823.

ON THE GEOMETRICAL INTERPRETATION OF CERTAIN FORMULÆ IN ELLIPTIC FUNCTIONS.

[From the *Johns Hopkins University Circulars*, No. 17 (1882), p. 238.]

I HAVE given in my *Elliptic Functions* expressions for the sn^2 of $u+\tfrac{1}{2}K$, $u+\tfrac{1}{2}iK'$, $u+\tfrac{1}{2}K+\tfrac{1}{2}iK'$; but it is better to consider the dn^2, sn^2, cn^2 of these combinations respectively, and to write the formulæ thus:

$$\operatorname{dn}^2(u+\tfrac{1}{2}K) = k'\frac{\operatorname{dn}u-(1-k')\operatorname{sn}u\operatorname{cn}u}{\operatorname{dn}u+(1-k')\operatorname{sn}u\operatorname{cn}u}, \quad = k'\frac{1-k^2x-(1-k')y}{1-k^2x+(1-k')y};$$

$$\operatorname{sn}^2(u+\tfrac{1}{2}iK') = \frac{1}{k}\frac{(1+k)\operatorname{sn}u+i\operatorname{cn}u\operatorname{dn}u}{(1+k)\operatorname{sn}u-i\operatorname{cn}u\operatorname{dn}u}, \quad = \frac{1}{k}\frac{(1+k)x+iy}{(1+k)x-iy};$$

$$\operatorname{cn}^2(u+\tfrac{1}{2}K+\tfrac{1}{2}iK') = \frac{-ik'}{k}\frac{\operatorname{cn}u-(k+ik')\operatorname{sn}u\operatorname{dn}u}{\operatorname{cn}u+(k+ik')\operatorname{sn}u\operatorname{dn}u}, \quad = \frac{-ik'}{k}\frac{1-x-(k+ik')y}{1-x+(k+ik')y},$$

where in the last set of values x, y are used to denote $\operatorname{sn}^2 u$ and $\operatorname{sn}u\operatorname{cn}u\operatorname{dn}u$ respectively; and the formulæ are thus brought into connexion with the cubic curve $y^2=x(1-x)(1-k^2x)$. The curve has an inflexion at infinity on the line $x=0$; and the three tangents from the inflexion are $x=0$, $x=1$, $x=\frac{1}{k^2}$, touching the curve at the points $x, y=(0, 0)$, $(1, 0)$, $\left(\frac{1}{k^2}, 0\right)$ respectively: hence these points are sextactic points. We may from any one of them, for instance the point $(0, 0)$, draw four tangents to the curve, $(1+k)x+iy=0$, $(1+k)x-iy=0$; $(1-k)x+iy=0$, $(1-k)x-iy=0$; where the first and second of these lines form a pair, and the third and fourth of them form a pair, viz. the two tangents of a pair touch in points such that the line joining them passes through the point of inflexion: in particular, for the first-mentioned pair, the equation of the line joining the points of contact is $1+kx=0$. The linear functions belonging to a pair of tangents are precisely those which present themselves in the formulæ; thus if $T_1=(1+k)x+iy$, $T_2=(1+k)x-iy$, the second of the three formulæ is $\operatorname{sn}^2(u+\tfrac{1}{2}K)=\frac{1}{k}\frac{T_1}{T_2}$; and the other two formulæ correspond in like manner to pairs of tangents from the sextactic points $\left(\frac{1}{k^2}, 0\right)$, and $(1, 0)$ respectively. The formulæ are connected with the fundamental equations expressing the functions sn, cn, dn as quotients of theta functions.