

21.

ON THE GENERAL SOLUTION (IN CERTAIN CASES) OF THE EQUATION $x^3 + y^3 + Az^3 = Mxyz$, &c.

[*Philosophical Magazine*, xxxi. (1847), pp. 467—471.]

I SHALL restrict the enunciation of the proposition I am about to advance to much narrower limits than I believe are necessary to the truth, with a view to avoid making any statement which I may hereafter have occasion to modify. Let us then suppose in the equation

$$x^3 + y^3 + Az^3 = Mxyz$$

that A is a *prime* number, and that $27A - M^3$ is *positive*, but exempt from positive prime factors of the form $6i + 1$. Then I say, and have succeeded in demonstrating, that all the possible solutions in integer numbers of the given equation may be obtained by explicit processes from one particular solution or system of values of x, y, z , which may be called the Primitive system.

This system of roots or of values of x, y, z is that system in which the value of the greatest of the three terms $x, y, A^{\frac{1}{3}}.z$ (which may be called the *Dominant*) is the least possible of all such dominants. I believe that in general the system of the least *Dominant* is identical with the system of the least *Content*, meaning by the latter term the product of the three terms out of which the *Dominant* is elected. I proceed to show the law of derivation.

To express this simply, I must premise that I shall have to employ such an expression as $S' = \phi(S)$ to indicate, not that a certain quantity, S' , is a function of S , but that a certain system of quantities disconnected from one another, denoted by S' , are severally functions of a certain other system of quantities denoted by S ; and, as usual, I shall denote $\phi\phi S$ by $\phi^2 S$, $\phi\phi^2 S$ by $\phi^3 S$, and so forth.

Let now P be the Primitive system of solution of the equation

$$x^3 + y^3 + Az^3 = Mxyz,$$

P denoting a certain system of values of and written in the order of the

letters x, y, z , which may always be found by a limited number of trials (provided that the equation admits of any solution). That this is the case is obvious, since we have only to give the Dominant every possible value from the integer next greatest to $A^{\frac{1}{3}}$ upwards, and combine the values of x^3, y^3, Az^3 so that none shall ever exceed at each step the cube of such dominant, and we must at last, if there exist any solution, arrive at the System of the Least Dominant.

Now, every system of solution is of one or the other of two characters. Either x and y must be odd and z even, or x and y must be one odd and the other even and z odd. That all three should be odd is inconsistent with the given conditions as to A being odd and M even; and if all three were even, by driving out the common factor we should revert to one or the other of the foregoing cases.

The systems of solution where z is even may be termed Reducible, those where z is odd Irreducible. Let ϕ denote a certain symbol of transformation hereafter to be explained.

Then the Reducible systems of the first order may be expressed by

$$\phi P, \phi^2 P, \phi^3 P, \text{ ad infinitum};$$

or in general by $\phi^{n_1} P$, n_1 being absolutely arbitrary. I will anticipate by stating that the function ϕ involves no variable constants; that is to say, $\phi(S)$ may be found explicitly from S without any reference to the particular equation to which S belongs. Let now ψ denote another symbol of transformation, also hereafter to be defined, and differing from ϕ insofar as it does involve as constants the three values of x, y, z contained in P : then the general representations of Irreducible systems of the first order will be denoted by $\psi \phi^{n_1} P$.

It is proper to state here that the symbol ψ is ambiguous; and $\psi \phi^{n_1} P$, when P and n_1 are given, will have two values, according to the way in which the terms represented by P are compared with x, y, z in the given equation

$$x^3 + y^3 + Az^3 = Mxyz;$$

for it is obvious that if $x=a, y=b, z=c$ satisfies the equation, so likewise will

$$x=b, \quad y=a, \quad z=c.$$

Each however of these values of $\psi \phi^{n_1} P$ gives a solution of the kind above designated.

Proceeding in like manner as before, the Reducible system of the second order may be designated by $\phi^{n_2} \cdot \psi \phi^{n_1} \cdot P$, the Irreducible by $\psi \phi^{n_2} \cdot \psi \phi^{n_1} \cdot P$; and in general every possible system of values of x, y, z satisfying the proposed equation, in which z is even, is comprised under the form

$$\phi^{n_r} \cdot \psi \phi^{n_{r-1}} \cdot \psi \dots \phi^{n_2} \cdot \psi \phi^{n_1} \cdot P;$$

and every possible system of such values, in which z is odd, is comprised under the form

$$\psi\phi^{n_r} \cdot \psi\phi^{n_{r-1}} \cdot \psi \dots \psi\phi^{n_1} \cdot P :$$

the quantities $n_1, n_2 \dots n_r$ being of course all independent of one another, and unlimited in number and value.

Thus then we may be said to have the general solution of the given equation in the same sense as an arbitrary sum of terms, each of a certain form, is in certain cases accepted as the complete solution of a partial differential equation.

As regards the value of the symbols ψ and ϕ , ϕ indicates the process by which a, b, c becomes transformed into α, β, γ , the relations between the two sets of elements being contained in the following equations :

$$\begin{aligned} a' &= a^3, & b' &= b^3, & c' &= Ac^3, \\ \alpha &= a'^2b' + b'^2c' + c'^2a' - 3a'b'c', \\ \beta &= a'b'^2 + b'c'^2 + c'a'^2 - 3a'b'c', \\ \gamma &= abc \{a'^2 + b'^2 + c'^2 - a'b' - a'c' - b'c'\}. \end{aligned}$$

Next, as to the effect of the Duplex symbol ψ . Let e, g, ι be the elements of the Primitive system P : ι being the value of z and e, g of x and y taken in either mode of combination, each with each, which satisfy the proposed equation

$$x^3 + y^3 + Az^3 = Mxyz.$$

Let l, m, n represent any system S ,

λ, μ, ν represent any system $\psi(S)$,

ψS has two values, which we may denote by $\psi'S, '\psi S$ respectively, and accentuating the elements λ, μ, ν accordingly to correspond, we shall have

$$\begin{aligned} \lambda' &= 3gm(gl - em) + 3Ain(il - en) - M(gil^2 - e^2lm), \\ \mu' &= 3Ain(im - gl) + 3el(em - gl) - M(eim^2 - g^2lm), \\ \nu' &= 3el(en - il) + 3gm(gn - im) - M(egn^2 - i^2lm): \end{aligned}$$

we have then

$$\psi'S \equiv \lambda', \mu', \nu',$$

and in like manner

$$' \psi S \equiv \lambda, \mu, \nu,$$

' ψS being derived from $\psi'S$ by the mere interchange of e and g one with the other.

I have stated that every possible solution of the proposed equation comes under one or the other of the orders, infinite in number and infinite to the power of infinity in variety of degree, above given: this is not strictly true, unless we understand that all systems of solution are considered to be equivalent which differ only in a multiplier common to all three terms of each; that is to say, which may be rendered identical by the expulsion of a common factor. So that $m\alpha, m\beta, m\gamma$ as a system is treated as identical with α, β, γ , which of course substantially it is; and it should be remarked that there is nothing to prevent the operations denoted by ϕ and ψ introducing a common factor into the systems which they serve to generate, and the latter in particular will have a strong tendency so to do.

I believe that this theorem may be extended with scarcely any modification to the case where A , instead of being a prime, is any power of the same, and to suppositions still more general. I believe also that, subject to certain very limited restrictions, the theorem may prove to apply to the case where the determinant $27A - M^3$ becomes negative.

The peculiarity of this case which distinguishes it from the former, is that it admits of all the three variables x, y, z in the equation

$$x^3 + y^3 + Az^3 = Mxyz$$

having the same sign, which is impossible when the determinant is positive; or in other words, the curve of the third degree represented by the equation $Y^3 + X^3 + 1 = \frac{M}{A^{\frac{1}{3}}}XY$ (in which I call the coefficient of XY the characteristic), which, as long as the quantity last named is less than 3, is a single continuous curve extending on both sides to infinity, as soon as the characteristic becomes equal to 3 assumes to itself an isolated point, the germ of an oval or closed branch, which continues to swell out (always lying apart from the infinite branch) as the characteristic continues indefinitely to increase.

I ought not to omit to call attention to the fact that the theorem above detailed is always applicable to the case of the equation

$$x^3 + y^3 + Az^3 = 0,$$

when A is any power of a prime number not of the form $6i + 1$; in other words, the above always belongs to the class of equations having Monogenous solutions, which for the sake of brevity may be termed themselves Monogenous Equations*.

* Thus the equation $x^3 + y^3 + 9z^3 = 0$ alluded to by Legendre is Monogenous, and the Primitive system of solution is $x=1, y=2, z=-1$, from which every other possible solution in Integers may be deduced.

On the probable existence of such a class of equations I hazarded a conjecture at the conclusion of my last communication to this *Magazine*. As I hope shortly to bring out a paper on this subject in a more complete form, I shall content myself at this time with merely stating a theorem of much importance to the completion of the theory of insoluble and of Monogenous equations of the third degree; to wit, that the equation in integers

$$a(x^3 + y^3 + z^3) + c(x^2y + y^2z + z^2x + xy^2 + yz^2 + zx^2) + exyz = 0$$

may always be transformed so as to depend upon the equation

$$fu^3 + gv^3 + hw^3 = (6a - e)uvw,$$

wherein $fgh = ae^2 - (c^2 + 3a^2)e + 9a^2 - 3ac^2 - 2c^3$.

By means of the above theorem, among other and more remarkable consequences, we are enabled to give a theory of the irresoluble and monogenous cases of the equation

$$x^3 + y^3 + m^3z^3 = Mxyz,$$

when m is some power of 2, or of certain other numbers.