

## 106.

## ON THE SINGULARITIES OF SURFACES.

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IN the following paper, for symmetry of nomenclature and in order to avoid ambiguities, I shall, with reference to plane curves and in various phrases and compound words, use the term "node" as synonymous with double point, and the term "spinode" as synonymous with cusp. I shall, besides, have occasion to consider the several singularities which I call the "flecnode," the "oscnode," the "fleflecnode," and the "tacnode:" the flecnode is a double point which is a point of inflexion on one of the branches through it; the oscnode is a double point which is a point of osculation on one of the branches through it; the fleflecnode is a double point which is a point of inflexion on each of the branches through it; and the tacnode is a double point where two branches touch. And it may be proper to remark here, that a tacnode may be considered as a point resulting from the coincidence and amalgamation of two double points (and therefore equivalent to twelve points of inflexion); or, in a different point of view, [?] as a point uniting the characters of a spinode and a flecnode. I wish to call to mind here the definition of conjugate tangent lines of a surface, viz. that a tangent to the curve of contact of the surface with any circumscribed developable and the corresponding generating line of the developable, are conjugate tangents of the surface.

Suppose, now, that an absolutely arbitrary surface of any order be intersected by a plane: the curve of intersection has not in general any singularities other than such as occur in a perfectly arbitrary curve of the same order; but as a plane can be made to satisfy one, two, or three conditions, the curve may be made to acquire singularities which do not occur in such absolutely arbitrary curve.

Let a single condition only be imposed on the plane. We may suppose that the curve of intersection has a node; the plane is then a tangent plane and the node is the point of contact—of course any point on the surface may be taken for

the node. We may if we please use the term "nodes of a surface," "node-planes of a surface," as synonymous with the points and tangent planes of a surface. And it will be convenient also to use the word node-tangents to denote the tangents to the curve of intersection at the node; it may be noticed here that the node-tangents are conjugate tangents of the surface.

Next let two conditions be imposed upon the plane: there are three distinct cases to be considered.

First, the curve of intersection may have a flecnode. The plane (which is of course still a tangent plane at the flecnode) may be termed a flecnode-plane; the flecnodes are singular points on the surface lying on a curve which may be termed the "flecnode-curve<sup>1</sup>," and the flecnode-planes give rise to a developable which may be termed the flecnode-developpe. The "flecnode-tangents" are the tangents to the curve of intersection at the flecnode; the tangent to the inflected branch may be termed the "singular flecnode-tangent," and the tangent to the other branch the "ordinary flecnode-tangent."

Secondly, the curve of intersection may have a spinode. The plane (which is of course still a tangent plane at the spinode) may be termed a spinode-plane; the spinodes are singular points on the surface lying on a curve which may be termed the "spinode-curve<sup>2</sup>." And the spinode-planes give rise to a developable which may be termed the "spinode-developpe." Also the "spinode-tangent" is the tangent to the curve of intersection at the spinode.

Thirdly, the curve of intersection may have two nodes, or what may be termed a "node-couple." The plane (which is a tangent plane at each of the nodes and therefore a double tangent plane) may be also termed a "node-couple-plane." The node-couples are pairs of singular points on the surface lying in a curve which may be termed the "node-couple-curve," and the node-couple-planes give rise to a developable which may be termed the "node-couple-developpe." The tangents to the curve of intersection at the two nodes of a node-couple might, if the term were required, be termed the "node-couple-tangents." Also one of the nodes of a node-couple may be termed a "node-with-node," and the tangents to the curve of intersection at such point will be the "node-with-node-tangents."

<sup>1</sup> The flecnode-curve, defined as the locus of the points through which can be drawn a line meeting the surface in four consecutive points, was, so far as I am aware, first noticed in Mr Salmon's paper "On the Triple Tangent Planes of a Surface of the Third Order" (*Journal*, t. iv. [1849], pp. 252—260), where Mr Salmon, among other things, shows that the order of the surface being  $n$ , the curve in question is the intersection of the surface with a surface of the order  $11n - 24$ .

<sup>2</sup> The notion of a spinode, considered as the point where the indicatrix is a parabola (on which account the spinode has been termed a parabolic point) may be found in Dupin's *Développements de Géométrie*: the most important step in the theory of these points is contained in Hesse's memoir "Ueber die Wendepuncte der Curven dritter Ordnung" (*Crelle*, t. xxviii. [1848], pp. 97—107), where it is shown that the spinode-curve is the curve of intersection of the surface supposed as before of the order  $n$ , with a certain surface of the order  $4(n - 2)$ . See also Mr Salmon's memoir "On the Condition that a Plane should touch a surface along a Curve Line" (*Journal*, t. iii. [1848], pp. 44—46).

It is hardly necessary to remark that the flecnode-curve is *not* the edge of regression of the flecnode-developpe, and the like remark applies *m.m.* to the spinode-curve and the node-couple curve.

Finally, let three conditions be imposed upon the plane: there are six distinct cases to be considered, in each of which we have no longer curves and developes, but only singular points and singular tangent planes determinate in number.

First, the curve of intersection may have an oscnode. The plane (which continues a tangent plane at the oscnode) is an "oscnode-plane." The "oscnode-tangents" are the tangents to the curve of intersection at the oscnode; the tangent to the osculating branch is the "singular oscnode-tangent;" and the tangent to the other branch the "ordinary oscnode-tangent."

Secondly, the curve of intersection may have a fleflecnode. The plane (which continues a tangent plane at the fleflecnode) is a "fleflecnode-plane." The "fleflecnode-tangents" are the tangents to the curve of intersection at the fleflecnode.

Thirdly, the curve of intersection may have a tacnode. The plane (which continues a tangent plane at the tacnode) is a "tacnode-plane." The "tacnode-tangent" is the tangent to the curve of intersection at the tacnode.

Fourthly, the curve of intersection may have a node and a flecnode, or what may be termed a node-and-flecnode. The plane (which is a tangent plane at the node and also at the flecnode, where it is obviously a flecnode-plane) is a "node-and-flecnode-plane." The "node-and-flecnode-tangents," if the term were required, would be the tangents to the curve of intersection at the node and at the flecnode of the node-and-flecnode. The node of the node-and-flecnode may be distinguished as the node-with-flecnode, and the flecnode as the flecnode-with-node, and we have thus the terms "node-with-flecnode-tangents," "flecnode-with-node-tangents," "singular flecnode-with-node-tangent," and "ordinary flecnode-with-node-tangent."

Fifthly, the curve of intersection may have a node and also a spinode, or what may be termed a "node-and-spinode." The plane (which is a tangent plane at the node, and is also a tangent plane at the spinode, where it is obviously a spinode-plane) is a "node-and-spinode-plane." The node-and-spinode-tangents, if the term were required, would be the tangents at the node and the tangent at the spinode of the node-and-spinode to the curve of intersection. The node of the node-and-spinode may be distinguished as the "node-with-spinode," and the spinode as the "spinode-with-node," and we have thus the terms "node-with-spinode-tangent," "spinode-with-node-tangent."

Sixthly, the curve of intersection may have three nodes, or what may be termed a "node-triplet." The plane (which is a triple tangent plane touching the surface at each of the nodes) is a "node-triplet-plane." The "node-triplet-tangents," if the term were required, would be the tangents to the curve of intersection at the nodes of the node-triplet. Each node of the node-triplet may be distinguished as a "node-

with-node-couple," and the tangents to the curve of intersection at such nodes are "node-with-node-couple-tangents." The terms "node-couple-with-node," "node-couple-with-node-tangent," might be made use of if necessary.

It should be remarked that the oscnodes lie on the flecnode-curve, as do also the flefnodes; these latter points are real double points of the flecnode-curve. The tacnodes are points of intersection and (what will appear in the sequel) points of contact of the flecnode-curve, the spinode-curve, and the node-couple-curve. The spinode-with-nodes are points of intersection of the spinode-curve and node-couple-curve, and the flecnode-with-nodes are points of intersection of the flecnode-curve and node-couple-curve; the node-with-node-couples are real double points (entering in triplets) of the node-couple-curve.

Consider for a moment an arbitrary curve on the surface; the locus of the node-tangents at the different points of this curve is in general a skew surface, which may however, in cases to be presently considered, degenerate in different ways.

Reverting now to the flecnode-curve, it may be shown that the singular flecnode-tangent coincides with the tangent of the flecnode-curve. For consider on a surface two consecutive points such that the line joining them meets the surface in two points consecutive to the first-mentioned two points. The line meets the surface in four consecutive points, it is therefore a singular flecnode-tangent; *each* of the first-mentioned two points must be on the flecnode-curve, or the singular flecnode-tangent touches the flecnode-curve. The two flecnode-tangents are by a preceding observation conjugate tangents. It follows that the skew surface, locus of the flecnode-tangents, breaks up into two surfaces, each of which is a developable, viz. the locus of the singular flecnode-tangents is the developable having the flecnode-curve for its edge of regression, and the locus of the ordinary flecnode-tangents is the flecnode-developable. Of course at the tacnode, the tacnode-tangent touches the flecnode-curve.

Passing next to the spinode-curve, the spinode-plane and the tangent-plane at a consecutive point along the spinode-tangent are identical<sup>1</sup>, or their line of intersection is indeterminate. The spinode-tangent is therefore the conjugate tangent to *any* other tangent line at the spinode, and therefore to the tangent to the spinode-curve. It follows that the surface locus of the spinode-tangents degenerates into a developable surface twice repeated, viz. the spinode-developable. Consider the tacnode as two coincident nodes; each of these nodes, by virtue of its constituting, in conjunction with the other, a tacnode, is on the spinode-curve; or, in other words, the tacnode-tangent touches the spinode-curve, and the same reasoning proves that it touches the node-couple-curve. It has already been seen that the tacnode-tangent touches the flecnode-curve; consequently the tacnode is a point, not of simple intersection only, but of contact, of the flecnode-curve, the spinode-curve, and the node-couple-curve.

In virtue of the principle of the spinode-plane being identical with the tangent plane at a consecutive point along the spinode tangent, it appears that the tacnode-

<sup>1</sup> It must not be inferred that the tangent plane at such consecutive point is a spinode-plane; this is obviously not the case.

plane is a stationary plane, as well of the flecnode-developpe as of the spinode-developpe, and it would at first sight appear that it must be also a stationary tangent plane of the node-couple-developpe. But this is not so; the node-with-node-planes envelope, not the node-couple-developpe, but the node-couple-developpe twice repeated: the tacnode-plane is in a sense a stationary plane on such duplicate developable, but not in any manner on the single developable. The tacnode-plane is an ordinary tangent plane of the node-couple-developpe.

Consider now a spinode-with-node, which we have seen is a point of intersection of the spinode-curve and node-couple-curve. The tangent plane at a consecutive point along the spinode-with-node-tangent, is *identical* with the spinode-with-node-plane; the curve of intersection of the tangent plane at such consecutive point has therefore a node at the node-with-spinode, or the tangent plane in question is a node-couple-plane, and the point of contact is a point on the node-couple-curve. Consequently the spinode-with-node-tangent touches the node-couple-curve, and thence also the spinode-with-node-plane is a stationary tangent plane of the node-couple-developpe.

It should be remarked that no circumscribed developable can have a stationary tangent plane except the tangent planes at the points where the curve of contact meets the spinode-curve, and any one of these planes is only a stationary plane when the curve of contact touches the spinode-tangent; and that the node-couple-curve and the flecnode-curve do not intersect the spinode-curve except in the points which have been discussed.

Recapitulating, the node-couple-curve and the spinode-curve touch at the tacnodes, and intersect at the spinode-with-nodes: moreover, the tacnode-planes are stationary planes of the spinode-developpe, and the spinode-with-node-planes are stationary planes of the node-couple-developpe. Besides this, the two curves are touched at the tacnodes by the flecnode-curve, and the tacnode-planes are stationary planes of the flecnode-developpe.