

SEQUEL TO THE THEOREMS RELATING TO "CANONIC ROOTS"  
GIVEN \* IN THE MARCH NUMBER OF THIS MAGAZINE.

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THE theorems kindly communicated from me by Mr Cayley in the March Number of this *Magazine* were originally designed to appear as a note or *excursus* to a memoir in preparation on the extension of Gauss's method of approximation from single to multiple integrals by a method which invariably leads to the construction of a *canonizant* whose roots are all real. To establish this reality, recourse may advantageously be had to a theorem of Jacobi, given at the end of his well known memoir "De Eliminatione Variabilis e duabus æquationibus," *Crelle*, vol. xv. p. 101, a very slight inspection of which at once leads to the further and interesting inference that the resultant of the canonizant of an odd-degreed function of  $x$  and *unity*, and of the canonizant of the second differential coefficient of that function in respect to  $x$ , is an exact power of the *catalecticant* of the first differential coefficient of  $x$  in respect to the same. This is the essence of the matter communicated by Mr Cayley; but subsequent successive generalizations of the theorem have led me on, step by step, to the discovery of a vast general theory of double determinants, that is, resultants of bipartite lineo-linear equations, constituting, I venture to predict, the dawn of a new epoch in the history of modern algebra and the science of pure tactic.

I will begin this note upon a note, by reproducing in brief the first of my two demonstrations of the simple theorem in question†. Let us write

$$X_0 = 1, \quad X_1 = \begin{vmatrix} 1, & x \\ a, & b \end{vmatrix}, \quad X_2 = \begin{vmatrix} 1, & x, & x^2 \\ a, & b, & c \\ b, & c, & d \end{vmatrix}, \quad X_3 = \begin{vmatrix} 1, & x, & x^2, & x^3 \\ a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \end{vmatrix},$$

and so on. And again, let

$$\lambda_1 = a, \quad \lambda_2 = \begin{vmatrix} a, & b \\ b, & c \end{vmatrix}, \quad \lambda_3 = \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix},$$

\* [Cayley, *Coll. Math. Papers*, vol. v. p. 104.]

† The second has been communicated by Mr Cayley in the March number of this *Magazine*.

and so on. The theorem in effect to be proved is simply this, that the resultant of  $X_i$  and  $X_{i-1}$  is an exact power of  $\lambda_i$ , which (as will at once be seen) is the coefficient of  $x^i$  in  $X_i$ . In what follows, I shall use  $R(P, Q)$  or  $R(Q, P)$  to denote indifferently the positive or negative resultant of any two functions  $P$  and  $Q$ , ignoring for greater simplicity all considerations as to the proper algebraical sign to be affixed to a resultant of two functions taken in assigned order.

Jacobi's theorem above referred to, stated so far as necessary for the purpose in hand, is as follows:

$$X_n = (Ax + B) X_{n-1} - \frac{\lambda_n^2}{\lambda_{n-1}^2} X_{n-2}.$$

Hence, by virtue of a general theorem of elimination\*,

$$R(X_n, X_{n-1}) = \lambda_{n-1}^2 R\left(-\frac{\lambda_n^2}{\lambda_{n-1}^2} X_{n-2}, X_{n-1}\right);$$

or, neglecting as premised all considerations of algebraical sign,

$$= (\lambda_{n-1})^2 \left(\frac{\lambda_n}{\lambda_{n-1}}\right)^{2(n-1)} \cdot R(X_{n-1}, X_{n-2}),$$

that is

$$\begin{aligned} \frac{R(X_n, X_{n-1})}{\lambda_n^{2(n-1)}} &= \frac{R(X_{n-1}, X_{n-2})}{\lambda_{n-1}^{2(n-2)}} = \frac{R(X_{n-2}, X_{n-3})}{\lambda_{n-2}^{2(n-3)}} \\ &= \&c. = \frac{R(X_1, X_0)}{\lambda_1} = 1; \end{aligned}$$

or if any one of my readers finds a difficulty in admitting that  $R(ax-b, 1)=a$ , he can stop short at  $\frac{R(X_2, X_1)}{\lambda_2^2}$ , which may easily be verified to be equal to unity. Hence

$$R(X_n, X_{n-1}) = \lambda_n^{2n-2}. \quad \text{Q. E. D.}$$

Thus we see that if  $X_n, X_{n-1}$  have one root in common,  $\lambda_n$  must vanish; but then, by the cited theorem of Jacobi, it follows that  $X_n$  completely contains  $X_{n-1}$ ; from this it was easy to infer the necessity of the function † of which  $X_n$  is the canonizant, having infinity for one of its "canonic roots"—or, in other words, of its being reducible to the form

$$k_1(x+h_1)^{2n-1} + k_2(x+h_2)^{2n-1} + \dots + k_{n-1}(x+h_{n-1})^{2n-1} + k_n.$$

\* This theorem is best seen by dealing in the first instance with  $U, V$ , any two homogeneous functions of  $x, y$  of degrees  $n, n-1$  respectively satisfying the identity  $U = (Ax + By)V + y^2W$ ; we have then

$$R(U, V) = R(V, y^2W) = R(V, W) \times \{R(V, y)\}^2,$$

where evidently  $R(V, y)$  is the coefficient of  $x^{n-1}$  in  $V$ ; let  $y$  become unity, then on calling  $U, V, R(V, y)$  respectively  $X_n, X_{n-1}, \lambda_{n-1}$ , and giving to  $W$  its corresponding value, we have the theorem as it is used in the text.

† For in general if  $X_n$  be the canonizant to  $F, X_{n-1}$  will be the canonizant to  $\frac{d^2F}{dx^2}$ .

And so it became natural to establish *a priori* the existence of this condition, and thus to obtain the proof virtually reproduced by Mr Cayley in the article referred to.

In what precedes,  $X_{n-1}$  was a *first principal minor* of  $X_n$ ; and it occurred to me to institute an inquiry into the form of the resultant of two functions related to each other as  $X_{n-1}$  is to  $X_n$ , with the sole but important difference that the constants in  $X_n$  are not to be contained in a concatenated order from one line to another, but to be taken perfectly independent as in the example

$$X_3 = \begin{vmatrix} 1, & x, & x^2, & x^3 \\ a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{vmatrix}$$

and

$$X_4 = \begin{vmatrix} 1, & x, & x^2, & x^3, & x^4 \\ a, & b, & c, & d, & e \\ a', & b', & c', & d', & e' \\ a'', & b'', & c'', & d'', & e'' \\ a''', & b''', & c''', & d''', & e''' \end{vmatrix}$$

Or, according to a suggestion of Mr Cayley, putting the question into a more general and simple form, we may inquire into the resultant of any two complete determinants, functions of  $x$  of the  $(n-1)$ th degree, which belong to the rectangular matrix

$$\begin{array}{cccc} 1, & x, & x^2 & \dots x^{n-1}, \\ a_{1,1}, & a_{1,2}, & a_{1,3} & \dots a_{1,n}, \\ a_{2,1}, & a_{2,2}, & a_{2,3} & \dots a_{2,n}, \\ \dots & \dots & \dots & \dots \\ a_{n-1,1}, & a_{n-1,2}, & a_{n-1,3} & \dots a_{n-1,n}, \\ a_{n,1}, & a_{n,2}, & a_{n,3} & \dots a_{n,n}; \end{array}$$

as, for instance, the resultant of the two determinants which may be obtained by suppressing successively the last and last but one line in the matrix above written: and by aid of the most elementary principles of the calculus of determinants the instructed reader will find no difficulty in proving that this resultant will resolve itself into two distinct parts—one a power of the determinant obtained by suppressing the uppermost (or  $x$ ) line in the above matrix, the other the Resultant of the matrix obtained by suppressing

simultaneously the two lowermost lines\*. This last suppression leaves a rectangular matrix which, written in a homogeneous form, becomes

$$\begin{array}{cccc}
 y^{n-1}, & y^{n-2}x, & y^{n-3}x^2 & \dots x^{n-1}, \\
 a_{1,1}, & a_{1,2}, & a_{1,3} & \dots a_{1,n}, \\
 a_{2,1}, & a_{2,2}, & a_{2,3} & \dots a_{2,n}, \\
 \dots & \dots & \dots & \dots \\
 a_{n-2,1}, & a_{n-2,2}, & a_{n-2,3} & \dots a_{n-2,n},
 \end{array}$$

consisting of  $n$  columns and  $(n - 1)$  lines †.

The Resultant of this matrix means the quantity  $R$  which, equated to zero, will indicate the possibility of the simultaneous nullity of *all* its first minors, so that  $R$  will be the factor common to the resultants of every couple of these minors. If we name the columns of the matrix taken in any arbitrary order  $C_1, C_2 \dots C_n$ , and call  $R'$  the resultant of

$$C_1 C_3 \dots C_{n-1} C_n, C_2 C_3 \dots C_{n-1} C_n,$$

it may readily be made out that  $\frac{R'}{R}$  is equal to a power of the determinant obtained by suppressing the uppermost (or  $x$ ) line of the rectangular matrix  $C_3 \dots C_{n-1} C_n$ .

To find  $R$ , we may proceed in the general case in the manner indicated in the example following, where  $n - 1$  is made 4. Taking the two extreme first minors and dividing them respectively by  $y$  and  $x$ , we have two equations of the following form for determining  $R$ , namely,

$$\begin{vmatrix} y^3, & y^2x, & yx^2, & x^3 \\ a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \end{vmatrix} = 0, \quad \begin{vmatrix} y^3, & y^2x, & yx^2, & x^3 \\ b, & c, & d, & e \\ b', & c', & d', & e' \\ b'', & c'', & d'', & e'' \end{vmatrix} = 0.$$

By rejecting, as we have done, the factors  $x$  and  $y$  from the above equations, certain factors, it is true, are lost to their resultant ( $R'$ ); but it will easily be seen that these factors are each of them powers of one and the same determinant, namely, the determinant

$$\begin{vmatrix} b, & c, & d \\ b', & c', & d' \\ b'', & c'', & d'' \end{vmatrix}$$

\* For, on making the last-named determinant referred to in the text zero, it may easily be shown, by aid of a familiar theorem in compound determinants, that the two determinants whose resultant is under investigation have all the coefficients of the one in the same ratio to each other as the corresponding coefficients of the other.

† The reader may notice that the real interest of the subject under consideration commences with the independent inquiry into the form of the Resultant of the above matrix—the original question, as to the quasi-canonzant, being important only as leading up to the appearance of this Resultant.

and that their product is contained in the irrelevant factor  $\frac{R'}{R}$ , itself a power of that determinant, as above explained. To find  $R$ , we may write down the oblong matrix

$$\begin{array}{ccc} y^2, & yx, & x^2, \\ b, & c, & d, \\ b', & c', & d', \\ b'', & c'', & d'', \end{array}$$

and make its three first minors respectively equal to  $u, v, w$ , that is,

$$\begin{vmatrix} y^2, & yx, & x^2 \\ b', & c', & d' \\ b'', & c'', & d'' \end{vmatrix} = u, \quad \begin{vmatrix} y^2, & yx, & x^2 \\ b'', & c'', & d'' \\ b, & c, & d \end{vmatrix} = v, \quad \begin{vmatrix} y^2, & yx, & x^2 \\ b, & c, & d \\ b', & c', & d' \end{vmatrix} = w;$$

then we shall obtain the equations following, of which the intermediate ones result solely from the equations last assumed, but the first and last from those combined with the original two given ones, namely,

$$(bu + b'v + b''w) y - (au + a'v + a''w) x = 0,$$

$$(cu + c'v + c''w) y - (bu + b'v + b''w) x = 0,$$

$$(du + d'v + d''w) y - (cu + c'v + c''w) x = 0,$$

$$(eu + e'v + e''w) y - (du + d'v + d''w) x = 0.$$

These equations may be satisfied by making simultaneously

$$u = 0, \quad v = 0, \quad w = 0,$$

all of which (since  $u, v, w$  are minors of the same rectangular matrix) may exist simultaneously, provided

$$\begin{vmatrix} b, & c, & d \\ b', & c', & d' \\ b'', & c'', & d'' \end{vmatrix} = 0.$$

Rejecting (as before) this irrelevant factor, it remains to find the resultant of the system of equations in  $x, y; u, v, w$ , above written, defined as the characteristic of the possibility of their coexistence for some particular system of values of  $x, y; u, v, w$ , but with joint and *several exclusion* of the system  $x = 0, y = 0$ , and of the system  $u = 0, v = 0, w = 0$ .

So, in like manner, in the general case we shall obtain a similar system of  $(m+1)$  homogeneous equations linear in  $x, y$ , and also in  $u_1, u_2, \dots, u_m$ ; and  $R$  will be the resultant of this system, subject to the same condition as to the exclusion of zero systems of  $x, y$ , and  $u_1, u_2, \dots, u_m$  as in the particular instance above treated. Such a resultant, as hinted at the outset, is entitled to the name of a double determinant. In general a double determinant will refer to two systems of variables, one  $p$ , the other  $q$  in number, and to  $(p+q-1)$  equations between them.

In the particular instance before us, one of these quantities, say  $q$ , is the number 2. There is, moreover, a further particularity (but which as it happens does not at all influence the form of the solution), consisting in the fact that the equations are of the *recurring* form

$$\begin{aligned} L_1 y - L_0 x &= 0, \\ L_2 y - L_1 x &= 0, \\ L_3 y - L_2 x &= 0, \\ \dots\dots\dots \\ L_{p+1} y - L_p x &= 0, \end{aligned}$$

where  $L_0, L_1, \dots, L_{p+1}$  are each of them linear homogeneous functions of  $u_1, u_2, \dots, u_p$ . This gives rise to an identification of the resultants of two matrices of very different appearance—one matrix, for example, being

$$\begin{matrix} y^4, & y^3 x, & y^2 x^2, & y x^3, & x^4, \\ a, & b, & c, & d, & e, \\ a', & b', & c', & d', & e', \\ a'', & b'', & c'', & d'', & e'', \end{matrix}$$

and the other being

$$\begin{matrix} au + a'v + a''w, & bu + b'v + b''w, & cu + c'v + c''w, & du + d'v + d''w, \\ bu + b'v + b''w, & cu + c'v + c''w, & du + d'v + d''w, & eu + e'v + e''w. \end{matrix}$$

I have ascertained, and hope shortly to publish, the method of obtaining the explicit value of double determinants in the most general case and under their most symmetrical form: for the particular case before our eyes, this resultant will be as follows:—

$$\begin{aligned} & \left| \begin{matrix} a, & b, & a', & a'' \\ b, & c, & b', & b'' \\ c, & d, & c', & c'' \\ d, & e, & d', & d'' \end{matrix} \right|, \quad \left| \begin{matrix} a, & b, & b', & a'' \\ b, & c, & c', & b'' \\ c, & d, & d', & c'' \\ d, & e, & e', & d'' \end{matrix} \right| + \left| \begin{matrix} a, & b, & a', & b'' \\ b, & c, & b', & c'' \\ c, & d, & c', & d'' \\ d, & e, & d', & e'' \end{matrix} \right|, \quad \left| \begin{matrix} a, & b, & b', & b'' \\ b, & c, & c', & c'' \\ c, & d, & d', & d'' \\ d, & e, & e', & e'' \end{matrix} \right| \\ & \left| \begin{matrix} a', & b', & a'', & a \\ b', & c', & b'', & b \\ c', & d', & c'', & c \\ d', & e', & d'', & d \end{matrix} \right|, \quad \left| \begin{matrix} a', & b', & b'', & a \\ b', & c', & c'', & b \\ c', & d', & d'', & c \\ d', & e', & e'', & d \end{matrix} \right| + \left| \begin{matrix} a', & b', & a'', & b \\ b', & c', & b'', & c \\ c', & d', & c'', & d \\ d', & e', & d'', & e \end{matrix} \right|, \quad \left| \begin{matrix} a', & b', & b'', & b \\ b', & c', & c'', & c \\ c', & d', & d'', & d \\ d', & e', & e'', & e \end{matrix} \right| \\ & \left| \begin{matrix} a'', & b'', & a, & a' \\ b'', & c'', & b, & b' \\ c'', & d'', & c, & c' \\ d'', & e'', & d, & d' \end{matrix} \right|, \quad \left| \begin{matrix} a'', & b'', & b, & a' \\ b'', & c'', & c, & b' \\ c'', & d'', & d, & c' \\ d'', & e'', & e, & d' \end{matrix} \right| + \left| \begin{matrix} a'', & b'', & a, & b' \\ b'', & c'', & b, & c' \\ c'', & d'', & c, & d' \\ d'', & e'', & d, & e' \end{matrix} \right|, \quad \left| \begin{matrix} a'', & b'', & b, & b' \\ b'', & c'', & c, & c' \\ c'', & d'', & d, & d' \\ d'', & e'', & e, & e' \end{matrix} \right| \end{aligned}$$

And it may be noticed that if we return to the original question, in which the coefficients are no longer independent, but where the column  $a'b'c'd'e'$  is

identical, term for term, with  $bcdef$ , and  $a''b''c''d''e''$  with  $cdefg$ , the above determinant becomes

$$\left| \begin{array}{cccc} * & & & \\ & * & & \\ & & \left| \begin{array}{cccc} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{array} \right| & \\ & & & * \\ * & & \left| \begin{array}{cccc} b, & c, & d, & a \\ c, & d, & e, & b \\ d, & e, & f, & c \\ e, & f, & g, & d \end{array} \right| & \\ & & & * \\ \left| \begin{array}{cccc} c, & d, & a, & b \\ d, & e, & b, & c \\ e, & f, & c, & d \\ f, & g, & d, & e \end{array} \right| & & * & & * \end{array} \right|$$

that is to say, it becomes a power of the determinant

$$\left| \begin{array}{cccc} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{array} \right|$$

as we know *à priori* it ought to do, by virtue of the theorem originating out of Jacobi's theorem stated at the beginning of this paper: in fact the two factors of the resultant of  $X_3, X_4$  each of them becomes equal to  $\lambda_4^3$ ; and so in general we shall find, if we use  $n$  instead of 4, each factor of the corresponding resultant becomes  $\lambda_n^{n-1}$ , giving  $\lambda_n^{2n-2}$  as the complete resultant for that singular case, as previously determined.

The author is conscious that some apology may appear due for the cursory mode of elucidation pursued in the preceding extended note, and for the absence as regards certain points of the appropriate proofs; but to have gone into all the details of demonstration would have swollen the paper to a length out of proportion to its importance. Let him be permitted also in all humility to add (as can be vouched by more than one contributor to this *Magazine*), that in consequence of the large arrears of algebraical and arithmetical speculations waiting in his mind their turn to be called into outward existence, he is driven to the alternative of leaving the fruits of his meditations to perish (as has been the fate of too many foregone theories, the still-born progeny of his brain, now for ever resolved back again into the primordial matter of thought), or venturing to produce from time to time such imperfect sketches as the present, calculated to evoke the mental cooperation of his readers, in whom the algebraical instinct has been to some extent developed, rather than to satisfy the strict demands of rigorously systematic exposition.