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ON THE MOTION AND REST OF FLUIDS.

[*Philosophical Magazine*, XIII. (1838), pp. 449—453.]

M. OSTROGRADSKY'S memoir on this subject inserted in the *Scientific Memoirs* seems to have excited much attention, and has been made the occasion of some annotations* by a distinguished writer in the *Philosophical Magazine*. Mr Ivory's recent papers in the same periodical must still more tend to invest with a new interest all such speculations. It seems to me desirable therefore to present the theory of fluids in all the simplicity of which it is susceptible.

I consider a fluid as a collection of particles subject to some law of relative position other than that of rigidity. These particles by their mutual actions maintain the connections of the system. As to the law of force between them we know nothing; but I assume it is a general principle of nature, that for each instant of time the sum of the internal actions (reckoned by the product of each particle into the square of the space due to the internal force acting on it) is a minimum. This in fact is Gauss's principle of least restraint. We may if we please split this principle into two parts; that is to say, assume that the internal system of forces is always such as if acting alone would keep the fluid at rest; and then again assume that any equilibrating system of forces must be subject to the law of virtual velocities. I say *assume*, because it is impossible *à priori* to prove this.

Lagrange's so-called demonstration is unworthy of his name, and (albeit sanctioned by the powerful oral authority of an ex-Cambridge Professor) contrary alike to sense and honesty. It is better therefore at once to proceed upon Gauss's principle. It might easily be shown that this is in effect tantamount in all cases to D'Alembert's and Lagrange's principles combined.

Before entering upon the investigation I may call attention to one point of great analytical interest, and relating to the difficult subject of the algebraical sign, viz. that if the density of a point (x, y) in any circumscribed space be expressed by the quantity $\frac{du}{dx} + \frac{dv}{dy}$ so that the mass is

$$\iint dx dy \left(\frac{du}{dx} \right) + \iint dx dy \left(\frac{dv}{dy} \right),$$

[* *Phil. Mag.* May, 1838, p. 385. ED.]

that is not equivalent to

$$\int (u dy + v dx),$$

that is if we please

$$\int \left(u \frac{dy}{ds} + v \frac{dx}{ds} \right) ds,$$

(where s is for clearness' sake and to avoid double limits taken an element of the bounding curve) as at first sight it might appear to be, but is in fact equal to

$$\int \left(u \frac{dy}{ds} - v \frac{dx}{ds} \right) ds.$$

I shall demonstrate this point in the next number* of the *Magazine*. It at first caused me some trouble in conducting the annexed inquiry. I shall also take occasion at some other time to revert to a new species (as I believe) of partial differential equations; that is to say, where there are fewer of them than of the principal variables, which may be called therefore Indeterminate Partial Differential Equations. A complete solution of one of these appears in the subjoined

Investigation.

For the sake of simplicity I take an incompressible fluid. The method is nowise different for a fluid of varying density.

Let Δx , Δy , Δz be any displacement undergone by a particle at the point x , y , z parallel to the axes x , y , z respectively; it is easily shown that to satisfy the condition of invariability of mass we must have

$$\frac{d\Delta x}{dx} + \frac{d\Delta y}{dy} + \frac{d\Delta z}{dz} = 0. \quad (\alpha)$$

One relation between u , v , w the velocities parallel to x , y , z is obtained immediately by putting $u\delta t$, $v\delta t$, $w\delta t$, for Δx , Δy , Δz , which gives

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0, \quad (1)$$

as usual.

Again, if X , Y , Z be the impressed forces, and X_1 , Y_1 , Z_1 the internal forces acting on any particle parallel to the axes, we have

$$X_1 + X = \frac{du}{dt} + \frac{du}{dx} u + \frac{du}{dy} v + \frac{du}{dz} w, \quad (2)$$

$$Y_1 + Y = \frac{dv}{dt} + \frac{dv}{dx} u + \frac{dv}{dy} v + \frac{dv}{dz} w, \quad (3)$$

$$Z_1 + Z = \frac{dw}{dt} + \frac{dw}{dx} u + \frac{dw}{dy} v + \frac{dw}{dz} w, \quad (4)$$

from the mere geometry of the question.

[* p. 36, below. ED.]

Finally, Gauss's principle teaches us that

$$\iiint dx dy dz \{X_1 \Delta X_1 + Y_1 \Delta Y_1 + Z_1 \Delta Z_1\} = 0. \quad (\beta)$$

Now

$$\frac{d(X + X_1)}{dx} + \frac{d(Y + Y_1)}{dy} + \frac{d(Z + Z_1)}{dz}$$

$$= \left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dy}\right)^2 + \left(\frac{dw}{dz}\right)^2 + 2 \left\{ \frac{dv}{dz} \frac{dw}{dy} + \frac{dw}{dx} \frac{du}{dz} + \frac{du}{dy} \frac{dv}{dx} \right\},$$

as appears from the equations (1), (2), (3), (4); and hence

$$\frac{d\Delta X_1}{dx} + \frac{d\Delta Y_1}{dy} + \frac{d\Delta Z_1}{dz} = 0,$$

the complete solution of which, free from the sign of integration, is

$$\Delta X_1 = \frac{d\psi}{dy} - \frac{d\phi}{dz},$$

$$\Delta Y_1 = \frac{d\omega}{dz} - \frac{d\psi}{dx},$$

$$\Delta Z_1 = \frac{d\phi}{dx} - \frac{d\omega}{dy},$$

ω, ϕ, ψ being any three independent functions of x, y, z .

On substituting these values in equation (β) we obtain

$$\iiint dx dy dz \left\{ X_1 \frac{d\psi}{dy} - Y_1 \frac{d\psi}{dx} \right\} + \iiint dx dy dz \left\{ Y_1 \frac{d\omega}{dz} - Z_1 \frac{d\omega}{dy} \right\}$$

$$+ \iiint dx dy dz \left\{ Z_1 \frac{d\phi}{dx} - X_1 \frac{d\phi}{dz} \right\} = 0.$$

This may be put under the form

$$\int dz \iint dx dy \left\{ \frac{d}{dy} (\psi X_1) - \frac{d}{dx} (\psi Y_1) \right\}$$

$$+ \int dx \iint dy dz \left\{ \frac{d}{dz} (\omega Y_1) - \frac{d}{dy} (\omega Z_1) \right\}$$

$$+ \int dy \iint dz dx \left\{ \frac{d}{dx} (\phi Z_1) - \frac{d}{dz} (\phi X_1) \right\}$$

$$- \iiint dx dy dz \cdot \psi \left(\frac{dX_1}{dy} - \frac{dY_1}{dx} \right)$$

$$- \iiint dx dy dz \cdot \omega \left(\frac{dY_1}{dz} - \frac{dZ_1}{dy} \right)$$

$$- \iiint dx dy dz \cdot \phi \left(\frac{dZ_1}{dx} - \frac{dX_1}{dz} \right) = 0.$$

Here it must be remembered that ω , ϕ , ψ are perfectly independent of each other. Also the values of the three first written quantities depend upon the values of X_1 , Y_1 , Z_1 at the bounding surface; the values of the three last-written depend upon the general values of X_1 , Y_1 , Z_1 . It is clear therefore that each system of three equations and each member of each system must be separately zero.

The three latter equations give

$$\left. \begin{aligned} \frac{dX_1}{dy} - \frac{dY_1}{dx} &= 0 \\ \frac{dY_1}{dz} - \frac{dZ_1}{dy} &= 0 \\ \frac{dZ_1}{dx} - \frac{dX_1}{dz} &= 0 \end{aligned} \right\} . \quad (\gamma)$$

The three former require that for each section of the surface parallel to the plane xy

$$\left. \begin{aligned} \int \psi (X_1 dx + Y_1 dy) &= 0, \\ \text{for each section parallel to } yz \\ \int \omega (Y_1 dy + Z_1 dz) &= 0, \\ \text{for each section parallel to } zx \\ \int \phi (Z_1 dz + X_1 dx) &= 0 \end{aligned} \right\} , \quad (\delta)^*$$

and these equations are to hold good whatever ψ , ϕ , ω may be. From the equations (γ) we derive

$$X_1 dx + Y_1 dy + Z_1 dz = df, \quad (5)$$

from equations (δ) we obtain

f = constant for all points in any section of the bounding surface parallel to the plane of xy ,

f = constant for all points in any section of the bounding surface parallel to the plane of yz ,

f = constant for all points in any section of the bounding surface parallel to the plane of zx .

Now by drawing through all the points in a plane parallel to xy , planes parallel to yz , we may cover the whole surface; hence f is constant all over the surface bounding the fluid.

* See remark at introduction.

Therefore $X_1 dx + Y_1 dy + Z_1 dz = 0,$ (6)

for all variations of dx, dy, dz taken upon the surface.

The equations (1, 2, 3, 4, 5, 6) are coincident with those obtained by the usual method; with this difference, that X_1, Y_1, Z_1 here take the place of

$$-\frac{dp}{dx}, -\frac{dp}{dy}, -\frac{dp}{dz}.$$

Thus then we have obtained all the conditions requisite for determining the motion of fluids from the universal principle of least constraint conjoined with the specific character of the system in question.

General Remarks.

In the case of equilibrium, that is in the case where no particle moves, we have $X_1 + X = 0, Y_1 + Y = 0, Z_1 + Z = 0$. Hence $Xdx + Ydy + Zdz$ is a complete differential always and zero for the surface.

The above results have been obtained upon the principles of the differential calculus, and the continuity of the forces has been tacitly assumed. If now we were to suppose forces of finite magnitude (as compared with the *whole sum* acting upon the entire system) to be applied to a layer of single particles or to a layer of a thickness of the same order of magnitude as the distances between the particles themselves, (which has been treated as an infinitesimal) it would appear that our results would be no longer applicable, just in the same manner as it would be erroneous to apply the principle of *vis-viva* (for example) without modification, to the case of impulsive forces, because we had deduced it by the calculus in the case of the motion being continuous. Hence the above equations ought not strictly to apply to the motion or rest of a fluid *contained between physical surfaces*; for the pressure afforded by these surfaces, whatever its actual value may be, we know *à priori* is commensurable with the whole amount of force acting on the fluid; but the immediate application of this pressure (*alias* repulsive force) is confined to the bounding layer of fluid particles, or at most extends to a distance bearing a low ratio to the distances between the particles themselves.

Accordingly, to the non-applicability of the equations for free fluids to the case of fluids confined at the boundaries, and to an independent investigation upon the minimum principle for this class of problems, it is, that I look for the true explanation of the phenomena of capillary attraction (vulgarly so called).