

MEMOIR ON THE DIALYTIC METHOD OF ELIMINATION.
PART I.

[*Philosophical Magazine*, XXI. (1842), pp. 534—539*.]

THE author confines himself in this part to the treatment of two equations, the final and other derivees of which form the subject of investigation.

The author was led to reconsider his former labours in this department of the general theory by finding certain results announced by M. Cauchy in *L'Institut*, March Number of the present year, which flow as obvious and immediate consequences from Mr Sylvester's own previously published principles and method.

Let there be two equations in x ,

$$U = ax^n + bx^{n-1} + cx^{n-2} + ex^{n-3} + \&c. = 0,$$

$$V = \alpha x^m + \beta x^{m-1} + \gamma x^{m-2} + \&c. = 0,$$

and let $n = m + \iota$, where ι is zero or any positive value (as may be).

Let any such quantities as $x^r U$, $x^s V$, be termed augmentatives of U or V .

To obtain the derivee of a degree s units lower than V , we must join s augmentatives of U with $s + \iota$ of V . Then out of $2s + \iota$ equations

$$x^0 U = 0, \quad x^1 U = 0, \quad x^2 U = 0, \quad \dots \quad x^{s-1} U = 0,$$

$$x^0 V = 0, \quad x^1 V = 0, \quad x^2 V = 0, \quad \dots \quad x^{\iota+s-1} V = 0,$$

we may eliminate linearly $2s + \iota - 1$ quantities.

Now these equations contain no power of x higher than $m + \iota + s - 1$; accordingly, all powers of x , superior to $m - s$, may be eliminated, and the derivee of the degree $(m - s)$ obtained in its prime form.

Thus to obtain the final derivee (which is the derivee of the degree zero), we take m augmentatives of U with n of V , and eliminate $(m + n - 1)$ quantities, namely,

$$x, \quad x^2, \quad x^3, \quad \dots \quad \text{up to } x^{m+n-1}.$$

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This process, founded upon the dialytic principle, admits of a very simple modification. Let us begin with the case where $\iota = 0$, or $m = n$. Let the augmentatives of U be termed $U_0, U_1, U_2, U_3, \dots$ and of $V, V_0, V_1, V_2, V_3, \dots$, the equations themselves being written

$$U = ax^n + bx^{n-1} + cx^{n-2} + \&c.$$

$$V = a'x^n + b'x^{n-1} + c'x^{n-2} + \&c.$$

It will readily be seen that

$$a'U_0 - aV_0,$$

$$(b'U_0 - bV_0) + (a'U_1 - aV_1),$$

$$(c'U_0 - cV_0) + (b'U_1 - bV_1) + (a'U_2 - aV_2), \&c.$$

will be each linearly independent functions of $x, x^2, \dots x^{m-1}$, no higher power of x remaining. Whence it follows, that to obtain a derivate of the degree $(m - s)$ in its prime form, we have only to employ the s of those which occur first in order, and amongst them eliminate $x^{m-1}, x^{m-2}, \dots x^{m-s+1}$. Thus, to obtain the final derivate, we must make use of n , that is, the entire number of them.

Now, let us suppose that ι is not zero, but $m = n - \iota$. The equation V may be conceived to be of n instead of m dimensions, if we write it under the form

$$0x^n + 0x^{n-1} + 0x^{n-2} + \dots + 0x^{m+1} + \alpha x^m + \beta x^{m-1} + \&c. = 0,$$

and we are able to apply the same method as above; but as the first ι of the coefficients in the equation above written are zero, the first ι of the quantities

$$(a'U_0 - aV_0), (b'U_0 - bV_0) + (a'U_1 - aV_1), \&c.$$

may be read simply

$$-aV_0, -bV_0 - aV_1, -cV_0 - bV_1 - aV_2, \&c.$$

and evidently their office can be supplied by the simple augmentatives themselves,

$$V_0 = 0, V_1 = 0, V_2 = 0 \dots V_{\iota-1} = 0;$$

and thus ι letters, which otherwise would be *irrelevant*, fall out of the several derivees.

The author then proceeds with remarks upon the general theory of simple equations, and shows how by virtue of that theory his method contains a solution of the identity

$$X_r U + Y_r V = D_r$$

where D_r is a derivate of the r th degree of U and V , and accordingly, X_r of the form

$$\lambda + \mu x + \nu x^2 + \dots + \theta x^{m-r-1},$$

and Y_r of the form

$$l + mx + \dots + tx^{n-r-1},$$

and accounts *à priori* for the fact of not more than $(n-r)$ simple equations being required for the determination of the $(m+n-2r)$ quantities $\lambda, \mu, \nu, \&c. l, m, n, \&c.$, by exhibiting these latter as *known* linear functions of no more than $(n-r)$ unknown quantities left to be determined.

Upon this remarkable relation may be constructed a method well adapted for the expeditious computation of numerical values of the different derivees.

He next, as a point of curiosity, exhibits the values of the secondary functions,

$$a'U_0 - aV_0,$$

$$b'U_0 - bV_0 + a'U_1 - aV_1,$$

$$c'U_0 - cV_0 + b'U_1 - bV_1 + a'U_2 - aV_2, \&c.$$

under the form of symmetric functions of the roots of the equations $U=0, V=0$, by aid of the theorems developed in the London and Edinburgh *Philosophical Magazine*, December 1839*, and afterwards proceeds to a more close examination of the final derivee resulting from two equations each of the same (any given) degree.

He conceives a number of cubic blocks each of which has two numbers, termed its *characteristics*, inscribed upon one of its faces, upon which the value of such a block (itself called an *element*) depends.

For instance, the value of the *element*, whose *characteristics* are r, s , is the difference between two products: the one of the coefficient r th in order occurring in the polynomial U , by that which comes s th in order in V ; the other product is that of the coefficient s th in order of the polynomial U , by that r th in order of V ; so that if the degree of each equation be n , there will be altogether $\frac{1}{2}n(n+1)$ such elements.

The blocks are formed into squares or flats (*plafonds*) of which the number is $\frac{n}{2}$ or $\frac{n+1}{2}$, according as n is even or odd. The first of these contains n blanks in a side, the next $(n-2)$, the next $(n-4)$, till finally we reach a square of four blocks or of one, according as n is even or odd. These flats are laid upon one another so as to form a regularly ascending pyramid, of which the two diagonal planes are termed the planes of separation and symmetry respectively. The former divides the pyramid into two halves, such that no element on the one side of it is the same as that of any block in the other. The plane of symmetry, as the name denotes, divides the pyramid into two exactly *similar* parts; it being a rule, that *all elements lying in any given line of a square (plafond) parallel to the plane of separation are identical*; moreover, the sum of the characteristics is the same, for *all elements lying anywhere in a plane parallel to that of separation.*

[* p. 40 above. ED.]

All the terms in the final derivee are made up by multiplying n elements of the pile together, under the sole restriction, that no two or more terms of the said product shall lie in any one plane out of the two *sets* of planes perpendicular to the sides of the squares. The *sign* of any such product is determined by the places of either set of planes parallel to a side of the squares and to one another, in which the elements composing it may be conceived to lie.

The author then enters into a disquisition relating to the *number* of terms which will appear in the final derivee, and concludes this first part with the statement of two general canons, each of which affords as many tests for determining whether a prepared combination of coefficients can enter into the final derivee of any number of equations as there are units in that number, but so connected as together only to afford double that number, *less* one, of independent conditions.

The first of these canons refers simply to the number of letters drawn out of *each of the given equations* (supposed homogeneous); the second to what he proposes to call the *weight* of every term in the derivee in respect to *each of the variables* which are to be eliminated.

The author subjoins, for the purpose of conveying a more accurate conception of his Pyramid of derivation, examples of the mode in which it is constructed.

When $n = 1$ there is one flat, viz.

1, 2

Let $n = 3$, there will be two flats:

2, 3

1, 2	1, 3	1, 4
1, 3	1, 4	2, 4
1, 4	2, 4	3, 4

When $n = 2$ there is one flat, viz.

2, 3	2, 4
2, 4	3, 4

Let $n = 4$, there will still be two flats only:

2, 3	2, 4
2, 4	3, 4

1, 2	1, 3	1, 4	1, 5
1, 3	1, 4	1, 5	2, 5
1, 4	1, 5	2, 5	3, 5
1, 5	2, 5	3, 5	4, 5

Let $n = 5$, there will be three flats:

3, 4

2, 3	2, 4	2, 5
2, 4	2, 5	3, 5
2, 5	3, 5	4, 5

1, 2	1, 3	1, 4	1, 5	1, 6
1, 3	1, 4	1, 5	1, 6	2, 6
1, 4	1, 5	1, 6	2, 6	3, 6
1, 5	1, 6	2, 6	3, 6	4, 6
1, 6	2, 6	3, 6	4, 6	5, 6

Let $n = 6$, there will be three flats:

3, 4	3, 5
3, 5	4, 5

2, 3	2, 4	2, 5	2, 6
2, 4	2, 5	2, 6	3, 6
2, 5	2, 6	3, 6	4, 6
2, 6	3, 6	4, 6	5, 6

1, 2	1, 3	1, 4	1, 5	1, 6	1, 7
1, 3	1, 4	1, 5	1, 6	1, 7	2, 7
1, 4	1, 5	1, 6	1, 7	2, 7	3, 7
1, 5	1, 6	1, 7	2, 7	3, 7	4, 7
1, 6	1, 7	2, 7	3, 7	4, 7	5, 7
1, 7	2, 7	3, 7	4, 7	5, 7	6, 7

Thus the work of computation reduces itself merely to calculating $n \frac{n+1}{2}$ elements, or the $n(n+1)$ cross-products out of which they are constituted, and combining them factorially after that law of the pyramid, to which allusion has been already made.