

289.

ON SOME NUMERICAL EXPANSIONS.

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THERE are in the theory of series and the calculus of finite differences and the differential calculus, various sets of numbers, such for instance as Bernoulli's Numbers, or the successive differences of 0^m , which from their more or less frequent occurrence and the complexity of their law it is desirable to tabulate. To these belong, I think, the coefficients of the successive powers of x in the expansion of a given power of $\frac{\log(1+x)}{x}$, or when the index of the power of the function is an indeterminate quantity r , then the coefficients of the several terms of the rational and integral functions of r which form the coefficients of the successive powers of x in the expansion in question. We have

$$\log \frac{\log(1+x)}{x} = -\frac{1}{2}x + \frac{5}{24}x^2 - \frac{1}{8}x^3 + \frac{251}{2880}x^4 - \frac{19}{288}x^5 + \frac{19087}{362880}x^6 - \&c.,$$

which is most easily verified by differentiating and multiplying by $\log(1+x)$, which gives on the left-hand side $\frac{1}{1+x}$, and on the right-hand side the expansion $1-x+x^2-\&c.$ And from the above, multiplying each side by r , and taking the exponential, we find

$$\begin{aligned} \left\{ \frac{(\log 1+x)}{x} \right\}^r &= 1 \\ &- \left(\frac{1}{2}r \right) \cdot x \\ &+ \left(\frac{1}{8}r^2 + \frac{5}{24}r \right) x^2 \\ &- \left(\frac{1}{48}r^3 + \frac{5}{48}r^2 + \frac{1}{8}r \right) x^3 \\ &+ \left(\frac{1}{384}r^4 + \frac{5}{192}r^3 + \frac{97}{1152}r^2 + \frac{251}{2880}r \right) x^4 \\ &- \left(\frac{1}{3840}r^5 + \frac{5}{1152}r^4 + \frac{61}{2304}r^3 + \frac{401}{5760}r^2 + \frac{19}{288}r \right) x^5 \\ &+ \left(\frac{1}{46080}r^6 + \frac{5}{9216}r^5 + \frac{49}{9216}r^4 + \frac{10543}{414720}r^3 + \frac{4075}{69120}r^2 + \frac{19087}{362880}r \right) x^6 \\ &\mp \&c. \end{aligned}$$

which may be verified by writing $r=1$.

As an instance of the use of the preceding table, let it be required to find expressions for the combinations without repetitions of the series of natural numbers 1, 2, 3, ... (n-1), or what is the same thing for the coefficients of the powers of k in $k(k-1)(k-2)\dots(k-n+1)$. We have

$$\begin{aligned} & k(k-1)(k-2)\dots(k-n+1) \\ &= k^n - A_1^n k^{n-1} + A_2^n k^{n-2} - \&c. \\ &= \Pi n \text{ coefficient } x^n \text{ in } (1+x)^k; \end{aligned}$$

whence

$$\begin{aligned} (-)^r A_r^n &= \Pi n \text{ coefficient } x^n k^{n-r} \text{ in } (1+x)^k \\ &= \Pi n \text{ coefficient } x^n k^{n-r} \text{ in } e^{k \log(1+x)} \\ &= \frac{\Pi n}{\Pi(n-r)} \text{ coefficient } x^n \text{ in } \{\log(1+x)\}^{n-r} \\ &= \frac{\Pi n}{\Pi(n-r)} \text{ coefficient } x^r \text{ in } \left\{ \frac{\log(1+x)}{x} \right\}^{n-r}. \end{aligned}$$

Thus

$$\begin{aligned} A_1^n &= n \left\{ \frac{1}{2}(n-1) \right\}, \\ A_2^n &= n(n-1) \left\{ \frac{1}{8}(n-2)^2 + \frac{5}{24}(n-2) \right\}, \\ A_3^n &= n(n-1)(n-2) \left\{ \frac{1}{48}(n-3)^3 + \frac{5}{48}(n-3)^2 + \frac{1}{8}(n-3) \right\}, \\ &\vdots \end{aligned}$$

and so on, as far as the expansion has been effected. It may be remarked that the general expression for the algebraical transcendent A_r^n is given in Dr Schläfli's paper, "Sur les coefficients du développement du produit $(1+x)(1+2x)\dots\{1+(n-1)x\}$ suivant les puissances ascendantes de x ." *Crelle*, t. XLIII. [1852], pp. 1-22, but the law is a very complicated one.

We have

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \frac{1}{7}x^7 - \&c.$$

and dividing by x , and taking the logarithm, we have as before

$$\log\left(\frac{1}{x} \log(1+x)\right) = -\frac{1}{2}x + \frac{5}{24}x^2 - \frac{1}{8}x^3 + \frac{251}{2880}x^4 - \frac{19}{288}x^5 + \frac{19087}{362880}x^6 - \&c.$$

which may be considered as the first of a series of logarithmic derivatives, viz., dividing by $-\frac{1}{2}x$ and taking the logarithm we have

$$\log\left(-\frac{2}{x} \log\left(\frac{1}{x} \log(1+x)\right)\right) = -\frac{5}{12}x + \frac{47}{288}x^2 - \frac{2443}{25920}x^3 + \frac{5303}{82944}x^4 - \frac{12631}{580608}x^5 + \&c.,$$

and by the like process

$$\begin{aligned} \log\left(-\frac{12}{5x} \log\left(-\frac{2}{x} \log\left(\frac{1}{x} \log(1+x)\right)\right)\right) = \\ -\frac{47}{120}x + \frac{11317}{86400}x^2 - \frac{439989}{5184000}x^3 + \frac{9390960319}{156764160000}x^4 - \&c., \end{aligned}$$

and so on.

Suppose in general that

$$\phi x = x + B_1 x^2 + C_1 x^3 + \dots,$$

and let it be required to find the r^{th} function $\phi^r x$. It is easy to see that the successive coefficients are rational and integral functions of r of the degrees 1, 2, 3, &c. respectively; we have in fact

$$\begin{aligned} \phi^r x = & \phi^0 x \\ & + \frac{r}{1} (\phi^1 x - \phi^0 x) \\ & + \frac{r \cdot r - 1}{1 \cdot 2} (\phi^2 x - 2\phi^1 x + \phi^0 x) \\ & + \frac{r \cdot r - 1 \cdot r - 2}{1 \cdot 2 \cdot 3} (\phi^3 x - 3\phi^2 x + 3\phi^1 x - \phi^0 x). \\ & \text{\&c.}, \end{aligned}$$

and by successive substitutions,

$$\begin{aligned} \phi^0 x &= x, \\ \phi^1 x &= x + B_1 x^2 + C_1 x^3 + D_1 x^4 + \dots, \\ \phi^2 x &= x + 2B_1 x^2 + (2B_1^2 + 2C_1) x^3 + (B_1^3 + 5B_1 C_1 + 2D_1) x^4 + \dots, \\ \phi^3 x &= x + 3B_1 x^2 + (6B_1^2 + 3C_1) x^3 + (9B_1^3 + 15B_1 C_1 + 3D_1) x^4 + \dots \\ & \vdots \end{aligned}$$

Whence

$$\begin{aligned} \phi^r x &= x \\ & + r B_1 x^2 \\ & + \{(r^2 - r) B_1^2 + r C_1\} x^3 \\ & + \{(r^3 - \frac{5}{2} r^2 + \frac{3}{2} r) B_1^3 + (\frac{5}{2} r^2 - \frac{5}{2} r) B_1 C_1 + r D_1\} x^4 \\ & + \text{\&c.} \end{aligned}$$

It would, I think, be worth while to continue the expansion some steps further.

2, Stone Buildings, W.C., Oct. 2nd, 1859.