

ON THE SHORTEST PATH CONSISTING OF  
STRAIGHT LINES BETWEEN TWO POINTS  
ON A RULED QUADRIC.

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IN a paper read before the Oxford Mathematical Society, a few years ago, by Professor Sylvester it was shown that the shortest path, consisting of three generators on a ruled quadric, from any point on the gorge (the principal elliptic section) to the diametrically opposite point, was of the same length for every point on the gorge. The following paper is an attempt to generalize this result by the aid of elliptic functions.

The coordinates of any point on the quadric

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

can be expressed as elliptic functions thus:

$$x = ak \operatorname{sn} \alpha \operatorname{sn} \beta,$$

$$y = \frac{b}{k_1} \operatorname{dn} \alpha \operatorname{dn} \beta,$$

$$z = \frac{ck}{k_1} \operatorname{cn} \alpha \operatorname{cn} \beta,$$

where  $k^2 + k_1^2 = 1$ .

It will be convenient to take  $k^2 = \frac{a^2 + c^2}{a^2 - b^2}$  and  $k_1^2 = \frac{b^2 + c^2}{b^2 - a^2}$ , since then the points where  $\alpha$  or  $\beta = \gamma$ , a constant, lie on the line of curvature which is the intersection of the given quadric

with

$$\frac{x^2}{a^2 + \lambda^2} + \frac{y^2}{b^2 + \lambda^2} - \frac{z^2}{c^2 - \lambda^2} = 1,$$

where (1)  $\alpha^2 + \lambda^2 = (a^2 + c^2) \operatorname{sn}^2 \gamma$ .

It will be noticed that  $k$  is greater than unity and therefore  $k_1$  is imaginary. This, however, does not constitute any real difficulty, and it will be shown that for all real points on the hyperboloid we may take  $\alpha$  and  $\beta - K$  as real quantities. The expressions obtained for  $x, y, z$  as real elliptic functions are not symmetrical.

It has been shown by Cayley (*Lond. Math. Soc.*, 1879) that the line joining two points  $\alpha_1, \beta_1$  and  $\alpha_2, \beta_2$  is a generator of one species if  $\alpha_1 + \beta_1 = \alpha_2 + \beta_2$ , and a generator of the other species if  $\alpha_1 - \beta_1 = \alpha_2 - \beta_2$ .

Along any generator of the first species then  $da + d\beta = 0$ , and along any generator of the second  $da - d\beta = 0$ .

Differentiating the expressions for  $x, y, z$ , we have

$$dx = ak (cn \alpha \operatorname{dn} \alpha \operatorname{sn} \beta \, d\alpha + \operatorname{sn} \alpha \operatorname{cn} \beta \operatorname{dn} \beta \, d\beta),$$

$$dy = \frac{-bk^2}{k_1} (\operatorname{sn} \alpha \operatorname{cn} \alpha \operatorname{dn} \beta \, d\alpha + \operatorname{dn} \alpha \operatorname{cn} \beta \operatorname{sn} \beta \, d\beta),$$

$$dz = \frac{-ck}{k_1} (\operatorname{sn} \alpha \operatorname{dn} \alpha \operatorname{cn} \beta \, d\alpha + \operatorname{cn} \alpha \operatorname{sn} \beta \operatorname{dn} \beta \, d\beta).$$

If we now square and add these expressions, remembering that  $a^2 k_1^2 + b^2 k^2 + c^2 = 0$ , we get, on extracting the square root,

$$(2) \quad ds = \pm k \sqrt{(a^2 + c^2)} (\operatorname{sn}^2 \alpha - \operatorname{sn}^2 \beta) \, d\alpha.$$

Hence, for a generator of the first system,

$$s = \pm k \sqrt{(a^2 + c^2)} \left[ \int_{\alpha_1}^{\alpha_2} \operatorname{sn}^2 \alpha \, d\alpha + \int_{\beta_1}^{\beta_2} \operatorname{sn}^2 \beta \, d\beta \right],$$

and for a generator of the second system

$$s = \pm k \sqrt{(a^2 + c^2)} \left[ \int_{\alpha_1}^{\alpha_2} \operatorname{sn}^2 \alpha \, d\alpha - \int_{\beta_1}^{\beta_2} \operatorname{sn}^2 \beta \, d\beta \right].$$

Now  $\lambda^2$ , being a root of  $\frac{x^2}{a^2 + \lambda^2} + \frac{y^2}{b^2 + \lambda^2} - \frac{z^2}{c^2 - \lambda^2} = 1$ , is necessarily real, and therefore by (1)  $\operatorname{sn}^2 \alpha$  and  $\operatorname{sn}^2 \beta$  are real. Knowing then that  $s$  is real, we see by (2) that the difference of the  $\alpha$ 's (and therefore of the  $\beta$ 's) of any two points is real. For the point  $a, 0, 0$  we may take  $\alpha = K - iK'$  and  $\beta = K$ . Now if  $H$  and  $H'$  are the corresponding quarter periods for the modulus  $h$ , where  $hk = 1$ , then it may be shown that  $kK = H + iH'$ , and  $kK' = H'$ ; hence,  $K - iK'$  is a real quantity, and  $K$  a complex quantity. We may therefore take  $\alpha$  and  $\beta - K$  to be real quantities all over the hyperboloid.

We can thus see how the hyperboloid is divided by the lines of curvature—

$\beta = K$  is the gorge;

$\beta = K + \gamma$ , where  $\gamma$  is a positive constant less than  $K - iK'$ ,

is the part of a closed (elliptic) line of curvature which lies above the gorge;

$\beta = K - \gamma$  the part of the same line of curvature below the gorge;

$\alpha = \gamma$ , where  $\gamma$  is a positive constant less than  $K - iK'$ , is a continuous branch of a hyperbolic line of curvature;

$\alpha = -\gamma$ ,  $\alpha = 2K - 2iK' - \gamma$ , and  $\alpha = 2K - 2iK' + \gamma$ , are the three other distinct parts of the same line of curvature.

These four branches of a hyperbolic line of curvature are respectively in the first, fourth, second, and third quadrants of the planes  $x, y$ .

It may be noticed here that we cannot have  $\text{sn}^2\alpha_1 = \text{sn}^2\beta_2$ , since then an elliptic and hyperbolic line of curvature would coincide; also if  $\alpha_1, \beta_1$  and  $\alpha_2, \beta_2$  are two points where a generator intersects an elliptic line of curvature, we have  $\text{sn}^2\beta_1 = \text{sn}^2\beta_2$ , and therefore (since  $\beta$  lies between  $iK'$  and  $2K - iK'$ )  $\beta_1 = \beta_2$ , or  $\beta_1 + \beta_2 = 2K$ . Now if  $\beta_1 = \beta_2$ , then  $\alpha_1 = \alpha_2$  (since  $\alpha_1 \pm \beta_1 = \alpha_2 \pm \beta_2$ ) and the points (1) and (2) coincide, which is impossible; we must therefore take the solution  $\beta_1 + \beta_2 = 2K$ . In a similar way it may be shown that, if (1) and (2) are two points where a generator intersects a hyperbolic line of curvature,  $\alpha_1 + \alpha_2 = 2p(K - iK')$ , where  $p$  may be 0, 1, or 2. Finally it may be noticed that the point diametrically opposite to  $\alpha, \beta$  is  $\alpha + 2(K - iK'), 2K - \beta$ .

Let now  $AP_1P_2\dots P_{n-1}B$  be a path consisting of  $n$  generators, and let the parameters of  $A, P_1, \dots, P_{n-1}, B$  be respectively  $(\alpha_0, \beta_0), (\alpha_1, \beta_1), \dots, (\alpha_{n-1}, \beta_{n-1}), (\alpha, \beta)$ .

Owing to the ambiguity of sign in the expression for  $ds$  we cannot write down a general expression for the length of this path; but it is not difficult to see that its differential is of the form

$$A_0 \text{sn}^2\alpha_0 d\alpha_0 + B_0 \text{sn}^2\beta_0 d\beta_0 + A \text{sn}^2\alpha d\alpha + B \text{sn}^2\beta d\beta \\ + 2(A_1 \text{sn}^2\alpha_1 d\alpha_1 + B_1 \text{sn}^2\beta_1 d\beta_1 + \dots + A_{n-1} \text{sn}^2\alpha_{n-1} d\alpha_{n-1} \\ + B_{n-1} \text{sn}^2\beta_{n-1} d\beta_{n-1}),$$

where  $A_0, B_0, A, B$  have the values  $\pm 1$ , and of the other letters  $A_r = 0$  and  $B_r = \pm 1$ , or  $A_r = \pm 1$  and  $B_r = 0$ .

For the sake of definiteness take  $n = 4$  though the reasoning is general, and assume further that this differential is of the form

$$(3) \quad \text{sn}^2\alpha_0 d\alpha_0 + \text{sn}^2\beta_0 d\beta_0 - \text{sn}^2\alpha d\alpha + \text{sn}^2\beta d\beta \\ + 2(-\text{sn}^2\alpha_1 d\alpha_1 - \text{sn}^2\beta_2 d\beta_2 + \text{sn}^2\alpha_3 d\alpha_3).$$

Cayley's equations are

$$\begin{aligned} \alpha_0 + \beta_0 &= \alpha_1 + \beta_1, \\ \alpha_1 - \beta_1 &= \alpha_2 - \beta_2, \\ \alpha_2 + \beta_2 &= \alpha_3 + \beta_3, \\ \alpha_3 - \beta_3 &= \alpha - \beta. \end{aligned}$$

Eliminating  $\beta_1, \alpha_2, \beta_3$  from these, we have

$$(4) \quad \alpha_0 + \beta_0 - \alpha + \beta + 2(-\alpha_1 - \beta_2 + \alpha_3).$$

It will be noticed that the signs in (3) and (4) follow the same law.

For the minimum path between  $\alpha_0, \beta_0$  and  $\alpha, \beta$ , we have

$$-\operatorname{sn}^2 \alpha_1 d\alpha_1 - \operatorname{sn}^2 \beta_2 d\beta_2 + \operatorname{sn}^2 \alpha_3 d\alpha_3 = 0,$$

and 
$$-d\alpha_1 - d\beta_2 + d\alpha_3 = 0,$$

therefore 
$$\operatorname{sn}^2 \alpha_1 = \operatorname{sn}^2 \beta_2 = \operatorname{sn}^2 \alpha_3.$$

Now we have seen that we can never have  $\operatorname{sn}^2 \alpha_1 = \operatorname{sn}^2 \beta_2$ , we conclude then that the expression for the differential of a path before it becomes critical cannot contain such a term as  $\operatorname{sn}^2 \alpha$ , if it contains such a term as  $\operatorname{sn}^2 \beta$ .

For a critical path then we must have

$$\operatorname{sn}^2 \alpha_1 = \operatorname{sn}^2 \alpha_2 = \dots \operatorname{sn}^2 \alpha_{n-1};$$

or else 
$$\operatorname{sn}^2 \beta_1 = \operatorname{sn}^2 \beta_2 = \dots \operatorname{sn}^2 \beta_{n-1}.$$

In other words a critical path must be such that all the turning points are either on the same hyperbolic line of curvature, or else all on the same elliptic line of curvature.

It is easily seen that the expression whose differential leads to the set of equations  $\operatorname{sn}^2 \alpha_1 = \operatorname{sn}^2 \alpha_2 = \dots \operatorname{sn}^2 \alpha_{n-1}$  must be (if  $n$  be odd the only case now considered)

$$\int_{\beta_0}^{\beta} \operatorname{sn}^2 \beta d\beta + \int_{\alpha_0}^{\alpha_1} \operatorname{sn}^2 \alpha d\alpha - \int_{\alpha_1}^{\alpha_2} \operatorname{sn}^2 \alpha d\alpha + \dots + \int_{\alpha_{n-1}}^{\alpha} \operatorname{sn}^2 \alpha d\alpha.$$

In order that this may be the true arithmetical length we must have  $\beta_0 < \beta_1, \beta_1 < \beta_2, \beta_2 < \beta_3, \dots$ , since  $\operatorname{sn}^2 \beta > \operatorname{sn}^2 \alpha$ , and therefore, by Cayley's equations,

$$\alpha_0 > \alpha_1, \alpha_1 < \alpha_2, \alpha_2 > \alpha_3, \dots,$$

or all letters  $\alpha$  with even suffix  $>$  letters adjacent with odd suffixes.

The system of equations

$$\operatorname{sn}^2 \alpha_1 = \operatorname{sn}^2 \alpha_2 = \dots$$

lead to

$$\alpha_1 + \alpha_2 = 2p (K - iK'),$$

$$\alpha_2 + \alpha_3 = 2q (K - iK'),$$

$$,, = ,,$$

$$,, = ,,$$

Now it may at once be seen that, if  $p=0$ , the curves  $\alpha_1$  and  $\alpha_2$  lie in the first and fourth quadrants; if  $p=1$  the curves lie in the first and second quadrants, or else in the third and fourth quadrants; if  $p=2$  the curves lie in the second and third quadrants.

If, then, the curve  $\alpha_1$  is in the first quadrant,  $\alpha_2$  must be in the second quadrant (since  $\alpha_2 > \alpha_1$ ), and therefore  $\alpha_3$  must lie in the first quadrant (since  $\alpha_3 < \alpha_2$ ). We thus see that  $p=q=\dots$

That is, if the turning points of a critical path lie on a hyperbolic line of curvature, they are alternately in the first and second quadrant, or else in the third and fourth quadrant, or else in the second and third, or else in the first and fourth.

From the equations  $\alpha_1 + \alpha_2 = \alpha_2 + \alpha_3 = \dots = 2p (K - iK')$ , combined with Cayley's, we deduce

$$\alpha_1 = \frac{\alpha_0 + \beta_0 - \alpha - \beta}{2(n-1)} + p(K - iK').$$

If the initial generator had been of the record species we should have had, for  $\beta_0$  and  $\beta$ ,  $-\beta_0$  and  $-\beta$  respectively.

For a path with turning points on an elliptic line of curvature, we have  $\beta_1 + \beta_2 = \beta_2 + \beta_3 = \dots = 2K$ , and can deduce

$$\beta_1 = \frac{\alpha_0 + \beta_0 - \alpha - \beta}{2(n-1)} + K.$$

Using the values obtained for  $\alpha_1, \alpha_2, \dots$ , we can see that the length of the path when the turning points are on a hyperbolic line of curvature is

$$k \sqrt{(a^2 + c^2)} \left\{ \int_{\beta_0}^{\beta} \operatorname{sn}^2 \beta \, d\beta + \int_{\alpha_0}^{\alpha} \operatorname{sn}^2 \alpha \, d\alpha + (n-1) \int_{\alpha_2}^{\alpha_1} \operatorname{sn}^2 \alpha \, d\alpha \right\},$$

with the conditions  $\alpha_0 > \alpha_1, \alpha_1 < \alpha_2, \alpha < \alpha_2$ ; or

$$k \sqrt{(a^2 + c^2)} \left\{ \int_{\beta_0}^{\beta} \text{sn}^2 \beta d\beta + \int_{\alpha_0}^{\alpha} \text{sn}^2 \alpha d\alpha - (n-1) \int_{p(K-iK')-\varpi}^{p(K-iK')+\varpi} \text{sn}^2 \alpha d\alpha \right\},$$

where  $\varpi = \frac{\alpha + \beta - \alpha_0 - \beta_0}{2(n-1)}$  is positive, and  $\alpha_0 + \omega > p(K - iK')$  and  $\alpha - \omega < p(K - iK')$ .

Similarly, when the turning points lie on an elliptic line of curvature, the length becomes  $k \sqrt{(a^2 + c^2)}$  into

$$\int_{\alpha_0}^{\alpha} \text{sn}^2 \alpha d\alpha + \int_{\beta_0}^{\beta} \text{sn}^2 \beta d\beta + (n-1) \int_{\beta_2}^{\beta_1} \text{sn}^2 \beta d\beta,$$

with the conditions  $\beta_1 > \beta_0, \beta_2 < \beta_1$ , and  $\beta > \beta_2$ ; that is  $k \sqrt{(a^2 + c^2)}$  into

$$\int_{\alpha_0}^{\alpha} \text{sn}^2 \alpha d\alpha + \int_{\beta_0}^{\beta} \text{sn}^2 \beta d\beta + (n-1) \int_{K-\omega}^{K+\omega} \text{sn}^2 \beta d\beta,$$

where  $\omega = \frac{\alpha_0 + \beta_0 - \alpha - \beta}{2(n-1)}$  is positive, and  $\beta_0 - \omega < K$  and  $\beta + \omega > K$ .

The question now arises whether these critical paths correspond to real maxima or real minima.

Consider the case when the turning points lie on an elliptic line of curvature. It may now be shown that a small variation from the critical path gives an increment of the length

$$2 \text{sn} \beta_1 \text{cn} \beta_1 \text{dn} \beta_1 (d\beta_1^2 + d\beta_2^2 + \dots + d\beta_{n-1}^2).$$

Now  $\beta_1$  lies between  $K$  and  $2K - iK'$ , and therefore  $\text{sn} \beta_1 \text{cn} \beta_1 \text{dn} \beta_1$  is positive, and the path is a minimum.

When the turning points lie on a hyperbolic line of curvature the path is similarly seen to be a maximum or a minimum according as  $\text{sn} \alpha_1 \text{cn} \alpha_1 \text{dn} \alpha_1$  is negative or positive; that is, according as  $\alpha_1$  lies in the second (or fourth); or lies in the first or third quadrant.

A geometrical statement of these results is 'the path is a minimum if (1) the projection of the points  $P_1, P_2, \dots$  lie on the closed curve, which is the projection on the plane  $z = 0$  of an elliptic line of curvature; (2) if the points are successively on opposite branches of the projection of a hyperbolic line of curvature; and is a maximum if the points are all on the same branch of the projection of a hyperbolic line of curvature.'

In the expression for the length of a critical path from  $A$  to  $B$  put  $\alpha = \alpha_0$ , and let  $\beta_0$  and  $\beta$  be given constants, and we have the theorem: 'If  $A$  and  $B$  are any two points, where the same branch of a variable hyperbolic line of curvature intersects two fixed elliptic lines of curvature, then the length of the critical path from  $A$  to  $B$  is independent of the position of the hyperbolic line.

Let  $A$  and  $B$  be opposite extremities of a diameter, then, since  $\alpha = \alpha_0 + 2(K - iK')$  and  $\beta = 2K - \beta_0$ , the length of the critical path from  $A$  to  $B$  depends only on  $\beta_0$ , and therefore the length of the critical path from any point on an elliptic line of curvature to the opposite point is constant.

By taking  $n = 3$  and  $\beta_0 = K$  this becomes the theorem proved by Sylvester.

It can also be seen that as  $n$  increases the shortest path grows less and less, until, in the limit when  $n$  is infinite,  $\beta_1 = K$ , and the elliptic line of curvature approaches the gorge on either side; so that the shortest path from  $A$  to  $B$  consists of the generator from  $A$ , up to the point where it crosses the gorge, and then consists of infinitesimal portions of the generators which intersect along the gorge, till it comes to the generator through  $B$ . Also it can at once be deduced that the length of this ultimate form is

$$k \sqrt{(a^2 + c^2)} \left\{ \int_{\alpha_0}^{\alpha} \operatorname{sn}^2 \alpha \, d\alpha + \int_{\beta_0}^{\beta} \operatorname{sn}^2 \beta \, d\beta + \alpha_0 + \beta_0 - \alpha - \beta \right\},$$

if  $\alpha_0 + \beta_0 > \alpha + \beta$  and  $\beta_0 < K < \beta$ .

The other two ultimate forms may be similarly deduced.

By deforming the hyperboloid till it becomes the focal conic

$$\frac{x^2}{a^2 + c^2} + \frac{y^2}{b^2 + c^2} = 1,$$

or by the projection  $\frac{\xi}{\sqrt{(a^2 + c^2)}} = \frac{x}{a}$ ,  $\frac{\eta}{\sqrt{(b^2 + c^2)}} = \frac{y}{b}$ ,

the problem can be reduced to that of finding the shortest distance between two points, consisting of tangents to a conic; and may be treated geometrically by the method employed by Prof. Mathews in the *Messenger*, XXII., p. 68.