

299.

MATHEMATICS, RECENT TERMINOLOGY IN.

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THE terms intended to be explained in the present article relate to subjects distinct indeed, but intimately connected together, as well logically as historically. *Determinants* were devised as a means to the solution of a system of simple equations, but the principle of their construction is contained in the rule of signs which belongs to the theory of *arrangements* (or permutations): this theory has been studied, as well for its own sake, as in reference to the theory of equations, and in it originated the notion of a *group*, the most outlying term of those which are here explained. Moreover, in a system of simple equations, if the coefficients arranged in the natural square order are considered apart by themselves, this leads to the theory of *matrices*, a theory which indeed might have preceded that of determinants; the matrix, is, so to speak, the matter of a determinant; the rule of signs giving the form. But when the rule of signs is applied to other matter, this leads to the function called *permutants*; these include *commutants* and *intermutants*, and also *Pfaffians*, which however were not originally so arrived at. The theory of elimination (according to one of the two ways in which it may be treated) is essentially dependent upon systems of linear equations, and is thus also connected with determinants. And all, or nearly all, the before-mentioned theories have an application to the theory of rational and integral homogeneous functions, or, as they have been termed, *forms* or *quantics*; they are thus connected with the "Calculus of Forms," and with "Quantics"; the last-mentioned expression (used as a singular), has been defined to denote the entire subject of rational and integral functions, and of the equations and loci to which those give rise. The theory of rational and integral functions was first studied in a general manner in the question of *linear transformations*, and it was this question which led to the discovery of the functions, called originally *hyperdeterminants*, but afterwards *invariants*, and of the more general functions called *covariants*: the theory of covariants is indeed the part which has been chiefly attended to of the Calculus of forms, or of Quantics.

The following list of terms may be convenient:

Rule of signs.

Group.

Determinant.

Minor,

Symmetric, skew, skew symmetric.

Commutant.

Pfaffian.

Permutant.

Intermutant.

Cumulant.

Matrix.

Resultant.

Discriminant.

Plexus.

Rational and integral functions (notation and nomenclature of).

Quantic, quadric, cubic, &c., binary, ternary, &c., facient, tantipartite, lineo-linear.

Emanant.

Linear transformations.

Modulus of transformation, unimodular.

Hyperdeterminant.

Invariant.

Covariant.

Contravariant, peninvariant, seminvariant, quadrinvariant, quadricovariant, &c., catalecticant, canonisant.

Canonical form.

Bezoutic matrix, &c.

Tactinvariant, reciprocal.

Functional determinant, Jacobian, Hessian.

Concomitant, cogredient, contragredient.

Combinant.

RULE OF SIGNS.—Any arrangement of a series of terms may be derived (and that in a variety of ways) from any other arrangement by successive interchanges of two terms; but in whatever way this is done, the number of interchanges will be constantly even or constantly odd; and the two arrangements are said to be of the same sign or of contrary signs accordingly. In particular, if any arrangement is selected as the primitive arrangement, and taken to be positive, then any other arrangement will be positive or negative according as it is derivable from the primitive arrangement by an even or an odd number of interchanges. The definition leads to the following theorem: any arrangement is positive or negative, according as the total number of times in which the several elements respectively precede (mediately or immediately) elements posterior to them in the primitive arrangement, is even or odd: it may be added, that the positive and negative arrangements are equal in number. Thus in the case of three terms, the primitive arrangement being 123; the positive arrangements are 123, 231, 312, the

negative arrangements, 132, 213, 321: in the case of four terms, the primitive arrangement being 1234, the arrangements 1234, 2341, 3412, 4123 are respectively positive, negative, positive, negative; there are in all twelve positive and twelve negative arrangements.

GROUP.—The term was originally used as applied to substitutions only, but the more general use of the term is as follows: let θ be a symbol operating on any number of terms x, y, z, \dots and producing as the result of the operation the same number of new terms X, Y, Z, \dots (where X, Y, Z, \dots may be each of them functions of all or any of the set, x, y, z, \dots ; if X, Y, Z, \dots are merely the terms, x, y, z, \dots in a different order, then θ is a substitution, which explains in what sense that term has just been used). Imagine a set of operative symbols $1, \theta, \phi, \chi, \dots$ (1 , as an operative symbol denotes, of course, a symbol which leaves the operand unaltered) such that the result of the operation of *any* two symbols θ, ϕ (the same or different, and if different, then in either order) is identical with that of the operation of *some* symbol χ of the set; as thus, $\theta\phi(x, y, z, \dots) = \theta(X, Y, Z, \dots) = (X', Y', Z', \dots) = \chi(x, y, z, \dots)$, say, $\theta\phi = \chi$; then the symbols $1, \theta, \phi, \chi, \dots$ form a *group*. It is to be remarked that 1 belongs to every group, and moreover, that if θ be any symbol of the group, then $\theta^2, \theta^3, \theta^4, \dots$ belong to the group: the most simple form of a group (and when the number of terms is prime, the only form) is $1, \theta, \theta^2, \dots, \theta^{n-1} [\theta^n = 1]$. More generally, if there are n terms in the group, then every symbol θ of the group is an operation periodic of the order n (if not of an order a submultiple of n) and thus satisfies the symbolic equation $\theta^n = 1$. The symbols of the group are, so to speak, the symbolic n -th roots of unity, and as in the above-mentioned example, they may, whether n is prime or composite, form a system precisely analogous to that of the ordinary n -th roots of unity; but when n is composite, then upon two grounds this is not of necessity the case. 1°. The symbols of a group need not be convertible (thus $n=6$, there is a group, $1, \beta, \beta^2, \alpha, \alpha\beta, \alpha\beta^2$ [$\alpha^2=1, \beta^3=1, \beta\alpha=\alpha\beta^2$ and $\therefore \beta^2\alpha=\alpha\beta$, this is in fact, the group of the substitutions of three things). 2°. There may be distinct n -th roots, thus $n=4$, there is a group, $1, \alpha, \beta, \alpha\beta$ [$\alpha^2=1, \beta^2=1, \alpha\beta=\beta\alpha$], in which α, β are distinct square roots of (the symbolical) unity, and which is thus wholly different from the group, $1, \alpha, \alpha^2, \alpha^3$ [$\alpha^4=1$].

The combination of a series of terms in the way of addition or subtraction, according to the rule of signs, gives rise to the class of functions called permutants, which include as a particular but the earliest discovered and most important case, the determinant:

DETERMINANT.—Imagine a square arrangement of terms, for example

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}$$

and taking this as the primitive arrangement, permute in every possible way entire columns (or, what would give the same results, entire lines) and for each such arrange-

ment form the product of the terms in the dexter diagonal (N.W. to S.E.) of the square, giving to such product the sign which belongs to the arrangement of the columns (or lines). The algebraical sum of these products is a determinant, and such determinant is or may be represented as above, by enclosing the terms within two vertical lines. Thus the developed value of the determinant in question is

$$ab'c'' - ab''c' + a'b''c + a''bc' - a''b'c - a'b''c.$$

The rule may be otherwise stated as follows: a determinant is the sum of a series of products each with its proper sign, such that in each product the factors are taken out of each line and out of each column, and if the factors are arranged according to the primitive arrangement of the columns in which they occur, then the sign is that corresponding to the resulting arrangement of the lines (or *vice versâ*): thus in the product $-ab''c'$, the factors a, b'', c' occur in the columns 1, 2, 3 (they are therefore arranged according to the primitive arrangement of the columns) and in the lines 1, 3, 2; such arrangement of the lines considered as derived from the primitive arrangement 1 2 3 is negative, and the product has therefore the sign $-$. A generalisation of this construction will be mentioned under the term commutant.

The word *resultant* was formerly used as synonymous with determinant, but it is now employed and is here explained in a more extended signification. The new synonym *eliminant* seems unnecessary.

A few of the numerous properties of determinants may be stated.

A determinant is a linear function (without constant term) of the terms in each of its columns, and also of the terms in each of its lines, or, more briefly expressed, it is a linear function of each column, and also of each line. Moreover, without altering the value of the determinant, the lines may be made columns, and the columns lines, and all the properties of the function exist equally with respect to the lines and to the columns. The absolute value of the determinant is not altered, but the sign is reversed, by an interchange of two columns, hence also if two columns become identical, the determinant vanishes. Moreover when the columns are permuted in any manner whatever, the absolute value is not altered, but the sign will be that corresponding to the arrangement of the columns. A determinant may be developed as a linear function of the terms in any line, thus

$$\begin{vmatrix} a, & b \\ a', & b' \end{vmatrix} = a \begin{vmatrix} b' \end{vmatrix} - a' \begin{vmatrix} b \end{vmatrix},$$

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix} = a \begin{vmatrix} b', & c' \\ b'', & c'' \end{vmatrix} + b \begin{vmatrix} c', & a' \\ c'', & a'' \end{vmatrix} + c \begin{vmatrix} a', & b' \\ a'', & b'' \end{vmatrix},$$

the signs being alternately positive and negative or else all positive, according as the number of columns is even or odd.

The square arrangement of terms out of which a determinant is formed, and generally any square or rectangular arrangement of terms, is called a matrix. Consider a determinant

$$\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \\ a'' & b'' & c'' & d'' \\ a''' & b''' & c''' & d''' \end{vmatrix}$$

and partitioning the lines in any manner, form with them the matrices

$$\begin{vmatrix} a & b & c & d \\ a' & b' & c' & d' \end{vmatrix}, \quad \begin{vmatrix} a'' & b'' & c'' & d'' \\ a''' & b''' & c''' & d''' \end{vmatrix},$$

and out of these matrices, with complementary columns thereof, a sum of products

$$\Sigma \pm \begin{vmatrix} a & b \\ a' & b' \end{vmatrix} \begin{vmatrix} c'' & d'' \\ c''' & d''' \end{vmatrix},$$

(the sign \pm being that corresponding to the product $ab'c''d'''$ of the terms in the dexter diagonals of the factor determinants, considered as a term of the original determinant), the sum of all the products so obtained in the original determinant.

It has been mentioned that the determinant is a linear function of each column; hence if the terms of any column are $\rho\alpha, \rho'\alpha, \dots$ the determinant is equal to ρ times a determinant in which the corresponding column is α, α', \dots and similarly if the column is $\alpha + b, \alpha' + b', \dots$ then the determinant is the sum of two other determinants in which the corresponding columns are α, α', \dots and b, b', \dots respectively. This property, in combination with some of those already mentioned, leads very simply to the rule for the multiplication of determinants; for example we have

$$\begin{vmatrix} \rho & \sigma \\ \rho' & \sigma' \end{vmatrix} \begin{vmatrix} \alpha & \beta \\ \alpha' & \beta' \end{vmatrix} = \begin{vmatrix} \rho\alpha + \sigma\beta & \rho'\alpha + \sigma'\beta \\ \rho\alpha' + \sigma\beta' & \rho'\alpha' + \sigma'\beta' \end{vmatrix},$$

from which the law is obvious. The product might also be expressed, and although it appears less simple, there is an advantage in expressing it, in the form

$$\begin{vmatrix} \rho\alpha + \sigma\beta & \rho'\alpha + \sigma'\beta \\ \rho\alpha' + \sigma\beta' & \rho'\alpha' + \sigma'\beta' \end{vmatrix}.$$

If we omit simultaneously any line and any column of a determinant, and with the terms which are left form a determinant, the determinants so obtained are the *first minors* of the given determinant. A similar process, but omitting pairs, triads, &c. of lines and columns, gives the *second minors*, *third minors*, &c. of the given determinant. But the first minors are the most important, and are sometimes spoken of simply as the minors.

A determinant

$$\begin{vmatrix} a, & h, & g, & \dots \\ h, & b, & f & \\ g, & f, & c & \\ \vdots & & & \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 11, & 12, & \dots \\ 21, & 22 & \\ \vdots & & \end{vmatrix},$$

where the corresponding terms on opposite sides of the dexter diagonal are equal to each other (say $rs = sr$) is said to be *symmetrical*.

But if the terms are equal in magnitude only, but have opposite signs (say $rs = -sr$, this relation not extending to the terms *in* the diagonal, for which $s = r$) the determinant is said to be *skew*; and if the relation extends to the case $s = r$, or what is the same thing, if the terms *in* the diagonal vanish, the determinant is said to be *skew symmetrical*. Skew determinants have an intimate connection with the functions called Pfaffians.

COMMUTANT.—The second rule for the construction of a determinant might have been thus stated, viz. for the determinant

$$\begin{vmatrix} 11, & 12, \dots, 1n \\ 21, & 22 \\ \vdots & \\ n1 & \end{vmatrix}$$

write down the expression

$$\begin{matrix} 11 \\ 22 \\ \vdots \\ nn \end{matrix}$$

and permute in every possible way the numbers in the first column, prefixing in each case the sign of the arrangement. Then reading off

$$\begin{matrix} \pm r1 \\ s2 \\ \vdots \\ zn. \end{matrix}$$

as meaning

$$\pm y1 . s2 \dots zn$$

the sum of all the terms so obtained is in fact the determinant in question. The same result would be obtained by permuting the numbers in the second column instead of those in the first column. And moreover, if the numbers in both columns are permuted, the sign being the sign $\pm \pm$ compounded of the signs corresponding to the separate arrangements, the only difference is, that the determinant will be multiplied by the numerical factor $1.2.3 \dots n$.

If instead of two we have three or more columns, the resulting function is a *commutant*. But a distinction is to be made according as the number of columns is even or odd. In the former case we may permute all but one of the columns, and it is indifferent which column is left unpermuted; and if all the columns are permuted, the effect is merely to introduce the numerical factor $1.2.3\dots n$. In the latter case, if all the columns are permuted, the result is zero, and it is therefore essential that one column should remain unpermuted; moreover, different results are obtained according to the column which is left unpermuted, and such column must therefore be distinguished; this is done by placing above it the mark †.

PFaffIAN.—Suppose that the terms 12, 13, 21, &c., are such that $21 = -12$, and generally that $sr = -rs$, then the Pfaffians 1234, 123456, &c., are defined by means of the equations

$$1234 = 12.34 + 13.42 + 14.23,$$

$$123456 = 12.3456 + 13.4562 + 14.5623 + 15.6234 + 16.2345,$$

(where of course $3456 = 34.56 + 35.64 + 36.45$, and so for 4562, &c.)

and so on. The functions in question occur in the solution of an important problem (including that of partial differential equations of the first order and of any degree) known as Pfaff's problem, and were named accordingly.

It may be noticed that a skew symmetrical determinant of any odd order is equal to zero; but that a skew symmetrical determinant of any even order is the square of a Pfaffian, e.g. if $12 = -21$, &c., as above, then

$$\begin{vmatrix} 0, & 12, & 13, & 14 \\ 21, & 0, & 23, & 24 \\ 31, & 32, & 0, & 34 \\ 41, & 42, & 43, & 0 \end{vmatrix} = (12.34 + 13.42 + 14.23)^2.$$

PERMUTANT.—A very simple instance of a permutant is as follows, viz.: V_{123}, V_{213} , &c. being any quantities whatever, then the permutant $((V_{123}))$ denotes the sum

$$V_{123} + V_{231} + V_{312} - V_{132} - V_{213} - V_{321}$$

and in like manner for any number of permutable suffixes, or if instead of a single set of permutable suffixes we have two or more sets of such suffixes. It will be at once obvious how a permutant includes a determinant, commutant, or Pfaffian; thus, if V_{123} denotes $\alpha_1\beta_2\gamma_3$ and therefore $V_{213} = \alpha_2\beta_1\gamma_3$, &c., then we have a determinant, so if V_{1234} denotes $\alpha_{12}.\alpha_{34}$ where $\alpha_{21} = -\alpha_{12}$, we have a Pfaffian.

INTERMUTANT is a special form of permutant which need not be here further explained.

CUMULANT.—The name has been given to the function which is the numerator or denominator of a continued fraction. Such function may be exhibited (and indeed

naturally presents itself) in the form of a determinant, thus the cumulant $(abcd)$ or numerator of the fractions $a + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$ is

$$\begin{vmatrix} a, & 1, & ., & . \\ -1, & b, & 1, & . \\ ., & -1, & c, & 1 \\ ., & ., & -1, & d \end{vmatrix},$$

and so for a greater number of terms. The developed expression is $abcd + ab + bc + cd + 1$ which is formed from the product $abcd$ by successively omitting each product (cd, bc, ab) , or set of products (cd, ab) of two consecutive letters; in like manner the cumulant $(abcde)$ is $abcde + abc + abc + a + c + e$.

MATRIX.—The term might be used to denote any arrangement of terms, but in a restricted sense it denotes a square or rectangular arrangement of terms, and it is thus employed in the theory of determinants.

To show further how the notion of a matrix is made use of, it may be remarked that a system of linear equations

$$\begin{aligned} \xi &= a x + b y + c z, \\ \eta &= a' x + b' y + c' z, \\ \zeta &= a'' x + b'' y + c'' z \end{aligned}$$

is in the notation of matrices represented by

$$(\xi, \eta, \zeta) = \begin{pmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{pmatrix} (x, y, z).$$

The corresponding set of equations which give (x, y, z) in terms of (ξ, η, ζ) is represented by

$$(x, y, z) = \left(\begin{pmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{pmatrix} \right)^{-1} (\xi, \eta, \zeta),$$

and we have thus the definition of the *inverse* or *reciprocal* matrix: it follows from the theory of determinants that the terms of the reciprocal matrix are the first minor determinants formed out of the original matrix, each of them divided by the determinant formed out of the original matrix; but in writing down the expression some attention is required with respect to the arrangement and signs of the terms.

Similar considerations lead to the notion of *multiplying* or *compounding* together two or more matrices. As an instance of such composition, take

$$\begin{pmatrix} \rho, & \sigma \\ \rho', & \sigma' \end{pmatrix} \begin{pmatrix} \alpha, & \beta \\ \alpha', & \beta' \end{pmatrix} = \begin{pmatrix} \rho\alpha + \sigma\alpha', & \rho\beta + \sigma\beta' \\ \rho'\alpha + \sigma'\alpha', & \rho'\beta + \sigma'\beta' \end{pmatrix}$$

where it is to be observed that the lines of the first or *further* component matrix are compounded with the columns of the second or *nearer* component matrix to form the lines of the compound matrix. The words *further*, *nearer*, are used in reference to a set (x, y) which is, or may be considered to be, understood at the right of each side of the equation. A matrix may be compounded with itself once or oftener, giving rise to a positive power of such matrix; the notion of the negative powers is deducible from that of the inverse or reciprocal matrix, and the same process of generalisation as is employed for powers of a single quantity leads to the notion of the fractional powers of a matrix. As a definition of *addition*, matrices are added together by the addition of their corresponding terms, and as a particular case of the multiplication or composition of matrices we have the multiplication of a matrix by a single quantity, effected by multiplying by such quantity each term of the matrix; all these notions together lead to the notion of *functions* of a matrix.

As an instance of the employment of the notation of matrices for another purpose, take

$$\begin{pmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{pmatrix} (x, y, z) (\xi, \eta, \zeta)$$

used to denote the lineo-linear function

$$\begin{aligned} &(a x + b y + c z) \xi \\ &(a' x + b' y + c' z) \eta \\ &+ (a'' x + b'' y + c'' z) \zeta \end{aligned}$$

which includes

$$\begin{pmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{pmatrix} (x, y, z)^2,$$

used to denote the quadric function

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy.$$

The last preceding notation is an instance of a *symmetrical* matrix: the terms *skew*, *skew symmetrical*, already explained with respect to determinants, apply also to matrices.

RESULTANT.—If there be a system of equations between the same number of unknown quantities (it is assumed that the several equations are of the form $U=0$, where U is a rational and integral homogeneous function), then the function of the coefficients which equated to zero expresses the result of the elimination of the unknown quantities from the several equations, or (what is the same thing) gives the condition for the existence of a set of values satisfying the equations simultaneously—is the *Resultant* of the equations, or of the functions which are thereby put equal to zero. In the case of two (non-homogeneous) equations involving a single unknown quantity,

we may say more briefly that the resultant is the function which equated to zero gives the condition for the existence of a common root. In the particular case of a system of linear equations between as many unknown quantities, the resultant is the determinant formed with the coefficients of the equations.

DISCRIMINANT.—If in a system of equations the functions equated to zero are the derived functions of a single rational and integral homogeneous function with respect to each of the variables thereof, the resultant of the system is said to be the discriminant of the single function. The definition is easily made applicable to the case of a non-homogeneous function, the functions equated to zero are here the function itself and its derived functions with respect to each of the several variables. For a single function, it may be said that the discriminant is the function which equated to zero gives the condition for a pair of equal roots of the equation obtained by putting the function equal to zero.

To fix the precise value of the discriminant of a given function, it is assumed that the coefficient of some one selected term is +1. Thus, the discriminant of $ax^2 + 2bxy + cy^2$ is $ac - b^2$: that of

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 \text{ is } a^2d^2 - 6abcd + 4ac^3 + 4b^3d - 3b^2c^2.$$

In quadratic forms (in the theory of numbers) the expression $b^2 - ac$, which is the determinant $\begin{vmatrix} a, & b \\ b, & c \end{vmatrix}$ with the sign reversed, is called the determinant of the form $ax^2 + 2bxy + cy^2$. And in like manner for ternary quadratic forms, there is the same reversal of sign. It may be said as a convenient definition, that the determinant is the discriminant taken negatively.

PLEXUS.—It frequently happens, in problems of elimination and in other problems, that a given number of relations existing between a system of quantities can only be completely expressed by means of a greater number of equations. Thus, to take a very simple instance, if the unknown quantities x, y , are to be eliminated between the three equations $ax + by = 0, a'x + b'y = 0, a''x + b''y = 0$: this implies two relations between the coefficients a, b, a', b', a'', b'' ; but these relations cannot be completely expressed otherwise than by means of the three equations $ab' - a'b = 0, a'b'' - a''b' = 0, a''b - ab'' = 0$; for taking any two of these equations, e.g. the first and second, these would be satisfied by $a' = 0, b' = 0$, which however do not satisfy the third equation and are not a solution. Such a system of equations, or generally the system of equations required for the complete expression of the relations existing between a set of quantities (and which are *in general* more numerous than the relations themselves) is said to be a *Plexus*.

RATIONAL AND INTEGRAL FUNCTIONS (Notation and Nomenclature of).—A rational and integral homogeneous function, such as the function $ax^2 + 2bxy + cy^2$ is denoted by

$$(*) (x, y)^2$$

where the coefficients are only indicated by the asterisk, but are not expressed. A non-homogeneous rational and integral function is considered as derived from a homo-

geneous function by putting one of the variables thereof equal to unity, and is represented accordingly: thus $ax^2 + 2bx + c$ is denoted by

$$(*) (x, 1)^2.$$

But it is often proper to express the coefficients, and in regard to this the following distinction is made, namely

$$(a, b, c) (x, y)^2$$

denotes $ax^2 + 2bxy + cy^2$; and in like manner $(a, b, c, d) (x, y)^3$ denotes $ax^3 + 3bx^2y + 3cxy^2 + dy^3$, &c., the numerical coefficients being those of the successive powers of a binomial. But when such numerical coefficients are not to be inserted, this is denoted by an arrowhead, or other distinctive mark; thus $(a, b, c) \dagger (x, y)^2$ denotes $ax^2 + bxy + cy^2$. A rational and integral function of any order is termed a *quantic*, and a function of the orders two, three, four, five, &c., is termed a *quadratic*, *cubic*, *quartic*, *quintic*, &c. respectively. The number of variables (the function being homogeneous) is denoted by the words *binary*, *ternary*, &c. As a correlative term to coefficients, the variables have been termed *facients*. A function which is linear in respect to several distinct sets of variables separately is said to be *tantipartite*: or, when there are two sets only, *lineo-linear*. Thus a determinant is a tantipartite function of the lines or of the columns; the function $axx' + bxy' + cx'y + dyy'$ is a lineo-linear function of (x, y) and (x', y') ; a notation for it is

$$\begin{pmatrix} a, & b \\ a', & b' \end{pmatrix} (x, y)(x', y')$$

such as has been spoken of in regard to matrices.

EMANANT.—The development of an expression such as

$$(*) (\lambda x + \mu x', \lambda y + \mu y')^m$$

is naturally written under the form

$$\begin{aligned} & (*) (x, y)^m \qquad \qquad \lambda^m \\ & + \frac{m}{1} (*) (x, y)^{m-2} (x', y') \lambda^{m-1} \mu \\ & \qquad \qquad \qquad \vdots \\ & + (*) (x', y')^m \qquad \qquad \mu^m, \end{aligned}$$

and the coefficients of the successive terms λ^m , $\lambda^{m-1}\mu$, &c. are said to be the emanants of the quantic $(*) (x, y)^m$. The coefficients of λ^m , or 0-th emanant, is the quantic itself, and the coefficient of μ^m , or ultimate emanant, is the quantic with (x', y') in the place of (x, y) ; but the intermediate emanants are functions of (x, y) and (x', y') , homogeneous in respect to the two sets separately. The coefficients may of course be expressed thus, the emanant (first emanant) of $(a, b, c) (x, y)^2$ is $(a, b, c) (x, y) (x', y')$ which stands for $axx' + b(xy' + x'y) + cyy'$.

LINEAR TRANSFORMATIONS.—In this theory the variables of a function are supposed to be respectively linear functions of a new set of variables, so that the function is transformed into a similar function of these new variables, with of course altered values of the coefficients, and the question was to find the relations which existed between the original and new coefficients and the coefficients of the linear equations. The determinant composed of the coefficients of the linear equations is said to be the *modulus of transformation*, and when this determinant is unity the transformation is said to be *unimodular*. It was observed that a certain function of the coefficients, namely, the discriminant, possessed a remarkable property, found afterwards to belong to it as one of a class of functions called originally *hyperdeterminants*, but now *invariants*, and it was in this manner that the problem of linear transformation led to the general theory of covariants.

INVARIANTS.—An invariant is a function of the coefficients of a rational and integral homogeneous function or quantic, the characteristic property whereof is as follows: namely, if a linear transformation is effected on the quantic, then the new value of the invariant is to a factor *près* equal to the original value; the factor in question (or quotient of the two values) being a power of the modulus of transformation, and the two values being thus equal when the transformation is unimodular. The easiest example is afforded by the quadric function $(a, b, c)(x, y)^2$; effecting upon it a linear transformation, suppose that we have identically

$$(a, b, c)(\alpha x' + \beta y', \gamma x' + \delta y')^2 = (a', b', c')(x', y')^2$$

then it may be easily verified that $a'c' - b'^2 = (\alpha\delta - \beta\gamma)^2(ac - b^2)$. The invariant $ac - b^2$ is however in this case nothing else than the discriminant; as another example take the quartic $(a, b, c, d, e)(x, y)^4$, for which $ae - 4bd + 3c^2$, $ace - ad^2 - b^2d + 2bcd - c^3$ are functions possessed of the like property of remaining to a factor *près* unaltered by the transformation, and are consequently invariants; it may be added that calling them I, J , respectively, the discriminant is here $= I^2 - 27J^2$, a rational and integral function of invariants of a lower degree.

COVARIANT.—Instead of a function of the coefficients only, we may have a function of the coefficients and variables, possessed of the like property of remaining unaltered to a factor *près* by the linear transformation: such function is termed a covariant. Thus, a covariant (the Hessian) of the quartic $(a, b, c, d, e)(x, y)^4$ is

$$(ac - b^2, 2ad - 2bc, ae + 2bd - 3c^2, 2be - 2cd, ce - d^2)(x, y)^4.$$

The quantic itself is one of its own covariants. The term covariant may be used in contradistinction to, or as including, invariant. The terms invariant, covariant, have been explained in reference to the simple case of a single quantic containing but one set of variables, but they apply equally to the case of a system of quantics, and to quantics which are homogeneous functions of two or more distinct sets of variables. There is one case which it is proper to mention; if in conjunction with a quantic $(*) (x, y, z, \dots)^m$ we consider a linear function $\xi x + \eta y + \zeta z + \dots$, the invariants of the system are functions of the coefficients of the quantic, and of the coefficients ξ, η, ζ, \dots of the linear function; and treating these as facients, the invariant is said to be a *contravariant* of the given quantic.

The foregoing definition gives the characteristic property of a covariant, but it does not directly show how the covariants of a given quantic are to be investigated. This is supplied as follows:—For any quantic with arbitrary coefficients, for example $(a, b, c, d)(x, y)^4$, there exist operators involving differentiations in respect to the coefficients, tantamount to the operators xd_y and yd_x in respect to the variables; thus the operator $ad_b + 2bd_c + 3cd_a$ is tantamount to yd_x , and $3bd_a + 2bd_c + cd_a$ is tantamount to xd_y . Or what is the same thing, denoting for shortness these operators by $\{yd_x\}$, $\{xd_y\}$ respectively, the quantic is reduced to zero by each of the operators $\{yd_x\} - yd_x$, $\{xd_y\} - xd_y$. Any function of the coefficients and the variables which, in like manner with the quantic itself, is reduced to zero by these two operators respectively, is said to be a *covariant*; or, if it contains the coefficients only, an *invariant* of the quantic. That the two definitions lead to the same result is of course a theorem to be proved.

The leading coefficient of a covariant, say the coefficient of x^m in any covariant of a binary quantic $(*)(x, y)^m$, possesses the property of being reduced to zero by the operator $\{yd_x\}$, and has been termed a *peninvariant* but a more appropriate term is *seminvariant*. An invariant is a function of a given degree in the coefficients, and a covariant is a function of a given degree in the coefficients and order in the variables, and they may be and are designated accordingly; thus, the above-mentioned invariants I, J of a binary quartic are called respectively the *quadrinvariant* and the *cubinvariant*, and the covariant of the same quartic is termed the *quadrucovariant*, or if the distinction were required it would be termed the *quadrucovariant quartic*. In these cases the designations are sufficient, but it is to be noticed that in general there is more than one invariant or covariant of the same degree or of the same degree and order, and that any such designation is only a generic, not a specific, name. An invariant or covariant may also be designated by a name referring to the mode of generation—for example, the discriminant. The name *catalecticant* denotes a certain invariant of a binary quantic of an even order: namely, it is a determinant, which, for the above-mentioned function, is

$$\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}$$

(being in this particular case the cubinvariant), and the name *canonisant* denotes a certain covariant of a binary quantic of an odd order, namely, it is a determinant the terms whereof are linear functions of the coefficients, and which for the cubic $(a, b, c, d)(x, y)^3$ is

$$\begin{vmatrix} ax + by, & bx + cy \\ bx + cy, & cx + dy \end{vmatrix}$$

(being for the particular case the Hessian or quadrucovariant).

CANONICAL FORMS.—A binary quantic of an odd order $2m + 1$ admits of being expressed as a sum of $(m + 1)$ powers of linear functions, for example, the cubic $(a, b, c, d)(x, y)^3$ can be expressed in the form $(lx + my)^3 + (l'x + m'y)^3$ —this is the

canonical form of a binary function of an odd order. And there is in like manner a form (not admitting, however, of a simple definition) which is taken as the *canonical form* of a binary quantic of an even order. The catalecticant and the canonisant present themselves in the problem of the reduction of a binary quantic to the canonical form.

BEZOUTIC MATRIX.—If V, W are any two binary quantics of the same order m , and V', W' are what V, W become when the variables (x, y) of the two quantics are changed into (x', y') ; then $(VW' - V'W) \div (xy' - x'y)$ is a rational and integral homogeneous function of the degree $m-1$ in each of the two sets $(x, y)(x', y')$, and the coefficients taken in their natural square arrangement constitute the Bezoutic matrix. The determinant formed out of this matrix is in fact the resultant of the two functions, or equated to zero it is the equation obtained by the elimination of the variables from the two equations $V=0, W=0$. If V, W are the derived functions of one and the same binary quantic of the order m , then the corresponding matrix, being of course of the order $m-2$, is the Bezoutoidal matrix, and the determinant is then the discriminant of the single quantic.

It would be too long to explain the allied terms Bezoutiant, Cobezoutiants, Bezoutoid, Cobezoutoids.

TACINVARIANT, RECIPROCAN.—A definition in the language of analytical geometry will be the most easily intelligible, and it can readily be converted into an analytical form and made applicable to any number of variables. The function of the coefficients which equated to zero expresses that the two curves $U=0, V=0$, touch each other, is an invariant, namely, it is the *tacinvvariant* of the two functions U, V . And in particular, if, instead of the curve $V=0$, we have the line $\xi x + \eta y + \lambda z = 0$, then the function which equated to zero expresses that this line touches the curve $U=0$, is a contravariant, namely, it is the *reciprocan* of the function U .

FUNCTIONAL DETERMINANT, JACOBIAN, HESSIAN.—If V, W be quantics, then the determinant—

$$\begin{vmatrix} d_x V, & d_y V, & \dots \\ d_x W, & d_y W, & \\ \vdots & & \end{vmatrix}$$

is the functional determinant, or Jacobian, of the quantics V, W, \dots . And if V, W, \dots are the derived functions of $d_x U, d_y U, \dots$ of one and the same quantic U , then the determinant in question is the *Hessian* of the single quantic: the Hessian is in fact to the Jacobian what the discriminant is to the resultant.

CONCOMITANT, COGREDIENT, CONTRAGREDIENT.—The theory of linear transformations has been considered from a different point of view; instead of the variables of a function being *put equal* to the linear functions of a new set of variables, they are considered as being *replaced* by a new set of variables, linear functions of the original variables. Two sets of variables may be so related that when the first set is thus replaced by a set of linear functions of themselves, the second set is also replaced by

a set of linear functions of themselves, the coefficients of the two sets of linear functions being related together in a definite manner; this is *concomitance*, or rather it is (what is alone here spoken of) *simple concomitance*. The two most important kinds of concomitance are, 1st. *Congrediency*, that is, when the substitution on the second set of variables is identical with that upon the first set; 2nd. *Contragrediency*, that is, when the substitution on the second set of variables is the inverse or reciprocal one to that on the first set; it will make the notion of contragrediency clearer to remark that if the variables x, y, \dots and ξ, η, \dots are contragredients, then $x', y' \dots$ (which are linear functions of ξ, η, \dots) are so related that $\xi'x' + \eta'y' + \dots$ is $= \xi x + \eta y + \dots$. It was from the consideration of contragredient variables that the notion of a contravariant was first derived, but as above remarked, the notion is really included in that of a covariant.

COMBINANT.—A combinant is a covariant (or invariant) of a set of quantics of the same order, which, besides being a covariant in the ordinary sense of the word, is, so to speak, a covariant *quoad* the system, that is, it remains a factor *près* unaltered, when the quantics of the system are replaced by linear functions of themselves; the factor in question being a power of the determinant formed with the coefficients of the linear functions. For instance, if $U = (a, b, c)(x, y)^2$ and $U' = (a', b', c')(x, y)^2$, then $ac' - 2bb' + ca'$ is a function which, when for the original coefficients are substituted those of $\lambda U + \mu U', \nu U + \rho U'$, is merely changed into $(\lambda\rho - \mu\nu)^2(ac' - 2bb' + ca')$, and it is therefore a combinant. It would appear that the notion of a combinant might be extended to the case of a system of quantics not of the same order, and that the resultant of the system of quantics could be brought under the extended definition of a combinant, but this is a point which has not been considered.

The principal text-books on the foregoing subjects, are—on determinants:—Spottiswoode's *Elementary Theorems relating to Determinants*, 4to. London, 1851; Brioschi, *Teorica dei Determinanti*, 4to. Pavia, 1854, translated into French by Combescure and into German by Schellbach; Baltzer, *Ueber die Determinanten*, 8vo. Leipzig, 1857 (especially valuable for its references to the original sources). On elimination: Faà de Bruno, *Théorie générale de l'élimination*, 8vo. Paris, 1859. And extending to nearly all the subjects: Salmon, *Lessons introductory to the modern higher Algebra*, 8vo. Dublin, 1858. The memoirs on the different subjects are very numerous, and it was not thought expedient to give a list of them.