f_*^{(i)}: H_i(X) \to H_i(Y)$ l'homomorphisme des groupes d'homologie, induit par f.

Soit $\varphi: S_{n-1} \to P_n$ une c. v. m. c. c. Désignons par $\Gamma = \Gamma_\varphi$ le graphique de la fonction φ et par $r: \Gamma \to S_{n-1}$, $s: \Gamma \to P_n$ ses projections sur S_{n-1} et $\varphi(S_{n-1})$ respectivement ($\Gamma_\varphi = \underset{z \in S_{n-1} \times P_n}{E} \{z = (x, y), y \in \varphi(x)\}$, $r(x, y) = x$, $s(x, y) = y$; $(x, y) \in \Gamma_\varphi$). L'ensemble Γ est compact dans $S_{n-1} \times P_n$ et la projection r verifie les conditions du Théorème de Vietoris [3] (toutes les images réciproques étant convexes) donc l'homomorphisme induit des groupes d'homologie $r_*^{(i)}: H_i(\Gamma) \to H_i(S_{n-1})$ est un isomorphisme sûr. Nous avons en particulier $r_* = r_*^{(n-1)}: H_{n-1}(\Gamma) \approx H_{n-1}(S_{n-1})$.

Posons $t(z) = \dfrac{s(z)}{\|s(z)\|}$ pour $z \in \Gamma$; nous avons $t: \Gamma \to S_{n-1}^0$

$$t_* = t_*^{(n-1)}: H_{n-1}(\Gamma) \to H_{n-1}(S_{n-1}^0), \qquad (r_*)^{-1}: H_{n-1}(S_{n-1}) \approx H_{n-1}(\Gamma)$$
$$\varphi_* = t_*(r_*)^{-1}: H_{n-1}(S_{n-1}) \to H_{n-1}(S_{n-1}^0).$$

Soient u et u^0 respectivement les générateurs des groupes $H_{n-1}(S_{n-1})$ et $H_{n-1}(S_{n-1}^0)$. Nous avons $\varphi_*(u) = \gamma \cdot u^0$ ou γ est un nombre entier.

Nous definissons la caractéristique du c. v. m. c. c. $\varphi: S_{n-1} \to P_n$ en posant $\gamma(\varphi, S_{n-1}) = \gamma$. La démonstration des propriétés I_n et II_n ne diffère pas essentiellement de celle de Jaworowski [4].

Les cas $a = \infty$. L'ensemble $\varphi(S_\infty)$ est fermé (voir lemme 2); désignons par ε un nombre plus petit que la demidistance d'ensemble $\varphi(S_\infty)$ du point 0. Selon le lemme 1 nous pouvons associer à ε un tel sous-espace $E_n \subset E_\infty$ et une telle transformation $\Phi_\varepsilon: S_\infty \to E_n$ que les inclusions (1) aient lieu pour chaque $x \in S_\infty$; nous pouvons supposer que $x_0 \in E_n$, donc $E_n \cap S_\infty = S_{n-1}$. Il en resulte qu'en posant $\varphi_\varepsilon(x) = x - \Phi_\varepsilon(x)$ pour $x \in S_{n-1}$ nous avons $\varphi_\varepsilon: S_{n-1} \to P_n$.

Nous définissons la caractèristique du champ $\varphi: S_\infty \to P_\infty$ en posant $\gamma(\varphi, S_\infty) = \gamma(\varphi_\varepsilon, S_{n-1})$. Le nombre $\gamma(\varphi, S_\infty)$ pour un E_n fixé ne dépend pas de la manière d'approximation de la fonction Φ par Φ_ε. En effet, soit Φ'_ε une autre fonction pour laquelle les inclusions (1) ont lieu et $\varphi'_\varepsilon(x) = x - \Phi'_\varepsilon(x)$. L'ensemble $V_\infty(\Phi(x), \varepsilon)$ est convexe, donc $x \,\bar{\in}\, \Psi(x, t) = t\Phi_\varepsilon(x) + (1-t)\Phi'_\varepsilon(x)$; en posant $\psi(x, t) = x - \Psi(x, t)$, $x \in S_{n-1}$ on obtient $\psi: S_{n-1} \times I \to P_n$, $\psi(x, 0) = \varphi_\varepsilon(x)$, $\psi(x, 1) = \varphi'_\varepsilon(x)$, $\varphi_\varepsilon \simeq \varphi'_\varepsilon$, $\gamma(\varphi_\varepsilon, S_{n-1}) = \gamma(\varphi'_\varepsilon, S_{n-1})$.

Le nombre $\gamma(\varphi, S_\infty)$ ne dépend pas du choix de E_n. Ceci résulte du lemme suivant:

Lemme 3. *Soit E_n un sous-espace de E_{n+1} contenant le centre d'une sphère $S_n \subset E_{n+1}$. Supposons que le c. v. m. c. c. $\varphi: S_n \to P_{n+1}$ satisfait à la*

condition suivante $\varphi(S_n^+) \subset E_{n+1}^+$, $\varphi(S_n^-) \subset E_{n+1}^-$ ***(donc*** $\varphi(S_{n-1}) \subset P_n$***),*** $S_{n-1} = S_n^+ \cap S_n^-$ ***et*** $\varphi_0 = \varphi|S_{n-1}$***. Alors*** $\gamma(\varphi, S_n) = \gamma(\varphi_0, S_{n-1})$.

Les propriétés I_∞ et II_∞ résultent de I_n et II_n respectivement.

Théorème 2_a. ***Si le c. v. m. c. c.*** $\varphi: (\overline{V}_a, S_{a-1}) \to (E_a, P_a)$ $\varphi(x) = x - \Phi(x)$ ***vérifie la condition*** $\gamma(\varphi_0, S_{a-1}) \neq 0$***, où*** $\varphi_0 = \varphi|S_{a-1}$***, il existe un point*** $x \in V_a$ ***tel, que*** $0 \in \varphi(x)$***, c'est à dire*** $x \in \Phi(x)$.

Notion du degré topologique. Soit $\psi: \overline{V}_a \to E_a$ un d. m. c. c. et $y_0 \overline{\in} \psi(S_{a-1})$. Nous définissons alors le ***degré topologique*** $d(\psi, V_a, y_0)$ en posant $d(\psi, V_a, y_0) = \gamma(\varphi, S_{a-1})$ où $\gamma(\varphi, S_{a-1})$ est la caractéristique du c. v. m. c. c. $\varphi: S_{a-1} \to P_a$ défini par $\varphi(x) = \psi(x) - y_0$ pour $x \in S_{a-1}$.

Théorème $2'_a$. ***Si*** $\psi: \overline{V}_a \to E_a$ ***est un d. m. c. c.,*** $y_0 \overline{\in} \psi(S_{a-1})$ ***et*** $d(\psi, V_a, y_0) \neq 0$***, il en résulte que*** $y_0 \in \mathrm{Int}(\psi(V_a))$.

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MATHEMATICS

Theorem on Antipodes and Theorems on Fixed Points for a Certain Class of Multi-valued Mappings in Banach Spaces

by

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1. Introduction. Denote by E_a an arbitrary Banach space, if the index $a = \infty$, and an n-dimensional subspace of space E_∞, if the index a is equal to a natural number n. Denote by P_a the space E_a without the point 0. If x_0 is a point of space E_a and ϱ a positive number, then we denote by $V_a(x_0, \varrho)$ an open full-sphere with centre x_0 and radius ϱ and by $S_{a-1}(x_0, \varrho)$ — the boundary of $V_a(x_0, \varrho)$.

The mapping φ defined on the set A and assigning to a point $x \in A$ a non-empty set $\varphi(x) \subset E_a$ is called *upper semicontinuous*, if the conditions $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $y_n \in \varphi(x_n)$ imply $y \in \varphi(x)$. In what follows we consider only upper semicontinuous mappings and assume their values to be *convex sets* in E_a. The notation $\varphi\colon A \to E_a$ denotes that φ is an upper semicontinuous mapping defined on A, whose every value $\varphi(x)$ is a convex set in E_a.

A multi-valued mapping $\Phi\colon A \to E_a$ is called *completely continuous* on A *), if its image $\Phi(A) = \underset{y\in E}{E}\ \underset{x\in A}{\exists} \{y \in \varphi(x)\}$ is relatively compact in E_a **).

A mapping $\varphi\colon A \to E_a$ is called *a completely continuous multi-valued vector field* on the set A, if it can be represented in the form:

(1) $\varphi(x) = x - \Phi(x)$, $x \in A$, where Φ is completely continuous on A ***).

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*) In the sequel we denote multi-valued completely continuous mappings by capital letters Φ, Ψ ...

**) The set $X \subset E_a$ is termed relatively compact in E_a if its closure is compact.

***) For any multi-valued functions $\varphi_1, \varphi_2 A \to E_a$ and real numbers t_1, t_2 we understand by $t_1\varphi_1(x) + t_2\varphi_2(x) = \varphi(x)$ a function which assigns to the point $x \in A$ the set $\varphi(x)$ of points of the form $s = t_1 y_1 + t_2 y_2$, where $y_1 \in \varphi_1(x)$, $y_2 \in \varphi_2(x)$.

We say that a completely continuous field $\varphi(x) = x - \Phi(x)$, $x \in A$ ***does not vanish*** and we write $\varphi\colon A \to P_a$, if the point 0 does not belong to the set $\varphi(x)$ for any $x \in A$, i. e. $x \bar{\in} \Phi(x)$.

We say that two non-vanishing completely continuous multi-valued fields $\varphi_1, \varphi_2\colon A \to P_a$, $\varphi_1(x) = x - \Phi_1(x)$, $\varphi_2(x) \triangleq x - \Phi_2(x)$ are *homotopic* and we write $\varphi_1 \simeq \varphi_2$, if there exists a function $\Psi(x, t)$ defined on the product $A \times J$ (J denoting the closed interval [01]) with values lying in E_a, which satisfies the conditions:

1° point 0 does not belong to any set $\psi(x, t) = x - \Psi(x, t)$, $x \in A$, $t \in J$,

2° $\Psi(x, 0) = \Phi_1(x)$, $\Psi(x, 1) = \Phi_2(x)$ for any $x \in A$,

3° the set $\Psi(A, J)$ is relatively compact in E_a.

Let $S_{a-1} = S_{a-1}(x_0, \varrho)$ be a sphere in the space E_a. We shall refer to the following [4]:

THEOREM. *To every completely continuous field $\varphi\colon S_{a-1} \to E_a$ we can assign a positive number $\gamma(\varphi, S_{a-1})$ — the characteristic of the field φ on the sphere S_{a-1} in such a way that the following conditions are satisfied:*

a) if the completely continuous fields $\varphi_1, \varphi_2\colon S_{a-1} \to P_a$ are homotopic, $\varphi_1 \simeq \varphi_2$, then their characteristics are equal, i. e. $\gamma(\varphi_1, S_{a-1}) = \gamma(\varphi_2, S_{a-1})$;

β) if a non-vanishing completely continuous field $\varphi_0\colon S_{a-1} \to P_a$ can be extended to a completely continuous field $\varphi\colon \overline{V}_a \to E_a$ defined on a full sphere $\overline{V}_a = \overline{V_a(x_0, \varrho)}$ and the characteristic $\gamma(\varphi_0, S_{a-1})$ of the field φ_0 is different from zero, then there exists a point $x_0 \in V_a$ whose image $\varphi(x_0)$ contains point 0, i. e. $0 \in \varphi(x_0)$.

The theorem quoted constitutes the general criterion regarding the existence of fixed points for completely continuous multi-valued mappings. In this note we give a few particular criteria, which are an extension to the case of multi-valued mappings of the basic theorems of the Leray-Schauder theory concerning the existence of fixed points for completely continuous single-valued mappings [7], [8].

2. Theorems on fixed points. The following theorem is an extension to the case of multi-valued functions of the well known theorem of Rothe (cf. [8], p. 186).

THEOREM 1. *Let a multi-valued completely continuous mapping Φ: $\overline{V}_\infty \to E_\infty$ carry a full sphere $\overline{V}_\infty = \overline{V_\infty(x_0, \varphi)}$ into a space E_∞.*

If for an arbitrary point $x \in S_\infty = \mathrm{Fr}(\overline{V}_\infty)$ the condition $\Phi(x) \subset \overline{V}_\infty$ is satisfied (i. e. $\Phi(S_\infty) \subset \overline{V}_\infty$), then there exists at least one fixed point of the mapping Φ i. e. $x_0 \in \Phi(x_0)$ for a certain $x_0 \in \overline{V}_\infty$.

Proof. Without reducing the generality of our considerations we can assume that the full sphere in question is the unit full sphere in the space E_∞, i. e. $\overline{V}_\infty = \overline{V_\infty(0, 1)} = \overline{V}^0_\infty$. Denote by S^0_∞ the boundary of $\overline{V}^0_\infty$. Consider a completely continuous field $\varphi\colon \overline{V}^0_\infty \to E_\infty$ defined by the formula

$\varphi(x) = x - \Phi(x)$ and let $\varphi_0 = \varphi|S^0_\infty$ *). We can assume that $\varphi_0\colon S^0_\infty \to P_\infty$ and by β) it suffices to prove that the characteristic $\gamma(\varphi_0, S^0_\infty)$ of the field φ_0 is different from zero. For this purpose let

$$\psi(x, t) = x - t\Phi(x)$$

for an arbitrary $x \in S^0_\infty$, $0 \leqslant t \leqslant 1$.

It follows from our assumption that for an arbitrary $y^* \in \psi(x, t)$, where $x \in S^0_\infty$, $0 \leqslant t < 1$ we have

$$\|y^*\| = \|x - ty\| \geqslant \|x\| - t\|y\| > 0,$$

and thus $\psi\colon S^0_\infty \times J \to P_\infty$. Further, $\psi(x, 0) = x$, $\psi(x, 1) = \varphi_0(x)$ for any $x \in S^0_\infty$, i. e. the fields $\psi(x, 0) = \psi_0(x)$ and $\varphi_0(x)$ are homotopic, whence, by α), $\gamma(\psi_0, S^0_\infty) = \gamma(\varphi_0, S^0_\infty)$. But the characteristic $\gamma(\psi_0, S^0_\infty)$ of the field ψ_0 is different from zero and Theorem 1 is thus proved.

Theorem 1 immediately implies the well known

Theorem of Bohnenblust and Karlin (cf. [2] p. 155). *If $\Phi\colon A \to A$ is a multi-valued mapping into itself of a compact convex set A lying in E_∞, then there exists a point $x_0 \in A$ for which $x_0 \in \Phi(x_0)$.*

Proof. Denote by $\overline{V}_\infty$ a closed full sphere in space E_∞ with centre 0 and radius $\varrho > 0$ so large that $A \cap \mathrm{Fr}(\overline{V}_\infty) = \theta$. Further denote by $r\colon \overline{V}_\infty \to A$ a retraction of the sphere $\overline{V}_\infty$ onto A (the existence of such a retraction can be proved in an elementary way) and let $\Phi'(x) = \Phi[r(x)]$ for any $x \in \overline{V}_\infty$. The function $\Phi'\colon \overline{V}_a \to A$ satisfies of course the conditions of Theorem 1, and thus there exists a point $x_0 \in \Phi'(x_0)$, $x_0 \in A$. We have $\Phi'(x_0) = \Phi(r(x_0)) = \Phi(x_0) \ni x_0$ and the proof is thus complete.

We shall now give a theorem, which in the finite-dimensional case of single-valued functions is known as the Kronecker-Hopf theorem **).

Let $\Phi\colon \overline{V}_a \to E_a$ be a completely continuous mapping of a sphere $\overline{V}_a$ in E_a. Suppose that the mapping Φ has a finite number of fixed points $x_1, x_2, \ldots, x_k \in V_a$, $x_i \in \Phi(x_i)$, $(i = 1, 2, \ldots, k)$.

Denote by $V^{(i)}_a$ a full sphere in the space E_a with centre x_i and radius $\varrho_i > 0$ so small that $\overline{V}^{(i)}_a \cap \overline{V}^{(j)}_a$ $(i \neq j,\ i, j = 1, 2, \ldots k)$, $\overline{V}^{(i)}_a \subset V_a$. Now, for an arbitrary $x \in \overline{V}_a$ let $\varphi(x) = x - \Phi(x)$, $\varphi_{(i)} = \varphi|S^{(i)}_{a-1}$ $(i = 1, 2, \ldots k)$, where $S^{(i)}_{a-1} = \mathrm{Fr}(\overline{V}^{(i)}_a)$. We have $\varphi_{(i)}\colon S^{(i)}_{a-1} \to P_a$, $\varphi\colon S_{a-1} \to P_a$, where $S_{a-1} = \mathrm{Fr}(V_a)$. The number $\gamma_i = \gamma(\varphi_i, S^{(i)}_a)$ equal to the characteristic of the completely continuous field φ_i on the sphere $S^{(i)}_{a-1}$ will be termed the *index of the fixed point x_i of the mapping Φ.*

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*) If $\varphi\colon A \to E_a$, $A_0 \subset A$, then the notation $\varphi_0 = \varphi|A$ denotes that φ_0 is a partial mapping on A_0.

**) Cf. [8] p. 188, for a single valued case in Banach spaces.

Theorem 2$_a$. *The characteristic $\gamma = \gamma(\varphi, S_{a-1})$ of a non-vanishing completely continuous multi-valued field φ: $S_{a-1} \to P_a$, $\varphi(x) = x - \Phi(x)$ is equal to the algebraic sum of the indices of the fixed points of the mapping Φ, i. e. we have the formula*

$$\gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_k . \tag{2}$$

The proof in the finite-dimensional case $a = n$ is carried out on the basis of the Vietoris Mappings Theorem ([1], theorem 1) and the case $a = \infty$ reduces to the finite-dimensional case.

3. Theorem on antipodes. The following theorem is an extension to multi-valued functions of the well-known theorem on antipodes [3], [6].

Theorem 3$_a$. *If a non-vanishing completely continuous multi-valued field φ: $S^0_{a-1} \to P_a$, defined on a unit sphere $S^0_{a-1} \subset E_a$, is odd*

$$\varphi(x) = -\varphi(-x), \tag{3}$$

then its characteristic $\gamma(\varphi, S^0_{a-1})$ is odd.

The proof in the case $a = n$ is based on the corollary 1 of paper [5]. The case $a = \infty$ reduces to the finite-dimensional case in the same way as in the case of single-valued mappings [6].

The theorem quoted can be formulated in a more general form on the grounds of property α):

Theorem 4$_a$. *If a non-vanishing completely continuous multi-valued field φ: $S^0_{a-1} \to P_a$ satisfies the condition*

$$\varphi(x) \frown \lambda\varphi(-x) = \theta \quad \text{for any} \quad \lambda > 0 , \tag{4}$$

*then its characteristic $\gamma(\varphi, S^0_{a-1})$ on the sphere S^0_{a-1} is odd **).*

Proof. For an arbitrary $(x, t) \in S^0_{a-1} \times J$ let

$$\Psi(x, t) = \frac{\Phi(x)}{1+t} - \frac{t\Phi(-x)}{1+t}, \ \psi(x, t) = x - \Psi(x, t) .$$

It follows from condition (4) that the function $\psi(x, t)$ satisfies all conditions of the definition of homotopy, and thus the field $\varphi(x) = \psi(x, 0)$ is homotopic with the odd field $\psi_0(x) = \psi(x, 1) = x - \frac{1}{2}[\Phi(x) - \Phi(-x)]$ whose characteristic is odd by Theorem 3$_a$. Hence, by α), we have $\gamma(\varphi, S^0_{a-1}) = \gamma(\psi, S^0_{a-1})$ and Theorem 4$_a$ is proved.

Theorem 5$_a$. *If a completely continuous multi-valued field φ: $\overline{V}^0_a \to E_a$, $\varphi(x) = x - \Phi(x)$ defined on a unit full sphere V^0_a satisfies for any $x \in S^0_{a-1}$*

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*) In the finite-dimensional case, i. e. for $a = n$ Theorems 4$_a$ and 5$_a$ can be proved in a more general form, cf. paper [5] corollary 1 and 2.

condition (3) *or* (4), *then there exists a fixed point* $x_0 \in V_a^0$ *of the completely continuous mapping* $\Phi: V_a^0 \to E_a$, *i. e.* $x_0 \in \Phi(x_0)$.

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MATHEMATICS

Some Theorems on Multi-Valued Mappings of Subsets of the Euclidean Space

by

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1. The aim of this note is to give other examples of generalization of some classical theorems concerning single-valued mappings to the case of multi-valued acyclic mappings. Many such examples are known as, for instance, a generalization of the Lefschetz fixed-point formula due to S. Eilenberg and D. Montgomery [3]; others are given in [6] and [7].

2. Let X and Y be topological spaces and F: $X \to Y$ – a multi-valued function from X to Y; i. e. for every $x \in X$, a non-empty subset $F(x)$ of Y is given. If $A \subset X$, then the set $F(A) = \bigcup_{a \in A} F(a)$ is called the *image* of A. The *graph* of F is the set

$$W(F) = \{(x, y) : y \in F(x)\} \subset X \times Y.$$

The single-valued function $f: X \to Y$ is said to be a *selector* of F if $W(f) \subset W(F)$, i. e. $f(x) \in F(x)$, for every $x \in X$.

The multi-valued function F: $X \to Y$ is said to be *continuous* if

1° the image of every compact subset of X is compact,

2° the graph of F is closed in $X \times Y$.

For compact Y, condition 1° may be dropped; in this case condition 2° is equivalent to the upper semi-continuity of F, when F is regarded as a single-valued mapping of X into the hyperspace of non-empty compact subsets of Y. If F is single-valued, i. e. for every $x \in X$, $F(x)$ is a single point $f(x)$, the continuity of F means the usual continuity of the single-valued mapping f.

3. We shall use the Vietoris-Čech homology groups with coefficients mod 2 and with *compact carriers* ([4], p. 255) of metric spaces, and we shall denote simply by $H_k(X)$ the k-th homology group of X. The space X is called *n-acyclic* if it is non-empty and if $H_k(X) = 0$ for $1 \leqslant k \leqslant n$; it is called *acyclic*, if it is n-acyclic for every $n \geqslant 1$. We shall often refer to the famous

VIETORIS MAPPING THEOREM [1]. *Let $f\colon X \to Y$ be a continuous mapping satisfying the following conditions*:

(i) *the counter-image $f^{-1}(C)$ of every compact subset of Y is compact*,

(ii) *$f^{-1}(y)$ is acyclic, for every $y \in Y$.*

Then f induces an isomorphism $f_*\colon \overset{c}{H}_k(X) \approx H_k(Y)$ for every k.

The condition (i) provides that the Vietoris Theorem remains true for homology with compact carriers even for non-compact spaces. In this note we confine ourselves to the case of homology mod 2; we do so for the sake of simplicity though some of the results are valid for homology with other coefficient groups for which the Vietoris Theorem holds. It is known, however, that this theorem does not hold when the integers are taken as coefficients.

4. The multi-valued mapping $F\colon X \to Y$ is said to be *acyclic* if the sets $F(x)$ are acyclic for every $x \in X$.

Let $F\colon X \to Y$ be a multi-valued continuous acyclic mapping and let W be the graph of F. Let $p\colon W \to X$, $q\colon W \to Y$ be the natural projections, i. e.

(1) $$p(x, y) = x\,, \qquad q(x, y) = y\,.$$

Then the sets $p^{-1}(x)$ are acyclic and, by the Vietoris Theorem, p induces an isomorphism $p_*\colon H_k(W) \approx H_k(X)$. The homomorphism

$$F_* = q_* p_*^{-1}\colon H_k(X) \to H_k(Y)$$

is said to be *induced* by the multi-valued mapping F. In particular, if F is single-valued, it coincides with the usual induced homomorphism of homology. More generally, if $f\colon X \to Y$ is a continuous selector of F, then $f_* = F_*$ ([7], p. 263).

5. The Vietoris Theorem enables us to define for multi-valued acyclic mappings many homology notions analogous to those concerning single-valued mappings with similar properties.

Let S^n denote the unit sphere in the Euclidean n-space E^{n+1}. Let 0 be the origin and let $P^{n+1} = E^{n+1} \setminus (0)$. Let $F\colon S^n \to E^{n+1}$ be a multi-valued continuous acyclic mapping.

If F does not vanish on S^n, i. e. if $F(S^n)$ does not contain the origin, then F may be regarded as a mapping $S^n \to P^{n+1}$. Such a mapping may also be looked upon as a multi-valued non-vanishing vector field on S^n. The notion of the Kronecker index of F may be defined as follows: Since the groups $H_n(S^n)$ and $H_n(P^{n+1})$ are both isomorphic with the coefficient group, the induced homomorphism $F_*\colon H_n(S^n) \to H_n(P^{n+1})$ may be identified with an integer mod 2; it is called the *index* of F and is denoted by $I(F)$ *).

*) The notion of index may be likewise defined for mappings of more general subsets of E^{n+1}, for instance, for mappings of n-dimensional manifold; it may also be defined for other coefficient groups, as has already been mentioned in [4].

In the case, when y_0 is an arbitrary point of E^{n+1} and $F(S^n)$ does not contain y_0, the *index* $I(F, y_0)$ *of* F *relative to* y_0 is defined by putting $I(F, y_0) = I(\bar{F})$, where $\bar{F}: S^n \to P^{n+1}$ is defined by $\bar{F}(x) = F(x) - y_0$.

The notion of homotopy of multi-valued continuous acyclic mappings was introduced in [7]. Although in [7] only the multi-valued mappings of compact spaces into compact ones were considered, condition 1° of par. 2 provides that all the results of [7] remain true in the general case, if homology with compact carriers is used. In particular, if $F, G: X \to Y$ are two multi-valued continuous acyclic mappings which are acyclically homotopic, then $F_* = G_*$. It follows that, *if* $F, G: S^n \to P^{n+1}$ *are two non-vanishing multi-valued continuous acyclic mappings which are acyclically homotopic (in* P^{n+1}*), then* $I(F) = I(G)$. If the single-valued mapping $f: S^n \to P^{n+1}$ is a continuous selector of $F: S^n \to P^{n+1}$, then $I(F) = I(f)$ and $I(f)$ is the usual Kronecker index of f.

Let Q^{n+1} be the closed spherical region in E^{n+1} bounded by S^n.

PROPOSITION. *Let* $G: Q^{n+1} \to E^{n+1}$ *be a multi-valued continuous acyclic mapping which does not vanish on* S^n. *If* $I(G|S^n) = 0$, *then* G *vanishes in some interior point of* Q^{n+1}, *i. e. there exists a point* $x_0 \in Q^{n+1}$ *such that* $0 \in G(x_0)$.

Proof. Let $F = G|S^n$. If G does not vanish on Q^{n+1}, then it may be regarded as a mapping $Q^{n+1} \to P^{n+1}$. Let W be the graph of G and V — the graph of F. Consider the following commutative diagram:

$$\begin{array}{ccccc} Q^{n+1} & \xleftarrow{p_1} & W & & \\ \uparrow i & & \uparrow j & \searrow q_1 & \\ & & & & P^{n+1}, \\ & & & \nearrow q_2 & \\ S^n & \xleftarrow{p_2} & V & & \end{array}$$

where i, j are injections, and p_1, q_1 and p_2, q_2 are the natural projections of the graphs W and V, respectively. By the Vietoris Theorem, p_{1*} and p_{2*} are isomorphisms. Since i_* is zero, it follows that $F_* = q_{2*} p_{2*}^{-1}$ is zero. Consequently, $I(F) = 0$.

6. The multi-valued mapping $\Phi: X \to X$ is called a multi-valued *involution*, if the graph of it is symmetric relative to the diagonal of $X \times X$, i. e. if $y \in \Phi(x)$ implies $x \in \Phi(y)$, for every $(x, y) \in X \times X$. The following theorem is a slight modification of Theorem 1 of [6]; it will be needed for proving Theorem 2 of par. 7 which is the main theorem of this note:

THEOREM 1. *Let* X *be an* n*-acyclic space and let* $\Phi: X \to X$ *be a multi-valued continuous acyclic involution. Let* $F: X \to E^{n+1}$ *be a multi-valued continuous acyclic mapping satisfying the following condition: no radius in* E^{n+1} *beginning at the origin intersects both the sets* $F(x)$ *and* $F(y)$*, for any* x, y *such that* $y \in \Phi(x)$. *Then* F *induces a non-zero homomorphism* $F: H_n(X) \to H_n(P^{n+1})$.

Proof. First, the homomorphism F is defined, since $F(X)$ cannot contain the origin. Let W be the graph of F and let

$$Z = \{(x, y, u, v) : u \in F(x), v \in F(y), v \in \Phi(u)\} \subset X \times X \times P^{n+1} \times P^{n+1}.$$

Consider the following commutative diagram:

$$\begin{array}{ccccc} Z & \xrightarrow{f} & P^{n+1} & \xrightarrow{r} & S^n \\ {\scriptstyle s}\downarrow & & \uparrow{\scriptstyle q} & & \\ X & \xleftarrow{p} & W & & \end{array}$$

where p and q are the natural projections defined by (1) in par. 4, and the mappings s, f and r are defined by formulae:

$$s(x, y, u, v) = x,$$

$$r(u) = \frac{u}{|u|},$$

$$f(x, y, u, v) = u - v.$$

Let us observe that f is well defined, for $f(x, y, u, v) = 0$ implies that the points $u \in F(x)$ and $v \in F(y)$ lie on the same radius beginning at the origin, which contradicts the assumption of the Theorem 1. Evidently, r_* is an isomorphism and, by the Vietoris Theorem, p_* is an isomorphism. The mapping s can be factored into three mappings: $(x, y, u, v) \to (x, y, u) \to (x, y) \to x$, each of which has as counter-images, the sets homeomorphic to $F(y)$, $F(x)$, and $\Phi(x)$, respectively. By the Vietoris Theorem, each of these three mappings induces a homology isomorphism; hence, s is an isomorphism. It follows that Z is n-acyclic.

Let $\varphi : Z \to Z$ be the involution defined by

$$\varphi(x, y, u, v) = (y, x, v, u).$$

Then the mapping $g = rf : Z \to S^n$ satisfies the condition

$$g\varphi(z) = -z$$

for every $z \in Z$. By theorem 6 of [5] it follows that the homomorphism $g_* : H_n(Z) \to H_n(S^n)$ is non-zero. It follows that $q_* \neq 0$, and then $F_* = q_* p_*^{-1} \neq 0$.

COROLLARY 1. *Let $F : S^n \to E^{n+1}$ be a multi-valued continuous acyclic mapping such that*

(2) *no radius beginning at the origin intersects both the sets $F(x)$ and $F(-x)$, for any $x \in S^n$.*

Then $I(F) \neq 0$.

COROLLARY 2. *Let $G : Q^{n+1} \to E^{n+1}$ be a multi-valued continuous acyclic mapping such that the mapping $F = G|S^n$ satisfies condition* (2). *Then G vanishes in some interior point of Q^{n+1}.*

7. Let A be a subset of E^{n+1}. The point $a \in A$ is an interior point of A if and only if the kernel of the homomorphism $H_n(A\setminus(a)) \to H_n(A)$ induced by injection is non-zero. The following lemma characterizes the interior points by means of the notion of Index:

LEMMA 1. *The point $a \in A$ is an interior point of A if and only if there exists a multi-valued continuous acyclic mapping $G: Q^{n+1} \to A$ such that $G(S^n) \subset A\setminus(a)$ and $I(G|S^n, a) \neq 0$.*

Proof. The necessity is obvious. To prove that the condition is sufficient, let us put $F = G|S^n$ and observe that the kernel of $H_n(A\setminus(a)) \to H_n(A)$ contains $F_*(H_n(S^n))$ which is non-zero by the assumption of Lemma.

The multi-valued mapping $F: X \to Y$ is said to be an *ε-mapping* ($\varepsilon > 0$) if $F(x_1) \cap F(x_2) \neq 0$ implies $\varrho(x_1, x_2) < \varepsilon$, for every $x_1, x_2 \in X$. The following theorem is a generalization of Borsuk's classical theorem on ε-mappings of the Euclidean space [2]:

THEOREM 2. *Let ε be a positive number and let $G: E^{n+1} \to E^{n+1}$ be a multi-valued continuous acyclic ε-mapping. Then $G(E^{n+1})$ is an open subset of E^{n+1}.*

The proof is based on the following

LEMMA 2. *Let $G: Q^{n+1} \to E^{n+1}$ be a multi-valued continuous acyclic 1-mapping. Let $F = G|S^n$ and let $y_0 \in G(0)$. Then $I(F, y_0) \neq 0$.*

Proof. Let $\bar{G}(x) = G(x) - y_0$; then $0 \in \bar{G}(0)$. Let W be the graph of $\bar{F} = \bar{G}|S^n$, i. e.

$$W = \{(x, u): x \in S^n;\ u \in \bar{G}(x)\}.$$

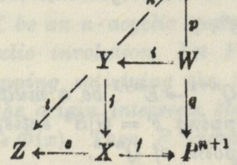
Let

$$Z = \{(x, y): x, y \in Q^{n+1};\ \varrho(x, y) = 1\},$$

$$X = \{(x, y, u, v): (x, y) \in Z;\ u \in \bar{G}(x);\ v \in \bar{G}(y)\},$$

$$Y = \{(x, y, u, v) \in X: y = 0\}.$$

Let us observe that if $(x, u) \in W$, then $(x, 0, u, 0) \in Y$. Therefore, $(x, u) \to (x, 0, u, 0)$ defines a mapping $i: W \to Y$. Consider the following commutative diagram:



In this diagram, p and q are the natural projections, defined as in par. 4, j is the injection and s, t, h and f are defined by:

$$s(x, y, u, v) = (x, y),$$
$$t(x, 0, u, v) = (x, 0),$$
$$h(x, 0, u, v) = x,$$
$$f(x, y, u, v) = u - v.$$

Notice that t is well defined, for $(x, 0, u, v) \in Y$ implies $\varrho(x, 0) = 1$; and so is f, for $f(x, y, u, v) = 0$ implies $u = v \in \bar{G}(x) \cap \bar{G}(y)$ which means that $G(x) \cap G(y) \neq 0$ and $\varrho(x, y) = 1$.

Since $F_* = q_* p_*^{-1}$, it should be proved that $q_*: H_n(W) \to H_n(P^{n+1})$ is non-zero, for p_* is an isomorphism. The mapping s can be factored into two mappings $(x, y, u, v) \to (x, y, u) \to (x, y)$ and each of them has acyclic counter-images. It follows from the Vietoris Theorem that s_* is an isomorphism. In the same way we prove that t_* and h_* are isomorphisms. It follows from the commutativity that i_* and j_* are isomorphisms. Since S^n is n-acyclic, therefore, X is n-acyclic.

Let $\varphi: X \to X$ be the (single-valued) involution defined by

$$\varphi(x, y, u, v) = (y, x, v, u).$$

The mapping f satisfies the condition

$$f\varphi(x, y, u, v) = -f(x, y, u, v).$$

Therefore, the assumptions of Theorem 1 of par. 6 are fulfilled, when φ is taken for Φ and f for F. It follows that $f_* \neq 0$ and then $q_* \neq 0$.

Proof of Theorem 2. Let $y_0 \in G(E^{n+1})$ and let $x_0 \in E^{n+1}$ be such that $y_0 \in G(x_0)$. Let Q_ε^{n+1} be the closed spherical region with the centre x_0 and the radius ε and $S_\varepsilon^n = \dot{Q}_\varepsilon^{n+1}$. Let $G_\varepsilon = G|Q_\varepsilon^{n+1}$, $F_\varepsilon = G_\varepsilon|S_\varepsilon^n$. Applying Lemma 2 to Q_ε^{n+1}, S_ε^n, G_ε, F_ε and ε, instead of to Q^{n+1}, S^n, G, F and 1 and then applying Lemma 1 we infer that y_0 is an interior point of $G(E^{n+1})$.

THEOREM 3 (ON THE INVARIANCE OF DOMAIN). *Let U be an open subset of E^{n+1} and let $G: U \to E^{n+1}$ be a multi-valued continuous acyclic mapping such that the sets $F(x_1)$ and $F(x_2)$ are disjoint, for every $x_1 \neq x_2$. Then $G(U)$ is open in E^{n+1}.*

Proof. It follows that G is an ε-mapping, for every $\varepsilon > 0$; let $y_0 \in G(x_0)$, where $x_0 \in U$. Let Q_ε^{n+1} be a closed spherical region, with the centre x_0 and the radius $\varepsilon > 0$, contained in U. Since $G|Q_\varepsilon^{n+1}$ is an ε-mapping, it follows by Lemma 2 and Lemma 1 that y_0 is an interior point of $G(U)$.

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GENERALIZING THE HOPF-LEFSCHETZ

FIXED POINT THEOREM FOR NON-COMPACT ANR-S

ANDRZEJ GRANAS

A continuous mapping $F: X \to Y$ between topological spaces X and Y is called *compact*, provided F maps X into a compact subset of Y. In this paper, we shall be concerned with the problem of the existence of fixed points for a compact map of an ANR for metric spaces into itself.*

Let X be an ANR for metric spaces. With a compact map $F: X \to X$ we shall associate an integer $\Lambda(F)$, defined in terms of the induced endomorphisms F_q of the homology of X and called the Lefschetz number of F.

In order to define the Lefschetz number $\Lambda(F)$, we shall invoke the theory of the trace, extended by J. Leray [9] for a class of endomorphisms of infinite-dimensional vector spaces. We shall not, however, state the definitions or properties needed in any more generality than will be necessary for our purposes.

The main result which we intend to present here is the following theorem: *if* $\Lambda(F) \neq 0$, *then the compact map* F *has a fixed point.*

This fact clearly implies several well-known fixed point theorems, both in functional analysis and topology, in particular, the Lefschetz Fixed Point Theorem for compact ANR-s and various forms of the Schauder Fixed Point Theorem.

* For a special kind of ANR-s (namely those which are r-dominated by subsets of Banach spaces) this problem was treated, in an implicit form, by F. Browder [3], in his paper concerned with fixed point theorems for Banach manifolds. The method used in that paper (based on the Mazur Lemma) could not be applied for arbitrary ANR-s.

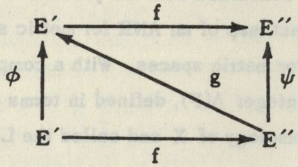
The proof is based on the Hopf Fixed Point Theorem for polyhedra [6] and the theorems of Kuratowski [7] and Wojdysławski [11] concerning the embedding of a metric space into a normed space.

§1. *The trace.* In what follows, we shall consider vector spaces only over the field Q of rational numbers.

Given an endomorphism $\phi: E \to E$ of a finite-dimensional vector space E, we denote by $tr(\phi)$ the trace of the endomorphism ϕ.

We recall the following well-known property of the trace:

(A) *assume that we are given finite-dimensional vector spaces* E', E'' *and a commutative diagram*



of linear maps. Then $tr(\phi) = tr(\psi)$.

A graded vector space $E = \{E_q\}_{q=0}^{\infty}$ is said to be of a *finite type*, provided all E_q are of finite dimension and $E_q = 0$ for almost all q. Let E be of a finite type and $\phi = \{\phi_q\}: E \to E$ be an endomorphism of degree 0. We let

$$tr(\phi) = \{tr(\phi_q)\}$$

$$\lambda(\phi) = \sum_{q=0} (-1)^q \, tr(\phi_q)$$

and call $\lambda(\phi)$ the Lefschetz number of ϕ.

Let E be an arbitrary vector space. Call an endomorphism $\phi: E \to E$ *finite-dimensional*, provided $\dim(\operatorname{Im}\phi) < +\infty$. For a finite-dimensional ϕ, let E' be a finite-dimensional subspace of E containing $\operatorname{Im}\phi$ and $\phi': E \to E'$ be the contraction* of ϕ to the pair (E', E').

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* Let $\mathcal{C}$ be the concrete category and let $f: X \to Y$ be a map in $\mathcal{C}$ such that $f(A) \subset B$, where $A \subset X$ and $B \subset Y$. By the *contraction* of f to the pair (A, B) we shall understand a map $f^*: A \to B$ in $\mathcal{C}$ defined by $f^*(a) = f(a)$, for $a \in A$. A contraction of f to the pair (A, Y) is simply the restriction $f|A$ of f to A.

We then define the (generalized) *trace* $Tr(\phi)$ of ϕ by putting

$$Tr(\phi) = tr(\phi') .$$

It follows clearly from (A) that $Tr(\phi)$ does not depend on the choice of the space E'.

Let $E = \{E_q\}$ be a graded vector space and

$$\phi = \{\phi_q\}: E \to E$$

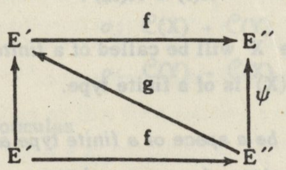
be an endomorphism of degree 0. We shall say that ϕ is a *Lefschetz endomorphism* or L-endomorphism, provided the graded vector space $\text{Im}\,\phi = \{\text{Im}\,\phi_q\}$ is of a finite type.

In this case we define the (generalized) *Lefschetz number* $\Lambda(\phi)$ of ϕ by putting

$$\Lambda(\phi) = \sum_{q=0} (-1)^q \, Tr(\phi_q) .$$

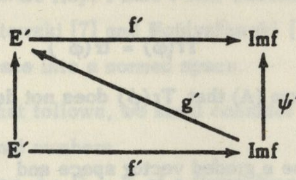
For notational convenience we let $Tr(\phi) = \{Tr(\phi_q)\}$. Clearly, if E is of a finite type and $\phi: E \to E$, then $tr(\phi) = Tr(\phi)$ and $\lambda(\phi) = \Lambda(\phi)$.

LEMMA 1. *Let $E' = \{E'_q\}$ be a graded vector space of a finite type and assume that we are given a commutative diagram of graded vector spaces and linear maps*



Then, ψ is a Lefschetz endomorphism, and we have in this case $\Lambda(\phi) = \Lambda(\psi)$.

Proof: From the commutativity of the above diagram, we have $\text{Im}\,\psi \subset \text{Im}\,f$. Since E' is of a finite type, this implies that ψ is a Lefschetz endomorphism. Consequently, we have the commutative diagram



in which f′, g′, ψ′ stand for the obvious contractions. In view of the property (A), we have $\operatorname{tr}(\phi) = \operatorname{tr}(\psi') = \operatorname{Tr}(\psi)$. Consequently, $\Lambda(\phi) = \Lambda(\psi)$ and the proof is completed.

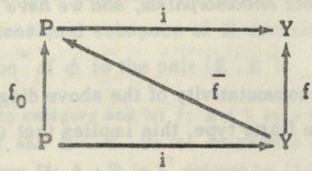
§2. *The Lefschetz maps.* In what follows, we shall denote by H the singular homology functor with rational coefficients from the category of topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree 0. Thus, $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q-dimensional singular homology group of a space X. For a map $f: X \to Y$, $H(f)$ is the induced linear map $f_* = \{f_q\}: H(X) \to H(Y)$, where $f_q: H_q(X) \to H_q(Y)$.

A map $f: X \to X$ will be called a *Lefschetz map* or L-map provided $f_*: H(X) \to H(X)$ is a Lefschetz endomorphism. In this case, we define the Lefschetz number $\Lambda(f)$ of f by putting

$$\Lambda(f) = \Lambda(f_*) .$$

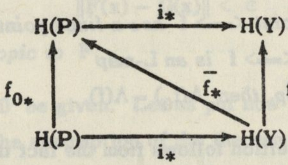
A topological space X will be called of a *finite type* provided the graded vector space $H(X)$ is of a finite type.

LEMMA 2. *Let* P *be a space of a finite type and assume that we are given a commutative diagram of spaces and maps*

$\mathfrak{D} =$ 

in which $i:P \to Y$ *stands for the inclusion. Then* f *is a Lefschetz map and we have in this case* $\Lambda(f) = \Lambda(f_0)$.

Proof. Applying the homology functor to the diagram $\mathfrak{D}$, we obtain the commutative diagram



which satisfies the assumption of Lemma 1.

Thus, $\Lambda(f_{0*}) = \Lambda(f_*)$ and, consequently, $\Lambda(f_0) = \Lambda(f)$. The proof is completed.

§3. *r-maps and s-maps.* Let $\mathcal{C}$ be a category and denote by $\mathcal{C}(X)$ the set of maps $X \xrightarrow{f} X$ in $\mathcal{C}$. Assume that we are given a pair of maps $s:X \to Y$ and $r:Y \to X$ in $\mathcal{C}$. Following Borsuk [1], we say that r is an *r-map for* s, and s an *s-map for* r provided $rs = 1_X$. A map $r:Y \to X$ (resp. $s:X \to Y$) for which there exists an s-map (resp. an r-map) is called simply an r-map (resp. an s-map).

Let $s:X \to Y$ be an s-map and $r:Y \to X$ an r-map for s. We define the maps between the sets

$$\sigma: \mathcal{C}(X) \to \mathcal{C}(Y)$$

$$\rho: \mathcal{C}(Y) \to \mathcal{C}(X)$$

by the following formulas

$$(1) \qquad \rho(f) = rfs \qquad \text{for } f \in \mathcal{C}(Y)$$

$$(2) \qquad \sigma(f_0) = sf_0r \qquad \text{for } f_0 \in \mathcal{C}(X)\,.$$

LEMMA 3. *Putting* $f = \sigma(f_0)$ *for each* $f_0 \in \mathcal{C}(X)$, *we establish* 1-1 *correspondence* $f_0 \longleftrightarrow f$ *between the sets* $\mathcal{C}(X)$ *and* $\sigma(\mathcal{C}(X)) \subset \mathcal{C}(Y)$.

If $\mathcal{C}$ *is the category of graded vector spaces, then*

1° f_0 *is an* L*-endomorphism* <==> f *is an* L*-endomorphism*

2° *if* f_0 *is an* L*-endomorphism, then* $\Lambda(f_0) = \Lambda(f)$.

If $\mathcal{C}$ *is the category of topological spaces, then*

3° f_0 *is compact* <==> f *is compact*

4° f_0 *has a fixed point* <==> f *has a fixed point*

5° f_0 *is an* L*-map* <==> f *is an* L*-map*

6° *if* f_0 *is an* L*-map, then* $\Lambda(f_0) = \Lambda(f)$.

Proof. The first assertion follows from the fact that ρ is an r-map for σ in the category of sets. 1° and 3° are evident. The proof of 2° is similar to that of Lemma 1.

To prove 4°, note that if x is a fixed point for f_0, then so is s(x) for f; if y is a fixed point for f, then so is r(y) for f_0.

5° and 6° follow from 1° and 2°, the same way as Lemma 2 follows from Lemma 1.

§4. *Compact maps into the special* ANR-s. In this and in the next two sections by a space we shall understand a metrizable space.

A space X is called an ANR (respectively AR) provided that for each embedding $h:X \to Y$ such that h(X) is closed in Y, there exists a retraction $r:U \to h(X)$, of an open set $U \subset Y$ onto h(X) (respectively a retraction $r:Y \to h(X)$).

A space Y will be called a *special* ANR provided Y is an open subset of a convex set Z contained in a normed space.*)

A mapping $F:X \to Y$ into a special ANR will be called *finite dimensional* provided the image F(X) is contained in a finite polyhedron $P \subset Y$. It turns out that every compact mapping into a special ANR can be uniformly approximated by finite dimensional mappings. More precisely, we have the following

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*) The fact that every special ANR is an ANR will not be used in our discussion.

THEOREM 1 *(Approximation Theorem). Let* Y *be a special* ANR *and* $F: X \to Y$ *be a compact mapping of a space* X *into* Y. *Then, for each* $\varepsilon > 0$, *there exists a finite dimensional mapping* $G: X \to Y$ *such that*

(i) *for all* $x \in X$ *we have*

$$\|F(x) - G(x)\| < \varepsilon$$

(ii) G *is homotopic to* F.

Proof. Let $\varepsilon > 0$ be given. Let us put now $Y_0 = \overline{F(X)} \subset Y$. Since Y_0 is compact (by the assumption of the theorem) and Y is open in a convex set Z, there exists a constant δ satisfying

(1) $$0 < \delta < \varepsilon$$

and such that each ball

(2) $$V(y_0, 2\delta) = \{y \in Z, \|y - y_0\| < 2\delta\}$$

with center $y_0 \in Y_0$ and radius 2δ is entirely contained in Y.

Now take an arbitrary $\frac{\delta}{2}$—net $N = \{y_1, y_2, \ldots, y_k\}$ of the compact set Y_0 and put for each $i = 1, 2, \ldots, k$

$$V_i = V(y_i, \frac{\delta}{2}), \quad V_i' = V(y_i, 2\delta) .$$

Evidently, all the balls V_i and V_i' are convex subsets of Y and $V = \bigcup_{i=1}^{k} V_i$ covers Y_0.

Define the partition of unity $\{\lambda_i\} \bigcup_{i=1}^{k}$ on V by putting

(3) $$\mu_i(y) = \max\{0, \frac{\delta}{2} - \|y - y_i\|\}$$

(4) $$\lambda_i(y) = \frac{\mu_i(y)}{\sum_{j=1}^{k} \mu_j(y)} \text{ for } y \in V$$

Let us put for each $x \in X$

(5) $$G(x) = \sum_{j=1}^{k} \lambda_i(F(x))\, y_i$$

It follows from (1) and the definition of G that

$$\|G(x) - F(x)\| < \varepsilon \text{ for all } x \in X \tag{6}$$

and that the values of G are in a simplicial complex P with the vertices $y_1, y_2, \ldots, y_k$.

It follows from (5) that the points y_i, which appear in a convex combination (5) for some x, belong to one of the balls V_i'. Since all V_i' are convex, we conclude that $P \subset Y$.

To prove the assertion (ii), note that for each $x \in X$, the points $G(x)$ and $F(x)$ belong to one of the balls V_i', say to V_j'. Since V_j' is convex subset of Y, the family of maps g_t defined by the formula

$$g_t(x) = tG(x) + (1-t)F(x) \text{ for } (x \cdot t) \in X \times I$$

has values in Y. Thus $g_t: X \to Y$ is a homotopy joining G with F and the proof is completed.

§5. *The main theorem.* In what follows we shall make use of the following elementary fact:

LEMMA 4. *Let* $F: Y \to Y$ *be a compact map of a metric space* Y *into itself and assume that* F *is a uniform limit of a sequence* $\{F_n\}$ *of maps* $F_n: Y \to Y$. *If each* F_n *has a fixed point, then so does the map* F.

Proof. Let $\{y_n\}$ be a sequence of points such that $F_n(y_n) = y_n$. From the assumption it follows that for almost all n we have

$$\rho(F(y_n), y_n) < \frac{1}{n}. \tag{1}$$

Since F is compact, we may assume, without loss of generality, that

$$\lim_{n=\infty} F(y_n) = y \in Y. \tag{2}$$

It follows from (1) and (2) that $\lim_{n=\infty} y_n = y$ and hence, by continuity of F, we have

$$\lim_{n=\infty} F(y_n) = F(y).$$

Comparing (2) and (3) we obtain $F(y) = y$ and the proof is completed.

Theorem 2. *Let* Y *be a special* ANR *and* $F:Y \to Y$ *be a compact map. Then* (i) F *is a Lefschetz map,* (ii) $\Lambda(F) \neq 0$ *implies that* F *has a fixed point.*

Proof. Theorem 1 implies that F is a uniform limit of a sequence $\{F_n\}$ of mappings $F_n:Y \to Y$ such that

(4) $$F \sim F_n \text{ for all } n$$

(5) $$F_n(Y) \subset P_n ,$$

where P_n is a finite polyhedron.

Denote by $\bar{F}_n:P_n \to P_n$ the obvious contraction of F_n. Lemma 2 implies that each F_n is a Lefschetz map, and hence, in view of (4), so is the map F. This completes the proof of (i).

To prove (ii), assume that $\Lambda(F) \neq 0$. In view of (4), we have $\lambda(F_n) \neq 0$. Lemma 2 implies that $\Lambda(\bar{F}_n) = \lambda(F_n) \neq 0$. This implies, in view of the Hopf Fixed Point Theorem [6], that for every n there exists a point $y_n \in P_n$ such that

$$y_n = \bar{F}_n(y_n) = F_n(y_n) .$$

It follows now from Lemma 4 that F has a fixed point and the proof of the theorem is completed.

Now we state the main result of the paper:

Theorem 3. *Let* X *be an arbitrary* ANR *and* $F_0:X \to X$ *be a compact map. Then* (i) F_0 *is a Lefschetz map,* (ii) $\Lambda(F_0) \neq 0$ *implies that* F_0 *has a fixed point.*

Proof. In view of the theorems of Kuratowski [7] and Wojdysławski [11], X is r-dominated by a special ANR Y. Using the notation of the Section 3, let us put $F = \sigma(F_0)$. By Lemma 3, $F:Y \to Y$ is compact and, hence, by Theorem 2, is a Lefschetz map. Applying Lemma 3 again, we conclude that F_0 is a Lefschetz map. Thus (i) is proved.

To prove (ii), assume that $\Lambda(F_0) \neq 0$. By Lemma 3, $\Lambda(F_0) = \Lambda(F) \neq 0$. This, in view of Theorem 2, implies that F has a fixed point. Applying

Lemma 3 for the last time, we conclude that F_0 has a fixed point, and the proof of the theorem is completed.

§6. *Corollaries.* We list now a few immediate consequences of the main theorem:

COROLLARY 1 (*The Lefschetz Fixed Point Theorem* [8]). *Let* X *be a compact* ANR *and* $f:X \to X$ *be a map. If* $\lambda(f) \neq 0$, *then* f *has a fixed point.*

A space X is said to have the *fixed point property in the narrow sense* (cf. [5]), provided every compact map $F: X \to X$ has a fixed point.

COROLLARY 2. *Acyclic* ANR-s *and in particular* AR-s *have the fixed point property in the narrow sense.*

COROLLARY 3 (*The Schauder Fixed Point Theorem* [10]). *Let* X *be a convex (not necessarily closed) subset of a normed space (or of a metrizable locally convex space). Then* X *has the fixed point property in the narrow sense.*

Proof. By the Theorem of Dugundji [4], X is an AR. Our assertion follows therefore from the Corollary 2.

COROLLARY 4 (*The Birkhoff-Kellogg Theorem* [2]). *Let* S *be the unit sphere in an infinite dimensional normed space* E *and* $F:S \to E$ *be a compact map such that*

(1) $$\|F(x)\| \geq \alpha > 0 \quad \text{for all } x \in S .$$

Then there is a positive λ; *such that* $F(x) = \lambda x$ *for some* $x \in S$.

Proof. Define $\bar{F}:S \to S$ by

$$\bar{F}(x) = \frac{F(x)}{\|F(x)\|} .$$

Because of (1), $\bar{F}$ is compact. Since the unit sphere S is an AR, by a theorem of Dugundji [4], there is a point $x \in S$ such that

$$x = \bar{F}(x) = \frac{F(x)}{\|F(x)\|} .$$

Thus, $F(x) = \lambda x$, with $\lambda = \|F(x)\|$, and the proof is completed.

§7. *Further remarks.* The main theorem given in the previous section can be generalized to non-metrizable case.

Call a topological space Y a *special Borsuk space* provided Y is an open set in a convex set Z lying in a locally convex linear topological space E. A topological space X is said to be a *Borsuk space* provided X is r-dominated by a special Borsuk space Y.

We have the following theorem:

THEOREM 4. *Let* $F: X \to X$ *be a compact map of a Borsuk space* X *into itself. Then* F *is a Lefschetz map and* $\Lambda(F) \neq 0$ *implies that* F *has a fixed point.*

A compact topological space X is called an ANR *for normal spaces* provided for each embedding $h: X \to Y$ into a normal space Y the set $h(Y)$ is a neighbourhood retract of Y. Every compact ANR for normal spaces (being homeomorphic with a neighbourhood retract of a Tychonoff cube) is evidently a Borsuk space. This implies, in view of Theorem 4, the following generalization of the Lefschetz fixed point theorem for non-metrizable compact ANR-s:

COROLLARY 1. *Let* X *be a compact* ANR *for normal spaces and* $f: X \to X$ *be a map. Then* f *is a Lefschetz map and* $\Lambda(f) \neq 0$ *implies that* f *has a fixed point.*

As a consequence of Theorem 4, we note also the following generalization of the Tychonoff fixed point theorem:

COROLLARY 2. *Let* X *be a convex (not necessarily closed) subset of a locally convex linear topological space. Then* X *has the fixed point property in the narrow sense.*

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MATHEMATICS
(TOPOLOGY)

Fixed Point Theorems for the Approximative ANR-s

by

A. GRANAS

Presented by K. BORSUK on October 10, 1967

It was shown by K. Borsuk [1] and S. Kinoshita [6] that there exist in E^3 locally connected and acyclic continua without fixed point property. It follows that the central fact in the fixed point theory known as the Hopf—Lefschetz Theorem (proved for manifolds, polyhedra and compact ANR-s in [7], [4] and [8], respectively) cannot be extended to orbitrary compacta.

Recently, K. Borsuk raised the question, whether the above theorem holds for a class of compacta introduced in 1953 by H. Noguchi [10] and called here the class of approximative ANR-s. It is the purpose of this paper to give an affirmative answer to the above question.

1. Algebraic preliminaries. In this paper we consider vector spaces only over the field of rational numbers. For an endomorphism $\varphi : E \to E$ of a finite dimensional vector space E we denote by $\mathrm{tr}(\varphi)$ the ordinary trace of φ.

A graded vector space $E = \{E_q\}_{q \geqslant 0}$ is said to be of a *finite type* provided (i) $\dim E_q < +\infty$ for all $q \geqslant 0$, (ii) $E_q = 0$ for almost all q. If $\varphi = \{\varphi_q\} : E \to E$ is an endomorphism of degree zero of such a space we let

$$\lambda(\varphi) = \sum_{q \geqslant 0} (-1)^q \mathrm{tr}(\varphi_q)$$

and call $\lambda(\varphi)$ the Lefschetz number of φ.

Let $\varphi : E \to E$ be an endomorphism of an arbitrary vector space. Call φ *finite dimensional* provided $\dim \mathrm{Im}\, \varphi < +\infty$. For such φ, we define the generalized trace $\mathrm{Tr}(\varphi)$ by putting

$$\mathrm{Tr}(\varphi) = \mathrm{tr}(\tilde{\varphi}),$$

where $\tilde{\varphi} : \mathrm{Im}\, \varphi \to \mathrm{Im}\, \varphi$ is defined by φ.

Let $E = \{E_q\}$ be an arbitrary graded vector space. An endomorphism $\varphi = \{\varphi_q\} : E \to E$ of degree zero is said to be a *Lefschetz endomorphism* provided

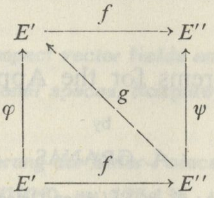
(i) φ_q is finite dimensional for all $q \geqslant 0$,

(ii) $\varphi_q = 0$ for almost all q. For such φ we define the (generalized) Lefschetz number $\Lambda(\varphi)$ by putting

$$\Lambda(\varphi) = \sum_{q \geqslant 0} (-1)^q \mathrm{Tr}(\varphi_q).$$

We have the following lemma:

(1.1) *Let $E' = \{E'_q\}$ be a graded vector space of a finite type and assume that we are given a commutative diagram of graded vector spaces and linear maps*



Then ψ is a Lefschetz endomorphism, and we have in this case $\Lambda(\varphi) = \Lambda(\psi)$.

For the proof see [2].

2. Lefschetz maps. In what follows we shall denote by H the Vietoris homology functor with rational coefficients from the category of metric spaces and continuous maps to the category of graded vector spaces and linear maps of degree 0. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q-dimensional Vietoris homology group of X. For a map $f: X \to Y$, $H(f)$ is the induced linear map $f_* = \{f_q\}: H(X) \to H(Y)$, where $f_q: H_q(X) \to H_q(Y)$.

A map $f: X \to X$ is called a *Lefschetz map* provided $f_*: H(X) \to H(X)$ is a Lefschetz endomorphism. For such f we define the Lefschetz number $\Lambda(f)$ by putting

$$\Lambda(f) = \Lambda(f_*).$$

A space X is said to be of a *finite type*, provided the graded vector space $H(X)$ is of a finite type. A map $f: X \to X$ of such a space is always a Lefschetz map and $\Lambda(f)$ coincides with the ordinary Lefschetz number $\lambda(f)$ of f.

Call a map $F: X \to Y$ *compact* provided $F(X)$ is contained in a compact subset of Y.

We shall make use of the following theorem:

(2.1) *Let X be an* ANR *for metric spaces and $F: X \to X$ be a compact map. Then F is a Lefschetz map and $\Lambda(F) \neq 0$ implies that F has a fixed point.*

For the singular homology, this theorem was proved in [2]. The case under consideration follows from the fact that for ANR-s the singular and the Vietoris homology theories are isomorphic [9].

3. Approximative ANR-s. Let (X, A) be a pair of metric spaces and ε be a positive number. A continuous map $r_\varepsilon: X \to A$ is called an *ε-retraction* provided $\varrho(r_\varepsilon(a), a) < \varepsilon$ for all $a \in A$.

A subspace $A \subset X$ is called an *approximative retract* of X (or simply an (a)-retract of X) provided for every $\varepsilon > 0$ there exists an ε-retraction $r_\varepsilon: X \to A$.

(3.1) *Assume that a compactum A is an approximative retract of a space X and γ is an infinite cycle in A. Then the relation $\gamma \sim 0$ in X implies $\gamma \sim 0$ in A.*

For the proof see [3].

A compactum X is said to be an *approximative* ANR (resp. approximative AR) provided for each embedding $h: X \to Y$ into a metric space Y, the set $h(X)$ is an (a)-retract of some open set U in Y (resp. an (a)-retract of Y).

(3.2) *Every compact approximative* ANR *is of a finite type.*

This implies that for every map $f: X \to X$ of a compact approximative ANR the Lefschetz number $\lambda(f)$ is defined.

Now we shall establish a lemma which will be used in the proof of the Main Theorem.

(3.3) *Let $f: X \to Y$ be a map into a compact approximative* ANR. *There exists an $\varepsilon > 0$ such that for each $g: X \to Y$ the condition*

$$\varrho\big(f(x), g(x)\big) < \varepsilon \quad \text{for all} \quad x \in X$$

implies that $f_* = g_*$.

Proof. In view of a theorem of Kuratowski [5], we may assume without loss of generality that Y is contained in a Banach space E and hence there exists an open set U in E, such that $Y \subset U$ is an approximative retract of U. Let $\varepsilon > 0$ be a number smaller than the distance dist $(Y, \dot{U})$ of the compact set Y to the boundary $\dot{U}$ of U in E. Let $g: X \to Y$ be a map such that

(1) $$\|g(x) - f(x)\| < \varepsilon \quad \text{for all} \quad x \in X.$$

and $\varkappa$ be an orbitrary infinite cycle in X. In order to prove the lemma it will be sufficient to prove that $f(\varkappa)$ is homologous to $g(\varkappa)$ in Y.

Now we denote by $j: Y \to U$ the inclusion and put $\bar{f} = jf$, $\bar{g} = jg$. It follows from (1) and the definition of ε that for each $x \in X$ the interval $t\bar{f}(x) + (1-t)\bar{g}(x)$ where $0 \leqslant t \leqslant 1$ is entirely contained in U. This implies that the maps $\bar{f}, \bar{g}: X \to U$ are homotopic and hence the cycle $\gamma = \bar{f}(\varkappa) - \bar{g}(\varkappa)$ is homologous to zero in U. Since $\bar{f}(\varkappa) = f(\varkappa)$, $\bar{g}(\varkappa) = f(\varkappa)$ and Y is an approximative retract of U, we conclude by **(3.1)** that $\gamma \sim 0$ in Y and hence $f(\varkappa) \sim g(\varkappa)$ in Y.

The proof is thus completed.

4. The Main Theorem. We shall state now the main result of this paper:

THEOREM. *Let X be an approximative compact* ANR *and $f: X \to X$ be a map. Then $\lambda(f) \neq 0$ implies that f has a fixed point.*

Proof. We may assume, without loss of generality, that X is an approximative retract of an open set U contained in a Banach space E. For each $n = 1, 2, \ldots$ let $r_n: U \to X$ be a $\frac{1}{n}$-retraction from U to X. We have

(1) $$\|x - r_n(x)\| < \frac{1}{n} \quad \text{for all} \quad x \in X.$$

Let $j: X \to U$ be the inclusion and define for each $n = 1, 2, \ldots$ a map $G_n: U \to U$ by putting

(2) $$G_n = jfr_n.$$

Consider now for each n the diagram

$$\mathcal{D}_n = \begin{array}{ccc} X & \xrightarrow{\;j\;} & U \\ {\scriptstyle f}\uparrow & \nwarrow{\scriptstyle fr_n} & \uparrow{\scriptstyle G_n} \\ X & \xrightarrow{\;j\;} & U \end{array}$$

and its image under the functor H in the category of graded vector spaces

$$H(\mathcal{D}_n) = \begin{array}{ccc} H(X) & \xrightarrow{\;j_*\;} & H(U) \\ {\scriptstyle f_*}\uparrow & \nwarrow{\scriptstyle (fr_n)_*} & \uparrow{\scriptstyle (G_n)_*} \\ H(X) & \xrightarrow{\;j_*\;} & H(U) \end{array}.$$

In view of (2) we have

(3) $$(G_n)_* = (j_*)(fr_n)_* \quad \text{for all} \quad n.$$

In view of (1) the identity map $1: X \to X$ is a uniform limit of the sequence $\{r_n j\}$ of maps $r_n j: X \to X$. Applying Lemma **(3.3)** to the map $1: X \to X$ we conclude that there exists an integer n_0 such that $1_* = (r_n j)_*$ for all $n \geqslant n_0$. This implies $f_* = (fr_n)_* \circ j_*$ for $n \geqslant n_0$, and hence in view of (3), the diagram $H(\mathcal{D}_n)$ commutes for $n \geqslant n_0$.

Since X is of a finite type we may apply now Lemma **(1.1)** to $H(\mathcal{D}_n)$ for $n \geqslant n_0$. By this Lemma G_n is a Lefschetz map and

(4) $$\lambda(f) = \Lambda(G_n) \quad \text{for} \quad n \geqslant n_0.$$

Now let us assume that $\lambda(f) \neq 0$. We shall prove that f has a fixed point.

In view of (2) and (4) each $G_n: U \to U$ is a compact map with $\Lambda(G_n) \neq 0$ for $n \geqslant n_0$.

Applying Theorem **(2.1)** we find a sequence $\{x_n\}$ of points in X such that

(5) $$G_n(x_n) = x_n \quad \text{for} \quad n \geqslant n_0.$$

Let $\{x_{k_n}\}$ be a subsequence of $\{x_n\}$ such that

(6) $$\lim_{n=\infty} x_{k_n} = x.$$

In view of (1) we have

(7) $$\varrho\left(x_{k_n}, r_{k_n}(x_{k_n})\right) < \frac{1}{k_n}$$

and hence, in view of (6),

(8) $$\lim_{n=\infty} r_{k_n}(x_{k_n}) = x.$$

By continuity of f, we have from (8)

$$\lim_{n=\infty} f(r_{k_n}(x_{k_n})) = f(x). \tag{9}$$

In view of (5) and (2), $x_{k_n} = G_{k_n}(x_{k_n}) = f(r_{k_n}(x_{k_n}))$ and, therefore, in view of (6), we have

$$\lim_{n=\infty} f(r_{k_n}(x_{k_n})) = x. \tag{10}$$

Comparing (9) and (10), we conclude that $x = f(x)$.

The proof of the Theorem is completed.

As an immediate consequence, we obtain the following:

COROLLARY. *Acyclic compact approximative* ANR-*s, and in particular approximative* AR-*s, have the fixed point property.*

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MATHEMATICS
(TOPOLOGY)

Some Theorems in Fixed Point Theory. The Leray—Schauder Index and the Lefschetz Number

by

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1. Introduction. In the first part of this note, an extension to the infinite dimensional case of the recent fixed point theory of A. Dold [6] is given. This leads to a version of the classical Leray—Schauder theory [10], which is suitable for establishing a relation between the Lefschetz number and the fixed point index of a compact map.

Let U be an open subset of a normed (or more generally locally convex) linear space E and let $f: U \to E$ be a compact map with a compact set of fixed points. To every such map f, we assign an integer Ind (f), called the Leray—Schauder index of f, which satisfies (as in [6]) all the naturally expected properties (Theorem 1); in particular, when $f: U \to U$, it is equal to the Lefschetz number $\Lambda(f)$ of f.

In the second part of this note we give some applications to the fixed point theory of the ANR spaces.

Let X be a space which is r-dominated by an open set U in E. Let $r: U \to X$ and $s: X \to U$ be the corresponding pair of maps with $rs = 1_X$. Let $g: X \to X$ be a compact map; then $sgr: U \to U$ is also compact and $\Lambda(g) = \Lambda(sgr) = \text{Ind}(sgr)$. Consequently, the Lefschetz Fixed Point Theorem holds for such a space X: if $\Lambda(g) \neq 0$, then g has a fixed point. Since every ANR for metric spaces is up to homeomorphism a neighbourhood retract of a normed space, this includes a fixed point theorem for ANR-s proved in [7].

Finally we mention that the theory of the Leray—Schauder index extends simply to the fixed point index theory for compact ANR-s. Such a theory was established previously by combinatorial means (and in a different form) by several authors [8], [5], [3], [4]

2. The Lefschetz number. In what follows we shall use in full generality the generalized theory of the trace as given by J. Leray in [9]. We shall consider vector spaces only over the field of rational numbers.

Let $\varphi : E \to E$ be an endomorphism of an arbitrary vector space E. Let us put

$$N(\varphi) = \bigcup_{n \geq 1} \ker \varphi^{(n)}, \quad \tilde{E} = E/N(\varphi)$$

and denote by $\tilde{\varphi} : \tilde{E} \to \tilde{E}$ the endomorphism induced by φ.

Assume that $\dim \tilde{E} < +\infty$. Following J. Leray, we define the *generalized trace* $\mathrm{Tr}(\varphi)$ of φ by putting $\mathrm{Tr}(\varphi) = \mathrm{tr}(\tilde{\varphi})$, where $\mathrm{tr}(\tilde{\varphi})$ is the ordinary trace of $\tilde{\varphi}$.

Let $\varphi = \{\varphi_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. Call φ a *Leray endomorphism* provided (i) $\dim \tilde{E}_q < \infty$ for all q (ii) $\tilde{E}_q = 0$ for almost all q.

For such φ we define the (generalized) *Lefschetz number* $\Lambda(\varphi)$ by putting

$$\Lambda(\varphi) = \sum_q (-1)^q \,\mathrm{Tr}(\varphi_q).$$

3. Lefschetz maps. Let H be the singular homology functor (with the rational coefficients) from the category of topological spaces and continuous mappings to the category of graded vector spaces and linear maps of degree 0. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q-dimensional singular homology group of X. For a continuous mapping $f : X \to Y$, $H(f)$ is the induced linear map $f_* = \{f_q\}$, where $f_q : H_q(X) \to H_q(Y)$.

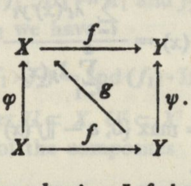
A continuous map $f : X \to X$ is called a *Lefschetz map* provided $f_* : H(X) \to H(X)$ is a Leray endomorphism. For such f we define the Lefschetz number $\Lambda(f)$ of f by putting

$$\Lambda(f) = \Lambda(f_*).$$

Clearly, if $f \sim g$, then $\Lambda(f) = \Lambda(g)$.

We shall make use of the following lemma:

(3.1). *Assume that in the category of topological spaces the following diagram commutes*



Then (a) *if one of the maps* φ *and* ψ *is a Lefschetz map, then so is the other and in that case* $\Lambda(\varphi) = \Lambda(\psi)$, (b) φ *has a fixed point if and only if* ψ *does.*

Proof. The first part follows (by applying the homology functor to the above diagram) from the corresponding property of the Leray endomorphisms [9]. The second part is obvious.

The following are the two instances in which the above lemma is used:

Example 1. Let $f: X \to X$ be a map such that $f(X) \subset K \subset X$. Then we have the commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\ \subset\ } & X \\ \uparrow{\scriptstyle f_K} & \nwarrow{\scriptstyle f} & \uparrow{\scriptstyle f} \\ K & \xrightarrow{\ \subset\ } & X \end{array}$$

with the obvious contractions.*)

Example 2. Let $r: Y \to X$, $s: X \to Y$ be a pair of continuous mappings such that $rs = 1_X$. In this case X is said to be r-dominated by Y and r is said to be an r-map. In this situation, given a map $\varphi: X \to X$, we have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\ s\ } & Y \\ \uparrow{\scriptstyle \varphi} & \nwarrow{\scriptstyle \varphi r} & \uparrow{\scriptstyle \psi} \\ X & \xrightarrow{\ s\ } & Y \end{array}$$

with $\psi = s\varphi r$.

4. Compact maps. A continuous map $f: X \to Y$ between topological spaces is called *compact* provided it maps X into a compact subset of Y. Let $h_t: X \to Y$ be a homotopy and $h: X \times I \to Y$ be defined by $h(x, t) = h_t(x)$ for $(x, t) \in X \times I$; then h_t is said to be a *compact homotopy* provided the map h is compact.

We shall make use of the following Approximation Theorem:

(4.1) *Let U be an open subset of a normed space E and let $f: X \to U$ be a compact mapping. Then for every $\varepsilon > 0$ there exists a finite polyhedron $K_\varepsilon \subset U$ and a mapping $f_\varepsilon: X \to U$, called an ε-approximation of f, such that*

(i) $\|f(x) - f_\varepsilon(x)\| < \varepsilon$ *for all* $x \in X$;

(ii) $f_\varepsilon(X) \subset K_\varepsilon$

(iii) f_ε *is homotopic to* f.

Proof. Given $\varepsilon > 0$ (which we may assume to be sufficiently small), $f(X)$ is contained in the union of a finite number of open balls $V(y_i, \varepsilon) \subset U$ $(i = 1, 2, \ldots, k)$. Putting for $x \in X$,

$$f_\varepsilon(x) = \frac{\sum_{i=1}^{k} \lambda_i(x) y_i}{\sum_{j=1}^{k} \lambda_j(x)},$$

where

$$\lambda_i(x) = \max\{0, \varepsilon - \|f(x) - y_i\|,$$

we obtain the required map f_ε.

From (4.1) and Lemma (3.1) (see Example 1) we obtain the following proposition:

(4.2) *Let $f: U \to U$ be a compact mapping of an open subset U of a normed space. Then f is a Lefschetz map.*

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*) Let $f: X \to Y$ be a map such that $f(A) \subset B$, where $A \subset X$ and $B \subset Y$. By the *contraction* of f to the pair (A, B), we understand a map $f': A \to B$ with the same values as f. A contraction of f to the pair (A, Y) is simply the restriction $f|A$ of f to A.

Remark. (4.1) and consequently (4.2) remain valid for a locally convex space E.

6. Fixed Point Index in R^n. Let $f: U \to X$ be a continuous map between topological spaces. Call f *admissible* provided U is an open subset of X and the fixed point set of f

$$\varkappa_f = \{x \in U, f(x) = x\} \subset U$$

is compact. A homotopy $h_t : U \to X$ will be called admissible provided the set $\bigcup_{0 \leqslant t \leqslant 1} \varkappa(h_t)$ is compact.

Let $\mathfrak{C}$ be a category of topological spaces in which a class $\mathfrak{A}$ of admissible maps and homotopies is distinguished. By a *fixed-point index on* $\mathfrak{C}$ we shall understand an integer-valued function $f \to \operatorname{Ind}(f)$ defined for all admissible maps $f \in \mathfrak{A}$, which satisfies the following conditions:

I (Excision). If $U' \subset U$ and $\varkappa_f \subset U'$, then for the restriction $f' = f | U' : U' \to X$ we have

$$\operatorname{Ind}(f) = \operatorname{Ind}(f').$$

II (Normalization). If f is a constant map, $f(x) = p$ for all $x \in U$, then

$$\operatorname{Ind} f = \begin{cases} 0 & \text{if} \quad p \notin U, \\ 1 & \text{if} \quad p \in U. \end{cases}$$

III (Fixed-points). If $\operatorname{Ind} f \neq 0$, then $\varkappa_f \neq 0$, i.e., the map f has a fixed point.

IV (Additivity). Assume that $U = \bigcup_{i=1}^{k} U_i$, $f_i = f | U_i$ and the fixed point sets $\varkappa_i = \varkappa_f \cap U_i$ are mutually disjoint, $\varkappa_i \cap \varkappa_j = \emptyset$ for $i \neq j$. Then

$$\operatorname{Ind} f = \sum_{i=1}^{k} \operatorname{Ind} f_i$$

V (Homotopy). Let $h_t : U \to X$, $0 \leqslant t \leqslant 1$, be an admissible homotopy. Then $\operatorname{Ind}(h_0) = \operatorname{Ind}(h_1)$.

VI (Multiplicativity). If $f_1 : U_1 \to X_1$ and $f_2 : U_2 \to X_2$ then for the product map $f_1 \times f_2 : U_1 \times U_2 \to X_1 \times X_2$ we have

$$\operatorname{Ind}(f_1 \times f_2) = \operatorname{Ind}(f_1) \cdot \operatorname{Ind}(f_2).$$

VII (Commutativity). Let $U \subset X$, $U' \subset X'$ be open and assume $f: U \to X'$, $g : U' \to X$ are maps. If one of the composites

$$gf : V = f^{-1}(U') \to X, \quad fg : V' = g^{-1}(U) \to X'$$

is admissible, then so is the other and we have in that case

$$\operatorname{Ind}(gf) = \operatorname{Ind}(fg).$$

VIII (Relation with the Lefschetz number). If $U = X$ and $f: X \to X$ is compact, then f is a Lefschetz map and $\operatorname{Ind}(f) = \Lambda(f)$.

We have the following theorem proved by A. Dold in [6]:

(6.1) (Fixed Point Index in R^n). *Let $\mathfrak{C}$ be the category of open subsets of Euclidean spaces and $\mathfrak{A}$ be the class of all continuous admissible maps in $\mathfrak{C}$. Then there is on $\mathfrak{A}$ a fixed point index function $f \to \mathrm{Ind}(f)$, which satisfies the properties* I—VIII.

We note that the excision and the commutativity implies the following property of the index:

IX (Contraction). Let U be open in R^{n+1} and $f: U \to R^{n+1}$ be an admissible map such that $f(U) \subset R^n$. Denote by $\tilde{f}: \tilde{U} \to R^n$ the contraction of f, where $\tilde{U} = U \cap R^n$. Then $\mathrm{Ind}(f) = \mathrm{Ind}(\tilde{f})$.

7. The Leray-Schauder Index. Let U be an open subset of a normed space E and let $f: U \to E$ be an admissible compact map. Take an open set $V \subset U$ such that $\varkappa_f \subset V$. Then the number $\varepsilon = \inf\{\|x - f(x)\| \text{ for } x \in \mathrm{Fr}(V)\}$ is positive.

Let $g = f|V: V \to E$. From the definition of ε, it follows that:

(i) every ε-approximation $g_\varepsilon: V \to E$ of g is admissible;

(ii) given two ε-approximations $g'_\varepsilon, g''_\varepsilon: V \to E$ of g, there exists an admissible finite dimensional compact homotopy $h_t: V \to E$, $0 \leqslant t \leqslant 1$, such that $h_0 = g'_\varepsilon$, $h_1 = g''_\varepsilon$.

DEFINITION. Let $f: U \to E$ be an admissible compact map and $g_\varepsilon: V \to E$ be an ε-approximation of $g = f|V$ as above. Denote by E^n a finite dimensional subspace of E such that $g_\varepsilon(V) \subset E^n$ and let $g'_\varepsilon: V_n \to E^n$, where $V_n = V \cap E^n$, be the contraction of g_ε. Using (6.1), we define the Leray—Schauder index of f by putting $\mathrm{Ind}(f) = \mathrm{Ind}(g'_\varepsilon)$.

It follows from (i), (ii), and the properties I, V and IX of the index in R^n, that $\mathrm{Ind}(f)$ is well defined.

THEOREM 1. *Let $\mathfrak{C}$ be the category of open subsets in linear normed spaces and let $\mathfrak{A}$ be the class of all admissible compact maps in $\mathfrak{C}$. Assume that all admissible homotopies are compact. Then, defined on $\mathfrak{A}$, the Leray—Schauder index $f \to \mathrm{Ind}(f)$ satisfies the properties* I—VIII. *In* VII *it is assumed that both f and g are compact.*

Proof. Properties I—VI follow from the corresponding properties of the index in R^n and (4.1). These once proved, property VII follows (similarly as in [6]) from I, V and VI.

Proof of property VIII. Given a compact map $f: U \to U$ let $f_\varepsilon: U \to U$ be an ε-approximation of f such that its values are in some finite dimensional subspace E^n of E and $U_n = U \cap E^n$.

Consider the following commutative diagram in which all the arrows represent either the obvious inclusions or the contractions of the map f.

By definition $\operatorname{Ind}(f) = \operatorname{Ind}(f'_*)$. From Lemma (3.1) (Example 1), we have $\Lambda(f'_*) = \Lambda(f_*)$ and, consequently, in view of (6.1), (property VIII), $\operatorname{Ind}(f) = \Lambda(f_*)$. Since f is homotopic to f_*, this implies that $\operatorname{Ind}(f) = \Lambda(f)$.

8. Compact maps of ANR-s. Denote by ANR (respectively AR) the class of metrizable absolute neighbourhood retracts (resp. absolute retracts).

The following simple characterization of the ANR-s permits to establish a link between the fixed point theory for such spaces and the Leray—Schauder index:

(8.1) *In order that $X \in$ ANR (respectively $X \in$ AR) it is necessary and sufficient that X be r-dominated by an open subset of a normed space (resp. by a normed space).*

Proof. Let $X \in$ ANR. By the theorem of Arens—Eells [1] there exists an embedding $\varphi : X \to E$ of X into a normed space E such that (X) is closed in E. Take a retraction $r : U \to \varphi(X)$ of an open set $U \supset \varphi(X)$. Then $\varphi^{-1} r : U \to X$ is an r-map. The converse follows from the general properties of the ANR-s [2]. The proof of the second part is similar.

THEOREM 2. *Let X be an* ANR *and $f : X \to X$ be a compact mapping. Assume further that U is open in a normed space E and $s : X \to U$, $r : U \to X$ is an arbitrary pair of maps such that $rs = 1_X$. Then we have* (i) *f is a Lefschetz map,* (ii) *the Lefschetz number of f is equal to the Leray—Schauder index of the map sfr, $\Lambda(f) = \operatorname{Ind}(sfr)$,* (iii) *if $\Lambda(f) \neq 0$, then the map f has a fixed point.*

Proof. (i) follows from Lemma (3.1) (Example 2) and (4.2); (ii) follows from the same Lemma and Theorem 1 (Property VIII); (iii) is a consequence of (ii), Theorem 1 (Property III) and again of the same Lemma (3.1).

Remark 1. The theory of the Leray—Schauder index (Theorem 1) extends to locally convex spaces. It follows that Theorem 2 remains valid, when X is an arbitrary space, which is r-dominated by a set U open in a locally convex space.

9. Fixed Point Index for compact ANR-s. The Leray—Schauder index permits to define a fixed point index in the category of compact ANR-s. Let X be a compact metric ANR and $f : U \to X$ be an admissible map. In order to define a fixed point index $\operatorname{Ind}(f)$ we take an open set V in a normed space E which r-dominates X. Let $s : X \to V$, $r : V \to X$ be a pair of maps such that $rs = 1$. Since the composite map

$$r^{-1}(U) \xrightarrow{r} U \xrightarrow{f} X \xrightarrow{s} V$$

is compact, its index is defined by Theorem 1, and we put

$$(*) \qquad \operatorname{Ind}(f) \underset{\mathrm{df}}{=} \operatorname{Ind}(sfr).$$

The excision and commutativity of the Leray—Schauder index imply that this definition is independent of the choices involved.

THEOREM 3. *Let $\mathfrak{C}$ be the category of compact metric* ANR-*s and $\mathfrak{A}$ be the class of all continuous admissible maps in $\mathfrak{C}$. Then the fixed-point index function $f \to$* Ind (f) *defined by* (*) *satisfies the properties* I—VIII.

Remarks. 1. The definition of the index and Theorem 3 remain valid for non-metrizable compact ANR-s (see the previous remarks).

2. Since in the definition of the fixed point index of f, only of importance is the behaviour of f in the neighbourhood of $\varkappa_f$. Theorem 3 remains valid when $\mathfrak{C}$ is the category of locally compact ANR-s.

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FIXED POINT THEOREMS FOR MULTI-VALUED MAPPINGS OF THE ABSOLUTE NEIGHBOURHOOD RETRACTS

By Lech **GÓRNIEWICZ** and Andrzej **GRANAS**.

1. Introduction. — In 1946, S. Eilenberg and D. Montgomery [4] made the important observation that, using an old theorem of L. Vietoris [21] as a tool, several results of the fixed-point theory for single-valued mappings could be carried over to the case of multi-valued acyclic maps, i. e., maps for which the image of every point is an acyclic compact set. Thus, the Lefschetz Fixed Point Theorem for compact ANR-*s* was extended by the above-named authors to arbitrary acyclic maps; some years later, similar generalizations of other topological theorems were given in [13], [14] and [10]. The important type of acyclic maps consists of those which are convex-valued. To this special type of maps various fixed-point theorems for compact operators were extended ([1], [6], [7]), as well as the basic facts of the Leray-Schauder theory in Banach spaces ([8], [11]). As in the single-valued case, fixed-point theorems for multi-valued mappings prove themselves useful in many branches of mathematics; they found, for instance, applications in the theory of games ([1], [6]) and more recently also in ordinary differential equations [16] and optimal control theory [17].

In the present note, we are concerned with the fixed-point theorems for multi-valued maps of non-compact spaces and our main result may be viewed as a generalization and application of the above Eilenberg-Montgomery theorem. Let X be a topologically complete absolute neighbourhood retract and let $\varphi : X \to X$ be a point-to-set transformation which is *continuous*, *acyclic* and *compact*. With the aid of the Vietoris Mapping Theorem (for homology with compact carriers) we define the induced homomorphism φ_* and then, in terms of the Leray trace [18], the generalized Lefschetz number $\Lambda(\varphi)$ of φ. Now, our principal theorem states : *If* $\Lambda(\varphi) \neq 0$, *then* φ *has a fixed point*, i. e., $x \in \varphi(x)$ *for some point* x *in* X.

The main line of our reasoning and several details have points in common with ([4], [3], [9], [10], [14]). The results presented in this note include several known fixed-point theorems both for single-valued and multi-valued maps ([4], [1], [3], [12], [14]). We remark that, for a single-valued map, the main theorem is valid without assuming X to be topologically complete [9]; the question whether the same can be proved for multi-valued maps remains open.

2. THE TRACE. — In what follows we shall use the generalized notion of the trace and the Lefschetz number as given by J. Leray in [18]. We shall consider vector spaces only over the field of rational numbers Q.

A graded vector space $E = \{E_q\}$ is of a *finite type* provided : (i) $\dim E_q < +\infty$ for all q, and (ii) $E_q = 0$ for almost all q. If $f = \{f_q\}$ is an endomorphism of such a space, then the Lefschetz number $\lambda(f)$ of f is given by

$$\lambda(f) = \sum_q (-1)^q \operatorname{tr}(f_q).$$

where tr stands for the ordinary trace function.

Let $f : E \to E$ be an endomorphism of an arbitrary vector space E. Let us put

$$N(f) = \bigcup_{n=1}^{\infty} \ker f^{(n)}, \qquad \tilde{E} = \frac{E}{N(f)},$$

where $f^{(n)}$ is the n-th iterate of f. Since $f(N(f)) \subset N(f)$, we have the induced endomorphism $\tilde{f} : \tilde{E} \to \tilde{E}$.

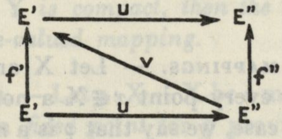
Assume that $\dim \tilde{E} < +\infty$; in this case, we define the *generalized trace* $\mathrm{Tr}(f)$ of f by putting $\mathrm{Tr}(f) = \mathrm{tr}(\tilde{f})$, where $\mathrm{tr}(\tilde{f})$ is the ordinary trace of $\tilde{f}$.

Now let $f = \{f_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. Call f the *Leray endomorphism* provided the graded space $\tilde{E} = \{\tilde{E}_q\}$ is of a finite type. For such an f we define the (generalized) *Lefschetz number* $\Lambda(f)$ by putting

$$\Lambda(f) = \sum_q (-1)^q \mathrm{Tr}(f_q).$$

The following important property of the Leray endomorphisms [18] is a consequence of the well-known formula $\mathrm{tr}(uv) = \mathrm{tr}(vu)$ for the ordinary trace :

(2.1) *Assume that in the category of graded vector spaces the following diagram commutes*



Then, if one of the linear maps f' or f'' is a Leray endomorphism, then so is the other and in that case $\Lambda(f') = \Lambda(f'')$.

3. Vietoris mappings. — In what follows only metrizable spaces will be considered. The category of such spaces and continuous mappings will be denoted by $\mathfrak{C}$.

By H we denote the Čech homology functor with compact carriers [5] and coefficients in Q from the category $\mathfrak{C}$ to the category $\mathfrak{A}$ of graded vector spaces and linear maps of degree zero. Thus, for a space X

$$H(X) = \{H_q(X)\}$$

is a graded vector space and, for a continuous mapping $f : X \to Y$, $H(f)$ is the induced linear map

$$f_* = \{f_q\} : H(X) \to H(Y),$$

where $f_q : H_q(X) \to H_q(Y)$.

A space X is *acyclic* provided : (i) X is non-empty; (ii) $H_q(X) = 0$ for all $q \geq 1$ and (iii) $H_0(X) \approx Q$.

(3.1) DEFINITION. — A continuous mapping $f: X \to Y$ is said to be a *Vietoris map* provided the following two conditions are satisfied :

(i) f is proper, i. e., for any compact C, the counter image $f^{-1}(C)$ is also compact;

(ii) the set $f^{-1}(y)$ is acyclic for every $y \in Y$.

In our subsequent considerations an essential use will be made of the following :

(3.2) *Vietoris Mapping Theorem. — If $f: X \to Y$ is a Vietoris map, then the induced map $f_*: H(X) \to H(Y)$ is invertible.*

Theorem (3.2) clearly follows from the original statement of the Vietoris Mapping Theorem for compacta (e. g. [21] and also [20]).

4. MULTI-VALUED MAPPINGS. — Let X and Y be two spaces and assume that for every point $x \in X$ a non-empty subset $\varphi(x)$ of Y is given; in this case, we say that φ is a *multi-valued mapping* from X to Y and we write $\varphi: X \multimap Y$. In what follows, the symbols φ, ψ, χ will be reserved for multi-valued mappings; the single-valued maps will be denoted by f, g, h, ...

Let $\varphi: X \multimap Y$ be a multi-valued map. We associate with φ the following diagram of continuous mappings

$$X \xleftarrow{p} \Gamma_\varphi \xrightarrow{q} Y$$

in which

$$\Gamma_\varphi = \{ (x, y) \in X \times Y, y \in \varphi(x) \}$$

is the *graph* of φ and the *natural projections* p and q *are given by*

$$p(x, y) = x \quad \text{and} \quad q(x, y) = y.$$

The point-to-set mapping φ extends to a set-to-set mapping by putting $\varphi(A) = \bigcup_{a \in A} \varphi(a) \subset Y$ for $A \subset X$; $\varphi(A)$ is said to be the *image* of A under φ. If $\varphi(A) \subset B \subset Y$, then the *contraction* of φ to the pair (A, B) is the multi-valued map $\varphi' : A \to B$ defined by $\varphi'(a) = \varphi(a)$ for each $a \in A$. A contraction of φ to the pair (A, Y) is simply the *restriction* $\varphi | A$ of φ to A.

(4.1) Definition. — A multi-valued mapping $\varphi : X \to Y$ is said to be *continuous* provided the graph Γ_φ of φ is closed in the product $X \times Y$; in other words, the conditions $x_n \to x$, $y_n \to y$, $y_n \in \varphi(x_n)$ imply $y \in \varphi(x)$.

We note that if $\varphi = f$ (i. e., φ is single-valued), then the above definition gives the ordinary continuity of f. In what follows only continuous multi-valued mappings will be considered.

(4.2) Definition. — A multi-valued mapping $\varphi : X \to Y$ is called *compact* provided the image $\varphi(X)$ of X under φ is contained in a compact subset of Y.

The following evident remark is of importance :

(4.3) *If* $\varphi : X \to Y$ *is compact, then the projection* $p : \Gamma_\varphi \to X$ *is proper as a single-valued mapping.*

(4.4) Definition. — Let $\varphi : X \to X$ be a multi-valued mapping. A point x is called a *fixed point* for φ provided $x \in \varphi(x)$.

5. Acyclic maps. — In this section, we recall the statement of the Eilenberg-Montgomery Theorem.

(5.1) Definition. — Let X and Y be two spaces. A multi-valued mapping $\varphi : X \to Y$ is said to be *acyclic* provided the set $\varphi(x)$ is acyclic for every point $x \in X$.

Assume that X and Y are compacta and $\varphi : X \to Y$ is an acyclic multi-valued mapping. We observe, that, since for every $x \in X$, $p^{-1}(x)$ is homeomorphic to $\varphi(x)$, the projection $p : \Gamma_\varphi \to X$ is a Vietoris map.

Using the Vietoris Mapping Theorem we define the linear map

$$\varphi_* : H(X) \to H(Y)$$

by putting

$$\varphi_* = q_* \circ p_*^{-1} ;$$

φ_* is said to be induced by the multi-valued mapping φ. It is easily seen that if $\varphi = f$ (i. e., φ is single-valued), then $\varphi_* = f_*$.

A compact space X is said to be of a *finite type* provided the graded space H(X) is of a finite type. We note that every compact ANR is of a finite type.

Let X be of a compact space of a finite type and $\varphi : X \to X$ be an acyclic multi-valued mapping of X into itself. We define the Lefschetz number $\lambda(\varphi)$ of φ by putting

$$\lambda(\varphi) = \lambda(\varphi_*).$$

(5.2) (Eilenberg-Montgomery Theorem). — *Let* X *be a compact* ANR *and* $\varphi : X \to X$ *an acyclic multi-valued mapping. Then* $\lambda(\varphi) \neq 0$ *implies that* φ *has a fixed point.*

6. The quasi-category $\tilde{\mathfrak{C}}$. — In the rest of this note, the symbol $f : X \Rightarrow Y$ will mean that either : (i) f is a Vietoris map or (ii) f is a homeomorphism; we remark that in either case, the induced map f_* is invertible.

(6.1) Definition. — A multi-valued mapping $\varphi : X \to Y$ is said to be *admissible* provided either : (i) φ is single-valued or (ii) φ is acyclic and compact. The class of all admissible maps will be denoted by $\tilde{\mathfrak{C}}$.

(6.2) *If a multi-valued mapping* $\varphi : X \to Y$ *is admissible, then the diagram of natural projections for* φ *has the form*

$$X \overset{p}{\Longleftarrow} \Gamma_\varphi \overset{q}{\longrightarrow} Y.$$

Proof. — If $\varphi = f$, the assertion is evident; if φ is acyclic and compact, our assertion is a consequence of (4.3) and the fact that $p^{-1}(x)$ is homeomorphic to $\varphi(x)$ for every $x \in X$.

(6.3) Definition. —. Two admissible mappings $\varphi : X \to Y$ and $\psi : Y \to Z$ are called *composable* provided either : (i) φ is single-valued or (ii) ψ is the inclusion; in either case, the composite $\psi \circ \varphi : X \to Z$ given by the assignment $x \to \psi(\varphi(x))$ is an admissible mapping from X to Z.

We note that, if one of the composites $(\varphi_3\varphi_2)\varphi_1$ or $\varphi_3(\varphi_2\varphi_1)$ is defined, then so is the other and, in that case $\varphi_3(\varphi_2\varphi_1) = (\varphi_3\varphi_2)\varphi_1$. It is not, however, true that the existence of both $\varphi_3\varphi_2$ and $\varphi_2\varphi_1$ implies that of $\varphi_3\varphi_2\varphi_1$.

Thus, $\tilde{\mathfrak{C}}$ is equipped with a partially defined operation of composition of maps. Next we show that the homology functor $H : \mathfrak{C} \to \mathfrak{A}$ can be extended over $\tilde{\mathfrak{C}}$ to a function $\tilde{H} : \tilde{\mathfrak{C}} \to \mathfrak{A}$ satisfying certain quasi-functorial properties; these turn out to be sufficient for the proofs of our main results.

(6.4) Definition. — Let $\varphi : X \to Y$ be an admissible map. Using (6.2) we define the linear map

$$\tilde{H}(\varphi) = \varphi_* : H(X) \to H(Y)$$

as the composite

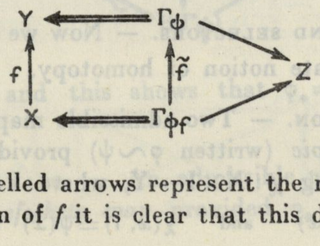
$$H(X) \xrightarrow{(p_*)^{-1}} H(\Gamma_\varphi) \xrightarrow{q_*} H(Y);$$

φ_* is said to be induced by φ; clearly, if $\varphi = f$, then $\varphi_* = f_*$.

(6.5) Theorem. — *Let* $\varphi : X \to Y$ *and* $\psi : Y \to Z$ *be two composable maps in* $\tilde{\mathfrak{C}}$. *Then we have* $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$; *in other words,* $\tilde{H}$ *sends commutaive triangles in* $\tilde{\mathfrak{C}}$ *into commutative triangles in* $\mathfrak{A}$.

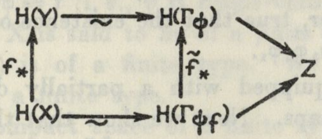
Proof. — (i) *The case* $\varphi = f$: Assume first that φ is single-valued and let $\varphi = f$. Then the product mapping $f \times \mathrm{id} : X \times Z \to Y \times Z$ maps $\Gamma_{\psi f} \subset X \times Z$ into $\Gamma_\psi \subset Y \times Z$ and therefore determines the map $\tilde{f} : \Gamma_{\psi f} \to \Gamma_\psi$.

Consider the following diagram :



in which all unlabelled arrows represent the natural projections. From the definition of f it is clear that this diagram commutes;

consequently, the diagram

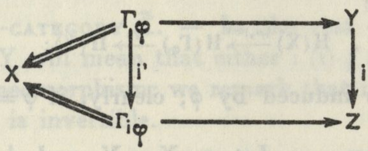


also commutes and this shows that $(\psi\circ f)_* = \psi_* \circ f_*$.

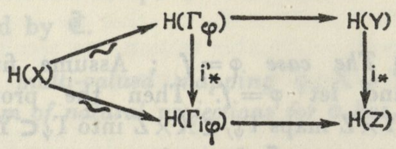
(ii) *The case* $\psi = i$: Now we assume that $\psi = i$ is the inclusion. In this case, we observe that the product map

$$\mathrm{id}\times i : X\times Y \to X\times Z$$

maps Γ_φ into $\Gamma_{i\varphi}$, and therefore determines the map $i' : \Gamma_\varphi \to \Gamma_{i\varphi}$. Consider now the diagram



in which unlabelled arrows stand for the natural projections. Clearly, this diagram commutes and, consequently, the diagram



also commutes. This shows that $(i\circ\varphi)_* = i_*\circ\varphi_*$. The proof is completed.

7. Homotopy and selectors. — Now we introduce for maps in $\tilde{\mathfrak{C}}$ an appropriate notion of homotopy.

(7.1) Definition. — Two admissible mappings $\varphi, \psi : X \to Y$ are called *homotopic* (written $\varphi \sim \psi$) provided there exists an admissible mapping $\chi : X\times I \to Y$, where $I = [0, 1]$, such that

$$\chi(x, 0) = \varphi(x) \quad \text{and} \quad \chi(x, 1) = \psi(x) \quad \text{for each } x\in X.$$

(7.2) **Theorem.** — *Let* $\varphi, \psi : X \to Y$ *be two admissible mappings. Then* $\varphi \sim \psi$ *implies* $\varphi_* = \psi_*$.

Proof. — Let $i_0, i_1 : X \to X \times I$ be two embeddings given by $x \to (x, 0)$ and $x \to (x, 1)$ respectively, and $\chi : X \times I \to Y$ be an admissible homotopy joining φ and ψ. Then

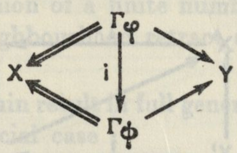
$$\varphi = \chi \circ i_0 \quad \text{and} \quad \psi = \chi \circ i_1.$$

From this, taking into account that $(i_0)_* = (i_1)_*$, we infer by theorem (6.5) that $\varphi_* = \psi_*$ and thus the proof is completed.

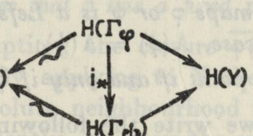
(7.3) **Definition.** — Let $\varphi, \psi : X \to Y$ be two multi-valued mappings such that $\Gamma_\varphi \subset \Gamma_\psi$, i. e., $\varphi(x) \subset \psi(x)$ for each $x \in X$; in this case, we say that φ is a *selector* of ψ and indicate this by writing $\varphi \subset \psi$.

(7.4) **Theorem.** — *Let* $\varphi, \psi : X \to Y$ *be two admissible mappings. Then* $\varphi \subset \psi$ *implies* $\varphi_* = \psi_*$.

Proof. — Assume that $\varphi \subset \psi$ and denote by $i : \Gamma_\varphi \to \Gamma_\psi$ the inclusion. Then it is easily seen that the diagram



with natural projections is commutative; consequently, its image under H



also commutes and this shows that $\varphi_* = \psi_*$. The proof is completed.

8. **Lefschetz maps.** — An admissible mapping $\varphi : X \to X$ is said to be a *Lefschetz map* provided $\varphi_* : H(X) \to H(X)$ is a

Leray endomorphism. Fur such φ we define the *Lefschetz number* $\Lambda(\varphi)$ of φ by putting $\Lambda(\varphi) = \Lambda(\varphi_*)$.

Note that if X is a compactum of a finite type, then any admissible $\varphi : X \to X$ is a Lefschetz map and $\Lambda(\varphi)$ coincides with the ordinary Lefschetz number $\lambda(\varphi)$ of φ.

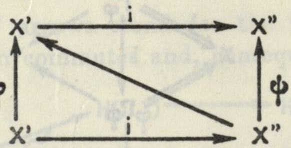
The following two theorems are immediate consequences of (7.2) and (7.4).

(8.1) Theorem. — *Let* $\varphi, \psi : X \to X$ *be two homotopic admissible maps. If* φ *is a Lefschetz map, then so is* ψ *and in this case* $\Lambda(\varphi) = \Lambda(\psi)$.

(8.2) Theorem. — *Let* $\varphi, \psi : X \to X$ *be two admissible maps such that* $\varphi \subset \psi$. *If one of them is a Lefschetz map, then so is the other and, in that case,* $\Lambda(\varphi) = \Lambda(\psi)$.

We turn now to the property of the Lefschetz maps which will be of importance in the proof of the main theorem.

(8.3) Lemma. — *Assume that we are given the following commutative diagram of spaces and admissible multi-valued maps*

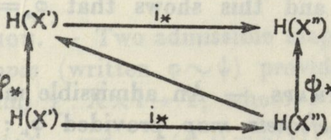


in which $i : X' \to X''$ *stands for the inclusion. Then*

(i) *if one of the maps* φ *or* ψ *is a Lefschetz map, then so is the other and, in that case,* $\Lambda(\varphi) = \Lambda(\psi)$;

(ii) φ *has a fixed point if and only if* ψ *does.*

Proof. — First we write the following diagram in the category of graded vector spaces



Then, taking into account the assumptions, definition (6.3) and theorem (6.5), we conclude that this diagram commutes. Consequently, by applying Lemma (2.1), our first assertion follows. The second assertion is evident.

9. The Main Theorem. — The proof of the main result of this note relies essentially on the following simple geometrical fact (*cf.* [3]) :

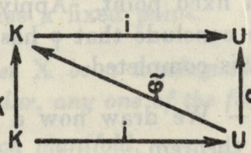
(9.1) Lemma. — *If* U *is open in a Banach space* E *and* $X \subset U$ *is compact, then there exists a compact absolute neighbourhood retract* K *such that* $X \subset K \subset U$.

Proof. — Cover X by a finite number of closed balls W_1, $W_2, \ldots, W_l \subset U$, and denote by K_i the convex closure of the compact set $X \cap W_i$. By the Mazur Lemma [19], every K_i is compact. From the inclusions $K_i \subset W_i \subset U$ we conclude that X is contained in the compact set $K = \bigcup_{i=1}^{l} K_i \subset U$. Now, taking into account the general properties of the ANR spaces [2], we infer that K as the union of a finite number of compact convex sets is an absolute neighbourhood retract and thus our assertion follows.

Before stating our main result in full generality, we shall consider first the following special case :

Theorem 1. — *Let* U *be open in a Banach space* E *and* $\varphi : U \to U$ *be an acyclic compact map. Then :* (i) φ *is a Lefschetz map,* *and* (ii) $\Lambda(\varphi) \neq 0$ *implies that* φ *has a fixed point.*

Proof. — By assumption, the closure $\overline{\varphi(U)} = X$ is compact and contained in U. By applying to X the preceeding lemma, we find a compact absolute neighbourhood retract K such that $\varphi(U) \subset K \subset U$. Consequently, we have the commutative diagram as in Lemma (8.3),



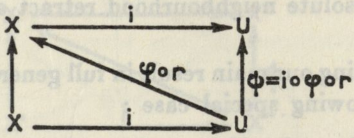
in which i is the inclusion, and φ_K, $\tilde{\varphi}$ stand for the obvious contractions of the map φ. Since K is a compact ANR, $\Lambda(\varphi_K)$ is defined; consequently, by Lemma (8.3), φ is a Lefschetz map and $\Lambda(\varphi_K) = \Lambda(\varphi)$.

To prove (ii) assume that $\Lambda(\varphi) \neq 0$. Then we have also $\Lambda(\varphi_K) \neq 0$ and, hence, by the Eilenberg-Montgomery theorem, there exists a point $x \in K$ such that $x \in \varphi_K(x) = \varphi(x)$. The proof is completed.

Now we are able to state the principal result of this note :

Theorem 2. — *Let* X *be a topologically complete* ANR *and* $\varphi : X \to X$ *be a compact acyclic multi-valued map. Then :* (i) φ *is a Lefschetz map and* (ii) $\Lambda(\varphi) \neq 0$ *implies that* φ *has a fixed point.*

Proof. — Since, by a theorem of Kuratowski [15], a metrizable space is embeddable into a Banach space, we may assume, by changing a metric if necessary, that X is a closed subset of a Banach space E. By assumption, there is a retraction $r : U \to X$ of an open set $U \subset E$ onto X. Denoting by $i : X \to U$ the inclusion we have the commutative diagram



as in Lemma (8.3). By assumption, the multi-valued map φ is compact; consequently, so is the map $\psi = i \circ \varphi \circ r$. Theorem 1 implies now that ψ is a Lefschetz map. Applying Lemma (8.3), we conclude that φ is also a Lefschetz map. Thus, the assertion (i) is proved.

To prove (ii), assume that $\Lambda(\varphi) \neq 0$. Applying Lemma (8.3) again, we have $\Lambda(\varphi) = \Lambda(\psi) \neq 0$. This, in view of Theorem 1, implies that ψ has a fixed point. Applying now Lemma (8.3) for the last time, we conclude that φ has a fixed point and the proof of the theorem is completed.

10. Corollaries. — We draw now a few immediate consequences of the main theorem.

Corollary 1. — *Let* X *be a topologically complete* ANR *and assume that* φ, $\psi : X \to X$ *are two compact admissible maps which satisfy one of the following conditions :*

(i) φ *is a selector of* ψ;

(ii) φ *and* ψ *are homotopic.*

Then both φ *and* ψ *are Lefschetz maps,* $\Lambda(\varphi) = \Lambda(\psi)$ *and* $\Lambda(\psi) \neq 0$ *implies that* φ *has a fixed point.*

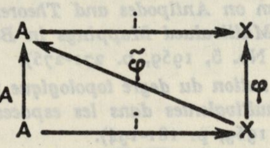
Corollary 2. — *Let* X *be a topologically complete* ANR *and* $\varphi : X \to X$ *be a compact admissible map. Assume further that one of the following conditions holds :*

(i) $\varphi(X)$ *is contained in an acyclic subset* A *of* X;

(ii) φ *is homotopic to an admissible constant map.*

Then $\Lambda(\varphi) = 1$ *and* φ *has a fixed point.*

Proof. — To prove (i) write the diagram



in which $i : A \to X$ stands for the inclusion and φ_A, $\hat{\varphi}$ denote the contractions of φ. By Lemma (8.3) and Theorem 2, $\Lambda(\varphi_A) = \Lambda(\varphi)$. By assumption, $H_0(A) = Q$ and $H_q(A) = 0$ for $q \geq 1$; it follows that $\Lambda(\varphi_A) = 1$ and thus our assertion follows from Theorem 2.

To prove (ii) denote by ψ an admissible constant map such that $\varphi \sim \psi$, and observe that ψ admits a single-valued constant selector ψ'. Since $\Lambda(\psi') = 1$, we have, by Corollary 1,

$$\Lambda(\psi') = \Lambda(\psi) = \Lambda(\varphi) = 1$$

and, consequently, φ has a fixed point.

Corollary 3. — *Let* X *be a topologically complete and acyclic* ANR *or, in particular, any one of the following :*

(i) *an acyclic Banach manifold;*

(ii) *a contractible open set in a Banach space;*

(iii) *a topologically complete* AR;

(iv) *a closed convex set in a Banach space.*

Then any compact admissible map $\varphi : X \to X$ *has a fixed point.*

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THE LERAY-SCHAUDER INDEX AND THE FIXED POINT THEORY FOR ARBITRARY ANRs [1]

BY

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1. Introduction

The Leray-Schauder theory of the degree or the equivalent notion of the fixed point index ([18], [19]) has played a basic role in non-linear functional analysis. In this note, we intend to show that the suitably modified and supplemented theory of the Leray-Schauder index belongs to topology and occupies in fact the central place in topological fixed point theory.

Let U be an open subset of a normed space E, and $f: U \to E$ be a compact map with a compact set of fixed points. To every such f, we assign an integer Ind (f), *the Leray-Schauder index of* f, which satisfies a number naturally expected properties; among those that supplement the classical ones the following two are of especial importance : (i) the Leray-Schauder index Ind (f) is topologically invariant, and (ii) when $f: U \to U$, it is equal to the (generalized) Lefschetz number $\lambda(f)$ of f [and hence $\lambda(f) \neq 0$ implies that f has a fixed point].

Now let X be a space which is r-dominated by an open set V in E [= a metric ANR [2]], $r: V \to X$, $s: X \to V$ a pair of maps with $rs = 1_X$. Let U be open in X and $f: U \to X$ be a compact map with a compact set of fixed points. Then the map $sfr : r^{-1}(U) \to V$ is also compact, and we define Ind (f) to be the Leray-Schauder index of sfr. Properties of the Leray Schauder index imply that Ind (f) is the extension of the

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[2] ANR = Absolute Neighbourhood Retract.

former over the larger category of spaces and thus we obtain the topological fixed point index theory for compact maps of arbitrary ANR-s. This theory contains several basic known results in topology (for example, the well-known theory of the fixed-point index for compact ANR-s) and non-linear functional analysis (e. g. the Schauder fixed point theorem, the Birkhoff-Kellogg theorem). It contains also the authors generalization [11] of the Lefschetz fixed point theorem to compact maps of arbitrary ANR-s.

The treatment of the fixed point index theory presented in this note has as its starting point the fixed point index in R^n due to A. Dold [8] and depends also on the notion of the generalized trace as given by J. Leray in [16]. A part of results presented here was announced earlier in some detail in [12].

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2. The Leray trace

In what follows an essential use will be made of the notion of the generalized trace and the Lefschetz number as given by J. Leray in [16]. We shall consider vector spaces only over the field of rational numbers Q.

A graded vector space $E = \{E_q\}$ is of a *finite type* provided : (i) $\dim E_q < \infty$ for all q, and (ii) $E_q = 0$ for almost all q. If $f = \{f_q\}$ is an endomorphism of such a space, then the Lefschetz number $\lambda(f)$ of f is given by

$$\lambda(f) = \Sigma_q (-1)^q \operatorname{tr}(f_q),$$

where tr stands for the ordinary trace function.

Let $f: E \to E$ be an endomorphism of an arbitrary vector space E. Let us put

$$N(f) = \cup_{n \geq 1} \ker f^n, \qquad \tilde{E} = E/N(f)$$

where $f^{(n)}$ is the n-th iterate of f. Since $f(N(f)) \subset N(f)$, we have the induced endomorphism $\tilde{f}: \tilde{E} \to \tilde{E}$. Assume that $\dim \tilde{E} < \infty$; in this case, we define the *generalized trace* $\operatorname{Tr}(f)$ of f by putting $\operatorname{Tr}(f) = \operatorname{tr}(\tilde{f})$.

Now let $f = \{f_q\}: E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. Call f the *Leray endomorphism* provided the graded space $\tilde{E} = \{\tilde{E}_q\}$ if of a finite type. For such an f, we define the (generalized) *Lefschetz number* $\Lambda(f)$ by putting

$$\Lambda(f) = \Sigma_q (-1)^q \operatorname{Tr}(f_q).$$

The following important property of the Leray endomorphisms [18] is a consequence of the well-known formula $\operatorname{tr}(uv) = \operatorname{tr}(vu)$ for the ordinary trace :

(2.1) LEMMA. — *Assume that in the category of graded vector spaces the following diagram commutes*

$$\begin{array}{ccc} E' & \xrightarrow{u} & E'' \\ {\scriptstyle f'}\uparrow & \nwarrow^{v} & \uparrow{\scriptstyle f''} \\ E' & \xrightarrow{u} & E'' \end{array}$$

Then, if f' or f'' is a Leray endomorphism, then so is the other and in that case $\Lambda(f') = \Lambda(f'')$.

3. Lefschetz maps

Let H be *the singular homology functor* (with the rational coefficients) from the category of topological spaces and continuous mappings to the category of graded vector spaces and linear maps of degree 0. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q-dimensional singular homology group of X. For a continuous mapping $f: X \to Y$, $H(f)$ is the induced linear map $f_* = \{f_q\}$, where $f_q : H_q(X) \to H_q(Y)$.

A continuous map $f: X \to X$ is called a *Lefschetz map* provided $f_* : H(X) \to H(X)$ is a Leray endomorphism. For such f, we define the Lefschetz number $\Lambda(f)$ of f by putting $\Lambda(f) = \Lambda(f_*)$.

Clearly, if f and g are homotopic, $f \sim g$, then $\Lambda(f) = \Lambda(g)$.

(3.1) LEMMA. — *Assume that in the category of topological spaces the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \varphi}\uparrow & \nwarrow^{g} & \uparrow{\scriptstyle \psi} \\ X & \xrightarrow{f} & Y \end{array}$$

Then :

(a) if one of the maps φ or ψ is a Lefschetz map, then so is the other and in that case $\Lambda(\varphi) = \Lambda(\psi)$;

(b) φ has a fixed point if and only if ψ does.

Proof. — The first part follows (by applying the homology functor to the above diagram) from lemma 2.1. The second part is obvious.

The following are the two instances in which the above lemma is used :

(3.2) *Example.* — Let $f: X \to X$ be a map such that $f(X) \subset K \subset X$. Then we have the commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\subset} & X \\ {\scriptstyle f_K}\uparrow & \nwarrow & \uparrow{\scriptstyle f} \\ K & \xrightarrow{\subset} & X \end{array}$$

with the obvious contractions (³).

(3.3) *Example.* — Let $r: Y \to X$, $s: X \to Y$ be a pair of continuous mappings such that $rs = 1_X$. In this case, X is said to be r-dominated by Y and r is said to be an r-map. In this situation, given a map $\varphi: X \to X$, we have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{s} & Y \\ {\scriptstyle \varphi}\uparrow & \nwarrow {\scriptstyle \varphi r} & \uparrow{\scriptstyle \psi} \\ X & \xrightarrow{s} & Y \end{array}$$

with $\psi = s\varphi r$.

4. Compact maps

A continuous map $f: X \to Y$ between topological spaces is called *compact* provided it maps X into a compact subset of Y. Let $h_t: X \to Y$ be a homotopy and $h: X \times I \to Y$ be defined by $h(x, t) = h_t(x)$ for $(x, t) \in X \times I$; then h_t is said to be a *compact homotopy* provided the map h is compact. Two compact maps $f, g: X \to Y$ are *compactly homotopic* provided there is a compact homotopy $h_t: X \to Y$ with $h_0 = f$ and $h_1 = g$. If Y *is a linear space then* f (resp. h_t) is said to be *finite dimensional* provided it is compact and the image $f(X)$ [resp. $h(X)$] is contained in a finite dimensional subspace of Y.

In what follows, we shall combine the Schauder approximation theorem [20] and a result of P. Aleksandrov concerning the maps of compacta into the polyhedra.

(4.1) Theorem (*cf.* [11], [18]). — *Let U be an open subset of a normed space E and let $f: X \to U$ be a compact mapping. Then for every suffi-*

(³) Let $f: X \to Y$ be a map such that $f(A) \subset B$, where $A \subset X$ and $B \subset Y$. By the *contraction* of f to the pair (A, B), we understand a map $f': A \to B$ with the same values as f. A contraction of f to the pair (A, Y) is simply the restriction $f \mid A$ of f to A.

ciently small $\varepsilon > 0$ *there exists a finite polyhedron* $K_\varepsilon \subset U$ *and a mapping* $f_\varepsilon : X \to U$, *called an* ε*-approximation of* f, *such that* :

(i) $\| f(x) - f_\varepsilon(x) \| < \varepsilon$ *for all* $x \in X$;

(ii) $f_\varepsilon(X) \subset K_\varepsilon$;

(iii) *the formula* $h_t(x) = t f_\varepsilon(x) + (1 - t) f(x)$ *defines a compact homotopy* $h_t : X \to U$ *joining* f_ε *with* f.

Proof. — Given $\varepsilon > 0$ (which we may assume to be sufficiently small) $f(X)$ is contained in the union of a finite number of open balls $V(y_i, \varepsilon) \subset U$ $(i = 1, 2, \ldots, k)$. Putting for $x \in X$,

$$f_\varepsilon(x) = (\Sigma_i \lambda_i(x) y_i)/(\Sigma_j \lambda_j(x)), \qquad 1 \leq i \leq k, \qquad 1 \leq j \leq k,$$

where

$$\lambda_i(x) = \max \{ 0, \varepsilon - \| f(x) - y_i \| \},$$

we obtain the map f_ε satisfying (i). Clearly, the values of f_ε are in a finite polyhedron $K_\varepsilon \subset U$ with vertices $y_1, y_2, \ldots, y_k$. Property (iii) is evident.

The proof of the following elementary fact is left as an easy exercise for the reader (*cf.* [10]).

(4.2) LEMMA. — *Let* V *be open in a normed space* E *and assume that* $f : \overline{V} \to E$ *is a compact map with no fixed points on the boundary* ∂V *of* V. *Then* :

(i) *the number* $\eta = \inf_{x \in \partial V} \| x - f(x) \|$ *is positive;*

(ii) *if* $\varepsilon < \eta$, *then any* ε*-approximation* f_ε *of* f *is fixed point free on* ∂V;

(iii) *given any two* ε*-approximations* f'_ε *and* f''_ε *of* f *with* $\varepsilon < (1/2)\eta$, *the formula*

$$h_t(x) = t f'_\varepsilon(x) + (1 - t) f''_\varepsilon$$

defines a finite dimensional η*-homotopy* (1) *joining* f' *and* f'' *which has no fixed points on* ∂V.

5. The axioms for the fixed point index

Let $f : U \to X$ be a continuous map between topological spaces. Call f *admissible* provided U is an open subset of X and the fixed point set of f.

$$\varkappa_f = \{ x \in U, f(x) = x \} \subset U$$

(1) A homotopy $h_t : X \to Y$ into a metric space (Y, φ) is said to be an ε*-homotopy* provided $\rho(h_t(x), h_{t'}(x)) < \varepsilon$ for all $x \in X$ and $t, t' \in (0, 1)$. If $f, g : X \to Y$ can be joined by an ε-homotopy, then we say that f is ε-homotopic to g and write $f \tilde{\varepsilon} g$; clearly $f \tilde{\varepsilon} g$ implies, in particular, that $\rho(f(x), g(x)) < \varepsilon$ for all $x \in X$.

is compact. A homotopy $h_t : U \to X$ will be called *admissible* provided the set $\varkappa(\{h_t\}) = \cup_{0 \leq t \leq 1} \varkappa(h_t)$ is compact.

DEFINITION (A. DOLD [8]). — Let $\mathfrak{C}$ be a category of topological spaces in which a class $\mathfrak{A} = \mathfrak{A}(\mathfrak{C})$ of admissible maps and homotopies is distinguished. By a *fixed-point index* on $\mathfrak{A}$ we shall understand a function $\text{Ind} : \mathfrak{A} \to Z$ which satisfies the following conditions :

(I) *Excision.* — If $U' \subset U$ and $\varkappa_f \subset U'$, then the restriction

$$f' = f \mid U' : U' \to X$$

is in $\mathfrak{A}$ and $\text{Ind}(f) = \text{Ind}(f')$.

(II) *Additivity.* — Assume that $U = \cup_i U_i$, $1 \leq i \leq k$, $f_i = f \mid U$ and the fixed point set $\varkappa_i = \varkappa_f \cap U_i$ are mutually disjoint, $\varkappa_i \cap \varkappa_j = \emptyset$ for $i \neq j$. Then

$$\text{Ind} f = \Sigma_i \text{Ind} f_i, \quad 1 \leq i \leq k.$$

(III) *Fixed points.* — If $\text{Ind} f \neq 0$, then $\varkappa_f \neq \emptyset$, i. e., the map f has a fixed point.

(IV) *Homotopy.* — Let $h : U \to X$, $0 \leq t \leq 1$, be an admissible homotopy in $\mathfrak{A}$. Then $\text{Ind}(h_0) = \text{Ind}(h_1)$.

(V) *Multiplicativity.* — *If* $f_1 : U_1 \to X_1$ *and* $f_2 : U_2 \to X_2$ *are in* $\mathfrak{A}$ *then so is the product map* $f_1 \times f_2 : U_1 \times U_2 \to X_1 \times X_2$ *and*

$$\text{Ind}(f_1 \times f_2) = \text{Ind}(f_1) . \text{Ind}(f_2).$$

(VI) *Commutativity.* — Let $U \subset X$, $U' \subset X'$ be open and assume $f : U \to X'$, $g : U' \to X$ are maps in $\mathfrak{C}$. If one of the composites

$$gf : V = f^{-1}(U') \to X \qquad \text{or} \qquad fg : V' = g^{-1}(U) \to X'$$

is in $\mathfrak{A}$, then so is the other and, in that case,

$$\text{Ind}(gf) = \text{Ind}(fg).$$

(VII) *Normalization.* — If $U = X$ and $f : X \to X$ is in $\mathfrak{A}$, then f is a Lefschetz map, and $\text{Ind}(f) = \Lambda(f)$.

6. The fixed point index in R^n

In the following definition H is the singular homology over the integers Z. Let us fix for each n an orientation $1 \in H_n(S^n)$ of the n-th sphere $S^n = \{x \in \mathbf{R}^{n+1};\ \| x \| = 1\}$ and accordingly identify $H_n(S^n) \approx Z$ with the integers Z.

DEFINITION (*cf.* A. DOLD [8]). — Let $f: U \to R^n$ be an admissible map. Denote by $K = \varkappa_f$ the fixed point set for f and by

$$(i - f): \quad (U, U - K) \to (R^n, R^n - \{0\})$$

the map given by $(i - f)(x) = x - f(x)$. The fixed point index Ind f of the map f is defined to be the image of 1 under the composite map

$$Z = H_n(S^n) \to H_n(S^n, S^n - K) \approx H_n(U, U - K) \xrightarrow{(i-f)_*} H_n(R^n, R^n - 0) \approx Z.$$

The following theorem established by A. DOLD [8] represents a modernized version of the classical result due essentially to H. HOPF.

(6.1) (*The fixed point index in* R^n). *Let* $\mathfrak{C}$ *be the category of open subsets of euclidean spaces and* $\mathfrak{A}(\mathfrak{C})$ *the class of all continuous admissible maps in* $\mathfrak{C}$. *Then the integer valued function* $f \to$ Ind f *defined above satisfies the properties* (I)-(VII). *In* (VII), *it is assumed that f is compact.*

We note that the excision and the commutativity implies the following property of the index :

(VIII) *Contraction.* — Let U be open in R^{n+1} and $f: U \to R^{n+1}$ be an admissible map such that $f(U) \subset R^n$. Denote by $f': U' \to R^n$ the contraction of f, where $U' = U \cap R^n$. Then Ind $(f) =$ Ind (f').

7. The Leray-Schauder index

Let U be an open subset of a normed space E and let $f: U \to E$ be an admissible compact map. Take an open set $V \subset U$ such that $\partial V \subset U$ and $\varkappa_f \subset V$. Then the number $\eta = (1/2) \inf \| x - f(x) \|$ for $x \in \partial V$ is positive.

Let $g = f \mid V: V \to E$. From the definition of η and lemma (4.2) it follows that :

(i) if $\varepsilon < \eta$, then every ε-approximation $g_\varepsilon: V \to E$ of g is admissible;

(ii) given two ε-approximations $g'_\varepsilon, g''_\varepsilon: V \to E$ of g with $\varepsilon < \eta$ there exists an admissible finite dimensional compact homotopy $h_t: V \to E$ $0 \leq t \leq 1$, such that $h_0 = g'$, $h_1 = g''$.

Let $f: U \to E$ be an admissible compact map and $g_\varepsilon: V \to E$ be an ε-approximation of $g = f \mid V$ as above. Denote by E^n a finite dimensional subspace of E which contains $g_\varepsilon(V)$ and by $g'_\varepsilon: E^n \cap V \to E^n$ the evident contraction of g_ε.

Let us put Ind $(f, V) =$ Ind (g'_ε). It follows from (i) and (ii) and the homotopy and contraction properties of the index in R^n, that Ind (f, V) does not depend on the choice g'_ε. Moreover, given V_1, V, with the same properties as V, we have

$$(7.1.1) \qquad \operatorname{Ind}(f, V_1) = \operatorname{Ind}(f, V_2).$$

For the proof of (7.1.1) we distinguish two cases :

(i) $V_1 \subset V_2$;

(ii) V_1 and V_2 are arbitrary.

In the first case, our assertion follows by the excision of the index in R^n and the second case reduces, évidently, to the first.

Definition. — For an admissible compact map $f: U \to X$, we define the *Leray-Schauder index* $\operatorname{Ind}(f)$ of f by putting

$$\operatorname{Ind}(f) = \operatorname{Ind}(f, V) = \operatorname{Ind}(g_\varepsilon).$$

It follows from (7.1.1) that $\operatorname{Ind}(f)$ is well defined.

We may state now the first main result of this note :

(7.1) Theorem. — *Let $\mathfrak{C}$ be the category of open subsets in linear normed spaces and let $\mathfrak{A}$ be the class of all admissible compact maps in $\mathfrak{C}$. Assume that all admissible homotopies are compact. Then, defined on $\mathfrak{A}$, the Leray-Schauder index* $f \to \operatorname{Ind}(f)$ *satisfies the properties* (I)-(VII). *In* (VI), *it is assumed that one of the maps f or g is compact.*

Proof. — Using the approximation theorem (4.1), lemma (4.2), properties (I)-(V) follow in a straightforward manner from the corresponding properties of the index in R^n. The proof of property (VI), which is somewhat more involved, will be given separately in section 8. It remains to establish the normalization property.

Proof of property (VII). — Given a compact map $f: U \to U$ let $\varepsilon > 0$ be smaller than dist $(x_f, \partial U)$, $f_\varepsilon: U \to U$ be an ε-approximation of f such that its values are in a finite dimensional subspace E^n of E and let $U_n = U \cap E^n$. Clearly every such f_ε is admissible and $f \sim f_\varepsilon$.

Consider the following commutative diagram in which all the arrows represent either the obvious inclusions or the contractions of the map f_ε :

$$\begin{array}{ccc} U_n & \xrightarrow{\subset} & U \\ {\scriptstyle f_\varepsilon}\uparrow & & \uparrow{\scriptstyle f_\varepsilon} \\ U_n & \xrightarrow{\subset} & U \end{array}$$

By the definition $\operatorname{Ind}(f) = \operatorname{Ind}(f_\varepsilon)$. By lemma (3.1) [Example (3.2)], we have $\lambda(f) = \lambda(f_\varepsilon)$ and, consequently, in view of theorem (7.1) (property VII), $\operatorname{Ind}(f) = \Lambda(f_\varepsilon)$. Since f is homotopic to f_ε, this implies that $\operatorname{Ind}(f) = \Lambda(f)$ and the proof is completed.

We remark that theorem (7.1) (the commutativity) contains the following important property of the Leray-Schauder index :

(IX) *Topological Invariance.* — Let $f: U \to E$ be an admissible compact map, and $h: E \to E'$ be a homeomorphism. Then $h \circ f \circ h^{-1}: h(U) \to E'$ is also an admissible compact map and

$$\operatorname{Ind}(h \circ f \circ h^{-1}) = \operatorname{Ind}(f).$$

8. Proof of the commutativity

In the proof of commutativity, we shall use the fact that the Leray-Schauder index satisfies properties (I), (IV) (V) and (VII).

Let $U \subset E$, $U' \subset E'$ be open in normed spaces E and E', respectively, $f: U' \to E$, $g: U \to E'$ be continuous and consider the composites

$$g \circ h: \ f^{-1}(U) \to E', \qquad f \circ g: \ g^{-1}(U') \to E.$$

We note that the maps $f: \varkappa(gf) \to \varkappa(fg)$ and $g: \varkappa(fg) \to \varkappa(gf)$ are inverse to each other and hence the fixed point sets $\varkappa(gf)$ and $\varkappa(fg)$ are homeomorphic; thus, if one of them is compact, then so is the other. In the proof of commutativity, we shall distinguish two cases :

Special case (both f and g are compact) : In this case, we proceed essentially as in Dold [8].

We let $\varphi: U' \times U \to E' \times E$ be given by

$$\varphi(x, y) = (g(y), f(x)). \tag{8.1.1}$$

And we define the homotopies :

$$\begin{aligned} h_t, h'_t &: \ U' \times U \to E' \times E, \\ H_t &: \ U' \times E \to E' \times E, \\ H'_t &: \ E' \times U \to E' \times E, \end{aligned}$$

by the following formulas :

$$\left|\begin{aligned} & h_t(x, y) = [tg f(x) + (1-t) g(y), f(x)], \\ & h'_t(x, y) = [g(y), tf g(y) + (1-t) f(x)], \end{aligned}\right. \quad (x, y, t) \in U' \times U \times I, \tag{8.1.2}$$

$$\left|\begin{aligned} & H_t(x, y) = [g f(x), (1-t) f(x)], && (x, y, t) \in U' \times E \times I, \\ & H'_t(x, y) = [(1-t) g(y), f g(y)], && (x, y, t) \in U \times E' \times I. \end{aligned}\right.$$

We have

(8.1.3) $$\varphi = h_0 = h'_0.$$

By assumption, f and g are compact; this implies that φ and all the above homotopies are compact; since the fixed point sets $\varkappa_\varphi$ and $\varkappa_{gf}$ are homeomorphic under $x \mapsto (x, f(x))$, $(x, y) \mapsto x$, φ is admissible.

Moreover, simple computation shows that the fixed point sets

$$\varkappa_\varphi = \varkappa(\{h_t\}) = \varkappa(\{h'_t\})$$

coincide and, therefore, the homotopies h_t and h'_t are admissible. By straightforward argument, one shows that H_t and H'_t are also admissible.

Therefore, by homotopy, we have, in view of (8.1.3),

(8.1.4) $$\text{Ind}(\varphi) = \text{Ind}(h_1) = \text{Ind}(h'_1).$$

On the other hand, since

$$h_1 = H_0 \mid U' \times U, \qquad h'_1 = H'_0 \mid U' \times U,$$

we get by excision and homotopy,

(8.1.5) $$\text{Ind}(\varphi) = \text{Ind}(H_1) = \text{Ind}(H'_1).$$

Both H_1 and H'_1 are product maps

(8.1.6) $$H_1 = (gf) \times (\text{Cte}), \qquad H'_1 = (\text{Cte}) \times (fg).$$

By multiplicativity, in view of (8.1.5) and (8.1.6),

$$\text{Ind}(gf) . \text{Ind}(\text{Cte}) = \text{Ind}(\text{Cte}) . \text{Ind}(fg)$$

and hence by normalization [since λ (Cte) = 1] we get Ind (gf) = Ind (fg).

General case: We assume now that $f : U' \to E$ is compact and $g : U \to E'$ continuous. To show that

$$\text{Ind}(gf \mid f^{-1}(U)) = \text{Ind}(fg \mid g^{-1}(U')),$$

we assume that gf (and hence fg) is admissible.

Take a smaller open set $O \subset U$ such that :

(i) O isbounded;

(ii) $\partial O \subset U$;

(iii) $\varkappa_{fg} \subset O$,

and put $O' = f^{-1}(O)$. Clearly $\varkappa(gf) \subset O'$, and we may assume that $\partial O' \subset U'$ ([5]).

Now both $gf : f^{-1}(O) \to E'$ and $fg : g^{-1}(O') \to E$ are compact. By excision, it is sufficient to show that

(8.1.7) $$\operatorname{Ind}(gf \mid f^{-1}(O)) = \operatorname{Ind}(fg \mid g^{-1}(O')).$$

Let us put

(8.1.8) $$\begin{cases} \eta_1 = \inf \| x - g f(x) \| & \text{for } x \in \partial f^{-1}(O), \\ \eta_2 = \inf \| y - f g(y) \| & \text{for } y \in \partial g^{-1}(O'), \\ \eta = \min(\eta_1, \eta_2). \end{cases}$$

By lemma 4.2, the number η is positive.

Let K be a compact set containing $f(U') \subset E$. Consider the map $g : U \to E'$ at points of a compact set $K \cap \overline{O} \subset U$. Continuity of g implies that for each $y \in K \cap \overline{O}$ there is a $\delta_y > 0$ such that :

(i) the open ball $V(y, \delta_y)$ with center y and radius δ_y is contained in U;

(ii) $y', y'' \in V(y, \delta_y) \Rightarrow \| (g(y') - g(y'') \| < \eta$.

From the compactness of $K \cap \overline{O}$ it follows that a finite number of balls $V(y_1, \delta_{y_1}), \ldots, V(y_k, \delta_{y_k})$ covers $K \cap \overline{O}$.

We let

(8.1.9) $$\begin{cases} \delta = \min(\delta_{y_1}, \ldots, \delta_{y_k}), \\ V = \cup_i V(y_k, \delta_{y_i}), \quad 1 \leq i \leq k, \\ \varepsilon = \min(\delta, \eta). \end{cases}$$

Clearly, from (8.1.9), it follows that

(8.1.10) $$\begin{cases} \text{if } y \in K \cap \overline{O} \text{ and } \| y' - y \| < \varepsilon, \text{ then} \\ \qquad \| g(y) - g(y') \| < \eta \\ \text{and} \\ ty + (1 - t) y' \in V \quad \text{for all } t \in [0, 1]. \end{cases}$$

Now let $f_\varepsilon : U' \to E$ be an ε-approximation of $f : U' \to E$ and $h_t : U' \to E$ be given by $h_t(x) = t f(x) + (1 - t) f_\varepsilon(x)$ clearly; h_t is an ε-homotopy joining compactly f anf f_ε. Since on $\overline{f^{-1}(O)} \subset f^{-1}(\overline{O})$ the values of h_t are in a compact subset of $V \subset U$, we may consider on $\overline{f^{-1}(O)}$ the compo-

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([5]) In view of the excision we may suppose (by taking slightly smaller open sets) that f and g are defined in fact on ∂U and $\partial U'$ respectively.

sition gh_t. It follows clearly from (8.1.10) that gf_t is an η-homotopy and therefore by lemma 4.2 (in view of the definition of η) it has no fixed points on $\partial f^{-1}(O)$; thus

$$gh_t : f^{-1}(O) \to E'$$

is an admissible homotopy joining gf_ε and gf on $f^{-1}(O)$.

Consequently, by homotopy, we have

$$\text{(8.1.11)} \qquad \operatorname{Ind}(gf \mid f^{-1}(O)) = \operatorname{Ind}(gf_\varepsilon \mid f^{-1}(O)).$$

Next, since f_ε is finite dimensional, $f_\varepsilon(U') \subset E''$, we may write the following contractions

$$\text{(8.1.12)} \qquad \begin{cases} \tilde{f}_\varepsilon : f^{-1}(O) \to E'' \cap O, \\ \tilde{g} : E'' \cap O \to E' \end{cases}$$

of f_ε and g respectively. On $f^{-1}(O)$, we have $g \circ f_\varepsilon = \tilde{g} \circ \tilde{f}_\varepsilon$ and, therefore

$$\text{(8.1.13)} \qquad \operatorname{Ind}(gf_\varepsilon \mid f^{-1}(O)) = \operatorname{Ind}(\tilde{g}\tilde{f}_\varepsilon \mid f^{-1}(O)).$$

Further, since $\tilde{g} \subset g \mid E'' \cap \overline{O}$ and O is bounded, we conclude that $\tilde{g}$ is compact. Thus, both $\tilde{f}_\varepsilon$ and $\tilde{g}$ being compact, we may apply the special case of commutativity. We have

$$\text{(8.1.14)} \qquad \operatorname{Ind}(\tilde{g}\tilde{f}_\varepsilon \mid f^{-1}(O)) = \operatorname{Ind}(\tilde{f}_\varepsilon \circ \tilde{g} \mid \tilde{g}^{-1} f^{-1}(O))$$

and finally, since $O' = f^{-1}(O)$, we obtain from (8.1.11), (8.1.13) and (8.1.14),

$$\text{(8.1.15)} \qquad \operatorname{Ind}(gf \mid f^{-1}(O)) = \operatorname{Ind}(f_\varepsilon \circ \tilde{g} \mid \tilde{g}^{-1}(O')).$$

On the other hand, consider the composition $h_t g$ on $\overline{g^{-1}(O')} \subset g^{-1}(\overline{O'})$. Clearly $h_t g$ is a compact ε-homotopy joining fg and $f_\varepsilon \circ g$; since $\varepsilon < \eta$, it is an η-homotopy and, hence, by lemma 4.2, it is fixed point free on $\partial g^{-1}(O')$. In other words, $h_t g : g^{-1}(O') \to E$ is an admissible compact homotopy joining $f_\varepsilon \circ g$ and fg on $g^{-1}(O')$, and consequently (by homotopy) we have

$$\text{(8.1.16)} \qquad \operatorname{Ind}(fg \mid g^{-1}(O')) = \operatorname{Ind}(f_\varepsilon \circ g \mid g^{-1}(O')).$$

Since the values of $f_\varepsilon \circ g$ are in $E'' \cap O$, we have, by the definition of the Leray-Schauder index,

$$\text{(8.1.17)} \qquad \operatorname{Ind}(f_\varepsilon \circ g \mid g^{-1}(O')) = \operatorname{Ind}(f_\varepsilon \circ g \mid g^{-1}(O') \cap O \cap E''),$$

and hence, because $g^{-1}(O') \cap O \cap E^n = \tilde{g}^{-1}(O')$, we get

$$\text{(8.1.18)} \qquad \operatorname{Ind}(fg \mid g^{-1}(O')) = \operatorname{Ind}(f_\varepsilon \circ \tilde{g} \mid \tilde{g}^{-1}(O')).$$

By comparing formulas (8.1.18) and (8.1.15), we obtain the desired conclusion (8.1.7), and thus, the proof of commutativity is completed.

9. Compact maps of the ANR-spaces

We propose now as the first consequence of the Leray-Schauder index a general fixed point theorem which on the one hand contains the classical Lefschetz theorem for compact ANR-s, and on the other hand contains various fixed point theorems of the non-linear functional analysis.

We denote by ANR (resp. AR) the class of metrizable absolute neighbourhood retracts (resp. absolute retracts). We recall (*cf.* [3]) that a metrizable space Y is an ANR (resp. AR) provided for any metrizable pair (X, A), with A closed in X and any continuous $f_0 : A \to Y$, there exists an extension $f : U \to Y$ of f_0 over a neighbourhood U of A in X (resp. an extension $f : X \to Y$ of f_0 over X).

(9.1) *Example.* — The following are some typical and important properties of the ANR spaces :

(i) If X is r-dominated by Y, then $Y \in$ ANR implies $X \in$ ANR;

(ii) If U is open in X, and $X \in$ ANR, then $U \in$ ANR;

(iii) If X is convex subset of a normed (or locally convex metrizable) linear space, then $X \in$ AR (J. Dugundji [9]).

(iv) A metrizable space which is locally ANR is an ANR; in particular manifolds, Banach manifolds are ANR-s.

In what follows, we shall use essentially the following fact from general topology :

(9.2) (Arens-Eells [1]) : *Every metrizable space can be embedded as a closed subset of a linear normed space.*

The above Arens-Eells embedding theorem permits to give the following simple characterization of the ANR-s :

(9.3) *In order that* $Y \in$ ANR (*resp.* $Y \in$ AR, *it is necessary and sufficient that* Y *be r-dominated by an open set of a normed space* (*resp. by a normed space*).

Proof. — Let $Y \in$ ANR. By theorem (9.2), there exists an embedding $\mathfrak{I} : Y \to E$ of Y into a normed space E such that $\mathfrak{I}(Y)$ is closed in E. Take a retraction $r : U \to \mathfrak{I}(Y)$ of an open set $U \supset \mathfrak{I}(Y)$. Then

$\mathfrak{s}^{-1} r : U \to Y$ is clearly an r-map. The converse follows from the general properties of the ANR-s [*cf.* Example (9.1)]. The proof of the second part is similar.

(9.4) THEOREM (*cf.* [12]). — *Let X be an* ANR *and* $f : X \to X$ *be a compact map. Assume further that* U *is open in a normed space E and* $s : X \to U$, $r : U \to X$ *be an arbitrary pair of maps with* $rs = 1_X$. *Then f is a Lefschetz map and the Lefschetz number of f is equal to the Leray-Schauder index of the map sfr,* $\Lambda(f) = \text{Ind}(sfr) = \Lambda(sfr)$.

Proof. — Theorem (9.4) clearly follows from theorem (7.1) and lemma (3.1) [Example (3.3)].

As a consequence of theorems (9.4), (7.1), we get the following generalization of the Lefschetz fixed point theorem, established by the author in [11].

(9.5) THEOREM. — *Let X be an* ANR *and* $f : X \to X$ *be a compact map. Then :*

(i) *f is a Lefschetz map;*

(ii) $\Lambda(f) \neq 0$ *implies that f has a fixed point.*

As an illustration, we list a number of well-known consequences of theorem (9.5) :

COROLLARY 1 (*Lefschetz fixed point theorem for compact* ANR-s). — *Let X be a compact* ANR *and* $f : X \to X$ *be continuous. Then* $\lambda(f) \neq 0$ *implies that f has a fixed point.*

COROLLARY 2. — *Let X be an acyclic* ANR *or, in particular, an* AR. *Then any compact map* $f : X \to X$ *has a fixed point.*

COROLLARY 3 (*Schauder fixed point theorem* [20]). — *Let X be a convex (not necessarily closed) subset of a normed (or locally convex metrizable) linear space. Then any compact* $f : X \to X$ *has a fixed point.*

Proof. — X is an AR [*cf.* Example (9.1)] and hence the assertion follows from corollary 3.

COROLLARY 4 (*Birkhoff-Kellog theorem* [2]). — *Let* $S = \{x \in E; \|x\| = 1\}$ *be the unit sphere in an infinite dimensional normed space E and* $f : S \to E$ *be a compact map satisfying*

$$\|f(x)\| \geq \alpha > 0 \quad \text{for all } x \in S. \tag{9.5.1}$$

Then there exists an invariant direction for f, i. e., for some $x_0 \in S$ *and* $\mu > 0$, *we have* $f(x_0) = \mu x_0$.

Proof. — Let us put for each $x \in S$,

$$\varphi(x) = f(x)/\| f(x) \|.$$

Then (9.5.1) implies that the map $\varphi : S \to S$ is compact. Since S is clearly an acyclic ANR (and even an AR, *cf.* [9]), φ has a fixed point, i. e.,

$$\varphi(x_0) = f(x_0)/\| f(x_0) \| = x_0.$$

for some x_0 and the proof of our assertion is competed.

COROLLARY 5 (BROWDER-EELLS [6]). — *Let X be a Banach (or more generally a Fréchet) manifold and* $f : X \to X$ *a compact map. Then* $\Lambda(f)$ *is defined, and* $\Lambda(f) \neq 0$ *implies that f has a fixed point.*

10. Fixed point index theory for arbitrary ANR-s

Now we turn to the main application of the Leray-Schauder index by establishing the existence of the fixed point index theory for compact maps of arbitrary ANR-s.

DEFINITION (*cf.* [7] and [8]). — Let X be an ANR, and $f : U \to X$ an admissible compact map. To define Ind (f), take an open set V in a normed space E which r-dominates X. Let $s : X \to V$, $r : V \to X$ be a pair of maps with $rs = 1$. Since the composite map

$$r^{-1}(U) \xrightarrow{r} U \xrightarrow{f} X \xrightarrow{s} V$$

is compact and admissible [because $\varkappa(f) = \varkappa(sfr)$] its Leray-Schauder index is defined by theorem (7.1), and we define

(10.1.1) $\quad \operatorname{Ind}(f) =_{\mathrm{df}} \operatorname{Ind}(sfr).$

Let $V' \subset E'$ be another open set in a normed space E', which r-dominates X, with $s' : X \to V'$, $r' : V' \to X$, $r's' = 1$. Then, since the second of the maps $sr' : V' \to V$, $s'fr : r^{-1}(U) \to V'$ is compact we may apply the commutativity property for the Leray-Schauder index and hence

$$\operatorname{Ind}((s'fr) \circ (sr') \mid (sr')^{-1}(r^{-1}(U)) = \operatorname{Ind}((sr') \circ (s'fr) \mid r^{-1}(U)).$$

Since $(s'fr) \circ (sr') = s'fr'$, $(sr') \circ (s'fr) = sfr$ and because

$$(sr')^{-1}(r^{-1}(U)) = r'^{-1}(U),$$

we get

$$\operatorname{Ind}(s'fr' \mid r'^{-1}(U)) = \operatorname{Ind}(sfr \mid r^{-1}(U)),$$

which proves that our definition is independent of the choices involved.

Now we may state our second main result :

(10.1) THEOREM. — *Let* $\mathfrak{C}$ *be a category of metric* ANR-s *and* $\mathfrak{A}$ *be a class of all admissible compact maps in* $\mathfrak{C}$. *Assume further that all admissible homotopies are compact. Then the fixed point index function* $f \to \mathrm{Ind}\,(f)$ *defined by formula* (10.1.1) *satisfies all the properties* (I)-(VII). *In* (VI), *it is assumed that one of the maps f or g is compact, which implies that the fixed point index is a topological invariant.*

Proof. — The normalization property was already established in the previous section. All the remaining properties follow easily from the corresponding properties of the Leray-Schauder index. Let us prove for instance property (VI). The proofs of other properties, being similar, are omitted.

Proof of property (VI). — Let $X, X' \in \mathrm{ANR}$, $f : U' \to X$, $g : U \to X'$ be admissible maps and assume that f is compact. Let V (resp. V') be an open set in a normed space E (resp. E') which r-dominates X (resp. X'); denote by $X \xrightarrow{s} V \xrightarrow{r} X$, $X' \xrightarrow{s'} V' \xrightarrow{r'} X'$ two pairs of maps satisfying $rs = 1_X$, $r' s' = 1_{X'}$.

Consider the following maps :

$$s\,fr' : \; r'^{-1}(U') \to V,$$
$$s'\,gr : \; r^{-1}(U) \to V',$$

and note that the first of them is compact. It follows by commutativity of the Leray-Schauder index (applied to the above maps) that

$$\mathrm{Ind}\,((sfr')(s'\,gr) \mid (s'\,gr)^{-1}\, r'^{-1}(U')) = \mathrm{Ind}\,((s'\,gr)(sfr') \mid (sfr')^{-1}\, r^{-1}(U))$$

and hence, in view of

$$(s'\,gr)^{-1}\, r'^{-1}(U') = r^{-1}\, g^{-1}(U'),$$
$$(sfr')^{-1}\, r^{-1}(U) = r'^{-1}\, f^{-1}(U);$$

we get

$$\mathrm{Ind}\,(sfgr \mid r^{-1}\, g^{-1}(U')) = \mathrm{Ind}\,(s'\,gfr' \mid r'^{-1}\, f^{-1}(U)).$$

From this (in view of the definition of the fixed point index), we get

$$\mathrm{Ind}\,(fg \mid g^{-1}(U')) = \mathrm{Ind}\,(gf \mid f^{-1}(U)),$$

and the proof is completed.

11. Remarks on the non-metrizable case

First, we note that the approximation theorem (4.1) extends (with appropriate modifications) to the case when U is open in locally convex topological space E.

This fact permits to extend the Leray-Schauder index to the case of locally convex spaces and to state theorem (7.1) in the following more general form :

(11.1) THEOREM. — *Let $\mathfrak{C}$ be the category of open subsets of locally convex topological spaces. Let $\mathfrak{A} = \mathfrak{A}(\mathfrak{C})$ be the class of all admissible compact maps and assume that all admissible homotopies are compact. Then, there exists a functions* Ind : $\mathfrak{A} \to Z$ *(the Leray-Schauder index) which satisfies the properties* (I)-(VII). *In* (VI), *it is assumed that one of the maps f or g is compact.*

Now, by proceeding as in the metrizable case, one gets from theorem (11.1) the following generalization of theorem (10.1) :

(11.2) THEOREM. — *Let $\mathfrak{C}$ be the category of spaces which are r-dominated by open sets in linear locally convex topological spaces. Let $\mathfrak{A} = \mathfrak{A}(\mathfrak{C})$ be the class of all admissible compact maps and assume that all admissible homotopies are compact. Then, there is on $\mathfrak{A}$ an integer valued function $f \to$* Ind (f), *which satisfies all the properties* (I)-(VII). *In* (VI), *it is assumed that one of the maps f or g is compact; in particular,* Ind (f) *is topologically invariant.*

Let X be a compact ANR for normal spaces and $h : X \to E'$ be an embedding of X into a locally convex space E'. It can be shown that the linear span E of the compact set $h(X)$ in E' is normal. It follows that X is r-dominated by a set open in a locally convex space. We obtain, therefore, as a special case of theorem (11.2) the following :

COROLLARY (*Fixed point index for compact non-metrizable* ANR-s). — *Let $\mathfrak{C}$ be the category of compact* ANR-s *for normal spaces and $\mathfrak{A}$ be the class of all continuous admissible maps in $\mathfrak{C}$. Then there is on $\mathfrak{A}$ the fixed point index which satisfies all the properties* (I)-(VII).

We remark that the fixed point index for compact (non metrizable) ANR-s was established previously by combinatorial means (and in a different form) by several authors (*cf.* J. LERAY [15], A. DELEANU [7], D. BOURGIN [4], F. BROWDER [5]).

12. Other generalizations

In the definition of the fixed point index of f, only of importance is the behaviour of f in the neighbourhood of the fixed point set $\varkappa_f$. This general remark indicates how to enlarge the class of maps for which the fixed point index is defined.

DEFINITION. — Let X be an ANR and $f : U \to X$ an admissible map satisfying the following condition :

(12.1.1) for some neighbourhood V of the fixed point set $\varkappa_f$, the restriction $f \mid V$ is compact.

For such f we define the fixed point index of f by putting

$$\text{(12.1.2)} \qquad \text{Ind}\,(f) = \text{Ind}\,(f \mid V).$$

[Example : every admissible map which is locally compact satisfies condition (12.1.1).]

With the above definition, we have the following generalization of theorem (10.1) :

(12.1) THEOREM. — *Let* $\mathfrak{C}$ *be the category of metric* ANR-s *and* $\mathfrak{A}$ *a class of all admissible maps satisfying condition* (12.1.1). *Assume further that, given an admissible homotopy* h_t *there is a neighbourhood* W *of* $\varkappa(\,;\, h_t\,;)$ *such that* h_t *is compact on* W. *Then the function* $f \to$ Ind (f) *defined by* (12.1.2) *satisfies properties* (I)-(VII). *In* (VI), *it is assumed that* f *is compact in some neighbourhood of* $\varkappa\,(gf)$ *and in* (VII) *it is assumed that* f *is compact.*

13. The uniqueness of the fixed point index

Let $\mathfrak{C}_0$ be the category of open sets in finite dimensional normed space and $\mathfrak{A}_0$ the class of admissible maps. It can be proved that the Dold index Ind : $\mathfrak{A}_0 \to Z$, defined in section 6, is determined *uniquely* by properties (I)-(VII). We indicate now how the uniqueness of Dold's index implies that of the other fixed point indices discussed in this paper.

Let $\mathfrak{C}_1$ (resp. $\mathfrak{C}_2$) be the category of open subsets of normed (resp. ANR) spaces, $\mathfrak{A}_1$ (resp. $\mathfrak{A}_2$) the class of admissible compact maps and assume that all admissible homotopies are compact. Let ind : $\mathfrak{A}_1 \to Z$ be an integer valued function satisfying properties (I)-(VII). The excision and commutativity imply that ind satisfies also the contraction property (similar to that in section 6). Since every compact map is compactly homotopic to a finite dimensional map, it follows by homotopy, excision and contraction, that the function ind is completely determined by its values on maps in $\mathfrak{A}_0$. Consequently, in view of the uniqueness of Dold's index on $\mathfrak{A}_0$, ind must coincide with the Leray-Schauder index Ind.

Next, let ind : $\mathfrak{A}_2 \to Z$ be defined on $\mathfrak{A}_2$ and assume that it satisfies properties (I)-(VII). Let $f : U \to X$ be a map in $\mathfrak{A}_2$, V be an open set in as normed space which r-dominates X with $s : X \to V$, $r : V \to X$, $rs = 1$. By commutativity applied to maps

$$s : X \to V, \qquad fr : r^{-1}(U) \to X$$

we get

$$\text{ind}\,(frs \mid s^{-1} r^{-1}(U)) = \text{ind}\,(f) = \text{ind}\,(sfr \mid r^{-1}(U)).$$

Thus if the function ind satisfies commutativity, then it is completely determined by its values on maps in $\mathfrak{A}_1$. Consequently, if it satisfies also properties (I)-(VI), it must necessarily be the unique extension of the Leray Schauder index from $\mathfrak{A}_1$ over $\mathfrak{A}_2$. Thus we get the uniqueness of the index constructed in section 7.

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THE LEFSCHETZ FIXED POINT THEOREM FOR SOME CLASSES OF NON-METRIZABLE SPACES (1)

By Gilles FOURNIER and Andrzej GRANAS

Let X be a topological Hausdorff space. Call X a Lefschetz space provided for any continuous compact map $f : X \to X$ the generalized Lefschetz number $\Lambda(f)$ is defined and $\Lambda(f) \neq 0$ implies that f has a fixed point. It is known [5] that any ANR (metric) is a Lefschetz space. In this note, being concerned with the extension of the above result to the non-metrizable case, we show that the following types of spaces are Lefschetz spaces :

(i) open subsets in admissible linear topological spaces (in the sense of Klee [10]);

(ii) all NES (compact) spaces (in the sense of Hanner [7]) and, in particular, all ANR (normal) spaces;

(iii) approchable NES (metric) spaces.

We remark that the problem of whether an arbitrary NES (metric) is a Lefschetz space remains open. The authors thank J. Dugundji for several helpful discussions.

1. Preliminaries. — In all that follows, by space we shall understand a Hausdorff topological space and by map a continuous transformation. For a space $Y \subset X$ we let $\mathrm{Cov}_X(Y)$ denote the directed set of all open coverings of Y in X.

Let Y be a space and $\alpha \in \mathrm{Cov}(Y)$. Two maps $f, g : X \to Y$ of a space X into Y are said to be *α-close* provided $f(x)$ and $g(x)$ belong to a common $U_x \in \alpha$ for each $x \in X$; a homotopy $h_t : X \to Y$ $(0 \leq t \leq 1)$ is said to be an *α-homotopy* if for each $x \in X$ the values of $h_t(x)$ belong to a common $U_x \in \alpha$ for all $t \in [0, 1]$. Two maps $f, g : X \to Y$ are *α-homotopic*

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(written $f \underset{\alpha}{\sim} g$) if there is an α-homotopy $h_t : X \to Y$ joining f and g. Clearly $f \underset{\alpha}{\sim} g$ implies that f and g are α-close.

Let $f : X \to X$ be a map and $\alpha \in \mathrm{Cov}(X)$. A point $x \in X$ is a *fixed point for* f if $f(x) = x$; x is said to be an *α-fixed point for* f provided x and $f(x)$ belong to a common $U_x \in \alpha$. Clearly, if $\alpha, \beta \in \mathrm{Cov}(X)$ and α refines β, then every α-fixed point for f is also a β-fixed point for f.

(1.1) *Let* $f : X \to X$ *be a map. The following statements are equivalent:*

(i) *f has a fixed point;*

(ii) *there is a cofinal family of coverings* $\mathcal{D} = \{\alpha\} \subset \mathrm{Cov}_X(Y)$ *of* $Y = f(X)$ *in* X *such that f has an α-fixed point for every* $\alpha \in \mathcal{D}$.

Proof. — (i) $\Rightarrow$ (ii) is evident. To show (ii) $\Rightarrow$ (i), assume that f has no fixed points. Then for each $y \in Y$ there are neighbourhoods V_y and $U_{f(y)}$ in X of y and $f(y)$ respectively such that $f(V_y) \subset U_{f(y)}$ and $V_y \cap U_{f(y)} = 0$. Putting $\beta = \{V_y\}$ we get a covering of Y in X such that f has no β-fixed point. If α is a member of $\mathcal{D}$ that refines β then f has no α-fixed point and thus we obtain a contradiction with (ii).

(1.2) *Let* $f : X \to X$ *be a map,* $Y = f(X)$ *and let* $\mathcal{D} = \{\alpha\} \subset \mathrm{Cov}_X(Y)$ *be a cofinal family of coverings of* Y *in* X. *Assume that for each* $\alpha \in \mathcal{D}$ *there is a map* $f_\alpha : X \to X$ *satisfying the following properties:*

(i) *f and f_α are α-close;*

(ii) *f_α has a fixed point.*

Then f has a fixed point.

Proof. — Clearly, because of (i), a fixed point for f_α is an α-fixed point for f. Hence (1.2) follows from (1.1).

2. The Leray trace. — In our considerations an essential use will be made of the notion of the generalized trace and the Lefschetz number as given by J. Leray in [11]. We shall consider vector spaces over a fixed field K.

Let $f : E \to E$ be an endomorphism of an arbitrary vector space E. Let us put $N(f) = \{x \in E : f^{(n)}(x) = 0 \text{ for some } n\}$ ($f^{(n)}$ is the n-th iterate of f) and $\tilde{E} = \dfrac{E}{N(f)}$. Since $f(N(f)) \subset N(f)$, we have the induced endomorphism $\tilde{f} : \tilde{E} \to \tilde{E}$. Call f admissible provided $\dim \tilde{E} < \infty$; for such f we define the *generalized trace* $\mathrm{Tr}(f)$ of f by putting $\mathrm{Tr}(f) = \mathrm{tr}(\tilde{f})$, where tr stands for the ordinary trace.

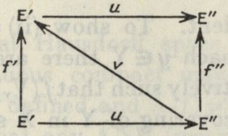
(2.1) Definition. — Let $f = \{f_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. Call f the *Leray endo-*

morphism if (i) all f_q are admissible; (ii) almost all $\tilde{E}_q$ are trivial. For such f we define the (generalized) *Lefschetz number* $\Lambda(f)$ by putting

$$\Lambda(f) = \sum_q (-1)^q \operatorname{Tr}(f_q).$$

The following important property of the Leray endomorphisms is a consequence of the well-known formula $\operatorname{tr}(uv) = \operatorname{tr}(vu)$ for the ordinary trace :

(2.2) *Assume that in the category of graded vector spaces the following diagram commutes*

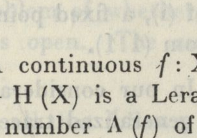


Then, if f' *or* f'' *is a Leray endomorphism, then so is the other and in that case* $\Lambda(f') = \Lambda(f'')$.

3. Lefschetz maps. — Let H be the singular homology functor (with coefficients in the field K) from the category of topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree 0. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q-dimensional singular homology group of X. For a continuous map $f: X \to Y$, $H(f)$ is the induced linear map $f_* = \{f_q\}$, where $f_q: H_q(X) \to H_q(Y)$.

(3.1) Definition. — A continuous $f: X \to X$ is called a *Lefschetz map* provided $f_*: H(X) \to H(X)$ is a Leray endomorphism. For such f we define the Lefschetz number $\Lambda(f)$ of f by putting $\Lambda(f) = \Lambda(f_*)$. Clearly, if f and g are homotopic, $f \sim g$, then $\Lambda(f) = \Lambda(g)$.

(3.2) Lemma. — *Assume that in the category of topological spaces the following diagram commutes :*

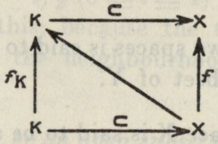


Then (i) *if one of the maps* f *or* g *is a Lefschetz map, then so is the other and in that case* $\Lambda(f) = \Lambda(g)$, (ii) f *has a fixed point if and only if* g *does.*

Proof. — The first part follows (by applying the homology functor to the above diagram) from (2.2). The second part is obvious.

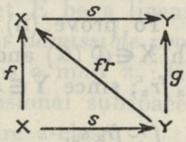
The following are typical instances in which the above lemma is used.

(3.3) Example. — Let $f: X \to X$ be a map such that $f(X) \subset K \subset X$. Then we have the commutative diagram :



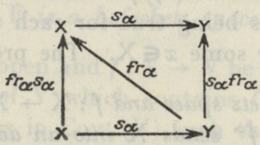
with the obvious contractions [2].

(3.4) Example. — Let $r: Y \to X$, $s: X \to Y$ be a pair of maps such that $rs = 1_X$; X is said to be *r-dominated* by Y and r is said to be an r-map. In this situation, given a map $f: X \to X$, we have the commutative diagram



with $g = sfr$.

(3.5) Example. — Let X and Y be two spaces and $\alpha \in \mathrm{Cov}(X)$. We say that X is *α-dominated* by Y provided there are maps $s_\alpha: X \to Y$, $r_\alpha: Y \to X$ such that $r_\alpha s_\alpha \underset{\alpha}{\sim} 1_X$. In this case, given an arbitrary $f: X \to X$, we have the commutative diagram



with $f \underset{\alpha}{\sim} fr_\alpha s_\alpha$.

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[2] Let $f: X \to Y$ be a map such that $f(A) \subset B$, where $A \subset X$ and $B \subset Y$. By the *contraction* of f to the pair (A, B), we understand a map $f': A \to A$ with the same values as f. A contraction of f to the pair (A, Y) is simply the restriction $f|A$ of f to A.

4. Lefschetz spaces.

Notation. — Given a class of spaces $\mathfrak{T}$ we denote by $\mathcal{R}(\mathfrak{T})$ and $\mathcal{D}(\mathfrak{T})$ two classes of spaces defined as follows :

$X \in \mathcal{R}(\mathfrak{T}) \Leftrightarrow$ there is a space $Y \in \mathfrak{T}$ which r-dominates X;

$X \in \mathcal{D}(\mathfrak{T}) \Leftrightarrow$ for each $\alpha \in \mathrm{Cov}(X)$ there is a space $Y_\alpha \in \mathfrak{T}$ which α-dominates X.

A map $f: X \to Y$ between two spaces is said to be *compact* provided $f(X)$ is contained in a compact subset of Y.

(4.1) Definition. — A space X is said to be a *Lefschetz space* provided any compact map $f: X \to X$ is a Lefschetz map and $\Lambda(f) \neq 0$ implies that f has a fixed point. We denote by $\mathcal{L}$ the class of all Lefschetz spaces.

Some general properties of Lefschetz spaces are summarized in the following two theorems ([3]).

(4.2) *Let* $\mathfrak{T}$ *be a class of Lefschetz spaces. Then* (i) $\mathcal{R}(\mathfrak{T}) \subset \mathcal{L}$ *and* (ii) $\mathcal{D}(\mathfrak{T}) \subset \mathcal{L}$.

Proof. — Clearly (ii) $\Rightarrow$ (i). To prove (ii) consider the commutative diagram in Example (3.5) with $X \in \mathcal{D}(\mathfrak{T})$ and $Y \in \mathfrak{T}$. The compactness of $f: X \to X$ implies that of $s_\alpha f r_\alpha$; since $Y \in \mathcal{L}$ and

(1) $$f \underset{\alpha}{\sim} f r_\alpha s_\alpha,$$

it follows, by Lemma (3.2), that f is a Lefschetz map and

(2) $$\Lambda(s_\alpha f r_\alpha) = \Lambda(f r_\alpha s_\alpha) = \Lambda(f).$$

Assume $\Lambda(f) \neq 0$; then (2) and $Y \in \mathcal{L}$ imply that $s_\alpha f r_\alpha$ has a fixed point; hence [by the second part of Lemma (3.2)] $f r_\alpha s_\alpha$ also has a fixed point. Now because [in view of (1)] f is α-close to $f r_\alpha s_\alpha$, it follows that f has an α-fixed point. This being true for each $\alpha \in \mathrm{Cov}(X)$, we conclude by (1.1) that $f(x) = x$ for some $x \in X$. The proof is completed.

(4.3) *Let X be a Lefschetz space and* $f: X \to X$ *be a compact map such that for some m the iterate* f^m *sends* X *into an acyclic subset of* X. *Then* $\Lambda(f) = 1$ *and hence f has a fixed point.*

For a proof *see* [6].

([3]) The second part of (4.2) is due to J. Dugundji (unpublished) and (in slightly less general form) to Jaworowski-Powers [8]; the prototype of theorem (4.3) was first established by Bourgin [1].

5. Fixed points in admissible linear spaces. — Let U be a neighbourhood of the origin in a linear topological space E. Then U is *shrinkable* provided for any $x \in \mathrm{cl}(U)$ and $0 < \lambda < 1$ the point λx lies in U. It is known (*cf.* Klee [9]) that the shrinkable neighbourhoods form a base of E at 0. It follows that given an arbitrary neighbourhood W of 0 there is a shrinkable neighbourhood V of 0 such that $V + V \subset W$ and such that any interval $tx + (1 - t)y$ $(0 \leq t \leq 1)$ with x and y in V is entirely contained in W. From this, because the topological structure of E is determined by a base of the neighbourhoods of the origin, we deduce the following :

(5.1) Lemma. — *Let* U *be an open subset of a linear topological space* E. *Then for each* $\alpha \in \mathrm{Cov}(U)$ *there exists a refinement* $\beta \in \mathrm{Cov}(U)$ *such that any two* β*-close maps of any space* X *into* U *are stationarily* α*-homotopic* (4).

Remark. — Proposition (5.1) implies that U is locally equiconnected, i. e. the diagonal $\Delta \subset U \times U$ is a strong neighbourhood deformation retract in $U \times U$ (*cf.* Dugundji [4]).

(5.2) Definition. — Let E be a linear topological space. We say (following Klee [9]) that E is *admissible* provided for any compact $K \subset E$ and any $\alpha \in \mathrm{Cov}_E(K)$ there is a map $\pi_\alpha : K \to E$ such that (i) $\pi_\alpha(K)$ is contained in a finite dimensional subspace of E and (ii) the inclusion $i : K \to E$ and $\pi_\alpha : K \to E$ are α-close.

(5.3) Examples. — (i) Every locally convex space is admissible [14]; (ii) the linear metric space S of measurable functions on [0, 1] is admissible [15]; (iii) the linear metric spaces $L_p(0, 1)$ $(0 < p < 1)$ are admissible [15].

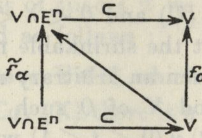
We may state now the first result of this note :

(5.4) Theorem. — *Let* E *be an admissible linear topological space. Then every open subset of* E *is a Lefschetz space.*

Proof. — Let $V \subset E$ be open and $f : V \to V$ be a compact map. Denote by K a compact subset of V which contains $f(V)$. Let $\mathcal{O} = \{\alpha\}$ be a cofinal family of coverings in $\mathrm{Cov}_V(K)$ such that each member of $\alpha \in \mathcal{O}$ is of the form $y + U$, where $y \in K$ and U is a shrinkable neighbourhood of the origin in E. Let $\alpha \in \mathcal{O}$; take $\pi_\alpha : K \to E$ satisfying $\pi_\alpha(K) \subset E^n \subset E$ for some n and such that $i : K \to E$ and $\pi_\alpha : K \to E$ are α-close. Define

(4) A homotopy $h_t : X \to Y$ joining f and g is *stationary* provided $h_t(x)$ is constant $(0 \leq t \leq 1)$ whenever $f(x) = g(x)$.

a compact map $f_\alpha : V \to V$ by putting $f_\alpha(x) = \pi_\alpha f(x)$. Clearly, we have the commutative diagram



with the obvious compact contractions. Since $V \cap E^n$ is a Lefschetz space (*cf.* [5]) it follows by Lemma (3.2) that $\tilde{f}_\alpha$ is a Lefschetz map and $\Lambda(\tilde{f}_\alpha) = \Lambda(f_\alpha)$. In view of Lemma (5.1) we may assume without loss of generality that f is homotopic to f_α for each α; consequently f is a Lefschetz map and $\Lambda(f) = \Lambda(f_\alpha)$. Now, assuming $\Lambda(f) \neq 0$, we have for each α, $\Lambda(f_\alpha) \neq 0$, and hence, because $V \cap E^n$ is a Lefschetz space, we get for each $\alpha \in \mathcal{D}$ a fixed point for f_α. Applying (1.2) we get a fixed point for f and the proof is completed.

(5.5) Corollary. — *The following types of spaces have the fixed point property within the class of all compact maps :*

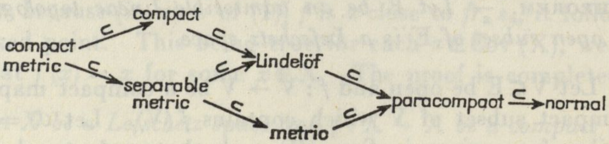
(i) *acyclic open sets in admissible spaces;*

(ii) *all admissible linear topological spaces;*

(iii) *closed shrinkable neighbourhoods in admissible spaces.*

Proof. — (i) and (ii) are evident. (iii) follows from (ii) because a closed shrinkable neighbourhood is a retract of the containing space (*cf.* Klee [9]).

6. Neighbourhood Extension Spaces. — In this section we recall the definition and some basic properties of the NES (Q) spaces. In all that follows Q (or Q′) will denote any of the classes of spaces appearing in the following diagram :

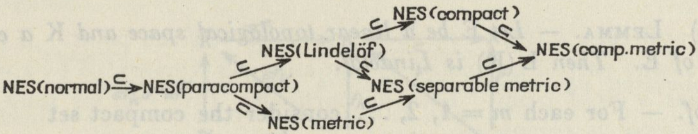


(6.1) Definition. — A space Y is an *extension* (resp. *neighbourhood extension*) *space* for Q provided for any pair (X, A) in Q with $A \subset X$ closed and any map $f_0 : A \to Y$ there is an extension $f : X \to Y$ (resp. neighbourhood extension $f : U \to Y$) of f_0 over X (resp. over a neighbourhood U of A in X). The classes of the extension spaces and the neighbourhood extension spaces for Q will be denoted by ES (Q) and NES (Q), respectively.

Clearly ES (Q) ⊂ NES (Q) and

$$Q \subset Q' \text{ implies } NES(Q') \subset NES(Q).$$

Thus, various inter-relations between different classes of NES-spaces may be displayed in the following diagram :



(6.2) Examples.

(i) the unit interval [0, 1] is an extension space for normal spaces (Tietze-Uryhson);

(ii) any convex subset of a locally convex linear space (or of a linear space with finite topology) is an extension space for metric spaces (Dugundji [2]);

(iii) any polytope with CW-topology is a neighbourhood extension space for metric spaces.

Other examples may be derived by means of the following theorem (*cf.* Hanner [7]) :

(6.3) *These are some basic properties of the NES-spaces :*

(i) *A retract of* ES (Q) *is an* ES (Q);

(ii) *A neighbourhood retract of* NES (Q) *is an* NES (Q);

(iii) *An open subset of* NES (Q) *is an* NES (Q);

(iv) *The product of any collection* (*resp. any finite collection*) *of* ES (Q) [*resp.* NES (Q)] *is* ES (Q) [*resp.* NES (Q)];

(v) *If* Q *consists only of paracompact spaces, then any local* NES (Q) *is* NES (Q).

(6.4) Definition. — A space Y is an *absolute retract* (resp. *absolute neighbourhood retract*) for a class of spaces Q provided (i) Y is in Q and (ii) whenever it is embedded in another space of Q as a closed subset it is a retract (resp. neighbourhood retract) of the containing space. The corresponding classes of spaces are denoted by AR (Q) and ANR (Q), respectively.

The following theorem describes the relation between the NES and the ANR spaces (*cf.* Hanner [7]).

(6.5) *Any* AR (Q) *is* ES (Q); *similarly, any* ANR (Q) *is* NES (Q).

7. Fixed Point Theorem for the NES (compact) spaces

Notation. — Given a linear space E and $K \subset E$ we denote by L (K) the linear span of K, i. e. the smallest linear subspace of E which contains K.

We begin with the following lemma (*cf.* [13]) :

(7.1) Lemma. — *Let* E *be a linear topological space and* K *a compact subset of* E. *Then* L (K) *is Lindelöf.*

Proof. — For each $m = 1, 2, \ldots$ consider the compact set

$$X_m = \underbrace{[-m, m] \times \ldots \times [-m, m]}_{m\text{-times}} \times \underbrace{K \times \ldots \times K}_{m\text{-times}}$$

the function $f_m : X_m \to E$ given by the assignment

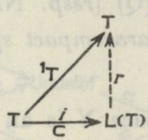
$$(\lambda_1, \ldots, \lambda_m, x_1, \ldots, x_m) \to \sum_{j=1}^{m} \lambda_i x_i$$

and put $K_m = f_m(X_m)$. Since f_m is continuous, each K_m is compact. It is clear that any element $y = \sum_{i=1}^{n} \mu_i x_i$ ($\mu_i \in R$, $x_i \in K$) of L (K) belongs to K_m for some m; thus $L(K) = \cup K_m$, i. e. L (K) is σ-compact.

Since L (K) is regular the assertion readily follows.

(7.2) Lemma. — *Let* E *be a linear topological space and* T *be a Tychonoff cube contained in* E. *Then* T *is a retract of* L (T).

Proof. — $T \in ES$ (normal). Hence the diagram :



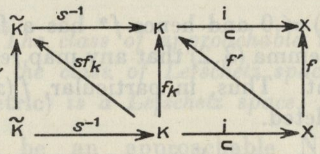
can be completed because, in view of Lemma (7.1), L (T) is normal.

We state now our next result.

(7.3) Theorem. — *Every* NES (compact) *is a Lefschetz space* ([5]).

([5]) In connection with this result *see* also R. Knill [11].

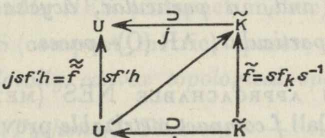
Proof. — Let X be a NES (compact) and $f : X \to X$ be a compact map : denote by K a compact set containing $f(X)$. We embed K into a Tychonoff cube T lying in a locally convex space E. Denote by $s : K \to \tilde{K}$ the homeomorphism of K onto $\tilde{K} \subset T$. We may write now the following commutative diagram :



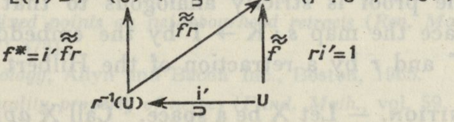
in which f' and f_k stand for the obvious compact contractions of f.

Next, consider the map $is^{-1} : \tilde{K} \to X$. Since $X \in$ NES (compact), there is an open set U in T containing $\tilde{K}$ and an extension $h : U \to X$ of is^{-1} over U; thus, if $j : K \to U$ is the inclusion we have $hj = is^{-1}$.

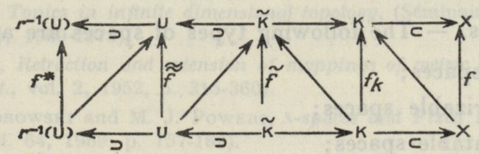
We note that the following diagram commutes :



Now take L (T) in a locally convex space E which contains T, the retraction $r : L(T) \to T$ [which exists in view of (6.2)] and write the commutative diagram :



Putting the above diagrams together, we have the following commutative diagram :



in which f^* is a compact self-map of an open set r^{-1} (U) in a locally convex space L (T). Since r^{-1} (U) is a Lefschetz space, $\Lambda(f^*)$ is defined. It follows now from Lemma (3.2) that all the vertical arrows in the above diagram represent Lefschetz maps and

$$\Lambda(f) = \Lambda(f_k) = \Lambda(\tilde{f}) = \Lambda(\tilde{\tilde{f}}) = \Lambda(f^*).$$

If $\Lambda(f) \neq 0$ then $\Lambda(f^*) \neq 0$ and hence f^* has a fixed point. If follows by the second part of Lemma (3.2) that any map represented by a vertical arrow has a fixed point. Thus, in particular, $f(x) = x$ for some $x \in X$ and the proof is completed.

The following are two immediate consequences of Theorem (7.3) :

(7.4) Corollary. — *Let* Q *be any of the following* : (i) *Lindelöf*; (ii) *paracompact*; (iii) *normal. Then any* NES (Q) *and in particular any* ANR (Q) *is a Lefschetz space.*

(7.5) Corollary. — *Let* Q *be as in* (7.4). *The following types of spaces have the fixed point property within the class of all compact maps :*

(i) *acyclic* NES (Q) *and, in particular, acyclic* ANR (Q)-*spaces;*

(ii) ES (Q) *and, in particular,* AR (Q)-*spaces.*

8. Fixed points in approachable NES (metric) spaces. — Let $f : X \to Y$ be a map. Call f *compact metrizable* provided $f(X)$ is contained in a compact metrizable subset of Y.

(8.1) Theorem. — *Let* $X \in$ NES (compact metric) *and* $f : X \to X$ *be a compact metrizable map. Then* (*i*) $\Lambda(f)$ *is defined and* (*ii*) $\Lambda(f) \neq 0$ *implies that* f *has a fixed point.*

Proof. — The proof is strictly analogous to that of Theorem (6.1) except we replace the map $s : K \to T$ by the embedding of K into the Hilbert cube I^∞ and r by a retraction of the Hilbert space onto I^∞.

(8.2) Definition. — Let X be a space. Call X *approachable* provided for any compact subset K of X and any $\alpha \in \mathrm{Cov}_X(K)$ there exists a map $\pi_\alpha : K \to X$ such that (i) $\pi_\alpha(K)$ is metrizable; (ii) the maps $\pi_\alpha : K \to X$ and $i : K \to X$ are α-close; (iii) π_α and i are homotopic.

(8.3) Examples. — The following types of spaces are approachable :

(i) metrizable spaces;

(ii) locally metrizable spaces;

(iii) second countable spaces;

(iv) admissible linear topological spaces (in the sense of Definition (5.3);

(v) linear spaces with finite topology and polytopes with CW-topology, [3];

(vi) open subsets of convex sets in linear locally convex topological spaces.

(8.4) THEOREM. — *The class of approachable* NES (compact metric)-*spaces is contained in the class of Lefschetz spaces. In particular, any approachable* NES (metric) *is a Lefschetz space.*

Proof. — Let X be an approachable NES (compact metric), $f: X \to X$ be a compact map and K be a compact set containing $f(X)$. For $\alpha \in \mathrm{Cov}_X(X)$ define $f_\alpha : X \to X$ by $f_\alpha = \pi_\alpha f$. Clearly, by approachability, we have (i) $f_\alpha \sim f$; (ii) f_α and f are α-close; (iii) f_α is compact metrizable. Now our assertion follows, in view of (1.2), from Theorem (8.1).

(8.5) COROLLARY. — *The following types of spaces have the fixed point property within the class of all compact maps* :

(i) *acyclic approachable* NES (compact metric) *spaces*;

(ii) *approachable* ES (compact metric) *spaces*;

(iii) *convex sets in locally convex topological spaces;*

(8.6) COROLLARY. (*Tychonoff Theorem.*) — *Any compact convex set in a locally convex topological space has the fixed point property.*

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60, 1981, p. 361 à 373.

SOME GENERAL THEOREMS IN COINCIDENCE THEORY. I

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1. Introduction

Given two continuous maps $p, q : \Gamma \to X$ of Hausdorff topological spaces the coincidence problem for (p, q) is concerned with conditions which guarantee that the pair (p, q) admits one or more coincidence points, that is points $y \in \Gamma$ such that $p(y) = q(y)$. The study of this problem (first treated in a topological setting in 1946 by Eilenberg-Montgomery [3]) was recently taken up by the first author in [6], where an extension of the Eilenberg-Montgomery coincidence Theorem to non-compact ANR-*s* was established. In this paper our aim is to initiate the systematic study of the coincidence problem in a framework of theory of retracts and under possibly general assumptions concerning p and q. Our approach has as its starting point the Vietoris Mapping Theorem and is based on some simple notions of the category theory in the sense of Eilenberg-MacLane; this approach was presented by the second author at the Conference on Fixed Point Theory in Halifax in 1975.

To formulate our main result we need some terminology. We recall that a topological space X is called a NES (compact) provided for each compact pair (Y, A) any map $f_0 : A \to X$ admits an extension $f : U \to X$ over some neighbourhood U of A. The class of NES (compact) spaces contains arbitrary metric ANR-*s* and also non-metrizable compact ANR-*s*.

By homology we shall understand the Čech homology with compact supports and rational coefficients. A continuous map $p : \Gamma \to X$ is said to be a Vietoris map provided p is proper and $p^{-1}(x)$ is acyclic for each $x \in X$. The Vietoris-Begle Mapping Theorem implies that if $p : \Gamma \to X$ is Vietoris, then the induced map $p_* : H(\Gamma) \to H(X)$ is an isomorphism.

Let X be a NES (compact) space $p, q : \Gamma \to X$ be a pair of maps such that p is Vietoris and q is a compact map [i.e. $q(\Gamma)$ is relatively compact in X]. Using the generalized trace theory of J. Leray [11] and properties of Vietoris maps we shall assign to the pair (p, q) a number:

$$\Lambda(p, q) = \sum_{n \geq 0} (-1)^n \operatorname{Tr}(q_{*n} \circ p_{*n}^{-1}),$$

defined in terms of the induced homomorphism p_* and q_* [and called the Lefschetz number of (p, q)]. The main result which we intend to present here is the following Theorem: *If* $\Lambda(p, q) \neq 0$ *then the maps* $p, q: \Gamma \to X$ *have a coincidence.*

This result implies several well-known coincidence and fixed point Theorems (for single valued as well as for multivalued maps) both in functional analysis and topology. Other coincidence results will be treated in the second part of this paper.

2. Algebraic preliminaries

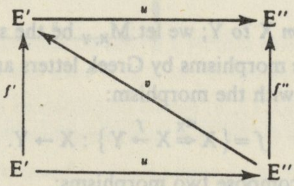
We begin by recalling the notion of the generalized Lefschetz number as given by J. Leray in [11]. We consider vector spaces only over the field of rational numbers Q.

Let $f: E \to E$ be an endomorphism of an arbitrary vector space E. Put $N(f) = \{x \in E \mid f^{(n)}(x) = 0 \text{ for some } n\}$ ($f^{(n)}$ is the n-th iterate of f) and $\tilde{E} = E/N(f)$. Since $f(N(f)) \subset N(f)$, we have the induced endomorphism $\tilde{f}: \tilde{E} \to \tilde{E}$. Call f admissible provided $\dim \tilde{E} < \infty$; for such f we define the *generalized trace* $\mathrm{Tr}(f)$ of f by putting $\mathrm{Tr}(f) = \mathrm{tr}(\tilde{f})$, where tr stands for the ordinary trace.

(2.1) DEFINITION. — Let $f = \{f_q\}: E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. Call f the *Leray-endomorphism* if: (i) all f_q are admissible; (ii) almost all E_q are trivial. For such f we define the (generalized) *Lefschetz number* $\Lambda(f)$ by putting $\Lambda(f) = \sum_q (-1)^q \mathrm{Tr}(f_q)$.

The following property of the Leray endomorphisms [11] is a consequence of the well-known formula $\mathrm{tr}(uv) = \mathrm{tr}(vu)$ for the ordinary trace:

(2.2) *Assume that in the category of graded vector spaces the following diagram commutes:*



If one of the vertical arrows is a Leray endomorphism, then so is the other and in that case $\Lambda(f') = \Lambda(f'')$.

3. Vietoris maps

In what follows by space we understand a Hausdorff topological space and by map a continuous transformation. By H we denote the Čech homology functor with compact carriers and coefficients in Q from the category $\mathcal{T}op$ of topological spaces to the category $\mathcal{E}$ of graded vector spaces over Q and linear maps of degree zero.

A map $p: \Gamma \to X$ between two spaces Γ and X is called *Vietoris* (written $p: \Gamma \Rightarrow X$) if: (i) p is proper (*) and (ii) for each $x \in X$ the set $p^{-1}(x)$ is acyclic.

(*) In this paper p is understood to be *proper* provided for each net $p(y_\alpha) \to x$ there is a subnet $\{y_{\alpha\beta}\}$ and a $y \in \Gamma$ such that $y_{\alpha\beta} \to y$ and $x = p(y)$.

Some important properties of Vietoris maps are summarized in the following.

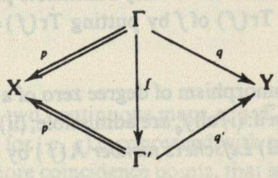
(3.1) THEOREM. – (i) (Vietoris-Begle). *If* $p:\Gamma\Rightarrow X$, *then the induced linear map* $p_*:H(\Gamma)\to H(X)$ *is an isomorphism;*

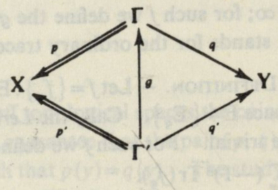
(ii) *if* $p:\Gamma\Rightarrow X$ *and* $p':X\Rightarrow Y$ *are Vietoris, then so also is the composite* $p'\circ p:\Gamma\Rightarrow Y$;

(iii) *the pull-back of a Vietoris map is also a Vietoris map.*

4. Category of morphisms

Given two spaces X and Y let $\mathscr{D}(X, Y)$ be the set of all diagrams of the form $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\rightarrow} Y$. Every such a diagram we denote briefly by (p, q). Given two diagrams $(p, q), (p', q') \in \mathscr{D}(X, Y)$, we write $(p, q) \sim (p', q')$ if there are maps $f:\Gamma\to\Gamma'$ and $g:\Gamma'\to\Gamma$ for which the following two diagrams commute:





Clearly $\sim$ is an equivalence relation in $\mathscr{D}(X, Y)$.

(4.1) DEFINITION. – The equivalence class of a diagram $(p, q)\in\mathscr{D}(X, Y)$ with respect to $\sim$ is denoted by:

$$\varphi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\rightarrow} Y\} : X \to Y$$

and is called a *morphism form* X *to* Y; we let $M_{X,Y}$ be the set of all such morphisms.

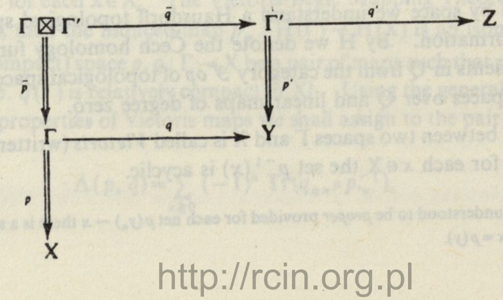
In what follows we denote morphisms by Greek letters and the ordinary maps by Latin letters; we identify $f:X\to Y$ with the morphism:

$$f = \{X \overset{id_X}{\Leftarrow} X \overset{f}{\rightarrow} Y\} : X \to Y.$$

(4.2) DEFINITION. – To compose two morphisms:

$$\varphi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\rightarrow} Y\} : X \to Y \quad \text{and} \quad \psi = \{Y \overset{p'}{\Leftarrow} \Gamma' \overset{q'}{\rightarrow} Z\} : Y \to Z$$

we write a commutative diagram:



in which $\Gamma \boxtimes \Gamma'$ is the fibre product of q and p' and $\bar{p}, \bar{q}$ is the pull-back of p', q; we define the composite $\psi \circ \varphi$ of φ and ψ by:

$$\psi \circ \varphi = \{ X \overset{p\bar{p}}{\Leftarrow} \Gamma \boxtimes \Gamma' \overset{q'\bar{q}}{\rightarrow} Z \} : X \rightarrow Z.$$

(4.3) *The composition law given above converts the collection $\mathscr{M}$ of morphisms into a category.*

In what follows the category of topological spaces $\mathscr{T}op$ is regarded in a natural way as a subcategory of $\mathscr{M}$.

(4.4) *Homology functor* H : $\mathscr{T}op \rightarrow \mathscr{E}$ *extends over* $\mathscr{M}$ *to a functor* $\hat{\mathrm{H}} : \mathscr{M} \rightarrow \mathscr{E}$.

Proof. – For a morphism:

$$\varphi = \{ X \overset{p}{\Leftarrow} \Gamma \overset{q}{\rightarrow} Y \} : X \rightarrow Y$$

we let $\hat{\mathrm{H}}(\varphi) = \varphi_* = q_* \circ p_*^{-1}$; it is easily seen that the definition of φ_* does not depend on the choice of a representative of φ and also that if $\varphi = f$ then $\hat{\mathrm{H}}(\varphi) = f_*$. Given two morphisms:

$$\varphi = \{ X \overset{p}{\Leftarrow} \Gamma \overset{q}{\rightarrow} Y \} : X \rightarrow Y \quad \text{and} \quad \psi = \{ Y \overset{p'}{\Leftarrow} \Gamma' \overset{q'}{\rightarrow} Z \} : Y \rightarrow Z,$$

consider the commutative diagram:

$$\begin{array}{ccccc} \Gamma \boxtimes \Gamma' & \xrightarrow{\bar{q}} & \Gamma' & \xrightarrow{q'} & Z \\ \Big\downarrow{\scriptstyle \bar{p}} & & \Big\downarrow{\scriptstyle p'} & & \\ \Gamma & \xrightarrow{q} & Y & & \\ \Big\downarrow{\scriptstyle p} & & & & \\ X & & & & \end{array}$$

We have by the definition:

$$(\psi \circ \varphi)_* = (q'\bar{q})_* (p\bar{p})_*^{-1} = q'_* \bar{q}_* (\bar{p})_*^{-1} p_*^{-1} = [q'_* (p')_*^{-1}] \circ [q_* p_*^{-1}] = \psi_* \circ \varphi_*.$$

This completes the proof.

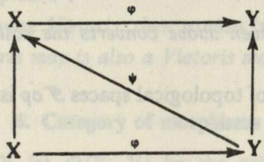
5. Lefschetz morphisms

Using the functor $\hat{\mathrm{H}}$ we may define for some self-morphisms a generalized Lefschetz number.

(5.1) Definition. – We say that $\varphi : X \rightarrow X$ is a *Lefschetz morphism* provided $\varphi_* : \mathrm{H}(X) \rightarrow \mathrm{H}(X)$ is a Leray endomorphism; for such φ we define the generalized Lefschetz number by putting $\Lambda(\varphi) = \Lambda(\varphi_*)$.

The following property of Lefschetz morphisms will be of importance:

(5.2) LEMMA. – *Assume that in the category of morphisms the following diagram commutes:*



Then: if one of the vertical arrows is a Lefschetz morphism, then so is the other and in that case $\Lambda(\Phi)=\Lambda(\Psi)$;

Proof. – By applying the functor $\hat{H}$ to the above diagram, our assertion follows at once from (2.2).

6. Coincidences

Let $p, q:\Gamma\to X$ be two maps and Cov(X) be the directed set of all open coverings of X. A point $y\in\Gamma$ is called a *coincidence point for* (p, q) if $p(y)=q(y)$; given $\alpha\in\mathrm{Cov}(X)$ we say that y is an *α-coincidence point for* (p, q) provided $p(y)$ and $q(y)$ belong to the same $U\in\alpha$.

(6.1) LEMMA. – *Let* $p, q:\Gamma\to X$ *be two maps such that p is proper and assume that for each* $\alpha\in\mathrm{Cov}(X)$ *there is an α-coincidence point for* (p, q). *Then there is a coincidence point for* (p, q).

Proof. – It is sufficient to show that if (p, q) has no coincidence point, then for each $x\in X$ there exists an open neighbourhood U_x of x in X such that: $p^{-1}(U_x)\cap q^{-1}(U_x)=\emptyset$.

Assume the contrary, i.e., that (p, q) has no coincidences and that there is a point $x_0\in X$ such that: $p^{-1}(U)\cap q^{-1}(U)\neq\emptyset$, for each open neighbourhood U of x_0 in X.

Let $\mathscr{B}(x_0)$ be the directed set of all open neighbourhood of x_0 in X with the partial order given by the inclusion. For each $U\in\mathscr{B}(x_0)$ we find a point $y_U\in\Gamma$ such that $p(y_U)\in U$ and $q(y_U)\in U$. Since $\lim_U p(y_U)=\lim_U q(y_U)=x_0$, and, because p is proper, there is a subnet of $\{y_U\}_{U\in\mathscr{B}(x_0)}$ converging to a point $y_0\in\Gamma$; by continuity of p and q we get $p(y_0)=q(y_0)=x_0$ and, with this contradiction, the proof is complete.

Let $\varphi: X\to X$ be a self-morphism and $(p, q), (p', q')$ be two representatives of φ. We let:

$$\chi_{(p,q)}=\{y\in\Gamma\mid p(y)=q(y)\},$$

$$\chi_{(p',q')}=\{y\in\Gamma'\mid p'(y)=q'(y)\}$$

and observe that:

$$p(\chi_{(p,q)})=p'(\chi_{(p',q')}).$$

(6.2) Definition. – We say that $\varphi : X \to X$ has a coincidence provided the set $\chi(\varphi) = p(\chi_{(p,q)})$ is not empty. Clearly φ has a coincidence if and only if for any representative (p, q) of φ the set $\chi_{(p,q)}$ is non-empty.

(6.3) Lemma. – *Let $\varphi : X \to Y$, $\psi : Y \to X$ be two morphisms. Then $\psi \circ \varphi$ has a coincidence if and only if so does $\varphi \circ \psi$.*

Proof. – Consider the following two commutative diagrams, which appear in the definition of $\psi \circ \varphi$ and $\varphi \circ \psi$.

$$\begin{array}{ccccc} \Gamma \boxtimes \Gamma' & \xrightarrow{\bar{q}} & \Gamma' & \xrightarrow{q'} & X \\ \downarrow{\scriptstyle \bar{p}} & & \downarrow{\scriptstyle p'} & & \\ \Gamma & \xrightarrow{q} & Y & & \\ \downarrow{\scriptstyle p} & & & & \\ X & & & & \end{array} \qquad \begin{array}{ccccc} \Gamma' \boxtimes \Gamma & \xrightarrow{\bar{q}'} & \Gamma & \xrightarrow{q} & Y \\ \downarrow{\scriptstyle \bar{p}'} & & \downarrow{\scriptstyle p} & & \\ \Gamma' & \xrightarrow{q'} & X & & \\ \downarrow{\scriptstyle p'} & & & & \\ Y & & & & \end{array}$$

It is enough to show that the set $\chi(p\bar{p}, q'\bar{q}) \neq \emptyset$ if and only if $\chi(p'\bar{p}', q\bar{q}') \neq \emptyset$. Let $(y, y') \in \Gamma \boxtimes \Gamma'$ and assume $(y, y') \in \chi(p\bar{p}, q'\bar{q})$. Then we have $p'(y') = q(y)$ and $p(y) = q'(y')$. Now, we see that $(y', y) \in \Gamma' \boxtimes \Gamma$ and $(y', y) \in \chi(p'\bar{p}', q\bar{q}')$. This completes the proof.

7. Coincidence spaces

Let $\varphi : X \to Y$ be a morphism and (p, q), be a representative of φ. We define $\varphi(X) \subset Y$ by $\varphi(X) = q[p^{-1}(X)]$. Observe that $\varphi(X)$ does not depend on the choice of the representative of φ.

(7.1) Definition. – A morphism $\varphi : X \to Y$ is called *compact* provided the set $\varphi(X)$ is relatively compact in Y.

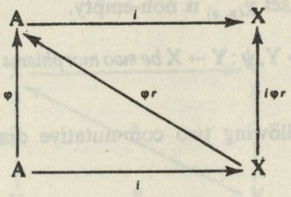
The following property of compact morphisms follows at once from the definitions:

(7.2) *Let $\varphi : X \to Y$ and $\psi : Y \to Z$ be two morphisms. If φ or ψ is compact, then so is the composite $\psi \circ \varphi : X \to Z$.*

(7.3) Definition. – A space X is said to be a *coincidence space* for compact morphisms, provided: (i) any compact morphism $\varphi : X \to X$ is a Lefschetz morphism and (ii) $\Lambda(\varphi) \neq 0$ implies that φ has a coincidence.

(7.4) Lemma. – *A retract of a coincidence space is a coincidence space.*

Proof. – Let $r: X \to A$ be a retraction and $\varphi: A \to A$ be an arbitrary compact morphism. Clearly, in the commutative diagram:

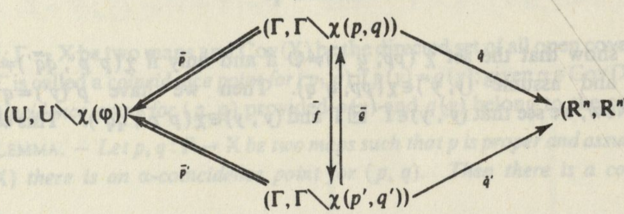


$i\varphi r$ is also compact and our assertion follows from (6.3).

We establish now two important special cases of the main result:

(7.5) Theorem. – *Every open* $U \subset R^n$ *is a coincidence space.*

Proof. – For the convenience of the reader we reproduce the main steps of the proof (*cf.* [7]). Let $U \subset R^n$ be open and $\varphi: U \to U$ be a compact morphism. It is evident that the set $\chi(\varphi)$ is compact. Consider two representatives $(p, q), (p', q')$ of φ and let $f: \Gamma \to \Gamma'$ and $g: \Gamma' \to \Gamma$ be two maps which establish the equivalence $(p, q) \sim (p', q')$. Then we have the following two commutative diagrams:



in which $\bar{p}, \bar{p}', \bar{f}, \bar{g}$ are mappings given by the same formulas as p, p', f, g respectively and:

$$\hat{q}(y) = p(y) - q(y), \qquad \hat{q}'(y') = p'(y') - q'(y').$$

Let $O_{\chi(\varphi)} \in H_n(U, U \setminus \chi(\varphi))$ be a fundamental class of the set $\chi(\varphi)$ (*see* Dold [2] and Górniewicz [7]) and let $I(p, q) = \hat{q}_*(p_*^{-1}(O_{\chi(\varphi)}))$.

From the commutativity of the above diagram it follows that $I(p, q) = I(p', q')$, i.e. $I(p, q)$ does not depend of a representative of φ. It is also evident that $I(p, q) \neq 0$ implies $\chi(\varphi) \neq \emptyset$ and therefore for the proof it is sufficient to show that $I(p, q) = \Lambda(\varphi)$.

Let K be a finite polyhedron such that $q(\Gamma) \subset K \subset U$. Let us consider the following maps:

$$\Delta: \quad (U, U \setminus K) \to (U, U \setminus K) \times U, \qquad \Delta(x) = (x, x),$$

$$d: \quad (U, U \setminus K) \times K \to (R^n, R^n \setminus \{0\}), \qquad d(x, y) = x - y,$$

$$t: \quad U \times K \to K \times U, \qquad t(x, y) = (y, x),$$

$$q_1: \quad \Gamma \to K, \qquad q_1(y) = q(y)$$

and the following homomorphisms :

$$\partial : \mathrm{H}(\mathrm{U}, \mathrm{U}\setminus\mathrm{K})\otimes\mathrm{H}(\mathrm{K}) \to (\mathrm{H}(\mathrm{K}))^*, \qquad \partial(u)(v) = d_*(u\otimes v),$$

where $(\mathrm{H}(\mathrm{K}))^* = \mathrm{Hom}(\mathrm{H}(\mathrm{K}), \mathrm{Q})$ is the dual of $\mathrm{H}(\mathrm{K})$:

$$e : \quad (\mathrm{H}(\mathrm{K}))^*\otimes\mathrm{H}(\mathrm{K}) \to \mathrm{Q}, \qquad e(f\otimes u) = f(u).$$

Let $a = (\partial\otimes\mathrm{Id})(\mathrm{Id}\otimes q_{1*}\, p_*^{-1})\,\Delta_*(\mathcal{O}_\mathrm{K}) \in (\mathrm{H}(\mathrm{K}))^*\otimes\mathrm{H}(\mathrm{K})$, where $\mathcal{O}_\mathrm{K}$ denotes the fundamental class of the compact set $\mathrm{K}\subset\mathrm{U}$.

From the commutativity of the diagram:

$$\begin{array}{ccccc}
\mathrm{H}(\mathrm{U},\mathrm{U}\setminus\mathrm{K}) & \xrightarrow{\Delta_*} & \mathrm{H}(\mathrm{U},\mathrm{U}\setminus\mathrm{K})\otimes\mathrm{H}(\mathrm{U}) & \xrightarrow{\mathrm{Id}_*\otimes q_{1*}\, p_*^{-1}} & \mathrm{H}(\mathrm{U},\mathrm{U}\setminus\mathrm{K})\otimes\mathrm{H}(\mathrm{K}) \\
 & \searrow^{q_*\circ p_*^{-1}} & \mathrm{I} \qquad \mathrm{II} & \swarrow_{d_*} & \downarrow{\scriptstyle \partial\otimes\mathrm{Id}_*} \\
 & & \mathrm{Q} \approx \mathrm{H}(\mathrm{R}^n, \mathrm{R}^n\setminus\{0\}) & \longleftarrow & \mathrm{H}(\mathrm{K}))^*\otimes\mathrm{H}(\mathrm{K})
\end{array}$$

we get:

(1) $$\mathrm{I}(p, q) = e(a).$$

Using the Dolds Lemma for Čech homology (*cf.* [7]) and the commutativity of the diagram:

$$\begin{array}{ccccc}
\mathrm{H}(\mathrm{U},\mathrm{U}\setminus\mathrm{K})\otimes\mathrm{H}(\mathrm{U})\otimes\mathrm{H}(\mathrm{K}) & \xrightarrow{\mathrm{Id}_*\otimes t_*} & \mathrm{H}(\mathrm{U},\mathrm{U}\setminus\mathrm{K})\otimes\mathrm{H}(\mathrm{K})\otimes\mathrm{H}(\mathrm{U}) & \xrightarrow{d_*\otimes\mathrm{Id}_*} & \mathrm{Q}\otimes\mathrm{H}(\mathrm{U}) \approx \mathrm{H} \\
\downarrow{\scriptstyle \partial\otimes q_{1*}\, p_*^{-1}\otimes\mathrm{Id}_*} & & \downarrow{\scriptstyle \partial\otimes\mathrm{Id}_*\otimes q_{1*}\, p_*^{-1}} & & \downarrow{\scriptstyle q_{1*}\, p_*^{-1}} \\
\mathrm{H}(\mathrm{K}))^*\otimes\mathrm{H}(\mathrm{K})\otimes\mathrm{H}(\mathrm{K}) & \xrightarrow{\mathrm{Id}_*\otimes t_*} & (\mathrm{H}(\mathrm{K}))^*\otimes\mathrm{H}(\mathrm{K})\otimes\mathrm{H}(\mathrm{K}) & \xrightarrow{e\otimes\mathrm{Id}_*} & \mathrm{Q}\otimes\mathrm{H}(\mathrm{K}) \approx \mathrm{H}(\mathrm{K})
\end{array}$$

we obtain:

(2) $$\Lambda(\varphi) = e(a).$$

By comparing (1) and (2) we get $\Lambda(\varphi) = \mathrm{I}(p, q)$ and the assertion of the theorem follows.

(7.6) THEOREM. — *Every open set* U *in a locally convex space* E *is a coincidence space.*

Proof. — Let $\varphi = \{\mathrm{U} \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} \mathrm{U}\} : \mathrm{U} \to \mathrm{U}$ be a compact morphism and let $\alpha \in \mathrm{Cov}(\mathrm{U})$. By the well-known approximation theorem (*cf.* [10]) there is a compact map $q_\alpha : \Gamma \to \mathrm{U}$ satisfying:

(i) for each $y \in \Gamma$ there is $\mathrm{V} \in \alpha$ such that:

$$q_\alpha(y) \in \mathrm{V} \quad \text{and} \quad q(y) \in \mathrm{V};$$

(ii) the maps q and q_α are homotopic;

(iii) there exists an open subset W of a finite dimensional subspace of E such that $\overline{q_\alpha(\Gamma)} \subset W \subset U$.

Consider now the following diagram:

$$\begin{array}{ccc} H(W) & \xrightarrow{\;i_*\;} & H(U) \\ \uparrow{\scriptstyle (\bar{q}_\alpha)_* \circ (p_1)_*^{-1}} & \nwarrow{\scriptstyle (q'_\alpha)_* p_*^{-1}} & \uparrow{\scriptstyle q_* \circ p_*^{-1}} \\ H(W) & \xrightarrow[\;i_*\;]{} & H(U) \end{array}$$

in which $\bar{q}_\alpha$, q'_α and p_1 are respectively contractions of q and p and $i : W \to U$ is the inclusion map.

From (ii) it follows that the above diagram is commutative. So by (2.2) and (7.5) $\Lambda((\bar{q}_\alpha)_*(p_1)_*^{-1}) = \Lambda(q_* \circ p_*^{-1})$ and thus φ is a Lefschetz morphism. If $\Lambda(\varphi) \neq 0$, then in view of (7.5), we obtain a coincidence point for $(p_1, \bar{q}_\alpha)$. By (i) this point is an α-coincidence point for (p, q) and hence our assertion follows from (6.1).

8. Neighbourhood extension spaces

We recall the definition and some basic properties of the NES-spaces. In what follows Q will denote either the class of compact spaces or the class of metric spaces.

(8.1) DEFINITION. – A space Y is an *extension* (resp. *neighbourhood extension*) space for Q provided for any pair (X, A) in Q with $A \subset X$ closed and any map $f_0 : A \to Y$ there is an extension $f : X \to Y$ (resp. neighbourhood extension $f : U \to Y$) of f_0 over X (resp. over a neighbourhood U of A in X). The classes of the extension spaces for Q will be denoted by ES(Q) and NES(Q), respectively.

Some basic properties of the NES-spaces are summarized in the following theorem (*cf.* Hanner [10]) :

(8.2) THEOREM. – (i) *A retract of* ES(Q) *is an* ES(Q); (ii) *a neighbourhood retract of* NES(Q) *is an* NES(Q); (iii) *an open subset of* NES(Q) *is an* NES(Q); (iv) *the product of any collection* (resp. *any finite collection*) *of* ES(Q) [resp. NES(Q)] *is* ES(Q) [resp. NES(Q)]; (v) *any local* NES(Q) *is* NES(Q).

(8.3) DEFINITION. – A space Y is an *absolute retract* (resp. *absolute neighbourhood retract*) for a class of spaces Q provided: (i) Y is in Q and (ii) whenever it is embedded in another space of Q as a closed subset it is a retract (resp. neighbourhood retract) of the containing space. The corresponding classes of spaces are denoted by AR(Q) and ANR(Q), respectively.

Some relations between the NES and ANR spaces are described in the following:

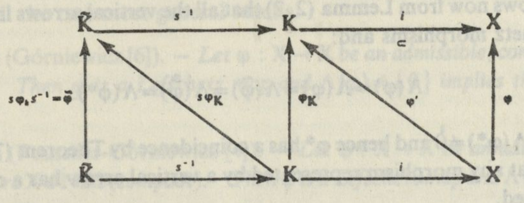
(8.4) THEOREM. – (i) (*Hanner*). *Any* ANR(Q) *is* NES(Q); (ii) *any* ANR (metric) *is a* NES (compact).

9. The main Theorem

We now state the main result of this paper:

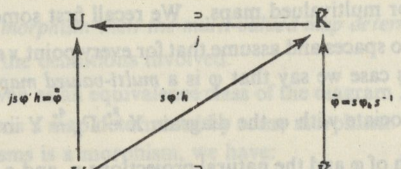
(9.1) THEOREM. — *Any* X ∈ NES (compact) *is a coincidence space.*

Proof. — Let X be NES (compact) and $\varphi : X \to X$ be a compact morphism. We are going to show that: (i) $\Lambda(\varphi)$ is defined and (ii) $\Lambda(\varphi) \neq 0$ implies that φ has a coincidence. Let (p, q) be a representative of φ and let K be a compact set containing $q(X)$. We embed K into a Tychonoff cube T and denote by $s : K \to \tilde{K}$ the homeomorphism of K onto $\tilde{K} \subset T$. We may write now the following commutative diagram in $\mathscr{U}$:

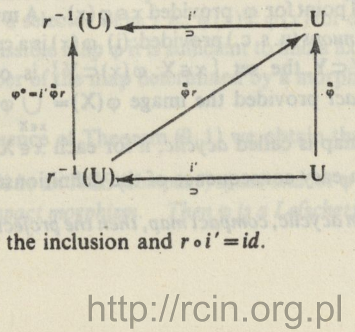


Next, consider the map $is^{-1} : \tilde{K} \to X$. Since X ∈ NES (compact) it follows that, there is an open set U in T containing $\tilde{K}$ and an extension $h : U \to X$ of is^{-1} over U; thus, if $j : \tilde{K} \to U$ is the inclusion we have $hj = is^{-1}$.

We note that the following diagram commutes:

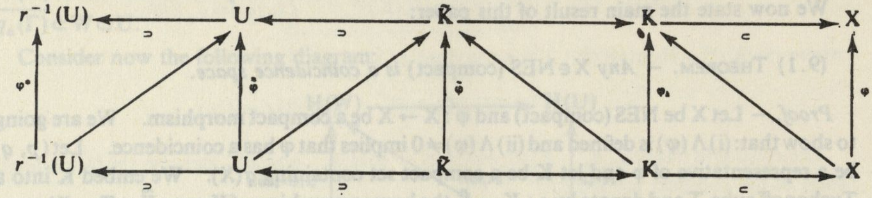


Now take the locally convex space E which contains T and such that T is a retract of E (*cf.* Fournier-Granas [5]); let $r : E \to T$ be a retraction and write the following commutative diagram in $\mathscr{M}$:



where $(i' : U \subset r^{-1}(U))$ is the inclusion and $r \circ i' = id$.

Putting the above diagrams together, we have the following commutative diagram:



in which φ^* is a compact morphism of an open set $r^{-1}(U)$ in E; by Theorem (7.6), $\Lambda(\varphi^*)$ is defined. It follows now from Lemma (2.2) that all the vertical arrows in the above diagram represent Lefschetz morphisms and:

$$\Lambda(\varphi)=\Lambda(\varphi)=\Lambda(\tilde{\varphi})=\Lambda(\tilde{\tilde{\varphi}})=\Lambda(\varphi^*).$$

If $\Lambda(\varphi)\neq 0$ then $\Lambda(\varphi^*)\neq 0$ and hence φ^* has a coincidence by Theorem (7.6). It follows by Lemma (6.3) that any morphism represented by a vertical arrow has a coincidence and the proof is completed.

10. Fixed point theorems for multivalued maps

In this section we give applications of the main Theorem to obtain a number of general fixed point results for multivalued maps. We recall first some terminology.

Let X and Y be two spaces and assume that for every point $x \in X$ a non-empty subset $\varphi(x)$ of Y is given; in this case we say that φ is a *multi-valued map* from X to Y and we write $\varphi : X \to Y$. We associate with φ the diagram $X \overset{p_\varphi}{\leftarrow} \Gamma_\varphi \overset{q_\varphi}{\rightarrow} Y$ in which $\Gamma_\varphi = \{(x, y) \in X \times Y;\ y \in \varphi(x)\}$ is the graph of φ and the natural projections p_φ and q_φ are given by $(x, y) \to x$ and $(x, y) \to y$.

If $\varphi : X \to Y$ and $\psi : Y \to Z$ are two multi-valued maps, then their *composition* is the map $\psi \circ \varphi : X \to Z$ defined by $(\psi \circ \varphi)(x) = \bigcup_{y \in \varphi(x)} \psi(y)$. Let $\varphi : X \to X$ be a multi-valued map; a point $x \in X$ is called a fixed point for φ, provided $x \in \varphi(x)$. A multi-valued map $\varphi : X \to Y$ is said to be *upper semi continuous* (u. s. c.) provided: (i) $\varphi(x)$ is a compact set for each $x \in X$ and (ii) for each open set $V \subset Y$ the set $\{x \in X;\ \varphi(x) \subset V\}$ is open. A multi-valued map $\varphi : X \to Y$ is called *compact* provided the image $\varphi(X) = \bigcup_{x \in X} \varphi(x)$ in a relatively compact subset of Y. An u.s.c. map is called *acyclic*, if for each $x \in X$ the set $\varphi(x)$ is acyclic.

The following fact is an easy consequence of the definitions:

(10.1) *If $\varphi : X \to Y$ is an acyclic, compact map, then the projection $p_\varphi : \Gamma_\varphi \Rightarrow X$ is a Vietoris map.*

A multi-valued map $\varphi : X \to Y$ is called *admissible* (*see* Górniewicz [6] and [7]) if there exists a diagram $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y$ such that $q(p^{-1}(x)) \subset \varphi(x)$, for each $x \in X$; in this case the pair (p, q) is called a *selected pair* for φ and we write $(p, q) \subset \varphi$.

It is easy to verify that any acyclic map and moreover any composition of acyclic maps is an admissible map (comp. [7]).

An admissible, compact map $\varphi : X \to X$ is called a *Lefschetz map* provided for each selected pair $(p, q) \subset \varphi$ the endomorphism $q_* \circ p_*^{-1} : H(X) \to H(X)$ is a Leray endomorphism. For a Lefschetz map $\varphi : X \to X$ we define a *Lefschetz set* $\Lambda(\varphi)$ by putting:

$$\Lambda(\varphi) = \{ (q_* \circ p_*^{-1}); (p, q) \subset \varphi \}.$$

The Lefschetz fixed point theorems for multi-valued maps were given by several authors. We recall the two most general results:

(10.2) Theorem (Górniewicz [6]). — *Let $\varphi : X \to X$ be an admissible, compact map, where* $X \in$ ANR (*metric*). *Then φ is a Lefschetz map and $\Lambda(\varphi) \neq \{0\}$ implies that φ has a fixed point.*

(10.3) Theorem (Fournier-Górniewicz [4]). — *Let $\varphi : X \to X$ be an admissible, compact and u. s. c. map, where* $X \in \dot{\text{N}}$ES (*compact*). *Then φ is a Lefschetz map and $\Lambda(\varphi) \neq \{0\}$ implies that φ has a fixed point.*

We shall discuss now the relationship between the multi-valued maps and the morphisms.

A multi-valued map $\varphi : X \to Y$ is said to be *determined* by a morphism $\{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\}$ provided $\varphi(x) = q[p^{-1}(x)]$ for each $x \in X$; the morphism which determines φ is also denoted by φ. Clearly every morphism determines a multi-valued map, but not conversely.

(10.4) *If $\varphi : X \to Y$ is a morphism then the multi-valued map determined by φ is u.s.c.*

This follows redily from the definitions involved.

If we take a morphism given as an equivalence class of the diagram $X \overset{p_*}{\Leftarrow} \Gamma_* \overset{q_*}{\to} Y$, then we obtain that any acyclic map is a map determined by some morphism. Moreover, because the composition of morphisms is a morphism, we have:

(10.5) *Any composition of acyclic maps is determined by some morphism.*

The definition of admissible maps can be reinterpreted in the terms of maps determined by morphisms as follows:

A map $\varphi : X \to Y$ is admissible if and only if there exists a morphism $\psi : X \to Y$ such that the map determined by ψ is a selector of φ, i. e., $\psi(x) \subset \varphi(x)$ for each $x \in X$. Therefore, to find a fixed point for an admissible map φ it is sufficient to find a fixed point for some selector of φ. By a Lefschetz number of the map determined by a morphism φ we will understand the Lefschetz number of φ.

As an immediate consequence of Theorem (9.1) we obtain the following.

(10.6) Theorem. — *Let X be an NES (compact) space and let $\varphi : X \to X$ be a multi-valued map determined by some compact morphism. Then φ is a Lefschetz map and $\Lambda(\varphi) \neq 0$ implies that φ has a fixed point.*

We observe, that Theorems (10.2) and (10.3) are special cases of (10.6). Finally we remark that the earlier Lefschetz-type fixed point results of Eilenberg-Montgomery [3], Granas [9], Górniewicz-Granas [8], Powers [12], Fournier-Granas [5], Górniewicz ([6], [7]), Fournier-Górniewicz [4], all are special cases of the Theorem (10.6).

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LE THÉORÈME DE LEFSCHETZ POUR LES ANR APPROXIMATIFS

PAR

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Dans [4] Clapp introduit la notion de ANR approximatifs (AANR) et démontre que tout ANR approximatif de type fini est un espace de Lefschetz (i.e. si X est un ANR approximatif de type fini, alors toute fonction continue $f\colon X \to X$ telle que $\lambda(f) \neq 0$ admet un point fixe). Nous présentons dans cet article une caractérisation des AANR, une nouvelle preuve du théorème de point fixe de Clapp ainsi qu'une généralisation de ce théorème à des AANR non nécessairement de type fini. Les espaces topologiques considérés sont tous métrisables et on utilise l'homologie de Čech à support compact et coefficients dans Q. Un espace X est dit de *type fini* si $H(X) = \{H_n(X)\}$ est de type fini.

1. Caractérisation des ANR approximatifs.

1.1. Définition (Clapp [4]). Un espace métrisable compact X est un *ANR approximatif* si pour tout plongement $h\colon X \to Y$, où Y est un espace métrique, on a: Pour tout $\varepsilon > 0$, il existe un voisinage ouvert U_ε de $h(X)$ dans Y et une fonction $r\colon U_\varepsilon \to h(X)$ tels que $\varrho(rj(y), y) < \varepsilon$ pour tout y dans $h(X)$, où $j\colon h(X) \to U_\varepsilon$ est l'inclusion canonique.

La définition suivante est souvent utilisée par la suite:

1.2. Définition. Soient Y un espace métrique et $\varepsilon > 0$. Deux fonctions f et g d'un espace X dans Y sont dites *ε-près* si $\varrho(f(x), g(x)) < \varepsilon$ pour tout x dans X.

1.3. THÉORÈME. *Soit X un espace métrisable compact et ϱ_X une métrique compatible. X est un ANR approximatif si et seulement si la condition suivante est satisfaite:*

Pour tout $\varepsilon > 0$ il existe un $A_\varepsilon \in ANR$ ($= ANR$ (métrisable) compact [1]) et des fonctions continues $\alpha\colon X \to A_\varepsilon$ et $\beta\colon A_\varepsilon \to X$ tels que $\beta\alpha$ et 1_X sont ε-près.

Démonstration. Soit X un ANR approximatif. X étant un espace métrisable compact il existe un plongement h de X dans le cube de Hilbert I^∞. Soit $\tilde{h}$ la contraction de h à la paire $(X, h(X))$. Pour $\varepsilon > 0$ soit $\delta(\varepsilon) > 0$ tel que si $x, y \in h(X)$ et $\varrho(x, y) < \delta(\varepsilon)$, alors $\varrho_X(\tilde{h}^{-1}(x), \tilde{h}^{-1}(y)) < \varepsilon$. Soit U_ε un voisinage ouvert de $h(X)$ dans I^∞ et soit $r: U_\varepsilon \to h(X)$ tels que rj et $1_{h(X)}$ sont $\delta(\varepsilon)$-près, où $j: h(X) \to U_\varepsilon$ est l'inclusion canonique. Soit A_ε un ANR tel que $h(X) \subset A_\varepsilon \subset U_\varepsilon$. Un tel A_ε existe par [1], p. 105, lemma 4.3.

Soit $i: h(X) \to A_\varepsilon$ l'inclusion canonique et $r': A_\varepsilon \to h(X)$ la restriction de r à A_ε. On a que $r'i$ et $1_{h(X)}$ sont $\delta(\varepsilon)$-près.

Posant $\alpha = i\tilde{h}: X \to A_\varepsilon$ et $\beta = \tilde{h}^{-1}r': A_\varepsilon \to X$, on a que $\beta\alpha$ et 1_X sont ε-près. Ce qui termine la démonstration de la première partie du théorème.

X étant un espace métrisable compact, on peut sans perte de généralité supposer que X est un sous-espace du cube de Hilbert. Soit $\varepsilon > 0$. Il existe un ANR A_ε et des fonctions continues $\alpha: X \to A_\varepsilon$ et $\beta: A_\varepsilon \to X$ tels que $\beta\alpha$ et 1_X sont ε-près. Puisque A_ε est un ANR la fonction $\alpha: X \to A_\varepsilon$ admet une extension continue $\tilde{\alpha}$ à un voisinage ouvert U_ε de X dans I^∞ ([1], p. 103, (2.19)). Posant $r = \beta\tilde{\alpha}$, on a que ri et 1_X sont ε-près, où $i: X \to U_\varepsilon$ est l'inclusion canonique. On a donc ([4], p. 118, theorem 2.1) que X est un ANR approximatif.

2. Le théorème de M. H. Clapp. Nous utilisons dans cette section le théorème suivant, dont on peut trouver une démonstration dans [5].

2.1. THÉORÈME. *Soit Z un espace métrique compact de type fini. Il existe $\varepsilon = \varepsilon(Z) > 0$ tel que pour tout espace compact X et pour toutes fonctions continues $f, g: X \to Z$, si f et g sont ε-près, alors*

$$f_* = g_*: H(X) \to H(Z).$$

Si X est un espace de type fini et $f: X \to X$ est une fonction continue, le nombre de Lefschetz de f est défini par

$$\lambda(f) = \sum_{n=0}^{\infty} (-1)^n \operatorname{tr}(f_{*n}),$$

où $f_{*n}: H_n(X) \to H_n(X)$ est l'homomorphisme induit par f et tr est la trace ordinaire.

2.2. THÉORÈME. *Soit X un ANR approximatif de type fini et soit $f: X \to X$ une fonction continue telle que $\lambda(f) \neq 0$. Alors f admet un point fixe.*

Démonstration. Soit ϱ une métrique compatible pour X et soit $\varepsilon' > 0$ tel que $f_* = g_*$ dès que f et g sont ε'-près (théorème 2.1), où f et g sont d'un domaine compact commun et à valeurs dans X. Soit ε tel que $0 < \varepsilon < \varepsilon'$. Soit $\delta(\varepsilon) > 0$ tel que si $x, y \in X$ et $\varrho(x, y) < \delta(\varepsilon)$. Alors

$\varrho(f(x), f(y)) < \varepsilon$. Par le théorème 1.3 il existe un ANR A_ε et des fonctions continues α: $X \to A_\varepsilon$ et β': $A_\varepsilon \to X$ tels que $\beta'\alpha$ et 1_X sont $\delta(\varepsilon)$-près.

Soit $\beta = f\beta'$. Puisque $\varepsilon < \varepsilon'$ on a que $(\beta\alpha)_* = f_*$ et $\lambda(f) = \lambda(\beta\alpha)$. Considérons $\alpha\beta$: $A_\varepsilon \to A_\varepsilon$. On a $\lambda(\alpha\beta) = \lambda(\beta\alpha) \neq 0$. Comme A_ε est un ANR, on a que $\alpha\beta$ admet un point fixe et il s'ensuit que $\beta\alpha$ admet aussi un point fixe. Pour tout $\varepsilon < \varepsilon'$, f admet un ε-point fixe (i.e. un x dans X tel que $\varrho(x, f(x)) < \varepsilon$). L'espace X étant compact il en résulte que f admet un point fixe.

2.3. Définition. Soient X et Y des espaces métrisables compacts et soit ϱ une métrique compatible pour Y. Une fonction f: $X \to Y$ est dite *presque-factorisable* si pour tout $\varepsilon > 0$ il existe un ANR A_ε et des fonctions continues α: $X \to A_\varepsilon$ et β: $A_\varepsilon \to X$ tels que $\beta\alpha$ et f sont ε-près par rapport à ϱ.

La propriété d'une fonction f d'être presque-factorisable ne dépend pas de la métrique compatible utilisée. On a la reformulation suivante du théorème 1.3:

2.4. *Soit X un espace métrisable compact. X est un ANR approximatif si et seulement si 1_X: $X \to X$ est presque-factorisable.*

2.5. *Soient X et Y des espaces métrisables compacts. Si 1_X: $X \to X$ est presque-factorisable, alors toutes les fonctions continues f: $Y \to X$ et g: $X \to Y$ sont presque-factorisables.*

Une démonstration en tout point semblable à celle du théorème 2.2 nous donne:

2.6. THÉORÈME. *Soit X un espace métrisable compact de type fini et soit f: $X \to X$ une fonction presque-factorisable. Si $\lambda(f) \neq 0$, alors f admet un point fixe.*

Dans [2] Borsuk introduit la notion de „NE-map". On peut montrer que les fonctions presque-factorisables sont des „NE-maps". Le théorème précédent est donc un cas particulier du théorème de point fixe de Borsuk concernant les „NE-maps" [3].

Soit Y^X l'espace métrique des fonctions continues de X dans Y avec la métrique ϱ définie par

$$\varrho(f, g) = \sup_{x \in X} \varrho(f(x), g(x)),$$

où X et Y sont des espaces métrisables compacts

2.7. Définition (Borsuk [2]). Soient X et Y des espaces métrisables compacts. Une fonction continue f: $X \to Y$ est *factorisable* si il existe un ANR A et des fonctions continues α: $X \to A$ et β: $A \to Y$ tels que $\beta\alpha = f$.

La situation des applications presque-factorisables parmis les „NE-maps" est la suivante:

2.8. La fermeture dans Y^X de l'ensemble des fonctions factorisables est l'ensemble des fonctions presque-factorisables. Il existe toutefois des „NE-maps" qui ne sont pas presque-factorisables [8].

3. ANR approximatifs non nécessairement de type fini. Nous présentons maintenant une généralisation du théorème 2.2 au cas où le ANR approximatif n'est pas nécessairement de type fini. Nous utilisons les notions de trace de Leray et de nombre de Lefschetz généralisé [7].

Soit $f\colon E \to E$ un endomorphisme d'espace vectoriel. Considérons

$$N(f) = \bigcup_{n>0} \operatorname{Ker} f^n \quad \text{et} \quad \tilde{E} = E|N(f).$$

Puisque $f(N(f)) \subset N(f)$, on a que f induit un endomorphisme $\tilde{f}\colon \tilde{E} \to \tilde{E}$. La fonction f est dit *admissible* si $\tilde{E}$ est de dimension finie. Si f est admissible, la trace généralisée $\operatorname{Tr}(f)$ de f est définie par $\operatorname{Tr}(f) = \operatorname{tr}(\tilde{f})$.

Soit $f = \{f_n\}_{n\in N}\colon E \to E$ un endomorphisme de degré zéro d'un espace vectoriel gradué $E = \{E_n\}_{n\in N}$. On dit que f est un *endomorphisme de Leray* si tous les f_n sont admissibles et tous les $\tilde{E}_n$ sauf un nombre fini sont triviaux. Si $f\colon E \to E$ est un endomorphisme de Leray, on définit le nombre de Lefschetz généralisé de f par

$$\Lambda(f) = \sum_{n=0}^{\infty} (-1)^n \operatorname{Tr}(f_n).$$

Une fonction continue f d'un espace X dans lui-même est dite une *fonction de Lefschetz* si $f_*\colon H(X) \to H(X)$ est un endomorphisme de Leray. Le nombre de Lefschetz généralisé de f est alors défini par $\Lambda(f) = \Lambda(f_*)$.

Nous utilisons les résultats suivants:

3.1. Théorème (voir [6]). *Soient X et Y des espaces topologiques, $u\colon X \to Y$ et $v\colon Y \to X$ des fonctions continues. Alors*:

(a) *si uv ou vu est une fonction de Lefschetz, alors les deux le sont et $\Lambda(uv) = \Lambda(vu)$;*

(b) *uv admet un point fixe si et seulement si vu admet un point fixe.*

Le théorème suivant est une généralisation du théorème 2.2 au cas où le ANR approximatif n'est pas nécessairement de type fini.

3.2. Théorème. *Soit X un ANR approximatif et soit $f\colon X \to X$ une fonction continue. S'il existe une partie compacte de type fini Y de X telle que $f(X) \subset Y$, alors f est une fonction de Lefschetz et $\Lambda(f) \neq 0$ entraîne qu'il existe un point fixe pour f.*

Démonstration. Soit ϱ une métrique compatible pour X. Puisque Y est un espace métrique compact de type fini, soit $\varepsilon' = \varepsilon'(Y) > 0$ tel que pour tout espace compact W et pour toutes fonctions continues g, h: $W \to Y$, si g et h sont ε'-près, alors $g_* = h_*$ (théorème 2.1). Soit $\varepsilon > 0$ tel que $0 < \varepsilon < \varepsilon'$ et soit $\delta(\varepsilon) > 0$ tel que si $x, y \in X$ et $\varrho(x, y) < \delta(\varepsilon)$, alors $\varrho(f(x), f(y)) < \varepsilon$.

X étant un ANR approximatif, il existe un $A_\varepsilon \in$ ANR et des fonctions continues $\alpha: X \to A_\varepsilon$ et $\beta: A_\varepsilon \to X$ tels que $\beta\alpha$ et 1_X sont $\delta(\varepsilon)$-près. Soient f' la contraction de f à la paire (X, Y), f_Y la contraction de f à la paire (Y, Y) et i l'inclusion canonique de Y dans X. Considérons

$$f_\varepsilon = f'\beta\alpha i\colon\ Y \to Y,$$

$$\varphi = \alpha i\colon\ Y \to A_\varepsilon \quad \text{et} \quad \psi = f'\beta\colon\ A_\varepsilon \to Y.$$

Puisque A_ε est un ANR, par le théorème de Lefschetz classique on a que $\varphi\psi$ admet un point fixe si $\Lambda(\varphi\psi) \neq 0$.

Montrons que $\Lambda(\varphi\psi) \neq 0$. Utilisant le théorème 3.1 on a que $\Lambda(\varphi\psi) = \Lambda(\psi\varphi) = \Lambda(f_\varepsilon)$. Les fonctions f_ε et f_Y étant ε-près on en déduit que $(f_\varepsilon)_* = (f_Y)_*$. L'espace Y étant de type fini, f_Y est une fonction de Lefschetz. Utilisant de nouveau le théorème 3.1 on a que f est une fonction de Lefschetz et que $\Lambda(f) = \Lambda(if') = \Lambda(f'i) = \Lambda(f_Y)$. On obtient $\Lambda(\varphi\psi) = \Lambda(f_\varepsilon) = \Lambda(f_Y) = \Lambda(f) \neq 0$.

Il s'ensuit que $f_\varepsilon = \psi\varphi$ admet aussi un point fixe. La fonction f_Y admet donc un ε-point fixe.

Puisque f_Y admet un ε-point fixe pour tout $\varepsilon < \varepsilon'$ elle admet un point fixe, lequel est aussi un point fixe pour f.

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A Proof of the Borsuk Antipodal Theorem

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By applying simple properties of the Brouwer degree, a new and self-contained proof of the Antipodal Theorem of Borsuk is presented. The proof uses elementary notions of the simplicial topology but has points in common with analytical arguments based on the application of Sard's Lemma.

1. PRELIMINARIES ON AFFINE MAPS

We begin by recalling some terminology and facts concerning the affine maps. By $L(R^n)$ we denote the normed linear space of linear operators $A: R^n \to R^n$ with $\|A\| = \mathrm{Sup}\{\|Ax\|;\ \|x\| \leqslant 1\}$. By $GL(n, R)$ we denote the subset of $L(R^n)$ consisting of invertible operators. In what follows, we shall repeatedly use a fact that $GL(n, R)$ is open and dense in $L(R^n)$.

We recall that a map $\phi: R^n \to R^n$ is *affine* provided

$$\phi(tx + (1-t)\,y) = t\phi(x) + (1-t)\,\phi(y),$$

for all $x, y \in R^n$ and $t \in R$. Given $A \in L(R^n)$ and $a \in R^n$, the formula $\phi(x) = A(x) + a$ defines an affine map. Furthermore, the assignment $(A, a) \mapsto \phi$ defines a bijective correspondence between $L(R^n) \times R^n$ and the set of all affine maps from R^n into R^n. An affine map $\phi: R^n \to R^n$ is *regular* provided $\phi(R^n) = R^n$. Note that if $\phi(x) = A(x) + a$, where $A \in L(R^n)$, $a \in R^n$, then ϕ is regular if and only if $A \in GL(n, R)$. From this remark we have:

(1.1) *Let* $\phi\colon R^n \to R^n$ *be an affine map. Then the set*

$$\mathscr{A} = \{A \in L(R^n) | \phi + A \text{ is regular}\}$$

is open and dense in $L(R^n)$.

Let $\sigma = \sigma(p_0, ..., p_k)$ be a k-simplex spanned by $k+1$ affinely independent points $p_0, ..., p_k \in R^n$. A map $\phi\colon \sigma \to R^n$ is *affine* provided

$$\phi(tx + (1-t)\,y) = t\phi(x) + (1-t)\,\phi(y)$$

for all $x, y \in \sigma$ and $t \in [0, 1]$. Note that in the case $k = n$, there is one-to-one correspondence between affine maps of σ into R^n and affine maps at R^n into R^n.

Let ϕ be an n-simplex of R^n. An affine map $\phi\colon \sigma \to R^n$ is *regular* provided $\phi(\sigma)$ is an n-dimensional simplex of R^n. Clearly $\phi\colon \sigma \to R^n$ is regular if and only if $\phi(x) = A(x) + a$, where $A \in GL(R, n)$, $a \in R^n$. This implies

(1.2) *Let* $\sigma \subset R^n$ *be an* n*-simplex and* $\phi\colon \sigma \to R^n$ *be an affine map. Then the set*

$$\mathscr{A} = \{A \in L(R^n) | \phi + A \text{ is regular}\}$$

is open and dense in $L(R^n)$.

(1.3) *Let* σ^{n-1} *be an* $(n-1)$*-simplex of* R^n, *with* $0 \notin \sigma^{n-1}$, $\phi\colon \sigma^{n-1} \to R^n$ *be an affine map and assume that* $0 \in \phi(\sigma^{n-1})$. *Then the set*

$$\mathscr{A} = \{A \in L(R^n) | 0 \notin (\phi + A)(\sigma^{n-1})\}$$

is open and dense subset of $L(R^n)$.

Proof. Clearly, $\mathscr{A}$ is open. To prove that $\mathscr{A}$ is dense, it will suffice to show that for any $\varepsilon > 0$ there exists an $A \in \mathscr{A}$ such that $\|A\| < \varepsilon$ and $0 \notin (\phi + A)(\sigma^{n-1})$. To prove this let L denote an $(n-1)$-dimensional linear subspace of R^n such that $\phi(\sigma^{n-1}) \subset L$ and choose $v \in R^n \backslash L$. Next because $0 \notin \sigma^{n-1}$ there is a linear functional $\eta\colon R^n \to R$ such that $\eta(x) > 0$ for all $x \in \sigma^{n-1}$. Choose a number $C > 0$ such that $C < (\|\eta\| \cdot \|v\|)^{-1}$. It is now clear that the linear map A given by $x \mapsto \varepsilon C \eta(x)\, v$ has the desired properties.

2. Simplicial Maps

We recall that a subspace $X \subset R^n$ is a *polyhedron* if there is a finite collection $T = \{\sigma\}$ of geometric simplexes such that (i) $X = \bigcup \{\sigma | \sigma \in T\}$, (ii) each face of a $\sigma \in T$ is also in T, (iii) if $\sigma_1, \sigma_2 \in T$, then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2. The collection T is called a *triangulation* of X. If k is an integer, then $X^k \subset X$ which is the union of all simplexes of dimension $\leqslant k$ is said to be the *k-skeleton* of X. A triangulation T' of X is a subdivision of T provided each simplex T is the union of simplexes in T'. In particular, subdividing barycentrically each $\sigma \in T$ we get a triangulation T' called the barycentric subdivision of T. In what follows by $T^{(m)}$ we denote the iterated barycentric subdivision of T of order m. We use the fact that the diameter of the simplexes of $T^{(m)}$ can be made arbitrarily small by taking m sufficiently large.

Let X be a polyhedron and T be a triangulation of X. A map $f\colon X \to R^n$ is said to be *T-simplicial* provided for any $\sigma \in T$ the restriction $f|\sigma\colon \sigma \to R^m$ is an affine map. Clearly such a map is determined by its behavior on vertices: if f is defined on the 0-skeleton X^0 of X, then it can be extended uniquely to a T-simplicial map from X to R^n.

Using this and the uniform continuity of f we obtain easily the following approximation result:

(2.1) *Let X be a polyhedron and $f\colon X \to R^n$ be continuous. Then for each $\varepsilon > 0$ there exists a triangulation T on X and a T-simplicial map $f_\varepsilon\colon X \to R^n$ such that $\|f(x) - f_\varepsilon(x)\| < \varepsilon$ for all $x \in X$.*

We introduce now an important notion of a regular value for a given simplicial map.

(2.2) Definition. Let (X, T) be a polyhedron and $f\colon X \to R^n$ be a simplicial map. We say that $y \in R^n$ is a *regular value* of f provided $y \notin f(X^{n-1})$, where X^{n-1} is the $(n-1)$-skeleton of X. A simplicial map $f\colon X \to R^n$ is said to be *normal* provided (i) the restriction $f|\sigma$ is regular for each n-simplex $\sigma \in T$ and (ii) 0 is a regular value of f.

The following basic lemma is of importance:

(2.3) Lemma. *Let X be a polyhedron with $0 \notin X$ and let $f\colon X \to R^n$ be a T-simplicial map. Then the set*

$$\mathscr{A} = \{A \in L(R^n) \,|\, f + A \text{ is normal}\}$$

is open and dense in $L(R^n)$.

Proof. Since $\mathscr{A} = \mathscr{A}_1 \cap \mathscr{A}_2$, where

$$\mathscr{A}_1 = \{A \in L(R^n) \,|\, (f + A)|_\sigma \text{ is regular for each } n\text{-simplex } \sigma \in T\},$$

$$\mathscr{A}_2 = \{A \in L(R^n) \,|\, 0 \text{ is a regular value for } f + A\},$$

it is enough to show that $\mathscr{A}_i$ $(i=1, 2)$ is open and dense in $L(R^n)$. If $i=1$, this follows from (1.2). If $i=2$, since for each $(n-1)$-simplex $\sigma^{n-1} \in T$ we have $0 \notin f(\sigma^{n-1})$, this is a consequence of (1.3).

3. The Topological Degree

We recall briefly the definition and some basic properties of the topological degree which will be used in the proof of the Antipodal Theorem. In the following definition we use the singular homology over the integers.

Let U be a bounded domain in R^n and $f\colon (U, \partial U) \to (R^n, R^n - 0)$ be continuous. Letting $K = f^{-1}(0)$ we have a diagram

$$\begin{array}{ccccc} S^n & \xrightarrow{i} & (S^n, S^n - K) & \xleftarrow{j} & (U, U-K) \\ & & & & \downarrow f \\ S^n & \xrightarrow{k} & (S^n, S^n - 0) & \xleftarrow{l} & (R^n, R^n - \{0\}), \end{array} \tag{1}$$

where all arrows but f denote inclusions and $S^n = R^n \cup \infty$. We have the corresponding diagram of homology groups

$$\begin{array}{ccccc} H_n(S^n) & \xrightarrow{i_*} & H_n(S^n, S^n - K) & \xrightarrow[\sim]{j_*^{-1}} & H_n(U,\ U-K) \\ & & & & \downarrow f_* \\ H_n(S^n) & \xleftarrow[\sim]{k_*^{-1}} & H_n(S^n,\ S^n - 0) & \xleftarrow{l_*} & H_n(R^n, R^n - 0). \end{array} \tag{1_*}$$

The topological degree $d(f, U)$ is defined by

$$k_*^{-1} l_* f_* j_*^{-1} i_*(v) = f(f, U)\, v, \tag{2}$$

where v is a generator of $H_n(S^n) = Z$.

Using a singular homology, the following basic properties of the degree can be easily verified:

(I) *Normalization.* Let $j\colon (\bar{U}, \partial U) \to (R^n, R^n - \{0\})$ be the inclusion. Then

$$\begin{aligned} d(j, U) &= 1 \quad \text{if} \quad 0 \in U, \\ &= 0 \quad \text{if} \quad 0 \notin U. \end{aligned}$$

(II) *Excision.* The degree depends only on the behaviour of f in the neighbourhood of $K = f^{-1}(0)$: if $K \subset V \subset U$, where V is open, then $d(f, U) = d(f \mid V, V)$.

(III) *Additivity*. Given f and $K = f^{-1}(0)$ let $\{V_j\}$ denote a finite open covering of K such that $V_j \subset U$ and $V_i \cap V_j = \emptyset$ when $i \neq j$. Then $d(f, U) = \sum_j d(f, V_j)$.

(IV) *Homotopy*. Let h_t: $(\bar{U}, \partial U) \to (R^n, R^n - \{0\})$ be a homotopy. Then $d(h_0, U) = d(h_1, U)$.

Remark 1. It follows from the homotopy property that the degree depends only on boundary values: if for f, g: $(\bar{U}, \partial U) \to (R^n, R^n - \{0\})$ we have $f | \partial U = g | \partial U$, then $d(f, U) = d(g, U)$, because the homotopy $h_t(x) = tf(x) - (1-t)\, g(x)$ joins f and g and does not vanish on ∂U.

Remark 2. Let $\bar{U} = \sigma(p_0, p_1, ..., p_n)$ be an n-simplex and f: $(\bar{U}, \partial U) \to (R^n, R^n - 0)$ an affine map. From the properties of the degree it follows easily that if f is regular (i.e., $f(x) = a + Ax$, where $A \in GL(n, R)$) and $0 \in f(U)$, then $d(f, U) = \text{sign}\det A$.

4. The Antipodal Theorem

Throughout this section we assume that U is a bounded domain in R^n which is centrally symmetric (i.e., $-U = U$), and that $\bar{U}$ is a polyhedron. We say that a map f: $\bar{U} \to R^n$ is *odd* if $f(-x) = -f(x)$ for $x \in \bar{U}$.

First we establish

(4.1) Lemma. *Let f: $(\bar{U}, \partial U) \to (R^n, R^n - \{0\})$ be a continuous odd map. If $0 \notin \bar{U}$, then $d(f, U)$ is even.*

Proof. Let $X = \bar{U}$. Since $0 \notin f(\partial U)$, $\varepsilon = \sup\{\|f(x)\|;\ x \in \partial U\} > 0$. By (2.1) there exist a triangulation T on X and a T-simplicial map $f_1: X \to R^n$ such that $\|f(x) - f_1(x)\| < \varepsilon/2$ for all $x \in X$. Without a loss of generality we may assume that T is symmetric (i.e., $\sigma \in T$ implies $-\sigma \in T$). Define $f_2: X \to R^n$ by $f_2(x) = \frac{1}{2}(f_1(x) - f_1(-x))$. Evidently f_2 is odd and T-simplicial. Moreover, $\|f(x) - f_2(x)\| = \|\frac{1}{2} f(x) - \frac{1}{2} f_1(x) - \frac{1}{2} f(-x) + \frac{1}{2} f_1(-x)\| \leqslant \frac{1}{2}\|f(x) - f_1(x)\| + \frac{1}{2}\|f(-x) - f_1(-x)\| \leqslant \varepsilon/2$ and thus f_2: $(\bar{U}, \partial U) \to (R^n, R^n - \{0\})$ is homotopic to f. Using (2.3), we find $A \in L(R^n)$ with $\|A\| < \varepsilon/2$ and such that $g = f_2 + A$ is normal; clearly, g: $(\bar{U}, \partial U) \to (R^n, R^n - \{0\})$, $f_2 \simeq g$ and hence $d(f, U) = d(g, U)$. Because g is normal $g^{-1}(0) = \{x_1, -x_1, ..., x_k, -x_k\}$, where each x_i is an inner point of an n-simplex $\sigma_i \in T$. By the additivity of the degree we have $d(g) = \sum_{i=1}^{k} (d(g, \text{Int}\,\sigma_i) - d(g, \text{Int}(-\sigma_i))$. On the other hand, by Remark 2 we have $d(g, \text{Int}\,\sigma_i) = d(g, \text{Int}(-\sigma_i)$, hence $d(g, U) = 2\sum_{i=1}^{K} d(g, \text{Int}\,\sigma_i)$ and the proof is complete.

(4.2) THEOREM (Antipodal Theorem). *Let* $f\colon (\bar{U}, \partial U) \to (R^n, R^n - \{0\})$ *be a continuous map such that* $f(-x) = -f(x)$ *for* $x \in \partial U$. *If* $0 \in U$, *then* $d(f, U)$ *is odd.*

Proof. Choose an $r > 0$ such that for $U_0 = \{x \in R^n, \|x\| < r\}$, $\bar{U}_0 \subset U$ and let $f_0 = f|_{\partial U}\colon \partial U \to R^n - \{0\}$. By Tietze's Extension Theorem there exists a continuous map $f_1\colon (\bar{U}, \partial U) \to (R^n, R^n - \{0\})$, such that $f_1(x) = f_0(x) = f(x)$ for $x \in \partial U$ and $f_1(x) = x$ for $x \in \bar{U}_0$. Let $g(x) = \frac{1}{2}(f_1(x) - f_1(-x))$. Then g is odd and $g(x) = f_2(x)$ for $x \in \partial U \cup \bar{U}_0$; consequently $g\colon (\bar{U}, \partial U) \to (R^n, R^n \setminus \{0\})$. Let $V = U \setminus \bar{U}_0$. By the additivity, the normalization and (4.1) we have $d(g, U) = d(g, U_0) + d(g, V) = 1 + d(g, V)$ and hence $d(g, U)$ is odd. Since $g|_{\partial U} = f|_{\partial U}$ we get $d(f, U) = d(g, U)$ and the proof is completed.

Note. The paper was suggested by [1], the difference between the proof given by Alexander and Yorke and our is that we replace differentiable maps by affine maps. The advantage is that this permits us to replace transversality arguments by a simpler fact that $GL(n, \mathbb{R})$ is an open and dense subset of $L(\mathbb{R}^n)$. The definition and the basic properties of the topological degree may be found in [2].

After this paper was submitted the authors were informed by H. O. Peitgen of his joint work with H. M. Siegberg [3] in which another type of proof of the Antipodal Theorem is given using *PL*-approach to the degree theory.

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