

ON THE TRINOMIAL UNILATERAL QUADRATIC EQUATION IN
MATRICES OF THE SECOND ORDER.

[*Quarterly Journal of Mathematics*, xx. (1885), pp. 305—312.]

IN the May number [p. 225 above] of the present year of the *London and Edinburgh Philosophical Magazine* (disfigured by numerous errors or inaccuracies) I investigated the number of the solutions of an equation in quaternions or matrices of the second order, belonging to what I term the unilateral class, meaning one in which the coefficients of any actual power of the unknown quantity lie on the same side of it; this number for the Jerrardian Trinomial form I proved *strictly* is $2i^2 - i$ (i being the degree of the equation) and with evidence little short of moral certainty $i^3 - i^2 + i^*$ in the general case where none of the terms are wanting†.

But it must be well borne in mind that these numbers only apply when the coefficients are left general, and that for special relations between them some or all of the roots may become either ideal or indeterminate, or some the one and some the other. In all cases of equations in matrices one principal feature of the investigation is, or should be, to determine the equation of condition between the coefficients, in order that the solution may lose or retain its normal form; if we wish to avoid being compelled to enter upon a complicated consideration of exceptions piled upon exceptions, it is necessary to presuppose a certain criterion function to be other than zero; otherwise it is like the opening of Pandora's box, letting loose an almost incalculable train of vexatious inquiries scarcely worth the trouble they give to answer correctly.

* This article was written and sent to the press many months ago. I have since shown that the number of roots of a general unilateral equation of degree i in matrices of the order ω is the number of combinations of $i\omega$ things taken ω and ω together, and consequently for the case of quaternions is $2i^2 - i$ for the general and not merely for the Jerrardian form. See [above, pp. 197, 233. Also] *Nature*, Nov. 13, 1884.

† I made the assumption that the required number is an analytical function of ω .

Take as an instance the subject of monothetic equations. I have defined a monothetic equation to be one in which all the coefficients are functions of a single matrix, which may be called the base. In such an equation of the degree i and of the order ω in the matrices, we may suppose the unknown quantity to be a function of the base, and then the general formula for expressing a function of a matrix as a rational and integral function of the matrix with the aid of its latent roots, shows that i^ω and no more of such roots exist. But this in no manner precludes the possibility of the existence of other roots which are not functions of the base. Thus, for example, in the very simple case of the equation $x^2 + px = 0$, where x and p are quaternions or matrices of the second order, I have shown in the *Comptes Rendus* [pp. 174, 179 above] that besides the four determinate ones, all of which (0 included) may be regarded as functions of p , there are two other *indeterminate* ones, each one containing an arbitrary constant, and neither of them (to use quaternion language) coplanar with the base. Here there is a sort of reversion to the normal case of 3 pairs of roots to an unilateral quadratic, with the modification of two of them having become indeterminate. It becomes then of importance to fix accurately the condition of this normal state of things ceasing to exist. Being intent on the Denumeration theory of the roots in the general quadratic, I did not in the paper cited do this explicitly for the unilateral quadratic, although I gave there my own form of solution. Moreover, there are other features of much interest belonging to the question, which, for the same reason, I omitted to notice. These omissions and shortcomings it is the object of this present article to supply.

Starting with the form $x^2 - 2px + q = 0$, and for convenience of comparison with Hamilton's formulæ treating p, q indifferently as matrices or as quaternions, and forming the equation $x^2 - 2Bx + D = 0$, where B, D are scalars to be determined, so that $B = Sx$ and $D = Tx^2$, we shall have

$$2x = (p - B)^{-1}(q - D).$$

If now we understand by b, c, d, e, f

$$Sp, Sq, Tp^2, S(VpVq), Tq^2 \text{ respectively,}$$

by means of the general formula

$$T\pi^2 \cdot (\pi^{-1}\chi)^2 - 2S(V\pi V\chi)(\pi^{-1}\chi) + T\chi^2 = 0^*,$$

[remembering that

$$T(p - B)^2 = d^2 - 2bB + B^2,$$

$$T(q - D)^2 = f^2 - 2cD + D^2,$$

* This formula, which I have not met with in Treatises on Quaternions, is a particular case only of the general Theorem in Matrices, that if

$$A\lambda^\omega + B\lambda^{\omega-1}\mu + \dots + L\mu^\omega$$

is the determinant to $(\lambda L + \mu M)$, where L and M are two matrices of the order ω and λ and μ two ordinary quantities, then

$$A(L^{-1}M)^\omega - B(L^{-1}M)^{\omega-1} \dots + (-)^\omega L = 0.$$

and $S\{V(p-B)V(q-D)\} = e - bD - cB + BD$,

we shall obtain [see p. 188 above]

$$4(d - 2bB + B^2)x^2 - 4(e - bD - cB + BD)x + (f - 2cD + D^2) = 0.$$

Hence, writing $B - b = u, D - c = v$,

$$d - b^2 = \alpha, \quad e - bc = \beta, \quad f - c^2 = \gamma,$$

and comparing with each other the two quadratic equations in x , we may write

$$u^2 + \alpha = \lambda, \quad uv + \beta = 2\lambda(u + b), \quad v^2 + \gamma = 4\lambda(v + c).$$

Eliminating v from the latter two equations there results

$$-(2\lambda u + 2b\lambda - \beta)^2 + 4\lambda(2\lambda u + 2b\lambda - \beta)u - (\gamma - 4c\lambda)u^2 = 0,$$

and finally writing $\lambda - \alpha$ for u^2 , we obtain

$$(4\lambda^2 + 4c\lambda - \gamma)(\lambda - \alpha) - (2b\lambda - \beta)^2 = 0.$$

There are thus 3 pairs of roots, for to each of the three values of λ correspond two values of u , namely

$$\pm(\lambda - d + b^2)^{\frac{1}{2}},$$

and to each value of λ and u one value of v , namely

$$2\lambda + \frac{2b\lambda + bc - e}{u}.$$

We have also $x = \frac{1}{2}\{(p - b - u)^{-1}(q - D)\}$,

consequently, since $p^2 - 2bp + d = 0$,

$$x = \frac{(p - b + u)(q - c - v)}{2(b^2 - d - u^2)} = -\frac{(p - b + u)(q - c - v)}{2\lambda}.$$

Thus then we see that x can only cease to have 6 determinate values when $\lambda = 0$, and consequently the *Criterion of Normality* is the last term in the equation to λ .

This equation, written out at length, is

$$4\lambda^3 + 4(c - b^2 - \alpha)\lambda^2 + (-4c\alpha + 4b\beta - \gamma)\lambda + \alpha\gamma - \beta^2 = 0,$$

that is, $4\lambda^3 + 4(c - d)\lambda^2 + (-4cd + 4be - f + c^2)\lambda + (d - b^2)(f - c^2) - (e - bc)^2$.

Hence the *Criterion* in question is $(d - b^2)(f - c^2) - (e - bc)^2$ or $df - c^2d - b^2f - e^2 + 2bce$, which is the discriminant to the quadratic form

$$\lambda^2 + 2b\lambda\nu + 2c\mu\nu + d\mu^2 + 2e\mu\nu + f\nu^2;$$

this, as I have elsewhere shown, is the *Criterion* of the matrices p, q^* being in *involution* †, that is, of a linear equation existing between the matrices $1, p, q, pq$; or if p, q are regarded as quaternions, it is the condition of the square of

* When p, q are regarded as matrices, then

$$p^2 - 2bp + d = 0, \quad q^2 - 2cq + f = 0, \quad \frac{1}{2}(pq + qp) - bq - cp + e = 0,$$

where

$$\lambda^2 + 2b\lambda\nu + 2c\mu\nu + d\mu^2 + 2e\mu\nu + f\nu^2$$

is the determinant to $\lambda + \mu p + \nu q$.

[† Above p. 116.]

the sine of the angle between the vectors of p and q vanishing; a condition which of course does not imply the coincidence of the vectors unless accompanied by the futile limitation of such vectors being real.

It admits of easy demonstration by virtue of the foregoing that in the case of the more general equation

$$px^2 + qx + r = 0,$$

the Criterion of Normality will be the discriminant of the ternary quadratic, which is the determinant of

$$pu + qv + rw;$$

this seems to me a very remarkable and noteworthy theorem. When this Criterion does not vanish, the quadratic equation above written must have 3 pairs of determinate roots.

Why they go in pairs and can be found by solving only a cubic instead of a sextic is best seen *à priori* by reverting to the original form $x^2 - 2px + q = 0$.

It follows from the nature of the process for finding B and D that they will be the same for that equation as for the equation $y^2 - 2yp + q = 0$.

But on writing $x + y = 2p$ these two equations pass into one another.

Hence each value of B , say B_1 , will be associated with another value, say B' , where $B_1 + B' = 2b^*$, that is to say, if u_1 , namely $B - b$ is one value of u , then $b - B$, that is, $-u_1$ will be another value of u , so that the equation in u^2 ought to be (as it has been shown to be) a cubic.

It might for a moment be supposed that $\lambda = \alpha = d - b^2$ would lead to a breach of normality on account of the equation $v - 2\lambda = \frac{2b\lambda + bc - e}{u}$, where $u^2 = 0$.

This, however, is not the case. For the equation

$$v^2 + \gamma = 4\lambda(v + c)$$

becomes, when $\lambda = \alpha$,

$$v^2 - 4(d - b^2)v + f - c^2 - 4cd + 4b^2c = 0,$$

so that v remains *finite*; consequently $2b\lambda + bc - e$, that is, $2bd - 2b^3 + bc - e$,

must vanish when $\lambda = d - b^2$, and $v - 2\lambda$ assumes the form $\frac{0}{0}$. Obviously then

in this case, to the one value $u = 0$ will be associated the two values of v , say v_1 and v_2 , given by the above quadratic, and to $\lambda = \alpha$ will still correspond two values of (u, v) , namely $(0, v_1)$, $(0, v_2)$; where, ideally speaking, the two zeros may be regarded as the same infinitesimal affected with opposite signs.

* In quaternion phrase, if $x + y = 2p$, $Sx + Sy = 2Sp$.

It should be observed, in order to understand what follows in the text, that $b - B_1 = B' - b$, and that the values of B must obviously be the same in the equation $x^2 - 2px + q = 0$ as in the equation $x^2 - 2xp + q = 0$.

The equation in λ may be made to undergo a useful linear transformation.

Let $\lambda = \mu + \alpha$, so that $\mu = u^2$.

Then

$$\mu \{4\mu^2 + (8\alpha + 4c)\mu + 4\alpha^2 + 4c\alpha - \gamma\} - (2b\mu + 2b\alpha - \beta)^2 = 0,$$

$$\text{that is } 4\mu^3 + \{4(c + 2d) - 12b^2\}\mu^2 + \{(c + 2d)^2 - 8(c + 2d)b^2 + 12b^4 + 4be - f\}\mu - \{b(c + 2d) - 2b^3 - e\}^2 = 0,$$

where it is noticeable that the number of parameters is reduced from 5 to 4, c and d only appearing together in the linear combination $c + 2d$. This is tantamount to the form obtained by Hamilton.

Let us make another linear transformation suggested by the preceding remark. Write $c + 2d = g$, and $\mu - b^2 = \gamma = \lambda - d$, the equation becomes

$$4\gamma^3 + 4g\gamma^2 + (g^2 + 4be - f)\gamma + 2beg - b^2f - e^2 = 0.$$

But obviously, notwithstanding this reduction of the parameters, λ itself is the most natural quantity to employ as the base of the solution, or, so to say, as the independent variable, and this admits of being determined by an equation of extraordinary simplicity.

For, let I be the discriminant of

$$\det. (\lambda + \mu p + \nu q) = I = df + 2bce - c^2d - b^2f - e^2.$$

Then it will be seen by actual inspection that the equation found for λ takes the following form

$$e^{\lambda(2\delta_c - \delta_d)} I = 0,$$

that is

$$I + \left(2 \frac{d}{dc} - \frac{d}{d \cdot d}\right) I \cdot \lambda + \frac{1}{2} \left(2 \frac{d}{dc} - \frac{d}{d \cdot d}\right)^2 I \cdot \lambda^2 + \frac{1}{1 \cdot 2 \cdot 3} \left(2 \frac{d}{dc} - \frac{d}{d \cdot d}\right)^3 I \cdot \lambda^3 = 0,$$

(the terms in the exponential function subsequent to the fourth term adding nothing to the value of the series).

If in the equation $x^2 - 2px + q = 0$, p and q be regarded as quaternions, then $\lambda = Sx^2 + Ip^2 - (Sp)^2$, $c = Sq$, $d = Ip^2$, and $I = \frac{1}{4}(pq - qp)^2$, which is a scalar quantity, and is to be regarded as an explicit function of $Sp, Sq; Tp^2, S(VpVq), Tq^2$; it is in fact the discriminant of the form

$$X^2 + 2SpXY + 2SqXZ + Tp^2Y^2 + 2S(VpVq)YZ + Tq^2Z^2,$$

an identity unknown I believe to the geometrical quaternionists.

[As an example of it, let $p = i, q = j$, then

$$Sp = 0, \quad Sq = 0, \quad S(VpVq) = 0, \quad Tp^2 = -1, \quad Tq^2 = -1,$$

$\frac{1}{4}(pq - qp)^2 = 1 =$ the discriminant of $X^2 - Y^2 - Z^2$.]

With these definitions $e^{\lambda(2\delta_c - \delta_d)} I$ becomes identically zero.

The equation $x^2 - 2px + q = 0$ having six roots it is natural to inquire as to the value of their sum. This may be readily found as follows. We have found

$$x = -\frac{(p-b+u)(q-c-v)}{2\lambda}.$$

Also, if

$$\begin{aligned} x + x' &= 2p, \\ x'^2 - 2x'p + q &= 0, \end{aligned}$$

and obviously

$$\Sigma x = \Sigma x'.$$

Hence

$$\Sigma x = -\Sigma \frac{(p-b+u)(q-c-v)}{2\lambda},$$

and

$$12p - \Sigma x = -\Sigma \frac{(q-c-v)(p-b+u)}{2\lambda}.$$

Therefore

$$\begin{aligned} \Sigma x &= 6p - \Sigma \frac{3}{2\lambda}(pq - qp) \\ &= 6p - 3I^{\frac{1}{2}}\Sigma \frac{1}{\lambda} \\ &= 6p + 3 \frac{I^{\frac{1}{2}}(2\delta_c - \delta_a)I}{I} \\ &= 6 \{p + (2\delta_c - \delta_a)I^{\frac{1}{2}}\}, \end{aligned}$$

where the sign of $I^{\frac{1}{2}}$ must be so taken that it shall be equal to $\frac{1}{2}(pq - qp)$.

So again

$$\begin{aligned} \Sigma x^2 &= 2p\Sigma x - 6q \\ &= 12p^2 - 6q + 12(2\delta_c - \delta_a)I^{\frac{1}{2}}p. \end{aligned}$$

Thus the mean value of each root is ϵ in excess, and that of each square root ϵp in excess, of what these means would be if p and q were nominal quantities, ϵ denoting $(2\delta_c - \delta_a)I^{\frac{1}{2}}p$. Of course Σx^i may be found by the formula of derivation

$$\Sigma x^{i+1} = 2p\Sigma x^i - 9\Sigma x^{i-1}.$$

In conclusion it may be observed in regard to the equation $x^2 - 2px + q = 0$, (since in writing $x + x_1 = 2p$, we have $x_1^2 - 2x_1p + q = 0$) it follows that (whatever be the order of the quantities p and q) the roots of either equation must be associated in pairs; because, if the identical equation to p is $p^\omega - \omega bp^{\omega-1} + \dots$ and to x is $x^\omega - \omega Bx^{\omega-1} + \dots$, the equation for finding B must be of the form $T(B-b)^2 = 0$.

P.S.—Since the above was sent to press I have discovered the general solution of the unilateral equation of any degree in matrices of any order; see the *Comptes Rendus* of the Institute for Oct. 20, 1884 [pp. 197, 233 above], and *Nature* for Nov. 13, 1884*.

[* This paper contains the Theorem "Every latent root of every root of a given unilateral function in matrices of any order, is an algebraical root of the determinant of that function taken as if the unknown were an ordinary quantity, and conversely every algebraical root of the determinant so taken is a latent root of one of the roots of the given function."]