# VIII. — q'fN.

## § 1. Σ, Π.

$u, v \,\varepsilon\, \mathrm{qfN} \,.\, m, n \,\varepsilon\, \mathrm{N} \,.\, \supset :$

**1**. $(\Sigma u)_n = u_1 + u_2 + \ldots + u_n = \sum_{r=1}^{r=n} u_r$ . Def.

2. $(\Pi u)_n = u_1 u_2 \ldots u_n = \prod_{r=1}^{r=n} u_r$ . Def.

3. $\Sigma u_n = (\Sigma u)_n \,.\, \Pi u_n = (\Pi u)_n$. Def.

3'. $(1 + u)_n = 1 + u_n \,.\, (u + v)_n = u_n + v_n \,.\, (u v)_n = u_n v_n$ . Def.

4. $\bar{\Sigma} u_1 = u_1 \,.\, \bar{\Sigma} u_{n+1} = u_{n+1} - u_n$. [1, 3]

5. $\bar{\Pi} u_1 = u_1 \,.\, \bar{\Pi} u_{n+1} = u_{n+1} / u_n$. [2, 3]

6. $\Sigma^2 u_1 = u_1 \,.\, \Sigma^2 u_2 = 2u_1 + u_2$ . [1, 3]

6'. $\Sigma^2 u_n = n u_1 + (n-1) u_2 + (n-2) u_3 + \ldots + u_n$ .

7. $\Sigma^m u_n = \binom{m+n-2}{m-1} u_1 + \binom{m+n-3}{m-1} u_2 + \ldots + \binom{m-1}{m-1} u_n$. [1, 3]

8. $\Pi^m u_n = u_1^{\binom{m+n-2}{m-1}} u_2^{\binom{m+n-3}{m-1}} \ldots u_n^{\binom{m-1}{m-1}}$ . [2, 3]

9. $\bar{\Sigma}^m u_{m+n} = u_{m+n} - \binom{m}{1} u_{m+n-1} + \binom{m}{2} u_{m+n-2} - \ldots \pm \binom{m}{m} u_n$. [4]

10. $\bar{\Pi}^m u_{m+n} = u_{m+n}\, u_{m+n-1}^{-\binom{m}{1}}\, u_{m+n-2}^{\binom{m}{2}} \ldots u_n^{(-1)^m \binom{m}{m}}$ . [5]

11. $u_{m+n} = u_m + n \bar{\Sigma} u_{m+1} + \binom{n}{2} \bar{\Sigma}^2 u_{m+2} + \ldots + \bar{\Sigma}^n u_{m+n}$ . [4]

12. $u_{m+n} = u_m (\bar{\Pi} u_{m+1})^{\binom{n}{1}} (\bar{\Pi}^2 u_{m+2})^{\binom{n}{2}} \ldots (\bar{\Pi}^n u_{m+n})^{\binom{n}{n}}$. [5]

13. $\Sigma_n^{n+m} u_r = \binom{m+1}{1} u_n + \binom{m+1}{2} \bar{\Sigma} u_{n+1} + \binom{m+1}{3} \bar{\Sigma}^2 u_{n+2}$

$+ \ldots + \binom{m+1}{m+1} \bar{\Sigma}^m u_{n+m}$ . [11]

14. $\Pi_n^{n+m} u_r = u_n^{\binom{m+1}{1}} (\bar{\Pi} u_{n+1})^{\binom{m+1}{2}} \ldots (\bar{\Pi}^m u_{n+m})^{\binom{m+1}{m+1}}$ . [12]

15. $\Sigma (u + v)_n = \Sigma u_n + \Sigma v_n \,.\, \Pi (uv)_n = (\Pi u_n)(\Pi v_n)$ . [1, 2, 3]

16. $\bar{\Sigma} (u + v)_n = \bar{\Sigma} u_n + \bar{\Sigma} v_n \,.\, \bar{\Pi} (uv)_n = (\bar{\Pi} u_n)(\bar{\Pi} v_n)$ . [3, 4, 5]

17. $r \,\varepsilon\, \mathrm{N} \,.\, \supset_r \,.\, u_r = \binom{m+r}{r} : \supset \,.\, \overline{\Sigma}^m u_{r+m} = 1 \,.$ [9]

18. $a, b \,\varepsilon\, \mathrm{q} : r \,\varepsilon\, \mathrm{N} \,.\, \supset_r \,.\, \overline{\Sigma}^m u_{r+m} = a \,.\, \overline{\Sigma}^n v_{r+n} = b : \supset \,.\, \overline{\Sigma}^{m+n} (uv)_{r+m+n} = \binom{m+n}{n} ab \,.$

[Hp. P4 . $\supset$ . $\overline{\Sigma} (uv)_{r+m+n} = u_{r+m+n} \overline{\Sigma} v_{r+m+n} + v_{r+m+n-1} \overline{\Sigma} u_{r+m+n}$ :
P16 . $\supset$ . $\overline{\Sigma}^{m+n} (uv)_{r+m+n} = \overline{\Sigma}^{m+n-1} (u_{r+m+n} \overline{\Sigma} v_{r+m+n}) +$
$\overline{\Sigma}^{m+n-1} (v_{r+m+n-1} \overline{\Sigma} u_{r+m+n})$ . Pi . $\supset$ . Ths.]

19. $\Pi (1+u)_n \,\Pi (1-u)_n = \Pi (1-u^2)_n \,.$

20. $v \,\varepsilon\, \mathrm{QfN} \,.\, \supset .\, (1+v_1)(1+v_2) \ldots (1+v_m) > 1 + v_1 + v_2 + \ldots + v_m .$

20'. $v \,\varepsilon\, \theta\, \mathrm{f N} \,.\, \supset .\, (1-v_1)(1-v_2) \ldots (1-v_m) > 1 - (v_1 + v_2 + \ldots + v_m) .$

21. $\dfrac{1}{\Pi (1+u)_n} = \Pi \left(1 - \dfrac{u}{1+u}\right)_n \,.\, \dfrac{1}{\Pi (1-u)_n} = \Pi \left(1 + \dfrac{u}{1-u}\right)_n \,.$

22. $u \,\varepsilon\, \theta\, \mathrm{f Z}_n \,.\, \supset : \Pi (1+u)_n < \dfrac{1}{\Pi (1-u)_n} \,.\, \Pi (1-u)_n < \dfrac{1}{\Pi (1+u)_n} \,.$ [19]

23. $u \,\varepsilon\, \mathrm{Q f Z}_m \,.\, \supset \,.\, \Pi (1+u)_m < 1 + \dfrac{\Sigma u_m}{1} + \dfrac{(\Sigma u_m)^2}{2!} + \ldots + \dfrac{(\Sigma u_m)^m}{m!} \,.$

24. $u \,\varepsilon\, \theta\, \mathrm{f N} \,.\, r \,\varepsilon\, \mathrm{N} \,.\, \supset \,.\, \Pi \left(\dfrac{1}{1-u}\right)_m > 1 + \dfrac{\Sigma u_m}{1} + \dfrac{(\Sigma u_m)^2}{2!} + \ldots + \dfrac{(\Sigma u_m)^r}{r!} \,.$

25. $u \,\varepsilon\, \theta\, \mathrm{f Z}_m \,.\, m \,\varepsilon\, (2+\mathrm{N}) \,.\, \supset \,.\, (1-u_1) \ldots (1-u_m) < 1 - (u_1 + \ldots + u_m) + (u_1 u_2 + u_1 u_3 + \ldots + u_{m-1} u_m) \,.$

26. $(1+u_1)(1+u_2) \ldots (1+u_m) = 1 + u_1 + (1+u_1) u_2 + (1+u_1)(1+u_2) u_3 + \ldots + (1+u_1)(1+u_2) \ldots (1+u_{m-1}) u_m \,.$

27. $u_1 + u_2 + \ldots + u_m = u_1 \left(1 + \dfrac{u_2}{u_1}\right) \left(1 + \dfrac{u_3}{u_1 + u_2}\right) \ldots \left(1 + \dfrac{u_m}{u_1 + u_2 + \ldots + u_{m-1}}\right) .$

28. $1 = \dfrac{u_1}{1+u_1} + \dfrac{u_2}{(1+u_1)(1+u_2)} + \ldots + \dfrac{u_n}{(1+u_1) \ldots (1+u_n)} + \dfrac{1}{(1+u_1) \ldots (1+u_n)} \,.$

28'. $a, b \,\varepsilon\, \mathrm{q} \,.\, (a+u_1)(a+u_2) \ldots (a+u_{n+1})(a-b) -= 0 \,.\, \supset \,.$

$$\frac{1}{a-b} = \frac{1}{a+u_1} + \frac{b+u_1}{(a+u_1)(a+u_2)} + \ldots + \frac{(b+u_1) \ldots (b+u_n)}{(a+u_1) \ldots (a+u_{n+1})} + \frac{(b+u_1) \ldots (b+u_n)(b+u_{n+1})}{(a+u_1) \ldots (a+u_{n+1})(a-b)} .$$

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§ 1. 23. Pringsheim, *Math. Ann.*, t. 33, a. 1889, pag. 142.
25. Cesàro. *Analisi algebrica*, a. 1894, pag. 106.
26, 27. Euler, *Novi comm. Acad. Petropolitanae*, t. V, pag. 76.
28'. Nicole. *Mém. de l'Acad. des Sciences de Paris*, 1727, pag. 257.

29. $(u+v) \varepsilon (\pm Q) \, f \, Z_n \, . \supset . \sum_{r=2}^{r=n} \frac{u_r \Pi v_{r-1}}{\Pi (u+v)_r} = \frac{v_1}{u_1+v_1} - \frac{\Pi v_n}{\Pi (u+v)_n}$ . [28]

30. $\sum_{m+1}^{m+n} u = u_{m+1} - u_{m+2} + 2 (u_{m+2} - u_{m+3}) + \ldots + (n-1)(u_{m+n-1} - u_{m+n}) + n \, u_{m+n}$ .

31. $\prod_{m+1}^{m+n} u = u_{m+1} / u_{m+2} (u_{m+2} / u_{m+3})^2 \ldots (u_{m+n-1} / u_{m+n})^{n-1} u^n{}_{m+n}$ .

32. $v_n u_n + (v_{n+1} - v_n) u_{n+1} + \ldots + (v_{n+m} - v_{n+m-1}) u_{n+m} = v_n (u_n - u_{n+1}) + \ldots + v_{n+m-1} (u_{n+m-1} - u_{n+m}) + v_{n+m} u_{n+m}$ .

33. $v_n^{u_n} (v_{n+1} / v_n)^{u_{n+1}} \ldots (v_{n+m} / v_{n+m-1})^{u_{n+m}} = v_n^{u_n - u_{n+1}} \ldots v_{n+m-1}^{u_{n+m-1} - u_{n+m}} v_{n+m}^{u_{n+m}}$ .

## § 2. $\Sigma u_n$.

$[\lim = \lim_{n=\infty}]$

$u, v \, \varepsilon \, q f N \, . \, m \, \varepsilon \, N \, . \supset :$

1. $\Sigma_1^\infty u = \lim \Sigma u_n$ . 1'. $\Sigma_m^\infty u = \lim \Sigma_m^n u$ . Def.

2. $u_1 + u_2 + \ldots = (\Sigma u)_\infty = \Sigma u_\infty = \Sigma_1^\infty u$ . Def.

3. $\Sigma u_\infty \, \varepsilon \, q \, . \supset . \lim u_n = 0 \, . \lim \Sigma_n^{n+m} u = 0$ . [§2 P2]

4. $\Sigma_1^\infty u \, \varepsilon \, q \, . = . \, \Sigma_m^\infty u \, \varepsilon \, q$ . [§2 P2]

4'. » $= \infty \, . = .$ » $= \infty$ . [ » ]

4''. $\Sigma_1^\infty u \, \varepsilon \, q : = : \varphi \, \varepsilon \, N f N \, . \supset_\varphi . \lim \Sigma_n^{n+\varphi_n} u = 0$ . [ » ]

5. $\Sigma u_\infty \, \varepsilon \, q \, . \, h \, \varepsilon \, Q \, . \supset .\!\cdot\, n \, \varepsilon \, N : r \, \varepsilon \, N \, . \supset_r . \bmod \Sigma_n^{n+r} u < h : - =_n \Lambda$ . [§2 P2]

6. $h \, \varepsilon \, Q \, . \supset_h .\!\cdot\, n \, \varepsilon \, N : r \, \varepsilon \, N \, . \supset_r . \bmod \Sigma_n^{n+r} u < h : - =_n \Lambda :: \supset . \Sigma u_\infty \, \varepsilon \, q$. [§2 P2]

7. $a \, \varepsilon \, q \, . \, u_1 + u_2 + \ldots \varepsilon \, q \, . \supset . \, a u_1 + a u_2 + \ldots = a (u_1 + u_2 + \ldots)$ . [ » ]

8. $\Sigma u_\infty, \Sigma v_\infty \, \varepsilon q . \supset . (u_1 + v_1) + (u_2 + v_2) + \ldots = (u_1 + u_2 + \ldots) + (v_1 + v_2 + \ldots)$ . [§2 P2]

9. $\Sigma (u - v)_\infty \varepsilon \, q \, . \, \varphi \, \varepsilon \, N f N . \supset . \lim \Sigma_n^{n+\varphi_n} u = \lim \Sigma_n^{n+\varphi_n} v$ . [ » ]

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§ 2. 3, 4, 4', 8, 14. Cauchy. *Analyse algébrique,* a. 1821, pag. 125, 144, 147.

10. $\lim u_n = 0 . \supset . u_1 = (u_1 - u_2) + (u_2 - u_3) + \ldots$

11. $\lim u_n = \infty . \supset . (u_2 - u_1) + (u_3 - u_2) + \ldots = \infty .$

12. $\Sigma u_\infty \varepsilon \mathrm{q} . r \varepsilon \mathrm{N f N} . \supset . \Sigma_1^{r_1} u + \Sigma_{r_1+1}^{r_1+r_2} u + \Sigma_{r_1+r_2+1}^{r_1+r_2+r_3} u + \ldots = \Sigma u_\infty .$

13. $\Sigma u_\infty = \infty .$ » » $= \infty .$

14. $u \varepsilon (\mathrm{QfN})\, \mathrm{decr} . \lim u_n = 0 . \supset . u_1 - u_2 + u_3 - u_4 + \ldots \varepsilon \mathrm{Q} \frown (u_1 - \mathrm{Q}).$

15. $a \varepsilon \mathrm{q} : n \varepsilon \mathrm{N} . \supset_n . u_{n+1} \varepsilon \mathrm{med}\,(u_n, a) : \supset . \lim u_n \varepsilon \mathrm{q} . \Sigma_1^\infty (u_n - u_{n+1}) = u_1 - \lim u_n .$

16. $u \varepsilon \mathrm{Q f N} . \Sigma \mathrm{mod}\, u_\infty \varepsilon \mathrm{Q} . \varphi \varepsilon (\mathrm{N f N})\, \mathrm{sim} . \supset . \Sigma_{n=1}^{n=\infty} u_{\varphi_n} = \Sigma_1^\infty u .$ [§2P4]

17. $\Sigma u_\infty \varepsilon \mathrm{q} . \Sigma \mathrm{mod}\, u_\infty = \infty . a \varepsilon \mathrm{q} . \supset : \varphi \varepsilon (\mathrm{N f N})\, \mathrm{sim} . \Sigma_{n=1}^{n=\infty} u_{\varphi_n} = a . - =_\varphi \Lambda .$

18. $u, v \varepsilon \mathrm{Q f N} . a, b \varepsilon \mathrm{Q} . b > a . \varphi, \psi \varepsilon \mathrm{N f N} . \supset \therefore a = \lim (\Sigma u_{\varphi_n} - \Sigma v_n) . b = \lim (\Sigma u_{\psi_n} - \Sigma v_n) : = : a = \lim (\Sigma u_{\varphi_n} - \Sigma v_n) . b - a = \lim \Sigma_{\varphi_n + 1}^{\psi_n} u .$

19. $u, v \varepsilon \theta \mathrm{f N} . a, b \varepsilon \mathrm{q} . \frac{b}{a} > 1 . \varphi, \psi \varepsilon \mathrm{N f N} . \supset \therefore a = \lim \Pi (1 + u)_{\varphi_n} \Pi (1 - v)_n . b = \lim \Pi (1 + u)_{\psi_n} \Pi (1 - v)_n : = : a = \lim \Pi (1 + u)_{\varphi_n} \Pi (1 - v)_n . \frac{b}{a} = \lim \Pi_{\varphi_n + 1}^{\psi_n} (1 + u) .$

20. $u \varepsilon \mathrm{Q f N} . \supset . \Sigma_1^\infty \frac{u_n}{(1 + u_1)(1 + u_2) \ldots (1 + u_n)} \varepsilon \theta .$ [§1 P28]

21. » $. \Sigma u_\infty = \infty . \supset .$ » $= 1 .$ [ » ]

22. $a \varepsilon \mathrm{q} . \supset . \frac{1}{a} = \frac{1}{a + u_1} + \frac{u_1}{(a+u_1)(a+u_2)} + \frac{u_1 u_2}{(a+u_1)(a+u_2)(a+u_3)} + \ldots + \frac{u_1 u_2 \ldots u_{n-1}}{(a+u_1)(a+u_2) \ldots (a+u_n)} + \frac{u_1 u_2 \ldots u_n}{(a+u_1)(a+u_2) \ldots (a+u_n)\, a} .$

23. $a \varepsilon \mathrm{Q} . u \varepsilon \mathrm{Q f N} . \supset . \frac{1}{a + u_1} + \frac{u_1}{(a+u_1)(a+u_2)} + \frac{u_1 u_2}{(a+u_1)(a+u_2)(a+u_3)} + \ldots \varepsilon \frac{\theta}{a} .$ [§1 P29]

24. $u, v \varepsilon \mathrm{Q f N} . \supset . \frac{u_1}{u_1 + v_1} + \Sigma_{n=2}^{n=\infty} \frac{u_n \Pi v_{n-1}}{\Pi (u+v)_n} \varepsilon \left(1 - \frac{\theta}{\Sigma (u/v)_\infty}\right) .$ [§1 P29]

25. $u \varepsilon \mathrm{Q f N} . \supset : \Sigma_{n=1}^{n=\infty} \left(u_{n+1} \Pi (1 + u)_n\right) = \infty . = . \Sigma u_\infty = \infty .$ [§1 P26]

26. $u \varepsilon \theta \mathrm{f N} . \supset : \lim \Pi (1 + u)_n = \infty . = . \lim \Pi (1 - u)_n = 0 .$ [§1 P22]

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17. RIEMANN's. *Werke*, a. 1892, pag. 235.

27. $r \varepsilon \mathrm{N} . \supset_r : \frac{u_{r+1}\, v_{r+1}}{u_r\, v_r} \varepsilon \mathrm{Q} . \lambda_r = v_r - v_{r+1} \frac{u_{r+1}}{u_r} : \supset .$

$$\frac{1}{u_1 v_1} \Sigma_1^m \lambda_n u_n = 1 - \mathrm{e}^{\Sigma_1^m \log\left(1 - \frac{\lambda_n}{v_n}\right)} = 1 - \mathrm{e}^{\Sigma_1^m \log \frac{u_{n+1}\, v_{n+1}}{u_n\, v_n}} .$$

28. $a \varepsilon (\mathrm{Q\,f\,N})\ \mathrm{cres} . \lim a_n = \infty . p \varepsilon \mathrm{Q} : \supset . \Sigma_1^\infty \frac{a_{n+1} - a_n}{\mathrm{e}^{p\, a_{n+1}}} \varepsilon \mathrm{Q} .$

29. $a \varepsilon \mathrm{Q} . \supset . \log \frac{a+m+1}{a+1} < \frac{1}{a+1} + \frac{1}{a+2} + \ldots + \frac{1}{a+m} < \log \frac{a+m}{a} .$

30. $a \varepsilon \mathrm{q} - (-\mathrm{N}) . \supset . \frac{1}{a+1} + \frac{1}{a+2} + \ldots = \Sigma_1^\infty \frac{1}{a+n} = \infty .$

31. $a \varepsilon \mathrm{Q} . p \varepsilon \mathrm{Q} . \supset . \frac{1}{(a+1)^{1+p}} + \frac{1}{(a+2)^{1+p}} + \ldots \varepsilon \mathrm{Q} \frown \left(\frac{1}{p\, a^p} - \mathrm{Q}\right) \frown$
$\left(\frac{1}{p\,(a+1)^p} + \mathrm{Q}\right) .$

32. $a , b , p \varepsilon \mathrm{Q} . \supset . \frac{1}{ab^p} < p\left(\frac{1}{b^{1+p}} + \frac{1}{(b+a)^{1+p}} + \frac{1}{(b+2\,a)^{1+p}} + \ldots\right)$
$< \frac{1}{ab^p} + \frac{p}{b^{1+p}} .$

33. $a \varepsilon \mathrm{Q\,f\,N} . \lim a_n = \infty . \lim a_{n+1}/a_n = 1 . \supset . \lim_{p=+0} p\, \Sigma_1^\infty \frac{a_{n+1} - a_n}{a_{n+1}\, a_n^p} = 1 .$

34. » » » $= c > 1 .$ » $\frac{c-1}{c \log c} .$

35. » » » $= \infty .$ » $= 0 .$

36. $x \varepsilon \mathrm{q} - (-\mathrm{N}) . \supset . \frac{1}{x+1} = \frac{1}{(x+1)(x+2)} + \frac{1}{(x+2)(x+3)} + \ldots$

37. $x \varepsilon \mathrm{q} - (-\mathrm{N}) . p \varepsilon \mathrm{N} . \supset . \Sigma_{n=1}^{n=\infty} \frac{1}{(x+n)\,(x+n+1)\ldots(x+n+p)} =$
$\frac{1}{p\,(x+1)\,(x+2)\ldots(x+p)} .$

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27. P. du Bois-Reymond. *Crelle's Journal*, t. 76, a. 1873, pag. 70.
28. Pringsheim. *Math. Ann.*, t. 35, a. 1889, pag. 329, 335.
30. Abel. *Œuvres*, a. MDCCCLXXXI, I, pag. 401.
32. Lejeune-Dirichlet. *Lezioni sulla teoria dei numeri*, tradotto da Faifofer, Venezia 1881, pag. 300; *Zahlentheorie*, a. 1894, pag. 301.
33-35. Pringsheim. *Math. Ann.*, t. 37. a. 1890, pag. 47.

38. $p \varepsilon \mathrm{N} \,.\supset .\, \frac{1}{1\,(p+1)} + \frac{1}{2\,(p+2)} + \ldots + \frac{1}{n\,(p+n)} + \ldots =$

$\frac{1}{p}\left(1 + \frac{1}{2} + \ldots + \frac{1}{p}\right).$

39. $b\,,a\, \varepsilon\, \mathrm{q} \,.\, a - \varepsilon\, (-\mathrm{N}_0 b) .\supset .\, \Sigma_{n=0}^{n=\infty} \frac{1}{(a+nb)\,(a+(n+1)b)\ldots(a+(n+m)b)}$

$= \frac{1}{mb} \frac{1}{a\,(a+b)\ldots(a+(m-1)\,b)}.$

40. $b\, \varepsilon\, \mathrm{Q} \,.\, a\, \varepsilon\, (b+\mathrm{Q}) \,.\supset .\, \frac{a}{a-b} = 1 + \frac{b}{a+b} + \frac{2\,!\,b^2}{(a+b)\,(a+2\,b)} +$

$\frac{3\,!\,b^3}{(a+b)\,(a+2\,b)\,(a+3\,b)} + \ldots$ [§2 P21]

41. $a\, \varepsilon\, \mathrm{q} \,.\, b\, \varepsilon\, \mathrm{Q} \,.\, p\, \varepsilon\, \mathrm{N} \,.\, \varphi\, \varepsilon\, \mathrm{N\,f\,N} \,.\supset .$

$\lim \Sigma_{\nu=n}^{\nu=n+\varphi_n} \left(\frac{b}{(a+\nu\, b)\log(a+\nu\, b)\ldots\log^{p-1}(a+\nu\, b)}\right.$

$\left. - \log^p (a + (\nu+1)\, b) + \log^p (a+\nu b)\right) = 0\,.$

42. $a\, \varepsilon\, \mathrm{q} \,.\, b\, \varepsilon\, \mathrm{Q} \,.\, p\, \varepsilon\, \mathrm{N} \,.\, \log^p (a+mb)\, \varepsilon\, \mathrm{Q} \,.\, \Sigma_{\nu=m}^{\nu=\infty}$

$\left(\frac{b}{(a+\nu\, b)\log(a+\nu\, b)\ldots\log^{p-1}(a+\nu\, b)} - \log^p (a+(\nu+1)\, b) + \right.$

$\left.\log^p (a+\nu\, b)\right) = \mathrm{E}_p \,.\supset : r\, \varepsilon\, (m+\mathrm{N}) \,.\supset_r .\, \log^p (a+mb) + \Sigma_{\nu=m}^{\nu=r}$

$\frac{b}{(a+\nu\, b)\log(a+\nu\, b)\ldots\log^{p-1}(a+\nu\, b)} - \log^p (a+rb) > \mathrm{E}_p >$

$\log^p (a+mb) + \Sigma_{\nu=m}^{\nu=r} \frac{b}{(a+\nu\, b)\log(a+\nu\, b)\ldots\log^{p-1}(a+\nu\, b)} -$

$\log^p (a+(r+1)\, b)\,.$

43. $\lim \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n\right) \varepsilon\, \mathrm{Q}\,.$ [§2 P42]

44. $\lim \left(\frac{1}{2\log 2} + \frac{1}{3\log 3} + \ldots + \frac{1}{n\log n} - \log^2 n\right) \varepsilon\, \mathrm{Q}\,.$ [ » ]

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39. $(m=1)$. Johannis Bernoulli. *Opera omnia*, a. MDCCXLII, t. 4, pag. 7.

41, 42, 46-50. Giudice. Giornale di Battaglini, t. 28, a. 1890, pag. 290-292 ; t. 27, a. 1889, pag. 342-351.

45. $\lim \left(\frac{1}{3 \log 3 \log^2 3} + \ldots + \frac{1}{n \log n \log^2 n} - \log^3 n\right) \varepsilon \mathrm{Q}$ . [§2 P42]

46. $a \varepsilon \mathrm{q\,f\,N} . \lim (a_n - a_{n-1}) \varepsilon \mathrm{Q} . \supset . \lim (a_n - a_{n-1}) = \lim (\log^m a_n - \log^m a_{n-1})\, a_n \log a_n \ldots \log^{m-1} a_n$ .

47. $p \varepsilon (m + \mathrm{N}) . \supset . \lim \left(\frac{1}{mn+1} + \frac{1}{mn+2} + \ldots + \frac{1}{pn}\right) = \log \frac{p}{m}$ .

48. » $\lim \left(\frac{1}{n^m \log n^m} + \frac{1}{(n^m+1) \log (n^m+1)} + \ldots + \frac{1}{n^p \log n^p}\right) = \log \frac{p}{m}$ .

49. $a , u \varepsilon \mathrm{q} . u -\varepsilon (-\mathrm{N}\, a) . k \varepsilon \mathrm{N} . \supset . \Sigma_{n=1}^{n=m} \frac{1}{u+na} - \Sigma_{n=1}^{n=k} \frac{1}{u+na} + \Sigma_{m+1}^{2m} \frac{1}{u+na} - \Sigma_{k+1}^{2k} \frac{1}{u+na} + \ldots = \frac{1}{a} \log \frac{m}{k}$ .

50. $k \varepsilon \mathrm{N} . \supset . \Sigma_1^m \frac{1}{2n-1} - \Sigma_1^k \frac{1}{2n} + \Sigma_{m+1}^{2m} \frac{1}{2n-1} - \Sigma_{k+1}^{2k} \frac{1}{2n} + \Sigma_{2m+1}^{3m} \frac{1}{2n-1} - \Sigma_{2k+1}^{3k} \frac{1}{2n} + \ldots = \log 2 + \frac{1}{2} \log \frac{m}{k}$ .

## § 3. Q f N.

$u \varepsilon \mathrm{QfN} . m \varepsilon \mathrm{N} . \supset :$

1. $\Sigma u \varepsilon (\mathrm{QfN})\, \mathrm{cres} . \Sigma u_\infty = \mathrm{l}' \Sigma u_\infty$ . [§2 P2]
2. $\Sigma u_\infty \varepsilon \mathrm{Q} \cup \iota \infty$ . [ » ]
3. $a \varepsilon \mathrm{Q} : n \varepsilon \mathrm{N} . \supset_n . \Sigma u_n < a : \supset . \Sigma u_\infty \varepsilon \mathrm{Q} . \Sigma u_\infty \varepsilon \theta a$ . [§3 P1]
4. $a \varepsilon \theta \cap - \iota 1 : n \varepsilon \mathrm{N} . \supset_n . \sqrt[n]{u_n} < a : \supset . \Sigma u_\infty \varepsilon \mathrm{Q} \cap \left(\frac{1}{1-a} - \mathrm{Q}\right)$ .
5. $a \varepsilon \mathrm{Q} : n \varepsilon \mathrm{N} . \supset_n . u_n / u_{n+1} > 1 + a : \supset . \Sigma u_\infty \varepsilon \mathrm{Q} \cap \left(u_1 + \frac{u_1}{a} - \mathrm{Q}\right)$ .
6. $\Sigma u_\infty \varepsilon \mathrm{Q} . \supset . 0 \varepsilon \lim n u_n . 0 = \min \lim n u_n$ .
7. $p \varepsilon \mathrm{Q} . \infty -\varepsilon \lim n^{1+p} u_n . \supset . \Sigma u_\infty \varepsilon \mathrm{Q}$ .

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§ 3. 4, 7, 14, 15, 40. Cauchy. *Anal. Alg.*, pag. 130, 132, 133, 134, 137.
5, 6, 20, 35, 36, 48, 49. Abel. *Oeuvres,* I, pag. 400; II, pag. 197, 198, 199, 201, 202.

8. $(v - u) \varepsilon \mathrm{Q f N} . \Sigma u_\infty = \infty . \supset . \Sigma v_\infty = \infty .$ [§3 P1]

9. $a \varepsilon \mathrm{Q} . \left(\frac{v}{u} - a\right) \varepsilon \mathrm{Q f N} . \Sigma u_\infty = \infty . \supset . \Sigma v_\infty = \infty .$

10. $\Sigma u_\infty = \infty . v \varepsilon \mathrm{Q f N} : n \varepsilon \mathrm{N} . \supset_n . \frac{v_{n+1}}{v_n} \geqq \frac{u_{n+1}}{u_n} . \supset . \Sigma v_\infty = \infty .$

11. $v \varepsilon \mathrm{QfN} . \Sigma v_\infty \varepsilon \mathrm{Q} : n \varepsilon \mathrm{N} . \supset_n . u_n \leqq v_n : \supset . \Sigma u_\infty \varepsilon \mathrm{Q} .$ [§3 P1]

12. » » $. \infty - \varepsilon \lim u_n / v_n . \supset .$ »

13. » » $: n \varepsilon \mathrm{N} . \supset_n . u_{n+1}/u_n < v_{n+1}/v_n . \supset .$ » .

14. $\max \lim u_n^{1/n} < 1 . \supset . \Sigma u_\infty \varepsilon \mathrm{Q} .$

15. $\min \lim u_n^{1/n} > 1 . \supset . \Sigma u_\infty = \infty .$

15'. $n \varepsilon (m + \mathrm{N}) . \supset_n . u_n^{1/n} \geqq 1 : \supset . \Sigma u_\infty = \infty .$

16. $v \varepsilon \mathrm{Q f N} . \min \lim \frac{v_n - v_{n+1}}{u_{n+1}} > 0 . \supset . \Sigma u_\infty \varepsilon \mathrm{Q} .$

17. » $. \lim v_n = \infty . \min \lim u_{n+1} v_{n+1} / (v_{n+1} - v_n) > 0 . \supset . \Sigma u_\infty = \infty .$

18. $v \varepsilon (\mathrm{Q f N}) \mathrm{cres} . \lim v_n = \infty . p \varepsilon \mathrm{Q} . \supset . \Sigma_1^\infty \frac{v_{n+1} - v_n}{v_{n+1}} = \infty . \Sigma_1^\infty \frac{v_{n+1} - v_n}{v_{n+1} v_n^p} \varepsilon \mathrm{Q} .$

19. $g \varepsilon (\mathrm{Q f N}) \mathrm{decr} . \lim g_n = 0 . p \varepsilon \mathrm{Q} . \supset . \Sigma_1^\infty \frac{g_n - g_{n+1}}{g_n^{1-p}} \varepsilon \mathrm{Q} . \Sigma_1^\infty$
$\frac{g_n - g_{n+1}}{g_n} = \infty .$ [§3 P18]

20. $\Sigma u_\infty = \infty . p \varepsilon \mathrm{Q} . \supset . \Sigma_1^\infty \frac{u_n}{(\Sigma u_n)^{1+p}} \varepsilon \mathrm{Q} . \Sigma_1^\infty \frac{u_n}{\Sigma u_n} = \infty .$ [ » ]

21. $p , \Sigma u_\infty \varepsilon \mathrm{Q} . \supset . \Sigma_1^\infty \frac{u_n}{(\Sigma_n^\infty u)^{1-p}} \varepsilon \mathrm{Q} . \Sigma_1^\infty \frac{u_n}{\Sigma_n^\infty u} = \infty .$ [§3 P19]

22. $p , \mathrm{l}^1 u_\infty \varepsilon \mathrm{Q} . \supset . \Sigma_1^\infty \frac{u_n}{(\Pi_1^{n-1} (1 + u))^p} \varepsilon \mathrm{Q} .$

23. $u \varepsilon \theta \mathrm{f N} . p \varepsilon \mathrm{Q} . \supset . \Sigma_1^\infty u_n (\Pi_1^{n-1} (1 - u))^p \varepsilon \mathrm{Q} .$

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9, 10, 12, 13, 45, 46, 46'. Bonnet. *Journal de Liouville,* VIII, a. 1843, pag. 73, 74, 95, 100.

16, 17, 19, 26, 27, 33, 34, 53, 53'. Dini. *Annali delle Università Toscane,* IX, a. 1867, pag. 43, 45, 46, 47, 61.

18, 22, 23. Pringsheim. *Mathematische Annalen,* XXXV, a. 1889, pag. 230, 330, 334.

21, 50, 51. Giudice. *Giornale di Battaglini,* XXVIII, a. 1890, pag. 301; Rendiconti Circolo Matematico di Palermo, IV, a. 1890, pag. 284.

24. $\Sigma u_\infty = \infty \,.\supset.\, \lim \frac{1}{\log \Sigma u_n}\left(1 + \frac{u_2}{\Sigma u_2} + \ldots + \frac{u_n}{\Sigma u_n}\right) = \lim 1/\log \left(1 + \frac{u_n}{\Sigma u_{n-1}}\right)^{\frac{\Sigma u_n}{u_n}}$ . [§1 P27, §3 P4]

25. $\Sigma u_\infty = \infty \,.\, \lim \frac{u_n}{\Sigma u_{n-1}} = 0 \,.\supset.\, \lim \frac{1 + \frac{u_2}{\Sigma u_2} + \ldots + \frac{u_n}{\Sigma u_n}}{\log \Sigma u_n} = 1$ .
» $a \varepsilon \mathrm{Q}$ . » $= a$ . » $a / (1+a) \log(1+a)$ .
» » $= \infty$ . » $= 0$ .
[§3 P24]

26. $\Sigma u_\infty = \infty .\supset.\, \Sigma_1^\infty \frac{u_{n+1}}{\Sigma u_n \log \Sigma u_n} = \infty \,.\, \Sigma_1^\infty \frac{u_{n+1}}{\Sigma u_n \log \Sigma u_n \log^2 \Sigma u_n} = \infty \,.\, \ldots$

27. » $p \varepsilon \mathrm{N} .\supset. \Sigma_1^\infty u_{n+1} \left[\Sigma u_n . \log \Sigma u_n . \log^2 \Sigma u_n . \ldots \log^p \Sigma u_n\right] = \infty$ .
[§3 P25]

28. $r \varepsilon \mathrm{N} \,.\, \log^r m \,\varepsilon\, \mathrm{Q} \,.\supset.\, \Sigma_m^\infty \frac{1}{n \log n \log^2 n \ldots \log^r n} = \infty$ . [§3 P27]

29. » » $. \, p \,\varepsilon\, \mathrm{Q} \,.\supset.\, \Sigma_m^\infty \frac{1}{n \log n \ldots (\log^{r-1} n)(\log^r n)^{1+p}} \,\varepsilon\, \mathrm{Q}$ .
[§3 P20, 25]

30. » $. \, \Sigma u_\infty \,\varepsilon\, \mathrm{Q} \,.\supset.\, 0 \,\varepsilon\, \lim u_n \, n \log n \log^2 n \ldots \log^r n$ . [§3 P9, 28]

31. » $. \, p \,\varepsilon\, \mathrm{Q} \,.\, \infty \,\mathrm{-}\varepsilon\, \lim u_n n \log n \log^2 n \ldots \log^{r-1} n (\log^r n)^{1+p} .\supset. \Sigma u_\infty \varepsilon \mathrm{Q}$
[§3 P12, 29]

32. $\Sigma u_\infty = \infty \,.\supset:\, \varphi \,\varepsilon\, \mathrm{N\,f\,N} \,.\, \lim \Sigma_n^{n+\varphi_n} u_n / [\Sigma u_n \log \Sigma u_n \ldots \log^m \Sigma u_n]$
$\mathrm{-}\varepsilon\, 0 \,.\, \mathrm{-}=_\varphi \Lambda$ . [§2 P4 - §3 P27]

33. $\Sigma u_\infty = \infty \,.\, p \,\varepsilon\, \mathrm{Q} \,.\supset:\, \varphi \,\varepsilon\, \mathrm{N\,f\,N} \,.\supset_\varphi .\, \lim \Sigma_n^{n+\varphi_n} u_n / [\Sigma u_n (\log \Sigma u_n)^{1+p}]$
$= 0 \,.\, \lim \Sigma_n^{n+\varphi_n} u_n / [\Sigma u_n \log \Sigma u_n (\log^2 \Sigma u_n)^{1+p}] = 0$ .

34. $\Sigma u_\infty = \infty \,.\, p \,\varepsilon\, \mathrm{Q} .\supset:\, \varphi \,\varepsilon\, \mathrm{N f N} .\supset_\varphi .\, \lim \Sigma_n^{n+\varphi_n} u_n / [\Sigma u_n \log \Sigma u_n \ldots \log^{m-1}$
$\Sigma u_n (\log^m \Sigma u_n)^{1+p}] = 0$ .

35. $\varphi \,\varepsilon\, \mathrm{Q\,f\,N} \,.\supset:\, v \,\varepsilon\, \mathrm{Q\,f\,N} \,.\, \min \lim \varphi_n v_n = 0 \,.\, \Sigma v_\infty = \infty \,.\, \mathrm{-}=_v \Lambda$ .

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24. Cesàro. *Analisi algebrica,* 1894, pag. 133.
28, 29, 47. Bertrand. *Journal de Liouville,* VII, a. 1842, p. 38, 43.
30-31. De Morgan. *Differential calculus,* a. 1839, pag. 323.

36. $\varphi \varepsilon \mathrm{Q f N} . \lim \varphi_n = \infty . \supset : v \varepsilon \mathrm{Q f N} . \max \lim \varphi_n v_n = \infty . \Sigma v_\infty \varepsilon \mathrm{Q} . - =_v \Lambda .$

37. $v \varepsilon (\mathrm{Q f N})\mathrm{cresc}. \lim v_n = \infty . p \varepsilon \mathrm{Q} . \max \lim \frac{v_{n+1} v_n^p}{v_{n+1} - v_n} u_n \varepsilon \mathrm{Q} . \supset . \Sigma u_\infty \varepsilon \mathrm{Q} .$ [§3 P12, 18]

38. » » $\min \lim \frac{v_n \log v_n \ldots \log^m v_n}{v_{n+1} - v_n} u_n . \supset . \Sigma u_\infty = \infty .$ [§3 P9, 27]

39. » » $p \varepsilon \mathrm{Q} . \max \lim \frac{v_n \log v_n \ldots \log^{m-1} v_n (\log^m v_n)^{1+p}}{v_n - v_{n-1}}$
$u_n \varepsilon \mathrm{Q} . \supset . \Sigma u_\infty \varepsilon \mathrm{Q} .$ [§3 P12, 34]

40. $\max \lim (u_{n+1}/u_n) < 1 . \supset . \Sigma u_\infty \varepsilon \mathrm{Q} .$

41. $\min \lim (n u_n / u_{n+1} - n - 1) > 0 . \supset . \Sigma u_\infty \varepsilon \mathrm{Q} .$
$\max \lim ($ » $) < 0 . \supset .$ » $= \infty .$

41′. $n \varepsilon (m + \mathrm{N}) . \supset_n . (n u_n / u_{n+1} - n - 1) \leqq 0 : \supset . \Sigma u_\infty = \infty .$

42. $v \varepsilon \mathrm{q f N} : n \varepsilon \mathrm{N} . \supset_n . u_{n+1}/u_n = 1 - v_n/n : \mathrm{l}' v_{\mathrm{N}} \leqq 1 : \supset . \Sigma u_\infty = \infty .$
$: \max \lim v_n < 1 : \supset .$ »
$: \mathrm{l}_1 v_{\mathrm{N}} > 1 : \supset . \Sigma u_\infty \varepsilon \mathrm{Q} .$
$: \min \lim v_n > 1 : \supset .$ »

43. $a \varepsilon \mathrm{Q} . p \varepsilon \mathrm{Q} . v \varepsilon \mathrm{q f N} . \infty - \varepsilon v_\infty : n \varepsilon \mathrm{N} . \supset_n . u_{n+1}/u_n = 1 + a/n + v_n/n^{1+p}$
$.\cdot. \supset : \lim u_n = \infty . \lim u_n / n^a \varepsilon \mathrm{Q} . \Sigma u_\infty = \infty .$
$\left[ c_n = \left(1 + \frac{a}{n} + \frac{v_n}{n^{1+p}}\right) \Big/ \left(1 + \frac{1}{n}\right)^a . \supset . \Pi c_\infty \varepsilon \mathrm{Q} . u_{n+1}/u_n = c_n \right.$
$\left. (n+1)^a / n^a \right]$

44. $a \varepsilon \mathrm{Q} . p \varepsilon \mathrm{Q} . v \varepsilon \mathrm{q f N} . \infty - \varepsilon v_\infty : n \varepsilon \mathrm{N} . \supset_n . u_{n+1}/u_n = 1 - a/n + v_n/n^{1+p}$
$.\cdot. \supset : \lim u_n = 0 . \lim n^a u_n \varepsilon \mathrm{Q} : a \leqq 1 . \supset . \Sigma u_\infty = \infty : a > 1 . \supset . \Sigma u_\infty \varepsilon \mathrm{Q} .$

45. $v \varepsilon \mathrm{q f N} : n \varepsilon \mathrm{N} . \supset_n . u_{n+1}/u_n = 1 - 1/n - v_n/(n \log n) : \mathrm{l}' v_{\mathrm{N}} \leqq 1 : \supset . \Sigma u_\infty = \infty .$
$: \max \lim v_n < 1 : \supset .$ »
$: \mathrm{l}_1 v_{\mathrm{N}} > 1 : \supset . \Sigma u_\infty \varepsilon \mathrm{Q} .$
$: \min \lim v_n > 1 : \supset . \Sigma u_\infty \varepsilon \mathrm{Q} .$

46. $v \varepsilon \mathrm{q f N} . p \varepsilon \mathrm{N} : n \varepsilon \mathrm{N} . \supset_n . u_{n+1}/u_n = 1 - 1/n - 1/(n \log n) - \ldots -$
$1/(n \log n \log^2 n \ldots \log^{p-1} n) - v_n/(n \log n \log^2 n \ldots \log^p n) :$
$\mathrm{l}' v_{\mathrm{N}} \leqq 1 : \supset . \Sigma u_\infty = \infty .$
$\mathrm{l}_1 v_{\mathrm{N}} > 1 : \supset . \Sigma u_\infty \varepsilon \mathrm{Q} .$

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41, 41′, 52, 52′. Kummer. *Crelle's Journal*, XIII, a. 1835, pag. 172, 173, 177.

42. Raabe-Duhamel. *Journal de Liouville*, IV, a. 1839, pag. 215; VI, a. 1840, pag. 85.

46'. $v \varepsilon \mathrm{q\,f\,N}.p \varepsilon \mathrm{N}:n \varepsilon \mathrm{N}.\supset_n.n\left(1-u_n^{\frac{1}{n}}\right)=\log n+\ldots+\log^{p-1}n+v_n \log^p n.$
$\mathrm{l}' v_{\mathrm{N}} \overline{\overline{<}} 1 . \supset . \Sigma u_\infty = \infty .$
$\mathrm{l}_1 v_{\mathrm{N}} > 1 . \supset :$ » $\varepsilon \mathrm{Q}$ .

47. $v \varepsilon \mathrm{qfN} . p \varepsilon \mathrm{N} : n \varepsilon \mathrm{N} . \supset_n . u_n/u_{n+1} = 1 + 1/n + 1/(n \log n) + \ldots +$ $1/(n \log n \log^2 n \ldots \log^{p-1} n) + v_n/(n \log n \log^2 n \ldots \log_p n) :$
$\mathrm{l}' v_{\mathrm{N}} \leqq 1 : \supset . \Sigma u_\infty = \infty .$
$\mathrm{l}_1 v_{\mathrm{N}} > 1 : \supset . \Sigma u_\infty \varepsilon \mathrm{Q}$ .

48. $\min\lim \dfrac{\log\left(\dfrac{1}{n\, u_n \log n \ldots \log^{m-1} n}\right)}{\log^{m+1} n} > 1 . \supset . \Sigma u_\infty \varepsilon \mathrm{Q}$ .

49. $\max\lim \dfrac{\log\left(\dfrac{1}{n u_n \log n \ldots \log^{m-1} n}\right)}{\log^{m+1} n} < 1 . \supset . \Sigma u_\infty = \infty$ .

50. $v \varepsilon \mathrm{Q\,f\,N} . \min\lim \dfrac{v_{n+1}}{v_n} = 0 . \Sigma v_\infty = \infty : - =_v \Lambda$ .

51. » $\max\lim v_{n+1}/v_n = \infty . \Sigma v_\infty \varepsilon \mathrm{Q} : - =_v \Lambda$ .

52. $v \varepsilon \mathrm{Q\,f\,N} . \min\lim (u_n v_n/v_{n+1} - u_{n+1}) > 0 . \supset . \Sigma v_\infty \varepsilon \mathrm{Q}$ .

52'. » $. \min\lim v_n u_n = 0 . \min\lim v_n u_n u_{n+1}/(u_n v_n - u_{n+1} v_{n+1}) > 0 .$
$\supset . \Sigma u_\infty = \infty$ .

53. » $. \Sigma_1^\infty \dfrac{1}{u} = \infty . \max\lim (u_n v_n/v_{n+1} - u_{n+1}) < 0 . \supset . \Sigma v_\infty = \infty$ .

53'. » . » $: n \varepsilon (m + \mathrm{N}) . \supset_n . (u_n v_n/v_{n+1} - u_{n+1}) \overline{\overline{<}} 0 . \supset . \Sigma v_\infty = \infty$ .

54. $v \varepsilon \mathrm{Q\,f\,N} : n \varepsilon \mathrm{N} . \supset_n . \dfrac{u_{n+1}}{u_n} \overline{\overline{<}} \dfrac{v_n}{1 + v_{n+1}} . \supset . \Sigma_1^\infty u \varepsilon \mathrm{Q}$ . [§2 P24, §3 P13]

55. $\Sigma_1^\infty u \varepsilon \mathrm{Q} . \supset : v \varepsilon \mathrm{Q\,f\,N} . n \varepsilon \mathrm{N} . \supset_n . \dfrac{u_{n+1}}{u_n} \overline{\overline{<}} \dfrac{v_n}{1 + v_{n+1}} : - =_v \Lambda$ .

56. $a , b \varepsilon \mathrm{q\,f\,Z}_m . n \varepsilon \mathrm{N} . \supset_n . \dfrac{u_n}{u_{n+1}} = \dfrac{n^m + a_1 n^{m-1} + \ldots + a_m}{n^m + b_1 n^{m-1} + \ldots + b_m} .$

$a_1 - b_1 > 1 . \supset . \Sigma u_\infty \varepsilon \mathrm{Q}$ .
» $\overline{\overline{<}} 1 . \supset .$ » $= \infty$ .

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56. Gauss. *Werke*, III, a. 1866, pag. 139.

57. $u_n/u_{n+1} = \lambda_n^{(0)} . n(u_n/u_{n+1} - 1) = \lambda_n^{(1)} . \log n\, (\lambda_n^{(1)} - 1) = \lambda_n^{(2)} . \log^2 n$
$(\lambda_n^{(2)} - 1) = \lambda_n^{(3)} \ldots \lim \lambda_n^{(0)} = \lim \lambda_n^{(1)} = \ldots = \lim \lambda_n^{(r-1)} = 1 .$
$\min\lim \lambda_n^{(m)} > 1 . \supset . \Sigma u_\infty \,\varepsilon\, Q .$
$\max\lim \lambda_n^{(m)} < 1 . \supset .$ » $= \infty .$ [§3 P47]

58. $v \,\varepsilon\, Q\,f\,N . \Sigma_1^\infty u = \infty : r \,\varepsilon\, N . \supset_r . \frac{u_{r+1}}{u_r} \leqq \left(1 + \frac{1}{v_r}\right) v_{r+1} . \supset . \Sigma_1^\infty v$
$= \infty . \lim \Sigma_1^n v / \Sigma_1^n u = 0 .$ [§2 P25]

59. $v \,\varepsilon\, Q\,f\,N : r \,\varepsilon\, N . \supset_r . \frac{u_{r+1}}{u_r} = \frac{v_{r+1}(v_r + 1)}{v_r} .$
$\lim u_n / v_n \,\varepsilon\, Q . \supset . \Sigma_1^\infty u \,\varepsilon\, Q .$
» $= \infty . \supset .$ » $= \infty .$ $\left[\frac{u_{r+1}}{v_{r+1}} - \frac{u_r}{v_r} = u_r .\right]$

60. $r \,\varepsilon\, N . \supset_r . \frac{u_{r+1}}{u_r} = \frac{v_{r+1}(v_r + 1)}{v_r} : \supset . v \,\varepsilon\, Q\,f\,N . -= \Lambda .$

61. $v \,\varepsilon\, Q\,f\,N : r \,\varepsilon\, N . \supset_r . \frac{u_{r+1}}{u_r} \leqq \frac{v_{r+1}(v_r + 1)}{v_r} : \max\lim \frac{u_n}{v_n} = \infty : \supset . \Sigma_1^\infty u = \infty .$

62. » » » » $\geqq$ » $\lim \frac{u_n}{v_n} \,\varepsilon\, Q : \supset .$ » $\varepsilon\, Q .$

63. $v \,\varepsilon\, Q\,f\,Z_{m+1} . a, b, (b - a) \,\varepsilon\, Q . \psi \,\varepsilon\, (a^{-}b)\, f\,Z_m : r \,\varepsilon\, Z_m . \supset_r . \frac{u_{r+1}}{u_r} =$
$\frac{v_{r+1}(\psi_r v_r + 1)}{v_r} : \supset . \frac{1}{b}\left(\frac{u_{m+1}}{v_{m+1}} - \frac{u_1}{v_1}\right) < \Sigma_1^m u < \frac{1}{a}\left(\frac{u_{m+1}}{v_{m+1}} - \frac{u_1}{v_1}\right) .$

64. $a, b, (b - a) \,\varepsilon\, Q . v \,\varepsilon\, Q\,f\,N . \psi \,\varepsilon\, (a^{-}b)\, f\,N . \Sigma_1^\infty u \,\varepsilon\, Q : r \,\varepsilon\, N . \supset_r . \frac{u_{r+1}}{u_r} =$
$\frac{v_{r+1}(\psi_r v_r + 1)}{v_r} : \supset . \lim \frac{u_n}{v_n} \,\varepsilon\, Q .$
$\frac{1}{b}\left(\lim \frac{u_n}{v_n} - \frac{u_1}{v_1}\right) < \Sigma_1^\infty u < \frac{1}{a}\left(\lim \frac{u_n}{v_n} - \frac{u_1}{v_1}\right) .$

## § 4. (Q f N) decr.

1. $u, v \,\varepsilon\, (Q\,f\,N)\,\mathrm{decr} . \Sigma u_\infty \,\varepsilon\, Q . \Sigma v_\infty = \infty . \varphi \,\varepsilon\, (N\,f\,N)\,\mathrm{cresc} . h \,\varepsilon\, Q :$
$r \,\varepsilon\, N . \supset_r . u_{\varphi_r} / v_{\varphi_r} > h .\therefore \supset . \infty \,\varepsilon\, \lim (\varphi_{n+1} - \varphi_n) .$

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§ 4. 1. GIUDICE. *Giornale Battaglini*, a. 1890, pag. 287.

2. $u \varepsilon (Q f N)$ decr . $\lambda \varepsilon (N f N)$ cresc . max lim $\frac{\lambda_n}{\lambda_{n+1}} < 1 . \Sigma u_\infty \varepsilon Q . \supset . \Sigma_1^\infty \lambda_n u_{\lambda_n} \varepsilon Q$ .

3. $u \varepsilon (Q f N)$ decr . $\lambda \varepsilon (N f N)$ cresc . min lim $\frac{\lambda_n}{\lambda_{n+1}} > 0 . \Sigma u_\infty = \infty . \supset . \Sigma_1^\infty \lambda_n u_{\lambda_n} = \infty$ .

4. $u \varepsilon (Q f N)$ decr . $a \varepsilon N . \supset : \Sigma u_\infty \varepsilon Q . = . \Sigma_1^\infty a^n u_{a^n} \varepsilon Q$ .

5. $v \varepsilon (Q f N)$ decr . $\Sigma v_\infty \varepsilon Q . u_1 = v_1 - v_2 . u_2 = 2(v_2 - v_3) . u_3 = 3(v_3 - v_4) . \ldots . \supset . \Sigma v_\infty = \Sigma_1^\infty \frac{u_r}{r} + \Sigma_2^\infty \frac{u_r}{r} + \Sigma_3^\infty \frac{u_r}{r} + \ldots = \Sigma_1^\infty u$ .

6. $u \varepsilon (Q f N)$ decr . max lim $nu_n > 0 . \supset . \Sigma u_\infty = \infty$ . [§4 P4]

7. » $. \supset : \Sigma u_\infty \varepsilon Q . = . \lim nu_n = 0 . \Sigma_1^\infty n(u_n - u_{n+1}) \varepsilon Q$ . [§1 P30 . §4 P6]

8. » $. \Sigma u_\infty \varepsilon Q . \supset . \Sigma u_\infty = \Sigma_1^\infty n(u_n - u_{n+1})$. [ » ]

9. $u \varepsilon Q f N . \Sigma u_\infty \varepsilon Q . \frac{1}{\lambda}, \alpha u \varepsilon (Q f N)$ decr . $\lim \lambda_n = \infty$ . max lim $\alpha_n (\lambda_n - \lambda_{n-1}) \varepsilon Q . \supset . \lim \lambda_n \alpha_n u_n = 0$ .
[Hp . $\supset . \Sigma_1^\infty (\lambda_n - \lambda_{n-1}) \alpha_n u_n \varepsilon Q . \lim \frac{\lambda_m}{\lambda_{m+n}} = 0 . (\lambda_{m+1} - \lambda_m) \alpha_{m+1} u_{m+1} + \ldots + (\lambda_{m+n} - \lambda_{m+n-1}) \alpha_{m+n} u_{m+n} > \lambda_{m+n} \left(1 - \frac{\lambda_m}{\lambda_{m+n}}\right) \alpha_{m+n} u_{m+n}$ : §2 P3 . $\supset$ . Ths.]

10. $u \varepsilon Q f N : n \varepsilon N . \supset_n . nu_n > (n+1) u_{n+1}$ : max lim $nu_n \log n > 0 . \supset . \Sigma u_\infty = \infty$ . [§4 P9 . §2 P46]

11. $u \varepsilon Q f N : n \varepsilon N . \supset_n . nu_n \log n > (n+1) u_{n+1} \log(n+1)$ : max lim $nu_n \log n \log^2 n > 0 . \supset . \Sigma u_\infty = \infty$ . [ » ]

12. $u \varepsilon Q f N : n \varepsilon N . \supset_n . nu_n \log n \log^2 n > (n+1) u_{n+1} \log(n+1) \log^2(n+1)$ : max lim $nu_n \log n \log^2 n \log^3 n > 0 . \supset . \Sigma u_\infty = \infty$ . [ » ]

13. $u \varepsilon Q f N . \Sigma u_\infty \varepsilon Q : n \varepsilon N . \supset_n . nu_n \log n \log^2 n \ldots \log^m n > (n+1) u_{n+1} \log(n+1) \ldots \log^m(n+1) . \supset . \lim nu_n \log n \log^2 n \ldots \log^{m+1} n = 0$ . [ » ]

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2, 3. Dini. *Annali Università Toscane*, a. 1867, pag. 78-80.
4. Cauchy. *Anal. Alg.*, pag. 135.

14. $u \varepsilon (\mathrm{Q\,f\,N})$ cresc . $d \varepsilon \mathrm{Q\,f\,N} . \Sigma_1^\infty \frac{1}{d} = \infty$ . max lim $\frac{u_{n+1} - u_n}{d_{n+1} - d_n} \varepsilon \mathrm{Q} . \supset .$
$\Sigma_1^\infty \frac{1}{u} = \infty$ .

15. $u \varepsilon (\mathrm{Q\,f\,N})$ cres . $c \varepsilon \mathrm{Q\,f\,N} . \Sigma_1^\infty \frac{1}{c} \varepsilon \mathrm{Q}$ . min lim $\frac{u_{n+1} - u_n}{c_{n+1} - c_n} \varepsilon \mathrm{Q} . \supset .$
$\Sigma_1^\infty \frac{1}{u} \varepsilon \mathrm{Q}$ .

16. $u \varepsilon (\mathrm{Q\,f\,N})$ cresc . max lim $\frac{u_{n+1} - u_n}{\log n \log^2 n \ldots \log^m n} \varepsilon \mathrm{Q} . \supset . \Sigma_1^\infty \frac{1}{u} = \infty$ .

17. $u \varepsilon (\mathrm{Q\,f\,N})$ cresc . $p \varepsilon \mathrm{Q}$ . min lim $\frac{u_{n+1} - u_n}{\log n \log^2 n \ldots \log^{m-1} n (\log^m n)^{1+p}}$
$\varepsilon \mathrm{Q} . \supset . \Sigma_1^\infty \frac{1}{u} \varepsilon \mathrm{Q}$ .

18. $u \varepsilon (\mathrm{Q\,f\,N})$ cresc . max lim $\frac{u_{n+1} - u_n}{\log u_n \log^2 u_n \ldots \log^m u_n} \varepsilon \mathrm{Q} . \supset . \Sigma_1^\infty \frac{1}{u} = \infty$ .

19. $u \varepsilon (\mathrm{Q\,f\,N})$ cresc . $p \varepsilon \mathrm{Q}$ . min lim $\frac{u_{n+1} - u_n}{\log u_n \log^2 u_n \ldots \log^{m-1} u_n (\log^m u_n)^{1+p}}$
$\varepsilon \mathrm{Q} . \supset . \Sigma_1^\infty \frac{1}{u} \varepsilon \mathrm{Q}$ .

20. $\varphi, \frac{1}{g}, \frac{1}{h} \varepsilon (\mathrm{Q\,f\,Q})$ decr . $(g-h) \varepsilon \mathrm{Q\,f\,Q}$ . $\lim_{x=\infty} h_x = \infty . p \varepsilon \mathrm{Q}$ . min
$\lim_{x=\infty} \frac{(g_{x+p} - g_x)\varphi(g_{x+p})}{(h_{x+p} - h_x)\varphi(h_x)} > 1 . \supset . \Sigma_1^\infty \varphi(n) = \infty$ .

21. $\varphi, \frac{1}{g}, \frac{1}{h} \varepsilon (\mathrm{Q\,f\,Q})$ decr . $(g-h) \varepsilon \mathrm{Q\,f\,Q}$ . $\lim_{x=\infty} h_x = \infty . p \varepsilon \mathrm{Q}$ . max
$\lim_{x=\infty} \frac{(g_{x+p} - g_x)\varphi(g_x)}{(h_{x+p} - h_x)\varphi(h_{x+p})} < 1 . \supset . \Sigma_1^\infty \varphi(n) \varepsilon \mathrm{Q}$ .

22. $\varphi, \frac{1}{g} \varepsilon (\mathrm{Q\,f\,Q})$ decr : $x \varepsilon \mathrm{Q} . \supset_x . g_x > x : \lim_{p=+0} \frac{g_{x+p} - g_x}{p} = g'_x$ .
$\lim \frac{g'_x \varphi(g_x)}{\varphi(x)} \begin{matrix} > 1 \\ < 1 \end{matrix} . \bigcap . \Sigma_1^\infty \varphi(n) \begin{matrix} = \infty . \\ \varepsilon \mathrm{Q} . \end{matrix}$

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14-21. PRINGSHEIM. *Mathematische Annalen,* XXXV, pag. 381-392.
22, 24, 25. ERMAKOF. *Bulletin de Darboux*, a. 1871, pag. 250, 255.

23. $\varphi\,,\,\frac{1}{h}\;\varepsilon\,(\mathrm{Q\,f\,Q})\,\mathrm{decr}\,.\,\lim_{x=\infty} h_x = \infty : x\,\varepsilon\,\mathrm{Q}\,.\,\supset_x .\, x > h_x : \lim_{p=+0}$

$$\frac{h_{x+p}-h_x}{p} = h'_x\,.$$

$$\lim_{x=\infty}\frac{\varphi(x)}{h'_x\,\varphi(h_x)}\begin{matrix}>1\,.\\<1\,.\end{matrix}\supset\begin{matrix}.\\.\end{matrix}\,\Sigma_1^\infty\,\varphi(n)\begin{matrix}=\infty\,.\\\varepsilon\,\mathrm{Q}\,.\end{matrix}$$

24. $\varphi\,\varepsilon\,(\mathrm{Q\,f\,Q})\,\mathrm{decr}\,.\,\lim\frac{e^n\varphi(e^n)}{\varphi(n)}\begin{matrix}>1\,.\\<1\,.\end{matrix}\supset\begin{matrix}.\\.\end{matrix}\,\Sigma_1^\infty\,\varphi(n)\begin{matrix}=\infty\,.\\\varepsilon\,\mathrm{Q}\,.\end{matrix}$

25. $\varphi\,\varepsilon\,(\mathrm{Q\,f\,Q})\,\mathrm{decr}\,.\,\lim\frac{n\,\varphi(n)}{\varphi(\log n)}\begin{matrix}>1\,.\\<1\,.\end{matrix}\supset\begin{matrix}.\\.\end{matrix}\,\Sigma_1^\infty\,\varphi(n)\begin{matrix}=\infty\,.\\\varepsilon\,\mathrm{Q}\,.\end{matrix}$

## § 5. $\Pi\, u_\infty$.

$u\,\varepsilon\,\mathrm{Q\,f\,N}\,.\supset :$

1. $\Pi\, u_\infty = \Pi_1^\infty\, u = u_1\, u_2\, u_3 \ldots = \lim \Pi\, u_n\,.$ [Def.]
2. $\Pi\, u_\infty\;\varepsilon\,\mathrm{Q}\,.=.\,\Sigma_1^\infty \log u\;\varepsilon\,\mathrm{q}\,.$
3. $u\,\varepsilon\,\mathrm{Q\,f\,N}\,.\supset : \Pi\, u_\infty = \infty\,.=.\,\Sigma_1^\infty \log u = \infty\,.=.\,\Pi_1^\infty \frac{1}{u} = 0\,.=.\,\Sigma_1^\infty$ $\log\frac{1}{u} = -\infty\,.$
4. $\Pi\, u_\infty\;\varepsilon\,\mathrm{Q}\,.\supset.\,\lim u_n = 1\,.$
5. $v\,\varepsilon\,\mathrm{Q f N}\,.\,\Sigma\, v_\infty\;\varepsilon\,\mathrm{Q}\,.\supset.\,\Pi_1^\infty\,(1+v_n)\;\varepsilon\,\mathrm{Q}\,.$
6. $\quad$ » $\quad$ » $= \infty\,.\supset.$ $\quad$ » $= \infty\,.$
7. $v\,\varepsilon\,\theta\,\mathrm{f N}\,.\,\Sigma\, v_\infty\;\varepsilon\,\mathrm{Q}\,.\supset.\,\Pi_1^\infty\,(1-v_n)\;\varepsilon\,\mathrm{Q}\,.$
8. $\quad$ » $= \infty\,.\supset.$ $\quad$ » $= 0\,.$
9. $v\,\varepsilon\,(-1+\mathrm{Q})\,\mathrm{f N}\,.\,\Sigma\, v_\infty\,,\,\Sigma\, v_\infty^2\;\varepsilon\,\mathrm{q}\,.\supset.\,\Pi_1^\infty\,(1+v_n)\;\varepsilon\,\mathrm{Q}\,.$
10. $\quad$ » $\quad \Sigma\, v_\infty\;\varepsilon\,\mathrm{q}\,.\,\Sigma\, v^2_\infty = \infty\,.\supset.$ » $= 0\,.$
11. $m\,\varepsilon\,\mathrm{N}\,.\supset : \Pi\, u_\infty = 0\,.=.\,\Pi_m^\infty\, u = 0\,.$ [§5 P1]
12. $m\,\varepsilon\,\mathrm{N}\,.\supset : \Pi\, u_\infty\;\varepsilon\,\mathrm{Q}\,.=.\,\Pi_m^\infty\, u\;\varepsilon\,\mathrm{Q}\,.$ [ » ]
13. $\Pi\, u_\infty\;\varepsilon\,\mathrm{q}\,.=.\,\varphi\,\varepsilon\,\mathrm{N\,f\,N}\,.\supset_\varphi.\,\lim \Pi_n^{n+\varphi n}\, u = 1\,.$ [ » ]

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§ 5. 2-10. CAUCHY. *Analyse algébrique*, pag. 561, 562, 563.

14. $\varphi, u \,\varepsilon\, (1+\mathrm{Q})\,\mathrm{f\,N} : n \,\varepsilon\, \mathrm{N} \,.\supset_n .\, \varphi_n / \varphi_{n+1} \overline{>} u_{n+1} : \supset .\, \Pi\, u_\infty \,\varepsilon\, \mathrm{Q}\,.$

15. » $.\, \lim \varphi_n = 1\,.$ » » $(\varphi_n / \varphi_{n+1})^{\frac{1}{\varphi_n - 1}} \overline{<} u_{n+1} : \supset .$ » $= \infty\,.$

16. $v \,\varepsilon\, (\mathrm{Q\,f\,N})\,\mathrm{decr}\,.\, \Pi\, v_\infty \,\varepsilon\, \mathrm{Q} : n \,\varepsilon\, \mathrm{N} \,.\supset_n .\, u_n = \left(\frac{v_n}{v_{n+1}}\right)^n : \supset .\, \Pi\, v_\infty = \Pi_1^\infty$

$u_r^{\frac{1}{r}}\, \Pi_2^\infty u_r^{\frac{1}{r}}\, \Pi_3^\infty u_r^{\frac{1}{r}} \ldots = \Pi\, u_\infty\,.$

17. $u \,\varepsilon\, (\mathrm{Q\,f\,N})\,\mathrm{decr}\,.\, \max\lim\, (1 + u_n)^n > 1 \,.\supset .\, \Pi\, (1+u)_\infty = \infty\,.$
[§4 P6 . §5 P2]

18. $\alpha \,\varepsilon\, \mathrm{Q\,f\,N}\,.\, \Pi\,(1+u)_\infty \,\varepsilon\, \mathrm{Q}\,.\, \frac{1}{\lambda}, (1+u)^\alpha \,\varepsilon\, (\mathrm{Q\,f\,N})\,\mathrm{decr}\,.\, \lim \lambda_n = \infty\,.$

$\max\lim\, \alpha_n (\lambda_n - \lambda_{n-1}) \,\varepsilon\, \mathrm{Q} \,.\supset .\, \lim\, (1+u_n)^{\lambda_n \alpha_n} = 1.$ [§4 P9 . §5 P2]

19. $a^2 < 1 \,.\supset .\, \Pi_1^\infty\, (1 + a^n) = \Pi_1^\infty \frac{1}{1 - a^{2n-1}}\,.$

20. $k \,\varepsilon\, (1+\mathrm{Q}) \,.\supset .\, \Sigma_1^\infty \frac{1}{n^k} = \Pi_1^\infty \left(1 - \frac{1}{\mathrm{Np}_n{}^k}\right).$

## § 6. q′ f N .

1. $\binom{\mathrm{q}'}{\mathrm{q}}$ [§ 1 . § 2, P1-13] $\binom{\mathrm{q}'}{\mathrm{Q}}$ [§ 5 P1 , 4 , 12 , 13]

2. $u \,\varepsilon\, \mathrm{q}'\mathrm{f\,N}\,.\, \Sigma_1^\infty \bmod u \,\varepsilon\, \mathrm{Q} \,.\supset .\, \Sigma\, u_\infty \,\varepsilon\, \mathrm{q}'\,.$

3. $u \,\varepsilon\, \mathrm{q}'\,\mathrm{f\,N}\,.\, \mathrm{l}' \bmod u_{\mathrm{N}} \,\varepsilon\, \mathrm{Q}\,.\, v \,\varepsilon\, \mathrm{Q\,f\,N}\,.\, \Sigma\, v_\infty \,\varepsilon\, \mathrm{Q} \,.\supset .\, \Sigma_1^\infty u_n v_n \,\varepsilon\, \mathrm{q}'\,.$
[§2 P4 . §6 P1]

4. $u \,\varepsilon\, \mathrm{q}'\mathrm{f\,N}\,.\, \Sigma \bmod u_\infty \,\varepsilon\, \mathrm{Q}\,.\, v \,\varepsilon\, \mathrm{q}'\mathrm{f\,N}\,.\, \mathrm{l}' \bmod v_{\mathrm{N}} \,\varepsilon\, \mathrm{Q} \,.\supset .\, \Sigma_1^\infty u_n v_n \,\varepsilon\, \mathrm{q}'\,.$
[ » ]

5. » $.\, \Sigma\, u_\infty \,\varepsilon\, \mathrm{q}'\,.\, v \,\varepsilon\, (\mathrm{Q\,f\,N})\,\mathrm{dec} \,.\supset .\, \Sigma_1^\infty u_n v_n \,\varepsilon\, \mathrm{q}'\,.$

6. » . » . » $\mathrm{cres}.\, v_\infty \overline{=} \infty \,.\supset .$ »

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14, 15. GIUDICE. *Giornale Battaglini*, a. 1890, pag. 305, 306.

20. EULER. *Introductio in Analysin inf.*, I, a. 1748, pag. 225.

§ 6. 2, 4, 15, 18, 19. CAUCHY. *Analyse Algébrique,* pag. 147, 274-277, 280, 281.

4. O. BONNET. *Liouville Journal*, a. 1843, pag. 73.

5, 6, 8, 9. ABEL. *Œuvres*, I, pag. 222. - DIRICHLET. *Teoria dei numeri*, pag. 368. - CAPELLI-GARBIERI. *Analisi algebrica*, a. 1886, pag. 190.

7. » $. \mathrm{l}' \bmod (\Sigma u)_N -= \infty . v \varepsilon (\mathrm{Q f N})\mathrm{dec.lim}\, v_n = 0 . \supset . \Sigma_1^\infty u_n v_n \varepsilon \mathrm{q}'.$

8. » » $. v \varepsilon \mathrm{q}'\mathrm{fN} . \Sigma_1^\infty \bmod (v_n - v_{n+1}) \varepsilon \mathrm{Q} .$
$\lim v_n = 0 . \supset . \Sigma_1^\infty u_n v_n \varepsilon \mathrm{q}' .$

9. » $. \Sigma u_\infty \varepsilon \mathrm{q}' . v \varepsilon \mathrm{q}'\mathrm{fN} . \Sigma_1^\infty \bmod (v_n - v_{n+1}) \varepsilon \mathrm{Q} . \supset . \Sigma_1^\infty u_n v_n \varepsilon \mathrm{q}'.$
[§1 P32 . §2 P4 . §6 P1]

10. $u \varepsilon \mathrm{q}'\mathrm{f N} . \Sigma u_\infty \varepsilon \mathrm{q}' . \supset .\because \varphi \varepsilon (\mathrm{N f N})\, \mathrm{sim} . \supset_\varphi . \sum_{n=1}^{n=\infty} u_{\varphi n} = \Sigma u_\infty :=: \Sigma_1^\infty$
$\bmod u \varepsilon \mathrm{Q} .$

11. $u \varepsilon \mathrm{q}'\mathrm{f N} . \mathrm{l}' \bmod u_N \varepsilon \mathrm{Q} . v \varepsilon (\mathrm{Q f N})\, \mathrm{decr.lim}\, v_n = 0 . \supset . \Sigma_1^\infty \bmod (u_n -$
$u_{n+1})\, v_n \varepsilon \mathrm{Q}_0 .$ [§1 P32]

12. $u \varepsilon \mathrm{q}'\mathrm{f N} . \lim u_n \varepsilon \mathrm{q}' . v \varepsilon \mathrm{q f N} . a \varepsilon \mathrm{q} : n \varepsilon \mathrm{N} . \supset_n . v_{n+1} \varepsilon \mathrm{med}\, (v_n , a) :$
$\supset . \Sigma \bmod (u_n - u_{n+1})\, v_n \varepsilon \mathrm{Q}_0 .$ [§1 P32]

13. $u \varepsilon \mathrm{q}'\mathrm{f N} . \bmod u \varepsilon \mathrm{Q f N} . \Pi u_\infty \varepsilon \mathrm{q}' . \supset .\because \varphi \varepsilon (\mathrm{N f N})\, \mathrm{sim} . \supset_\varphi . \prod_{n=1}^{n=\infty} u_{\varphi n} =$
$\Pi u_\infty :=: \Sigma_1^\infty \bmod (u - 1) \varepsilon \mathrm{Q} .$

14. $u \varepsilon \mathrm{q}'\mathrm{f N} . \Sigma u_\infty , \Pi (1 + u)_\infty \varepsilon \mathrm{q}' . \supset . \bmod \Sigma u_\infty \overline{<} \Sigma_1^\infty \bmod u . \bmod \Pi$
$(1 + u)_\infty = \Pi_1^\infty \bmod (1 + u) .$

$u , v \varepsilon \mathrm{q}'\mathrm{f N} : n \varepsilon \mathrm{N} . \supset_n . w_n = \sum_{m=1}^{m=n} u_m v_{n-m+1} : \supset$

15. $\Sigma \bmod u_\infty , \Sigma \bmod v_\infty \varepsilon \mathrm{Q} . \supset . \Sigma w_\infty = (\Sigma u_\infty)(\Sigma v_\infty) .$

16. $\Sigma u_\infty , \Sigma v_\infty , \Sigma w_\infty \varepsilon \mathrm{q}' . \supset .$ »

17. $\Sigma \bmod u_\infty , \Sigma v_\infty \varepsilon \mathrm{q}' . \supset .$ »

18. $u , v \varepsilon \mathrm{q f N} . \supset : \Sigma_1^\infty (u_n + \mathrm{i}\, v_n) \varepsilon \mathrm{q}' . = . \Sigma u_\infty , \Sigma v_\infty \varepsilon \mathrm{q} .$

19. » $. \Sigma u_\infty , \Sigma v_\infty \varepsilon \mathrm{q} . \supset . \Sigma_1^\infty (u_n + \mathrm{i}\, v_n) = \Sigma u_\infty + \mathrm{i}\, \Sigma v_\infty .$

20. » $: n \varepsilon \mathrm{N} . \supset_n . u_n u_{n+1} , v_n v_{n+1} \varepsilon \mathrm{Q} : \supset : \Pi_1^\infty (1 + u_n + \mathrm{i}\, v_n)$
$\varepsilon \mathrm{q}' . = . \Sigma_1^\infty \bmod (u_n + \mathrm{i}\, v_\iota) \varepsilon \mathrm{Q} .$

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10. Dirichlet. *Mathem. Abhandl. der Königl. Akademie der Wissenschaften zu Berlin.* a. 1837, pag. 48.

13. Dini. *Annali di Matematica,* II, a. 1868-69, pag. 35.

16. Abel. *Œuvres,* I, pag. 226. - Cesàro. *Bulletin Darboux,* a. 1890, pag. 114.

17. Mertens. *Crelle's Journal,* t. 79, a. 1875, pag. 182.

20. Pringsheim. *Mathematische Annalen,* XXXIII, a. 1889, p. 139.

21. $a \varepsilon q' . \mathrm{mod}\, a < 1 . \supset . (1 + 2 \Sigma_1^\infty a^{4n^2}) \Sigma_1^\infty a^{(2n-1)^2} = \frac{a}{1-a^2} - \frac{a^3}{1-a^6}$

$+ \frac{a^5}{1-a^{10}} - \frac{a^7}{1-a^{14}} + \ldots$

22. $\psi \varepsilon q' f N . \Sigma_1^\infty \mathrm{mod}\, \psi(n) \varepsilon Q . \psi(1) = 1 : n , n' \varepsilon N . \supset_{n, n'} . \psi(n) \psi(n')$

$= \psi(n n') : : \supset . \Pi_1^\infty \frac{1}{1-\psi(Np_n)} = \Sigma_1^\infty \psi(n) .$

F. Giudice.

21, 22. Dirichlet. *Teoria dei numeri*, versione italiana Faifofer, pag. 225, 336.