

XIX.

RESEARCHES RESPECTING VIBRATION CONNECTED
WITH THE THEORY OF LIGHT

[1839.]

[Note Book 52.]

[The dynamical system consists of a number $(n+2)$ of particles $(P_0, P_1, \dots, P_{n+1})$ each of unit mass and in equilibrium, spaced at unit distances along a straight line. The end particles are fixed and each particle is attracted by a force (a^2) by the one immediately before and immediately after. The system executes small transverse vibrations and these are studied in five Problems. Each Problem is worked out in great detail with examples and Hamilton is led to various results, some of which must have been independent discoveries such as sequence equations and asymptotic values of Bessel Functions and others were many years ahead of their time such as the Reciprocal Theorem in Dynamics and the distinction between Phase-velocity and other types of velocity. The idea of a "fluctuating" function is first mentioned also here.

Problem I (pp. 451-463). P_0, P_{n+1} fixed, P_1, P_2, \dots, P_n having any assigned initial displacements and velocities.

Problem II (pp. 463-487). All initial displacements and velocities zero except for $P_j, P_{j+1}, \dots, P_{j-1}, P_j$ and their displacements and velocities to correspond to the i th mode of vibration.

Problem III (pp. 487-503). The initial displacements and velocities of a number of particles to correspond to those of a progressive sinusoidal wave.

Problem IV (pp. 503-510). Discussion of previous case for large values of t .

Problem V (pp. 511-526). A single particle is constrained to move in an assigned manner.]

Problem I.

1. A finite number $(n+2)$ of equal particles $(P_0, P_1, \dots, P_l, \dots, P_n, P_{n+1})$ being supposed to be arranged in one plane, and nearly in one straight line, at finite and very nearly equal intervals (each = 1); the two extreme particles $(P_0$ and $P_{n+1})$ being also supposed to be fixed and each of the (n) intermediate particles (as P_l) to be acted on only by the attractions (each = a^2) of the two $(P_{l-1}$ & $P_{l+1})$ which immediately precede and follow it in the series; it is required to determine the laws of the transversal vibrations of the system: that is, to express the transversal displacement $(y_{l,t})$, at any time (t) , of any intermediate and moveable particle (P_l) from the right line or axis (of x) connecting the two extreme and fixed particles $(P_0$ and $P_{n+1})$, for any given but arbitrary set of (n) small initial transversal displacements $(y_{l,0})$, and of (n) small initial transversal velocities $(y'_{l,0})$.

2. (*Solution.*) This problem is equivalent to that of integrating generally a system of n simultaneous differential equations, of the second order, and of the form

$$y''_{l,t} = a^2 (y_{l+1,t} - 2y_{l,t} + y_{l-1,t}), \tag{1}$$

or

$$\left(D_t^2 - \frac{a^2 \Delta_l^2}{1 + \Delta_l} \right) y_{l,t} = 0; \tag{2}$$

the integer l taking in succession all values from 1 to n ; and $y_{0,t}, y_{n+1,t}$ being supposed to be each equal to zero. It is easy to effect this integration by the known methods. We have only to assume

$$L_{l,k} = \sin \frac{kl\pi}{n+1},$$

$$r_k = 2a \sin \frac{\frac{1}{2}k\pi}{n+1},$$

$$Y_{k,t} = L_{1,k} y_{1,t} + \dots + L_{l,k} y_{l,t} + \dots + L_{n,k} y_{n,t},$$

k being an integer which takes in succession all values from 1 to n ; and to observe that these assumptions give

$$-r_k^2 L_{l,k} = a^2 (L_{l+1,k} - 2L_{l,k} + L_{l-1,k}),$$

$$L_{0,k} = 0, \quad L_{n+1,k} = 0,$$

$$y_{l,t} = \frac{2}{n+1} (L_{l,1} Y_{1,t} + \dots + L_{l,k} Y_{k,t} + \dots + L_{l,n} Y_{n,t}).$$

For thus we easily transform the differential system (1) into another, which may be thus denoted,

$$Y''_{k,t} + r_k^2 Y_{k,t} = 0, \tag{3}$$

and which gives, by integration,

$$Y_{k,t} = Y_{k,0} \cos tr_k + Y'_{k,0} r_k^{-1} \sin tr_k; \tag{4}$$

so that the sought expression for $y_{l,t}$ may be thus written

$$y_{l,t} = \frac{2}{n+1} \left\{ L_{l,1} Y_{1,0} \cos tr_1 + \dots + L_{l,k} Y_{k,0} \cos tr_k + \dots + L_{l,n} Y_{n,0} \cos tr_n \right. \\ \left. + L_{l,1} Y'_{1,0} \frac{\sin tr_1}{r_1} + \dots + L_{l,k} Y'_{k,0} \frac{\sin tr_k}{r_k} + \dots + L_{l,n} Y'_{n,0} \frac{\sin tr_n}{r_n} \right\}, \tag{5}$$

or, more concisely,

$$y_{l,t} = \frac{2}{n+1} \sum_{(k)1}^n L_{l,k} (Y_{k,0} \cos tr_k + Y'_{k,0} r_k^{-1} \sin tr_k); \tag{6}$$

in which we are to remember that

$$Y_{k,0} = \sum_{(l)1}^n L_{l,k} y_{l,0}, \quad Y'_{k,0} = \sum_{(l)1}^n L_{l,k} y'_{l,0}.$$

3. (*Corollary 1.*) If there be but one particle, P_j , displaced at the time 0, and if no particle have at that time any velocity, we may write $Y_{k,0} = y_{j,0} L_{j,k}$, $Y'_{k,0} = 0$, and the expression for the displacement $y_{l,t}$ of any particle P_l at the time t becomes

$$y_{l,t} = \frac{2y_{j,0}}{n+1} \sum_{(k)1}^n L_{j,k} L_{l,k} \cos tr_k \\ = \frac{2y_{j,0}}{n+1} \sum_{(k)1}^n \sin \frac{jk\pi}{n+1} \sin \frac{lk\pi}{n+1} \cos \left(2at \sin \frac{\frac{1}{2}k\pi}{n+1} \right).$$

4. (*Corollary 2.*) In like manner, if only one particle, P_j , have an initial velocity, $y'_{j,0}$, and if no particle have any initial displacement, we may write

$$Y_{k,0} = 0, \quad Y'_{k,0} = y'_{j,0} L_{j,k},$$

and

$$\begin{aligned} y_{i,t} &= \frac{2y'_{j,0}}{n+1} \sum_{(k)1}^n L_{j,k} L_{i,k} \int_0^t dt \cos tr_k \\ &= \frac{2y'_{j,0}}{n+1} \sum_{(k)1}^n \sin \frac{jk\pi}{n+1} \sin \frac{lk\pi}{n+1} \int_0^t dt \cos \left(2at \sin \frac{\frac{1}{2}k\pi}{n+1} \right). \end{aligned}$$

5. (*Corollary 3.*) The general solution (6) may therefore be put under the form

$$y_{i,t} = \frac{2}{n+1} \sum_{(j)1}^n \left(y_{j,0} + y'_{j,0} \int_0^t dt \right) \sum_{(k)1}^n \sin \frac{jk\pi}{n+1} \sin \frac{lk\pi}{n+1} \cos \left(2at \sin \frac{\frac{1}{2}k\pi}{n+1} \right). \quad (7)$$

6. (*Corollary 4.*) If the initial displacements and velocities be of the forms

$$y_{i,0} = \eta_i \sin \frac{i\pi}{n+1}, \quad y'_{i,0} = \eta'_i \sin \frac{i\pi}{n+1},$$

i being any integer from 1 to n and η_i, η'_i being constants, we shall have $Y_{i,0} = \frac{n+1}{2} \eta_i$, $Y'_{i,0} = \frac{n+1}{2} \eta'_i$, and all the other values of $Y_{k,0}$ and $Y'_{k,0}$ will vanish; therefore, in this case, the general expression (6) reduces itself to the following:

$$y_{i,t} = \sin \frac{i\pi}{n+1} \left(\eta_i + \eta'_i \int_0^t dt \right) \cos \left(2at \sin \frac{\frac{1}{2}i\pi}{n+1} \right).$$

7. (*Corollary 5.*) By taking

$$\eta_i = \frac{2}{n+1} \sum_{(j)1}^n y_{j,0} \sin \frac{j\pi}{n+1}, \quad \eta'_i = \frac{2}{n+1} \sum_{(j)1}^n y'_{j,0} \sin \frac{j\pi}{n+1},$$

we may express any arbitrary initial displacements $y_{i,0}$ and velocities $y'_{i,0}$ by developements of the forms

$$y_{i,0} = \sum_{(j)1}^n \eta_j \sin \frac{j\pi}{n+1}, \quad y'_{i,0} = \sum_{(j)1}^n \eta'_j \sin \frac{j\pi}{n+1};$$

if then we had found otherwise the expression given in the last corollary for $y_{i,t}$, corresponding to the particular suppositions

$$y_{i,0} = \eta_i \sin \frac{i\pi}{n+1}, \quad y'_{i,0} = \eta'_i \sin \frac{i\pi}{n+1},$$

we might have thence deduced the general expression (6) under the form

$$y_{i,t} = \sum_{(j)1}^n \sin \frac{j\pi}{n+1} \left(\eta_j + \eta'_j \int_0^t dt \right) \cos \left(2at \sin \frac{\frac{1}{2}j\pi}{n+1} \right). \quad (8)$$

8. (*Corollary 6.*) If we write, according to a notation already employed,

$$r_i = 2a \sin \frac{\frac{1}{2}i\pi}{n+1}$$

and introduce two new constants, B_i and β_i , such that

$$B_i \cos \beta_i = \eta_i, \quad B_i \sin \beta_i = \eta'_i r_i^{-1},$$

we may employ this other expression

$$y_{l,t} = \sum_{(i)1}^n B_i \cos(tr_i - \beta_i) \sin \frac{il\pi}{n+1}; \quad (9)$$

in which r_i is a known function of the index (or integer) i , but B_i and β_i are in general arbitrary functions of that index.

9. (*Corollary 7.*) The general system of total displacements $y_{l,t}$ may be considered as the sum of n component systems of partial displacements,

$$y_{l,t} = B_i \cos(tr_i - \beta_i) \sin \frac{il\pi}{n+1},$$

of which each is separately possible, & of which all are mutually superposed. Each system of displacements, by itself, may be called a *simple movement* or *mode of simple vibration*. It corresponds to some one integer value of i (from 1 to n inclusive), and to one corresponding *periodic time*

$$\frac{2\pi}{r_i} = \frac{\pi}{a} \operatorname{cosec} \frac{\frac{1}{2}i\pi}{n+1};$$

involving also two arbitrary constants, or arbitrary functions of i , namely η_i and η'_i , or B_i and β_i , which latter may be called *constants of amplitude and of epoch*.

10. (*Corollary 8.*) In any one such simple movement, corresponding to any one value of i , the displacements all attain extreme values when $t = r_i^{-1}\beta_i$; and these simultaneous and extreme values are all expressed by the formula

$$y_{l,r_i^{-1}\beta_i} = B_i \sin \frac{il\pi}{n+1}.$$

If $i = 1$, these extreme displacements (relatively to t) increase in magnitude with l from $l = 1$ till $l = \frac{n+1}{2}$ if n be odd, or till $l = \frac{n}{2}$ if n be even; and afterwards decrease from $l = \frac{n+1}{2}$ or from $l = \frac{n+2}{2}$ to $l = n$; being all of the same sign as B_1 . But if $i = 2$, the displacements $B_2 \sin \frac{2l\pi}{n+1}$ increase in magnitude with l from $l = 1$ till $l = \frac{n+1}{4}$, or $= \frac{n}{4}$, or $= \frac{n-1}{4}$, or $= \frac{n+2}{4}$, according as n is of the form $4\nu - 1$, or 4ν , or $4\nu + 1$, or $4\nu + 2$, ν being an integer & ≤ 0 ; they afterwards decrease and become negative when l is between $\frac{n+1}{2}$ and $n+1$, if $B_2 > 0$.

In general the formula $B_i \sin \frac{il\pi}{n+1}$ may be considered as corresponding to $i - 1$ nodes N_1, N_2, \dots, N_{i-1} , for which l (though integer for each actual particle) is supposed to receive the (perhaps fractional) values $\frac{(n+1)}{i}, \frac{2(n+1)}{i}, \dots, \frac{(i-1)(n+1)}{i}$. Between P_0 and N_1 the sine of $\frac{il\pi}{n+1}$ is positive; between N_1 and N_2 negative; and so on alternately. The i intermediate points $V_1, V_2, \dots, V_{i-1}, V_i$ for which $l = \frac{n+1}{2i}, \frac{3(n+1)}{2i}, \dots, \frac{(2i-3)(n+1)}{2i}, \frac{(2i-1)(n+1)}{2i}$ are *venters* or points of extreme excursion, alternately positive and negative (if $B_i > 0$).

11. (*Example 1.*) If $n = 1$, so that there is but one moveable particle P_1 , attracted equally to two fixed centres, P_0 and P_2 , and slightly and transversally displaced from the middle of the line ($= 2$) which joins them; then the variable displacement of this particle P_1 at the time t is represented by the formula

$$y_{1,t} = B_1 \cos (tr_1 - \beta_1),$$

because $\sin \frac{\pi}{2} = 1$; in which formula $r_1 = 2a \sin \frac{\pi}{4} = a\sqrt{2}$. The extreme displacement is B_1 and the law is that of the cycloidal pendulum.

$$\eta_1 = y_{1,0} = B_1 \cos \beta_1, \quad \eta'_1 = y'_{1,0} = r_1 B_1 \sin \beta_1;$$

and

$$y_{1,t} = y_{1,0} \cos (at\sqrt{2}) + y'_{1,0} \frac{\sin (at\sqrt{2})}{a\sqrt{2}}.$$

12. (*Example 2.*) If $n = 2$, so that there are two moveable particles P_1 and P_2 between two fixed particles P_0 and P_3 ; then the variable displacements $y_{1,t}$ and $y_{2,t}$ of P_1 and P_2 are

$$y_{1,t} = \frac{\sqrt{3}}{2} \{B_1 \cos (tr_1 - \beta_1) + B_2 \cos (tr_2 - \beta_2)\}, \quad y_{2,t} = \frac{\sqrt{3}}{2} \{B_1 \cos (tr_1 - \beta_1) - B_2 \cos (tr_2 - \beta_2)\},$$

because

$$\frac{\sqrt{3}}{2} = \sin \frac{\pi}{3} = \sin \frac{2\pi}{3} = -\sin \frac{4\pi}{3};$$

also

$$r_1 = 2a \sin \frac{\pi}{6} = a, \quad r_2 = 2a \sin \frac{\pi}{3} = a\sqrt{3}.$$

The two simple modes of vibration, which are here superposed, are

$$\text{1st} \quad y_{1,t} = y_{2,t} = \frac{\sqrt{3}}{2} B_1 \cos (at - \beta_1),$$

and

$$\text{2nd} \quad y_{1,t} = -y_{2,t} = \frac{\sqrt{3}}{2} B_2 \cos (at\sqrt{3} - \beta_2).$$

The periodic time of the first mode is greater than that of the 2nd in the ratio of $\sqrt{3}$ to 1. The displacements of the two particles P_1 and P_2 are equal and on the same side in the 1st mode, but equal and opposite in the 2nd.

$$B_1 \cos \beta_1 = \eta_1 = \frac{1}{\sqrt{3}} (y_{1,0} + y_{2,0}), \quad B_1 \sin \beta_1 = a^{-1} \eta'_1 = \frac{1}{a\sqrt{3}} (y'_{1,0} + y'_{2,0}),$$

$$B_2 \cos \beta_2 = \eta_2 = \frac{1}{\sqrt{3}} (y_{1,0} - y_{2,0}), \quad B_2 \sin \beta_2 = (a\sqrt{3})^{-1} \eta'_2 = \frac{1}{3a} (y'_{1,0} - y'_{2,0}).$$

13. (*Example 3.*) If $n = 3$, then

$$y_{1,t} = \sqrt{\frac{1}{2}} B_1 \cos (tr_1 - \beta_1) + B_2 \cos (tr_2 - \beta_2) + \sqrt{\frac{1}{2}} B_3 \cos (tr_3 - \beta_3),$$

$$y_{2,t} = B_1 \cos (tr_1 - \beta_1) - B_3 \cos (tr_3 - \beta_3),$$

$$y_{3,t} = \sqrt{\frac{1}{2}} B_1 \cos (tr_1 - \beta_1) - B_2 \cos (tr_2 - \beta_2) + \sqrt{\frac{1}{2}} B_3 \cos (tr_3 - \beta_3);$$

$$r_1 = 2a \sin \frac{\pi}{8} = a\sqrt{2 - \sqrt{2}}, \quad r_2 = 2a \sin \frac{2\pi}{8} = a\sqrt{2}, \quad r_3 = 2a \sin \frac{3\pi}{8} = a\sqrt{2 + \sqrt{2}};$$

$$\begin{aligned}
 B_1 \cos \beta_1 &= \frac{1}{2}(\sqrt{\frac{1}{2}}y_{1,0} + y_{2,0} + \sqrt{\frac{1}{2}}y_{3,0}), & B_1 \sin \beta_1 &= \frac{1}{2r_1}(\sqrt{\frac{1}{2}}y'_{1,0} + y'_{2,0} + \sqrt{\frac{1}{2}}y'_{3,0}), \\
 B_2 \cos \beta_2 &= \frac{1}{2}(y_{1,0} - y_{3,0}), & B_2 \sin \beta_2 &= \frac{1}{2r_2}(y'_{1,0} - y'_{3,0}), \\
 B_3 \cos \beta_3 &= \frac{1}{2}(\sqrt{\frac{1}{2}}y_{1,0} - y_{2,0} + \sqrt{\frac{1}{2}}y_{3,0}), & B_3 \sin \beta_3 &= \frac{1}{2r_3}(\sqrt{\frac{1}{2}}y'_{1,0} - y'_{2,0} + \sqrt{\frac{1}{2}}y'_{3,0});
 \end{aligned}$$

there are three simple modes of vibration, with periods which are respectively $\frac{2\pi}{r_1}, \frac{2\pi}{r_2}, \frac{2\pi}{r_3}$, that is $\frac{\pi}{a} \operatorname{cosec} \frac{\pi}{8}, \frac{\pi}{a} \operatorname{cosec} \frac{2\pi}{8}, \frac{\pi}{a} \operatorname{cosec} \frac{3\pi}{8}$, or finally $\frac{\pi}{a} \frac{2}{\sqrt{2}-\sqrt{2}}, \frac{\pi}{a} \sqrt{2}, \frac{\pi}{a} \frac{2}{\sqrt{2}+\sqrt{2}}$; they may also be

thus written $\frac{\pi\sqrt{2}}{a} \sqrt{2+\sqrt{2}}, \frac{\pi\sqrt{2}}{a}, \frac{\pi\sqrt{2}}{a} \sqrt{2-\sqrt{2}}$; in the 1st or slowest mode, the 3 displacements have all the same sign & are proportional to $\sqrt{\frac{1}{2}}, 1, \sqrt{\frac{1}{2}}$, that is, to 1, $\sqrt{2}, 1$, the second particle being a *venter*; in the 2nd mode, the 1st & 3rd displacements are equal and opposite, & the 2nd displacement vanishes, so that the middle particle P_2 remains at rest and forms what is called a *node*, the first and third particles being venters; in the 3rd or quickest mode, the 1st and 3rd displacements are equal and of a common sign, while the 2nd is of an opposite sign and greater in the ratio of $\sqrt{2}$ to 1; so that, in this mode there may be considered to be *two nodes*, one between P_1 and P_2 but nearer to P_1 and the other between P_2 and P_3 but nearer to P_3 ; in fact the abscissae of these two nodes are $\frac{4}{3}$ and $\frac{8}{3}$ respectively, the abscissae of the 3 vibrating particles P_1, P_2, P_3 being 1, 2, 3; and in the same third mode, there are *three venters* of which the first and third have for abscissae $\frac{2}{3}$ and $\frac{10}{3}$, so that they are near P_1 & P_3 , but between P_0 and P_1 and P_3 and P_4 respectively, while the second venter coincides with the particle P_2 .

14. (*Corollary 9.*) If there be but one particle P_j which at the time 0 has any displacement or velocity, we shall have

$$B_i \cos \beta_i = \frac{2y_{j,0}}{n+1} \sin \frac{ij\pi}{n+1}, \quad B_i \sin \beta_i = \frac{2r_i^{-1}y'_{j,0}}{n+1} \sin \frac{ij\pi}{n+1};$$

and therefore

$$B_i = \frac{2}{n+1} \sqrt{y_{j,0}^2 + r_i^{-2}y'_{j,0}^2} \sin \frac{ij\pi}{n+1}, \quad \tan \beta_i = \frac{r_i^{-1}y'_{j,0}}{y_{j,0}}.$$

15. (*Example 4.*) If $n=2$, then

$$B_i \cos \beta_i = \frac{2}{3}y_{j,0} \sin \frac{ij\pi}{3}, \quad B_i \sin \beta_i = \frac{2}{3}r_i^{-1}y'_{j,0} \sin \frac{ij\pi}{3};$$

therefore, more particularly,

$$B_1 \cos \beta_1 = \frac{2}{3}y_{j,0} \sin \frac{j\pi}{3}, \quad B_1 \sin \beta_1 = \frac{2}{3a}y'_{j,0} \sin \frac{j\pi}{3},$$

$$B_2 \cos \beta_2 = \frac{2}{3}y_{j,0} \sin \frac{2j\pi}{3}, \quad B_2 \sin \beta_2 = \frac{2}{3a\sqrt{3}}y'_{j,0} \sin \frac{2j\pi}{3}.$$

Still more particularly, if $j=1$,

$$B_1 \cos \beta_1 = \frac{1}{\sqrt{3}}y_{1,0}, \quad B_1 \sin \beta_1 = \frac{1}{a\sqrt{3}}y'_{1,0},$$

$$B_2 \cos \beta_2 = \frac{1}{\sqrt{3}}y_{1,0}, \quad B_2 \sin \beta_2 = \frac{1}{3a}y'_{1,0},$$

and, if $j=2$,

$$B_1 \cos \beta_1 = \frac{1}{\sqrt{3}} y_{2,0}, \quad B_1 \sin \beta_1 = \frac{1}{a\sqrt{3}} y'_{2,0},$$

$$B_2 \cos \beta_2 = -\frac{1}{\sqrt{3}} y_{2,0}, \quad B_2 \sin \beta_2 = -\frac{1}{3a} y'_{2,0}.$$

In this manner we determine the coefficients of the two coexisting simple modes of vibration in the system $P_0 P_1 P_2 P_3$, corresponding to any initial displacement and velocity of P_1 alone, or of P_2 alone.

16. (Example 5.) In the system $P_0 P_1 P_2 P_3 P_4$, $n=3$ and

$$B_i \cos \beta_i = \frac{1}{2} y_{j,0} \sin \frac{ij\pi}{4}, \quad B_i \sin \beta_i = \frac{1}{2} r_i^{-1} y'_{j,0} \sin \frac{ij\pi}{4};$$

that is,

$$B_1 \cos \beta_1 = \frac{1}{2} y_{j,0} \sin \frac{j\pi}{4}, \quad B_1 \sin \beta_1 = \frac{\sqrt{2+\sqrt{2}}}{2a\sqrt{2}} y'_{j,0} \sin \frac{j\pi}{4},$$

$$B_2 \cos \beta_2 = \frac{1}{2} y_{j,0} \sin \frac{2j\pi}{4}, \quad B_2 \sin \beta_2 = \frac{1}{2a\sqrt{2}} y'_{j,0} \sin \frac{2j\pi}{4},$$

$$B_3 \cos \beta_3 = \frac{1}{2} y_{j,0} \sin \frac{3j\pi}{4}, \quad B_3 \sin \beta_3 = \frac{\sqrt{2-\sqrt{2}}}{2a\sqrt{2}} y'_{j,0} \sin \frac{3j\pi}{4}.$$

17. (Corollary 10.) By last corollary or by corollary 3, article 5, the whole effect at the time t on the particle P_l , of the initial state of P_j , is

$$\left(y_{j,0} + y'_{j,0} \int_0^t dt \right) f(j, l, t),$$

in which

$$f(j, l, t) = \frac{2}{n+1} \sum_{(i)1}^n \sin \frac{ij\pi}{n+1} \sin \frac{il\pi}{n+1} \cos \left(2at \sin \frac{1}{2} i\pi \right).$$

If $\frac{n+1}{2}$ be much larger than j or l , this finite sum is nearly = the definite integral

$$\frac{4}{\pi} \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos (2at \sin \theta).$$

If then we consider the case of a very numerous system of particles, we shall have, *nearly*, for those which are much *nearer to one end than to the middle*,

$$y_{l,t} = \frac{4}{\pi} \sum_{(i)1}^{\infty} \left(y_{j,0} + y'_{j,0} \int_0^t dt \right) \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos (2at \sin \theta); \quad (10)$$

$y_{j,0}$ and $y'_{j,0}$ being supposed = 0 unless $\frac{2j}{n+1}$ be small; and this expression corresponds *rigorously* to the limit $n = \infty$, j and l remaining finite.

18. (Corollary 11.) If nothing be neglected, we have

$$4 \sin \alpha \sin \beta \cos \gamma = 2 (\cos \alpha - \beta - \cos \alpha + \beta) \cos \gamma$$

$$= \cos (\alpha - \beta + \gamma) + \cos (\alpha - \beta - \gamma) - \cos (\alpha + \beta - \gamma) - \cos (\alpha + \beta + \gamma);$$

therefore*

$$f(j, l, t) = \frac{1}{2(n+1)} \sum_{(i)1}^n \left\{ \cos \left(2at \sin \frac{\frac{1}{2}i\pi}{n+1} + \frac{i(l-j)\pi}{n+1} \right) \right. \\ + \cos \left(2at \sin \frac{\frac{1}{2}i\pi}{n+1} - \frac{i(l-j)\pi}{n+1} \right) \\ - \cos \left(2at \sin \frac{\frac{1}{2}i\pi}{n+1} + \frac{i(l+j)\pi}{n+1} \right) \\ \left. - \cos \left(2at \sin \frac{\frac{1}{2}i\pi}{n+1} - \frac{i(l+j)\pi}{n+1} \right) \right\};$$

and therefore

$$y_{l,t} = \frac{1}{2(n+1)} \sum_{(j)1}^n \left(y_{j,0} + y'_{j,0} \int_0^t dt \right) \sum_{(i)1}^n \left\{ \cos \left(2at \sin \frac{\frac{1}{2}i\pi}{n+1} + \frac{i(l-j)\pi}{n+1} \right) \right. \\ + \cos \left(2at \sin \frac{\frac{1}{2}i\pi}{n+1} - \frac{i(l-j)\pi}{n+1} \right) - \cos \left(2at \sin \frac{\frac{1}{2}i\pi}{n+1} + \frac{i(l+j)\pi}{n+1} \right) \\ \left. - \cos \left(2at \sin \frac{\frac{1}{2}i\pi}{n+1} - \frac{i(l+j)\pi}{n+1} \right) \right\}. \quad (11)$$

If, now, j, l and n all tend to ∞ but so that $\frac{l+j}{n+1}$ is nearly = 1, and that $\frac{l-j}{n+1}$ is nearly = 0, or in other words so that $\frac{2l}{n+1}$ and $\frac{2j}{n+1}$ are each nearly = 1, though $2l-n-1$ and $2j-n-1$ may both be large numbers positive or negative; in short, if we consider only particles P_j and P_l which are much *nearer to the middle than to the ends* of the very long line P_0P_{n+1} , although they are not necessarily near to one another; we may then neglect those sums of rapidly fluctuating cosines which involve $\frac{i(l+j)\pi}{n+1}$ † and may transform the other sums into definite integrals by making $\frac{\frac{1}{2}i\pi}{n+1} = \theta$; and thus we obtain, as a very approximate formula,

$$y_{l,t} = \frac{2}{\pi} \sum_{(h)-\infty}^{\infty} \left(y_{l+h,0} + y'_{l+h,0} \int_0^t dt \right) \int_0^{\frac{\pi}{2}} d\theta \cos 2h\theta \cos (2at \sin \theta). \quad (12)$$

Accordingly this expression gives

$$y_{l\pm 1,t} = \frac{2}{\pi} \sum_{(h)-\infty}^{\infty} \left(y_{l+h,0} + y'_{l+h,0} \int_0^t dt \right) \int_0^{\frac{\pi}{2}} d\theta \cos (2h\theta \mp 2\theta) \cos (2at \sin \theta),$$

* $[f(j, l, t) = \{J_{2(l-j)}(2at) - J_{2(l+j)}(2at)\}]$.

The following note appears on the opposite page of the manuscript. "It is remarkable that this function $f(j, l, t)$ is symmetric relatively to j and l , even if n be not large. Indeed each part, corresponding to any one value of i , or to any one mode of simple vibration, is symmetric also. Thus, the effect (and even that part of the effect which corresponds to any given number i of *venters*) of the initial state of P_j or the state of P_l at the time t , is the same as the effect of a like initial state of P_l on the state of P_j at the time t ; even though P_j may be near one extremity and P_l near the middle of the system. It will be important to try whether a similar result holds good for other attracting or repelling systems."

† [This can be inferred from the value given in the previous note for $f(j, l, t)$. Hamilton's paper on Fluctuating Functions did not appear until 1843, *Trans. R.I.A.* XIX, pp. 264-321, although there is a short note in *Proceedings R.I.A.* I (1841), pp. 475-477.]

therefore

$$y_{l+1,t} + y_{l-1,t} = \frac{4}{\pi} \sum_{(h)-\infty}^{\infty} \left(y_{l+h,0} + y'_{l+h,0} \int_0^t dt \right) \int_0^{\frac{\pi}{2}} d\theta \cos 2h\theta \cos 2\theta \cos (2at \sin \theta),$$

and

$$y_{l+1,t} - 2y_{l,t} + y_{l-1,t} = -\frac{8}{\pi} \sum_{(h)-\infty}^{\infty} \left(y_{l+h,0} + y'_{l+h,0} \int_0^t dt \right) \int_0^{\frac{\pi}{2}} d\theta \cos 2h\theta \sin^2 \theta \cos (2at \sin \theta);$$

the function (12) therefore satisfies rigorously and indefinitely the equation in mixed differences (1), and is the complete integral of that indefinite equation because it reproduces the arbitrary initial data $y_{l,0}$ and $y'_{l,0}$, as the values of $y_{l,t}$ and $y'_{l,t}$, for $t=0$.

19. (*Remark.*) Thus the expression (10) corresponds rigorously to the transversal vibrations of an indefinite line of equal particles extending in one direction from the fixed point P_0 ; or if in both directions, then so that $y_{-l,t} = -y_{l,t}$; and the expression (12) corresponds rigorously to the transversal vibrations of an indefinite line of particles extending in two opposite directions, & having no point fixed.

As applied to the theory of light, the expression (12) seems adapted to illustrate the *internal propagation* of luminiferous vibration, and the expression (10) to illustrate the *reflexion* of such vibration. And this expression (10) may be thus written

$$y_{l,t} = \frac{2}{\pi} \sum_{(j)-\infty}^{\infty} \left(y_{j,0} + y'_{j,0} \int_0^t dt \right) \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos (2at \sin \theta), \tag{13}$$

if we consider $y_{-j,0}$ and $y'_{-j,0}$ as equal to $-y_{j,0}$ and $-y'_{j,0}$.

20. When n is finite, if we put for abridgement

$$\phi = \frac{\pi}{2(n+1)}, \quad \text{and therefore} \quad r_i = 2a \sin i\phi,$$

we have, for any simple vibration, the formula

$$y_{l,t} = B_i \cos (2at \sin i\phi - \beta_i) \sin 2il\phi,$$

which may be put under the form

$$y_{l,t} = \frac{1}{2} B_i \sin (2il\phi - 2at \sin i\phi + \beta_i) + \frac{1}{2} B_i \sin (2il\phi + 2at \sin i\phi - \beta_i).$$

It may therefore be considered as the sum (or resultant) of two *conjugate* simple movements, of which the *phases* are respectively $2il\phi - 2at \sin i\phi + \beta_i$ and $2il\phi + 2at \sin i\phi - \beta_i$; the *amplitudes* are each $= \frac{1}{2} B_i$; and the *velocities of transmission of phase* (from particle to particle) are respectively $\frac{a \sin i\phi}{i\phi}$ and $-\frac{a \sin i\phi}{i\phi}$; that is, they are equal in amount but opposite in direction. The positive velocity is $< a$ and $> \frac{2a}{\pi}$, because $i\phi > 0$ but $< \frac{\pi}{2}$. The *epochs* β_i and $-\beta_i$ are, in like manner, equal and opposite.

21. Each of the two conjugate simple movements, described in the last article, satisfies the indefinite equation in mixed differences (1) whatever i and ϕ may be; but the advantage of combining them, & of supposing $\phi = \frac{\pi}{2(n+1)}$, is that we thereby satisfy also the conditions

$y_{0,t}=0$, $y_{n+1,t}=0$, for all integer values of i . If we omit the last condition ($y_{n+1,t}=0$) we may take any values for i and ϕ ; but we must still combine the two conjugate formulae. If we omit both of the extreme conditions, we may use either formula alone, and may assign any value to i and ϕ (between $\phi=0$ and $\phi=\frac{\pi}{2}$).

22. In this manner then we might perceive that at the limit considered in article 17, which corresponds to the integration of the original equation (1), subject only to the one condition $y_{0,t}=0$, we may write

$$y_{i,t} = \int_{\theta_1}^{\theta_2} d\theta B_\theta \cos(2at \sin \theta - \beta_\theta) \sin 2l\theta, \quad (14)$$

the limits θ_1 and θ_2 being arbitrary quantities and B_θ and β_θ being arbitrary functions of θ . But in order to reproduce in this case the initial values of $y_{i,t}$ and $y'_{i,t}$ we must (if possible) determine these arbitraries so as to have

$$y_{i,0} = \int_{\theta_1}^{\theta_2} d\theta B_\theta \cos \beta_\theta \sin 2l\theta, \quad y'_{i,0} = \int_{\theta_1}^{\theta_2} 2a d\theta B_\theta \sin \beta_\theta \sin \theta \sin 2l\theta;$$

and these conditions accordingly are satisfied, as in the formula (13), by supposing

$$\theta_1 = 0, \quad \theta_2 = \frac{\pi}{2}, \quad B_\theta \cos \beta_\theta = \frac{2}{\pi} \sum_{(j)-\infty}^{\infty} y_{j,0} \sin 2j\theta,$$

$$y_{-j,0} = -y_{j,0}, \quad y'_{-j,0} = -y'_{j,0}, \quad B_\theta \sin \beta_\theta = \frac{1}{a\pi} \sum_{(j)-\infty}^{\infty} y'_{j,0} \sin 2j\theta \operatorname{cosec} \theta.$$

23. We might also, in like manner, have perceived, that at the other limit considered in article 18, corresponding merely to the indefinite integration of the equation (1), we may write

$$y_{i,t} = \int_{\theta_1}^{\theta_2} d\theta B_\theta \sin(2l\theta - 2at \sin \theta + \beta_\theta) + \int_{\iota_1}^{\iota_2} d\iota C_\iota \sin(2l\iota + 2at \sin \iota + \gamma_\iota), \quad (15)$$

the limits θ_1 , θ_2 and ι_1 , ι_2 being arbitrary quantities, while B_θ , β_θ are arbitrary functions of θ , and C_ι , γ_ι are arbitrary functions of ι . To reproduce the initial values we must endeavour to determine these arbitrary quantities and functions, so as to have

$$y_{i,0} = \int_{\theta_1}^{\theta_2} d\theta B_\theta \sin(2l\theta + \beta_\theta) + \int_{\iota_1}^{\iota_2} d\iota C_\iota \sin(2l\iota + \gamma_\iota),$$

$$y'_{i,0} = -2a \int_{\theta_1}^{\theta_2} d\theta B_\theta \cos(2l\theta + \beta_\theta) \sin \theta + 2a \int_{\iota_1}^{\iota_2} d\iota C_\iota \cos(2l\iota + \gamma_\iota) \sin \iota;$$

conditions which may be satisfied, as in the formula (12), by supposing

$$B_\theta \cos \beta_\theta = \frac{1}{\pi} \sum_{(h)-\infty}^{\infty} \left(y_{h,0} \sin 2h\theta - \frac{y'_{h,0} \cos 2h\theta}{2a \sin \theta} \right),$$

$$B_\theta \sin \beta_\theta = \frac{1}{\pi} \sum_{(h)-\infty}^{\infty} \left(y_{h,0} \cos 2h\theta + \frac{y'_{h,0} \sin 2h\theta}{2a \sin \theta} \right),$$

$$C_\iota \cos \gamma_\iota = \frac{1}{\pi} \sum_{(h)-\infty}^{\infty} \left(y_{h,0} \sin 2h\iota + \frac{y'_{h,0} \cos 2h\iota}{2a \sin \iota} \right),$$

$$C_\iota \sin \gamma_\iota = \frac{1}{\pi} \sum_{(h)-\infty}^{\infty} \left(y_{h,0} \cos 2h\iota - \frac{y'_{h,0} \sin 2h\iota}{2a \sin \iota} \right),$$

and $\theta_1 = 0, \theta_2 = \frac{\pi}{2}, \iota_1 = 0, \iota_2 = \frac{\pi}{2}$. In fact these last suppositions give

$$B_\theta \sin(2l\theta + \beta_\theta) = \frac{1}{\pi} \sum_{(h)-\infty}^{\infty} \left(y_{h,0} \cos(2h\theta - 2l\theta) + \frac{y'_{h,0}}{2a \sin \theta} \sin(2h\theta - 2l\theta) \right),$$

$$C_\iota \sin(2l\iota + \gamma_\iota) = \frac{1}{\pi} \sum_{(h)-\infty}^{\infty} \left(y_{h,0} \cos(2h\iota - 2l\iota) - \frac{y'_{h,0}}{2a \sin \iota} \sin(2h\iota - 2l\iota) \right);$$

and therefore

$$B_\theta \sin(2l\theta + \beta_\theta) + C_\theta \sin(2l\theta + \gamma_\theta) = \frac{2}{\pi} \sum_{(h)-\infty}^{\infty} y_{h,0} \cos(2h\theta - 2l\theta),$$

so that, performing on this the operation $\int_0^{\frac{\pi}{2}} d\theta$ we get $y_{l,0}$; also

$$B_\theta \cos(2l\theta + \beta_\theta) = \frac{1}{\pi} \sum_{(h)-\infty}^{\infty} \left(y_{h,0} \sin(2h\theta - 2l\theta) - \frac{y'_{h,0}}{2a \sin \theta} \cos(2h\theta - 2l\theta) \right),$$

$$C_\iota \cos(2l\iota + \gamma_\iota) = \frac{1}{\pi} \sum_{(h)-\infty}^{\infty} \left(y_{h,0} \sin(2h\iota - 2l\iota) + \frac{y'_{h,0}}{2a \sin \iota} \cos(2h\iota - 2l\iota) \right),$$

and therefore

$$2a \sin \theta \{ -B_\theta \cos(2l\theta + \beta_\theta) + C_\theta \cos(2l\theta + \gamma_\theta) \} = \frac{2}{\pi} \sum_{(h)-\infty}^{\infty} y'_{h,0} \cos(2h\theta - 2l\theta),$$

so that the operation $\int_0^{\frac{\pi}{2}} d\theta$, performed on this, reduces it to $y'_{l,0}$; the initial values are therefore reproduced. At the same time, the expression (15) becomes

$$\begin{aligned} y_{l,t} &= \int_0^{\frac{\pi}{2}} d\theta [\cos(2at \sin \theta) \{ B_\theta \sin(2l\theta + \beta_\theta) + C_\theta \sin(2l\theta + \gamma_\theta) \} \\ &\quad + \sin(2at \sin \theta) \{ -B_\theta \cos(2l\theta + \beta_\theta) + C_\theta \cos(2l\theta + \gamma_\theta) \}] \\ &= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} d\theta \sum_{(h)-\infty}^{\infty} \cos(2h\theta - 2l\theta) \left(y_{h,0} \cos(2at \sin \theta) + y'_{h,0} \frac{\sin(2at \sin \theta)}{2a \sin \theta} \right) \\ &= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} d\theta \sum_{(h)-\infty}^{\infty} \cos(2h\theta - 2l\theta) \left(y_{h,0} + y'_{h,0} \int_0^t dt \right) \cos(2at \sin \theta) \\ &= \frac{2}{\pi} \sum_{(h)-\infty}^{\infty} \left(y_{l+h,0} + y'_{l+h,0} \int_0^t dt \right) \int_0^{\frac{\pi}{2}} d\theta \cos 2h\theta \cos(2at \sin \theta), \end{aligned}$$

so that the formula (12) is re-deduced.

24. One element in the solution of the problem of article 1 has been the theorem that

$$\sum_{(k)1}^n \sin \frac{jk\pi}{n+1} \sin \frac{lk\pi}{n+1} = 0, \text{ or } = \frac{n+1}{2},$$

according as j and l , being both integer numbers > 0 & $< n+1$, are unequal or equal to each other. As we shall have several analogous summations to perform in these researches, it may be well to give here the process of proof in full.

The equation $2 \sin \alpha \cos(2k\alpha + \beta) = \sin(2k\alpha + \alpha + \beta) - \sin(2k\alpha - \alpha + \beta)$ gives, when it is summed with reference to k ,

$$2 \sin \alpha \sum_{(k)k_1}^{k_2} \cos(2k\alpha + \beta) = \sin(2k_2\alpha + \alpha + \beta) - \sin(2k_1\alpha - \alpha + \beta).$$

Let

$$\alpha = \alpha_1 \pm \alpha_2, \quad \beta = 0,$$

then $\sum_{(k)k_1}^{k_2} \cos(2k\alpha_1 \pm 2k\alpha_2) = \frac{\sin(2k_2+1)(\alpha_1 \pm \alpha_2) - \sin(2k_1-1)(\alpha_1 \pm \alpha_2)}{2 \sin(\alpha_1 \pm \alpha_2)};$

$\therefore \sum_{(k)k_1}^{k_2} \cos 2k\alpha_1 \cos 2k\alpha_2 = \frac{\sin(2k_2+1)(\alpha_1 - \alpha_2) - \sin(2k_1-1)(\alpha_1 - \alpha_2)}{4 \sin(\alpha_1 - \alpha_2)} + \frac{\sin(2k_2+1)(\alpha_1 + \alpha_2) - \sin(2k_1-1)(\alpha_1 + \alpha_2)}{4 \sin(\alpha_1 + \alpha_2)},$

and $\sum_{(k)k_1}^{k_2} \sin 2k\alpha_1 \sin 2k\alpha_2 = \frac{\sin(2k_2+1)(\alpha_1 - \alpha_2) - \sin(2k_1-1)(\alpha_1 - \alpha_2)}{4 \sin(\alpha_1 - \alpha_2)} - \frac{\sin(2k_2+1)(\alpha_1 + \alpha_2) - \sin(2k_1-1)(\alpha_1 + \alpha_2)}{4 \sin(\alpha_1 + \alpha_2)}.$

Hence

$$\begin{aligned} & \sum_{(k)1}^n \sin 2k\alpha_1 \sin 2k\alpha_2 \\ &= \frac{\sin(2n+1)(\alpha_1 - \alpha_2) - \sin(\alpha_1 - \alpha_2)}{4 \sin(\alpha_1 - \alpha_2)} - \frac{\sin(2n+1)(\alpha_1 + \alpha_2) - \sin(\alpha_1 + \alpha_2)}{4 \sin(\alpha_1 + \alpha_2)} \\ &= \frac{\sin(\alpha_1 + \alpha_2) \sin(2n+1)(\alpha_1 - \alpha_2) - \sin(\alpha_1 - \alpha_2) \sin(2n+1)(\alpha_1 + \alpha_2)}{4 \sin(\alpha_1 + \alpha_2) \sin(\alpha_1 - \alpha_2)} \\ &= \frac{\cos(2n\alpha_1 - 2n + 2\alpha_2) - \cos(2n + 2\alpha_1 - 2n\alpha_2) + \cos(2n + 2\alpha_1 + 2n\alpha_2) - \cos(2n\alpha_1 + 2n + 2\alpha_2)}{8 \sin(\alpha_1 + \alpha_2) \sin(\alpha_1 - \alpha_2)} \\ &= \frac{\sin 2n\alpha_1 \sin 2n + 2\alpha_2 - \sin 2n\alpha_2 \sin 2n + 2\alpha_1}{2(\cos 2\alpha_2 - \cos 2\alpha_1)}, \end{aligned}$$

this sum therefore vanishes, if $\alpha_1 = \frac{j\pi}{2n+2}$, $\alpha_2 = \frac{l\pi}{2n+2}$, unless $\cos 2\alpha_2 = \cos 2\alpha_1$, that is, in the present question, unless $j=l$. But, for that particular case, the sum may be found by differentiating numerator and denominator relatively to α_1 , & then making $\alpha_1 = \alpha_2$; it is \therefore

$$= -\frac{n+1}{2} \cos l\pi \sin \frac{n l \pi}{n+1} \operatorname{cosec} \frac{l\pi}{n+1} = \frac{n+1}{2}.$$

25. The same theorem of summation shows that, in the notation of articles 7 and 8,

$$\sum_{(i)1}^n B_i^2 \cos \beta_i^2 = \sum_{(i)1}^n \eta_i^2 = \frac{2}{n+1} \sum_{(i)1}^n y_{i,0}^2;$$

$$\sum_{(i)1}^n r_i^2 B_i^2 \sin \beta_i^2 = \sum_{(i)1}^n \eta_i'^2 = \frac{2}{n+1} \sum_{(i)1}^n y_{i,0}'^2.$$

It is interesting to calculate also $\sum_{(i)1}^n r_i^2 B_i^2 \cos \beta_i^2 = \sum_{(i)1}^n r_i^2 \eta_i^2$. Since $r_i = 2a \sin \frac{\frac{1}{2}i\pi}{n+1}$, we have

$r_i^2 = 2a^2 - 2a^2 \cos \frac{i\pi}{n+1}$; we have therefore only to calculate

$$\left(\frac{n+1}{2}\right)^2 \sum_{(i)1}^n \eta_i^2 \cos \frac{i\pi}{n+1} = \sum_{(i)1}^n \cos \frac{i\pi}{n+1} \left(\sum_{(i)1}^n y_{i,0} \sin \frac{i l \pi}{n+1}\right)^2.$$

For this purpose we have to calculate

$$2 \sum_{(i)1}^n \cos \frac{i\pi}{n+1} \sin \frac{il\pi}{n+1} \sin \frac{j\pi}{n+1} = \sum_{(i)1}^n \left(\sin \frac{i(j+1)\pi}{n+1} + \sin \frac{i(j-1)\pi}{n+1} \right) \sin \frac{il\pi}{n+1} = 0,$$

unless $j=l \pm 1$. But if $j=l \pm 1$, j & l being each > 0 and $< n+1$, then the last sum $= \frac{n+1}{2}$.

Hence

$$\sum_{(i)1}^n \cos \frac{i\pi}{n+1} \left(\sum_{(l)1}^n y_{l,0} \sin \frac{il\pi}{n+1} \right)^2 = \frac{n+1}{2} \sum_{(l)1}^{n-1} y_{l,0} y_{l+1,0};$$

therefore

$$\sum_{(i)1}^n \eta_i^2 \cos \frac{i\pi}{n+1} = \frac{2}{n+1} \sum_{(l)1}^{n-1} y_{l,0} y_{l+1,0};$$

and finally, because $y_{n+1,0} = 0$,

$$\sum_{(i)1}^n r_i^2 B_i^2 \cos \beta_i^2 = \sum_{(i)1}^n r_i^2 \eta_i^2 = \frac{4a^2}{n+1} \sum_{(l)1}^n y_{l,0} (y_{l,0} - y_{l+1,0}).$$

Hence

$$\sum_{(i)1}^n r_i^2 B_i^2 = \frac{2}{n+1} \sum_{(l)1}^n \{y_{l,0}^2 + 2a^2 y_{l,0} (y_{l,0} - y_{l+1,0})\}.$$

26. (Examples.) When $n=1$, then

$$r_1 = a\sqrt{2}, \quad B_1 \cos \beta_1 = \eta_1 = y_{1,0}, \quad r_1 B_1 \sin \beta_1 = \eta'_1 = y'_{1,0}.$$

When $n=2$, then

$$r_1 = a, \quad r_2 = a\sqrt{3}, \quad B_1^2 \cos \beta_1^2 + B_2^2 \cos \beta_2^2 = \frac{2}{3} (y_{1,0}^2 + y_{2,0}^2),$$

$$r_1^2 B_1^2 \sin \beta_1^2 + r_2^2 B_2^2 \sin \beta_2^2 = \frac{2}{3} (y_{1,0}^2 + y_{2,0}^2),$$

$$r_1^2 B_1^2 \cos \beta_1^2 + r_2^2 B_2^2 \cos \beta_2^2 = \frac{4a^2}{3} (y_{1,0}^2 + y_{2,0}^2 - y_{1,0} y_{2,0}).$$

When $n=3$, then

$$r_1 = a\sqrt{2-\sqrt{2}}, \quad r_2 = a\sqrt{2}, \quad r_3 = a\sqrt{2+\sqrt{2}},$$

$$B_1^2 \cos \beta_1^2 + B_2^2 \cos \beta_2^2 + B_3^2 \cos \beta_3^2 = \frac{1}{2} (y_{1,0}^2 + y_{2,0}^2 + y_{3,0}^2),$$

$$r_1^2 B_1^2 \sin \beta_1^2 + r_2^2 B_2^2 \sin \beta_2^2 + r_3^2 B_3^2 \sin \beta_3^2 = \frac{1}{2} (y_{1,0}^2 + y_{2,0}^2 + y_{3,0}^2),$$

$$r_1^2 B_1^2 \cos \beta_1^2 + r_2^2 B_2^2 \cos \beta_2^2 + r_3^2 B_3^2 \cos \beta_3^2 = a^2 (y_{1,0}^2 + y_{2,0}^2 + y_{3,0}^2 - y_{1,0} y_{2,0} - y_{2,0} y_{3,0}).$$

27. The non-periodical part of $\sum_{(i)1}^n y_{l,i}^2$ is $\frac{n+1}{4} \sum_{(i)1}^n r_i^2 B_i^2$; this non-periodical part is therefore equal to the sum

$$Q = \sum_{(i)1}^n \left\{ \frac{1}{2} y_{l,0}^2 + a^2 y_{l,0} (y_{l,0} - y_{l+1,0}) \right\}. \tag{16}$$

This part Q appears to be in some sense a measure* of the *quantity of vibration* of the system (the mass of each particle being unity).

Problem II.

28. It is required to apply the general solution of the 1st Problem to the case where, at the time 0, all the displacements & velocities vanish except those of the $j-j, +1$ consecutive particles $P_j, P_{j+1}, \dots, P_{j-1}, P_j$; supposing also that the initial displacements and velocities of

* [Sum of initial kinetic and potential energies.]

these are such as to agree with a simple mode of vibration, in such a manner that, if l be $> j, -1$ and $< j+1$, we have

$$y_{l,0} = B_k \cos \beta_k \sin 2kl\phi, \quad y'_{l,0} = B_k r_k \sin \beta_k \sin 2kl\phi,$$

ϕ being $= \frac{\frac{1}{2}\pi}{n+1}$, and $r_k = 2a \sin k\phi$; but that

$$y_{l,0} = 0, \quad y'_{l,0} = 0,$$

if l be $< j$, or $> j$.

29. (Solution.) The general expressions

$$B_i \cos \beta_i = \eta_i = \frac{2}{n+1} \sum_{(i)1}^n y_{i,0} \sin 2i\phi,$$

$$r_i B_i \sin \beta_i = \eta'_i = \frac{2}{n+1} \sum_{(i)1}^n y'_{i,0} \sin 2i\phi,$$

become now

$$\eta_i = \frac{2}{n+1} B_k \cos \beta_k \sum_{(i)j}^j \sin 2i\phi \sin 2kl\phi,$$

$$\eta'_i = \frac{2r_k}{n+1} B_k \sin \beta_k \sum_{(i)j}^j \sin 2i\phi \sin 2kl\phi.$$

But, by article 24,

$$\sum_{(i)j}^j \sin 2i\phi \sin 2kl\phi = \frac{\sin (2j+1)(i-k)\phi - \sin (2j-1)(i-k)\phi}{4 \sin (i-k)\phi} \\ - \frac{\sin (2j+1)(i+k)\phi - \sin (2j-1)(i+k)\phi}{4 \sin (i+k)\phi} = \frac{F(j) - F(j-1)}{2};$$

$$F(j) = \frac{\sin (2j+1)(i-k)\phi}{2 \sin (i-k)\phi} - \frac{\sin (2j+1)(i+k)\phi}{2 \sin (i+k)\phi} = \frac{\sin 2ji\phi \sin \overline{2j+1}k\phi - \sin 2jk\phi \sin \overline{2j+1}i\phi}{\cos 2k\phi - \cos 2i\phi};$$

therefore

$$2 \sum_{(i)j}^j \sin 2i\phi \sin 2kl\phi = \frac{1}{\cos 2k\phi - \cos 2i\phi} \{ \sin 2ji\phi \sin \overline{2j+1}k\phi - \sin 2jk\phi \sin \overline{2j+1}i\phi \\ - \sin 2j,k\phi \sin \overline{2j-1}i\phi + \sin 2j,i\phi \sin \overline{2j-1}k\phi \}; \\ 2 \sum_{(i)j}^j (\sin 2kl\phi)^2 = j-j+1 + \frac{\cos 2j,k\phi \sin \overline{2j-1}k\phi - \sin 2jk\phi \cos \overline{2j+1}k\phi}{\sin 2k\phi}.$$

Hence, by the general formula (9), we have, in the present question,

$$y_{l,t} = \frac{B_k}{n+1} \sum_{(i)1}^n \left(\cos \beta_k \cos (2at \sin i\phi) + \sin \beta_k \frac{\sin k\phi}{\sin i\phi} \sin (2at \sin i\phi) \right) \frac{\sin 2i\phi}{\cos 2k\phi - \cos 2i\phi} \\ \times \{ \sin 2ji\phi \sin \overline{2j+1}k\phi - \sin 2jk\phi \sin \overline{2j+1}i\phi \\ - \sin 2j,k\phi \sin \overline{2j-1}i\phi + \sin 2j,i\phi \sin \overline{2j-1}k\phi \}; \quad (17)$$

and the part corresponding to $i = k$ is

$$\frac{B_k'}{n+1} \cos(2at \sin k\phi - \beta_k') \sin 2kl\phi \left\{ j - j + 1 + \frac{\cos 2j, k\phi \sin \overline{2j-1}k\phi - \sin 2jk\phi \cos \overline{2j+1}k\phi}{\sin 2k\phi} \right\}$$

30. If $j = 1$, so that the j first particles P_1, \dots, P_j are all disturbed originally in the way above supposed, then

$$y_{l,t} = \frac{B_k'}{n+1} \sum_{(i)1}^n \left(\cos \beta_k' \cos(2at \sin i\phi) + \sin \beta_k' \frac{\sin k\phi}{\sin i\phi} \sin(2at \sin i\phi) \right) \times \frac{\sin 2il\phi \{ \sin 2ji\phi \sin \overline{2j+1}k\phi - \sin 2jk\phi \sin \overline{2j+1}i\phi \}}{\cos 2k\phi - \cos 2i\phi}; \quad (18)$$

and the part corresponding to $i = k$ is

$$\frac{B_k'}{n+1} \cos(2at \sin k\phi - \beta_k') \sin 2kl\phi \left\{ j - \frac{\sin 2jk\phi \cos \overline{2j+1}k\phi}{\sin 2k\phi} \right\}$$

31. If j and l be each much smaller than n , so that n is treated as infinite, while j and l , though perhaps large, are finite, then the expression in article 30 becomes a definite integral, namely

$$y_{l,t} = \frac{2}{\pi} B_k' \int_0^{\frac{\pi}{2}} d\theta \left(\cos \beta_k' \cos(2at \sin \theta) + \sin \beta_k' \frac{\sin \alpha}{\sin \theta} \sin(2at \sin \theta) \right) \times \frac{\sin 2l\theta \{ \sin 2j\theta \sin 2(j+1)\alpha - \sin 2j\alpha \sin 2(j+1)\theta \}}{\cos 2\alpha - \cos 2\theta}; \quad (19)$$

in which $\alpha = k\phi$. This expression satisfies the equation in mixed differences (1); and gives $y_{l,0} = 0, y'_{l,0} = 0$, if $l > j$; but $y_{l,0} = B_k' \cos \beta_k' \sin 2l\alpha, y'_{l,0} = 2\alpha \sin \alpha B_k' \sin \beta_k' \sin 2l\alpha$, if $l < j + 1, l$ being a positive integer: it gives also $y_{0,t} = 0$.*

32. In the formula (18), making $i\phi = \theta$ and $k\phi = \alpha$, we are led to consider the product

$$\begin{aligned} & \sin 2l\theta \{ \sin 2j\theta \sin \overline{2j+1}\alpha - \sin 2j\alpha \sin \overline{2j+1}\theta \} \\ &= \frac{1}{2} \sin 2l\theta \{ \cos(2j\alpha - \theta + 2\alpha) - \cos(2j\alpha + \theta + 2\alpha) - \cos(2j\theta - \alpha + 2\theta) + \cos(2j\theta + \alpha + 2\theta) \} \\ &= \sin 2l\theta \{ \sin(\theta + \alpha) \sin(2j+1)(\theta - \alpha) - \sin(\theta - \alpha) \sin(2j+1)(\theta + \alpha) \} \\ &= \frac{1}{2} \sin(\theta + \alpha) \{ \cos(\overline{2j-2l+1}\theta - \overline{2j+1}\alpha) - \cos(\overline{2j+2l+1}\theta - \overline{2j+1}\alpha) \} \\ & \quad - \frac{1}{2} \sin(\theta - \alpha) \{ \cos(\overline{2j-2l+1}\theta + \overline{2j+1}\alpha) - \cos(\overline{2j+2l+1}\theta + \overline{2j+1}\alpha) \}. \end{aligned}$$

If now we suppose, as in article 18, that $\frac{j+l}{n+1}$ is nearly = 1 but that $\frac{j-l}{n+1}$ is nearly = 0, we may neglect those sums which involve cosines of $2(j+l)\theta + \text{const.}$, unless they be divided by something which vanishes or becomes very small in the course of the summation; and may reduce (under the sign of summation) the recent product to

$$\begin{aligned} & \cos(2j-2l+1)\theta \cdot \cos(2j+1)\alpha \cdot \cos \theta \cdot \sin \alpha + \sin(2j-2l+1)\theta \cdot \sin(2j+1)\alpha \cdot \sin \theta \cdot \cos \alpha \\ &= \frac{1}{2} \sin(\theta + \alpha) \cos \{ (2j-2l+1)\theta - (2j+1)\alpha \} - \frac{1}{2} \sin(\theta - \alpha) \cos \{ (2j-2l+1)\theta + (2j+1)\alpha \}; \end{aligned}$$

reserving, however, the part

$$-\frac{1}{2} \sin(\theta + \alpha) \cos \{ (2j+2l+1)\theta - (2j+1)\alpha \}$$

* $\left[\frac{\sin 2j\theta \sin 2(j+1)\alpha - \sin 2j\alpha \sin 2(j+1)\theta}{\cos 2\alpha - \cos 2\theta} = \sum_{s=0}^{j-1} 2 \sin 2(j-s)\theta \sin 2(j-s)\alpha \right]$

for special consideration. It may however be instructive, before thus passing to these limits, to resume the formula (18), & to study first the consequences of it in the case when the number n of moveable particles is finite but even, & when $j = \frac{n}{2}$.

33. The expression, $j - \frac{\sin 2j\alpha \cos 2(j+1)\alpha}{\sin 2\alpha}$, which occurs in article 30, may be put under the form $j + \frac{1}{2} - \frac{\sin 2(2j+1)\alpha}{2 \sin 2\alpha}$; it reduces itself therefore to $\frac{n+1}{2}$, if $j = \frac{n}{2}$, (because $\alpha = \frac{\frac{1}{2}k\pi}{n+1}$), that is, if exactly half of the whole number n of moveable particles have such original displacements and velocities as correspond to a simple movement of any one kind; & consequently, in this case, that part of the whole resultant movement which is of the same period is

$$\frac{1}{2} B_k \cos(2at \sin \alpha - \beta_k) \sin 2l\alpha;$$

it is therefore exactly half of that other movement

$$B_k \cos(2at \sin \alpha - \beta_k) \sin 2l\alpha$$

which would reproduce the initial displacements and velocities, not only for half but for the whole of the system of moveable particles. In other words, we have the theorem:

If the initial state of half the system $P_1, P_2, \dots, P_{\frac{n}{2}}$ correspond to one simple movement

$$\frac{1}{2} B_k \cos(2at \sin \alpha - \beta_k) \sin 2l\alpha,$$

and if the initial state of the other half $P_{\frac{n}{2}+1}, \dots, P_{n-1}, P_n$ correspond to another simple movement of the same period and amplitude, but with an epoch differing by an odd multiple of π ,

$$-\frac{1}{2} B_k \cos(2at \sin \alpha - \beta_k) \sin 2l\alpha,$$

in which $\alpha = \frac{\frac{1}{2}k\pi}{n+1}$, then the resultant vibration of the system will be composed entirely of simple movements of other orders, that is, with other periodic times. (The next article will show that the indices i which mark these orders differ by odd numbers from the index k .)

34. To express this resultant vibration, we may employ the formula (18), under the form

$$y_{i,t} = \frac{B_k}{2j+1} \Sigma_{(i)} \left(\cos \beta_k \cos(2at \sin \theta) + \sin \beta_k \frac{\sin \alpha}{\sin \theta} \sin(2at \sin \theta) \right) \\ \times \frac{\sin 2l\theta \{ \sin(\theta + \alpha) \sin(2j+1)(\theta - \alpha) - \sin(\theta - \alpha) \sin(2j+1)(\theta + \alpha) \}}{\cos 2\alpha - \cos 2\theta}; \quad (20)$$

in which the part corresponding to $i = k$ is now to be omitted; we have also, now,

$$\theta = \frac{\frac{1}{2}i\pi}{2j+1}, \quad \alpha = \frac{\frac{1}{2}k\pi}{2j+1},$$

\therefore

$$(2j+1)(\theta \mp \alpha) = \frac{1}{2}(i \mp k)\pi;$$

we need therefore attend only to those values of i which differ from k by odd numbers, positive or negative. Let $l = j + h$; then

$$\sin 2l\theta = \sin(2j\theta + 2h\theta) = \sin \left(\frac{i\pi}{2} + (2h-1)\theta \right) \\ = \sin \left(\frac{(i-k)\pi}{2} + \frac{k\pi}{2} + (2h-1)\theta \right) = \sin \frac{(i-k)\pi}{2} \cos \left(\frac{k\pi}{2} + (2h-1)\theta \right),$$

because $\cos \frac{(i-k)\pi}{2} = 0$. For the same reason

$$\left(\sin \frac{(i-k)\pi}{2}\right)^2 = 1, \quad \sin \frac{(i-k)\pi}{2} \sin \frac{(i+k)\pi}{2} = \cos k\pi;$$

$$\begin{aligned} \therefore \frac{\sin 2l\theta \{ \sin(\theta + \alpha) \sin(2j+1)(\theta - \alpha) - \sin(\theta - \alpha) \sin(2j+1)(\theta + \alpha) \}}{\cos 2\alpha - \cos 2\theta} \\ = \frac{\cos\left(\frac{k\pi}{2} + (2h-1)\theta\right)}{\cos 2\alpha - \cos 2\theta} \{ \sin(\theta + \alpha) - \cos k\pi \sin(\theta - \alpha) \} \\ = \frac{1}{2} \cos\left(\frac{k\pi}{2} + (2h-1)\theta\right) \left(\frac{1}{\sin(\theta - \alpha)} + \frac{(-1)^{k+1}}{\sin(\theta + \alpha)} \right). \end{aligned}$$

(If k be odd, this becomes

$$\frac{1}{2} (-1)^{\frac{k+1}{2}} \sin(2h-1)\theta \left(\frac{1}{\sin(\theta - \alpha)} + \frac{1}{\sin(\theta + \alpha)} \right);$$

if k be even,

$$\frac{1}{2} (-1)^{\frac{k}{2}} \cos(2h-1)\theta \left(\frac{1}{\sin(\theta - \alpha)} - \frac{1}{\sin(\theta + \alpha)} \right).$$

Hence the expression for $y_{l,t}$ becomes

$$\begin{aligned} y_{j+h,t} = \frac{1}{2} B_k \sum_{(i)} \left(\cos \beta_k \cos(2at \sin \theta) + \sin \beta_k \frac{\sin \alpha}{\sin \theta} \sin(2at \sin \theta) \right) \\ \times \cos\left(\frac{k\pi}{2} + (2h-1)\theta\right) \left(\frac{1}{\sin(\theta - \alpha)} + \frac{(-1)^{k+1}}{\sin(\theta + \alpha)} \right); \quad (21) \end{aligned}$$

in which the summation is to be performed relatively to i for all values of that index which differ from k by odd (integer) differences, being also > 0 and $< n+1$; and

$$\theta = \frac{\frac{1}{2}i\pi}{2j+1}, \quad \alpha = \frac{\frac{1}{2}k\pi}{2j+1}.$$

35. For example, if $j=1, k=1$, we must take $i=2, \theta = \frac{\pi}{3}, \alpha = \frac{\pi}{6}, \theta - \alpha = \frac{\pi}{6}, \theta + \alpha = \frac{\pi}{2}$, $\sin(\theta - \alpha) = \frac{1}{2}, \sin(\theta + \alpha) = 1, \sin \alpha = \frac{1}{2}, \sin \theta = \frac{\sqrt{3}}{2}$, and

$$y_{1+h,t} = \frac{1}{2} B_1 \left(\cos \beta_1 \cos at\sqrt{3} + \sin \beta_1 \frac{\sin at\sqrt{3}}{\sqrt{3}} \right) \sin(1-2h)\theta;$$

that is,

$$y_{1,t} = B_1 \frac{\sqrt{3}}{4} \left(\cos \beta_1 \cos at\sqrt{3} + \sin \beta_1 \frac{\sin at\sqrt{3}}{\sqrt{3}} \right) = -y_{2,t}.$$

These values accordingly result from the more general formulae of article 12 by supposing

$$\begin{aligned} y_{1,0} = \frac{\sqrt{3}}{4} B_1 \cos \beta_1, \quad y'_{1,0} = \frac{a\sqrt{3}}{4} B_1 \sin \beta_1, \\ y_{2,0} = -\frac{\sqrt{3}}{4} B_1 \cos \beta_1, \quad y'_{2,0} = -\frac{a\sqrt{3}}{4} B_1 \sin \beta_1. \end{aligned}$$

And if we had supposed $j = 1, k = 2$, we should have been obliged to take $i = 1, \alpha = \frac{\pi}{3}, \theta = \frac{\pi}{6}$,
 $\theta - \alpha = -\frac{\pi}{6}, \theta + \alpha = \frac{\pi}{2}$,

$$y_{1+h,t} = \frac{1}{2} B_2 (\cos \beta_2 \cos at + \sqrt{3} \sin \beta_2 \sin at) \cos (2h - 1) \frac{\pi}{6};$$

that is,

$$y_{1,t} = y_{2,t} = \frac{\sqrt{3}}{4} B_2 (\cos \beta_2 \cos at + \sqrt{3} \sin \beta_2 \sin at).$$

Accordingly these expressions result from the formulae of article 12 by supposing

$$y_{1,0} = y_{2,0} = \frac{\sqrt{3}}{4} B_2 \cos \beta_2, \quad y'_{1,0} = y'_{2,0} = \frac{3a}{4} B_2 \sin \beta_2.$$

36. Whatever j may be, if we take k odd and $= 2\kappa - 1$, we must take i even, and of the form 2ι ; κ and ι being each some one of the integers $1, 2, \dots j$. Hence, in this case,

$$y_{j+h,t} = \frac{2(-1)^\kappa}{2j+1} B_{2\kappa-1} \sum_{(\iota)1}^j \left\{ \cos \beta_{2\kappa-1} \cos \left(2at \sin \frac{\iota\pi}{2j+1} \right) \sin \frac{\iota\pi}{2j+1} \right. \\ \left. + \sin \beta_{2\kappa-1} \sin \left(2at \sin \frac{\iota\pi}{2j+1} \right) \sin \frac{(\kappa - \frac{1}{2})\pi}{2j+1} \right\} \frac{\sin \frac{(2h-1)\iota\pi}{2j+1} \cos \frac{(\kappa - \frac{1}{2})\pi}{2j+1}}{\cos \frac{(2\kappa-1)\pi}{2j+1} - \cos \frac{2\iota\pi}{2j+1}}. \quad (22)$$

And if, on the other hand, we take $k = 2\kappa$ and $i = 2\iota - 1$, we have

$$y_{j+h,t} = \frac{2(-1)^\kappa}{2j+1} B_{2\kappa} \sum_{(\iota)1}^j \left\{ \cos \beta_{2\kappa} \cos \left(2at \sin \frac{(\iota - \frac{1}{2})\pi}{2j+1} \right) \sin \frac{(\iota - \frac{1}{2})\pi}{2j+1} \right. \\ \left. + \sin \beta_{2\kappa} \sin \left(2at \sin \frac{(\iota - \frac{1}{2})\pi}{2j+1} \right) \sin \frac{\kappa\pi}{2j+1} \right\} \frac{\cos \frac{(2h-1)(\iota - \frac{1}{2})\pi}{2j+1} \sin \frac{\kappa\pi}{2j+1} \cotan \frac{(\iota - \frac{1}{2})\pi}{2j+1}}{\cos \frac{2\kappa\pi}{2j+1} - \cos \frac{(2\iota-1)\pi}{2j+1}}. \quad (23)$$

And these formulae may be considered as rigorous with reference to the present question.

37. Supposing now that j increases without limit, but that k so increases with it as to leave $\alpha =$ some finite arc, between 0 and $\frac{\pi}{2}$; we shall have, as the limits of the two last formulae, the following:

$$y_{j+h,t} = \frac{2}{\pi} (-1)^\kappa B_{2\kappa-1} \int_0^{\frac{\pi}{2}} d\theta \{ \cos \beta_{2\kappa-1} \cos (2at \sin \theta) \sin \theta + \sin \beta_{2\kappa-1} \sin (2at \sin \theta) \sin \alpha \} \\ \times \frac{\sin (2h\theta - \theta) \cos \alpha}{\cos 2\alpha - \cos 2\theta}; \quad (24)$$

and

$$y_{j+h,t} = \frac{2}{\pi} (-1)^\kappa B_{2\kappa} \int_0^{\frac{\pi}{2}} d\theta \{ \cos \beta_{2\kappa} \cos (2at \sin \theta) \sin \theta + \sin \beta_{2\kappa} \sin (2at \sin \theta) \sin \alpha \} \\ \times \frac{\cos (2h\theta - \theta) \sin \alpha \cotan \theta}{\cos 2\alpha - \cos 2\theta}; \quad (25)$$

the first corresponding to the case $k = 2\kappa - 1$, & the second to $k = 2\kappa$. It is evident that both these expressions satisfy the equation of mixed differences; & to show that they also reproduce the initial displacements and velocities, we must show that they give, according as the integer h is $>$ or not > 0 ,

$$y_{j+h,0} = \mp \frac{1}{2} B_k \cos \beta_k \sin 2(j+h)\alpha,$$

$$y'_{j+h,0} = \mp a B_k \sin \beta_k \sin 2(j+h)\alpha \sin \alpha;$$

in which $\alpha = \frac{\frac{1}{2}k\pi}{2j+1}$, so that

$$\sin 2(j+h)\alpha = \sin \left\{ \frac{1}{2}k\pi + (2h-1)\alpha \right\} = \sin \frac{1}{2}k\pi \cos (2h-1)\alpha + \cos \frac{1}{2}k\pi \sin (2h-1)\alpha$$

$$= (-1)^{\kappa+1} \cos (2h-1)\alpha, \quad \text{or} \quad = (-1)^\kappa \sin (2h-1)\alpha,$$

according as k is of the form $2\kappa - 1$, or of the form 2κ .

38. There are, therefore, for a verification, or for an *à posteriori* proof of the formulae of the last article, the 2 following equations to be proved:*

$$\pm \frac{\pi \cos (2h\alpha - \alpha)}{4 \cos \alpha} = \int_0^{\frac{\pi}{2}} d\theta \frac{\sin \theta \sin (2h\theta - \theta)}{\cos 2\alpha - \cos 2\theta};$$

$$\mp \frac{\pi \sin (2h\alpha - \alpha)}{4 \sin \alpha} = \int_0^{\frac{\pi}{2}} d\theta \frac{\cos \theta \cos (2h\theta - \theta)}{\cos 2\alpha - \cos 2\theta};$$

the upper signs corresponding to positive values, and the lower signs corresponding to negative values, of the odd integer $2h - 1$.

Now, if we put

$$c_h = \frac{2}{\pi} \sin 2\alpha \int_0^{\frac{\pi}{2}} \frac{\cos 2h\theta}{\cos 2\theta - \cos 2\alpha} d\theta,$$

we shall have, for *all* values of h ,

$$c_{h+1} + c_{h-1} - 2 \cos 2\alpha c_h = \frac{4}{\pi} \sin 2\alpha \int_0^{\frac{\pi}{2}} \cos 2h\theta d\theta = \frac{2}{h\pi} \sin 2\alpha \sin h\pi;$$

therefore this function vanishes, if h be any integer $>$ or < 0 ; but $c_1 + c_{-1} - 2 \cos 2\alpha c_0 = 2 \sin 2\alpha$. Again $c_h = c_{-h}$; and $c_0 = 0$. To prove this last relation, we may set out with the evident relation † $0 = \int_{-\infty}^{\infty} \frac{dx}{x}$, which gives $0 = \int_{-\infty}^{\infty} \frac{dx}{x-a}$, a being real; (though the complete discussion of the value of this definite integral belongs to the theory of singular integrals, considered first by Cauchy;) therefore

$$0 = \int_0^{\infty} \left(\frac{dx}{x-a} - \frac{dx}{x+a} \right) = 2a \int_0^{\infty} \frac{dx}{x^2 - a^2},$$

* [The integrals which follow are to be interpreted as being Cauchy's Principal Values, i.e.

$$\int_0^{\frac{\pi}{2}} = \lim_{\epsilon \rightarrow 0} \left(\int_0^{\alpha - \epsilon} + \int_{\alpha + \epsilon}^{\frac{\pi}{2}} \right).$$

†

$$\left[\text{I.e. } \int_{\epsilon}^{\infty} \frac{dx}{x} + \int_{-\infty}^{-\epsilon} \frac{dx}{x} = 0. \right]$$

and therefore $0 = \int_0^\infty \frac{dx}{x^2 - a^2}$, if a be real and different from 0. Make $x = \tan 2\theta$, $a = \tan 2\alpha$, and suppose $(\cos 2\alpha)^2 > 0$; then

$$0 = 2 \int_0^{\frac{\pi}{4}} \frac{d\theta (\sec 2\theta)^2 \sec 2\alpha}{(\sec 2\theta)^2 - (\sec 2\alpha)^2} = \int_0^{\frac{\pi}{4}} \left(\frac{d\theta}{\cos 2\alpha + \cos 2\theta} + \frac{d\theta}{\cos 2\alpha - \cos 2\theta} \right) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos 2\alpha - \cos 2\theta},$$

which is what was to be proved. (However, it is to be observed that we have here supposed $\cos 2\alpha$ to be different from 0. Yet even if it were $= 0$, so that we had to consider the integral

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos 2\theta}, \text{ we might consider this as being } = \int_0^{\frac{\pi}{4}} (\sec 2\theta - \sec 2\theta) d\theta, \text{ and therefore as being } = 0.)$$

Admitting then that $c_0 = 0$, we have $c_{-1} = c_1 = \sin 2\alpha$. Hence $c_h = \sin 2h\alpha$, if h be any integer not less than 0; and $c_h = -\sin 2h\alpha$, if h be any integer not greater than 0. That is,

$$\int_0^{\frac{\pi}{2}} d\theta \frac{\cos 2h\theta}{\cos 2\theta - \cos 2\alpha} = \pm \frac{\pi \sin 2h\alpha}{2 \sin 2\alpha}, \text{ or } = 0,$$

according as the integer h is ≥ 0 , or $= 0$. Hence, if $h > 0$,

$$\int_0^{\frac{\pi}{2}} d\theta \frac{\cos 2h\theta - \cos (2h\theta - 2\theta)}{\cos 2\theta - \cos 2\alpha} = \frac{\pi \sin 2h\alpha - \sin (2h\alpha - 2\alpha)}{2 \sin 2\alpha};$$

that is, dividing by $+2$,

$$\int_0^{\frac{\pi}{2}} d\theta \frac{\sin \theta \sin (2h\theta - \theta)}{\cos 2\alpha - \cos 2\theta} = \frac{\pi \cos (2h\alpha - \alpha)}{4 \cos \alpha},$$

if the integer $2h - 1$ be > 0 ; from which, without any new calculation, we see that

$$\int_0^{\frac{\pi}{2}} d\theta \frac{\sin \theta \sin (2h\theta - \theta)}{\cos 2\alpha - \cos 2\theta} = -\frac{\pi \cos (2h\alpha - \alpha)}{4 \cos \alpha},$$

if the integer $2h - 1$ be < 0 .

In like manner, if h be > 0 (being integer), we have

$$\int_0^{\frac{\pi}{2}} d\theta \frac{\cos 2h\theta + \cos (2h\theta - 2\theta)}{\cos 2\theta - \cos 2\alpha} = \frac{\pi \sin 2h\alpha + \sin (2h\alpha - 2\alpha)}{2 \sin 2\alpha},$$

that is,

$$\int_0^{\frac{\pi}{2}} d\theta \frac{\cos \theta \cos (2h\theta - \theta)}{\cos 2\alpha - \cos 2\theta} = -\frac{\pi \sin (2h\alpha - \alpha)}{4 \sin \alpha},$$

if the integer $2h - 1$ be > 0 . And hence, without any new calculation, we see that

$$\int_0^{\frac{\pi}{2}} d\theta \frac{\cos \theta \cos (2h\theta - \theta)}{\cos 2\alpha - \cos 2\theta} = \frac{\pi \sin (2h\alpha - \alpha)}{4 \sin \alpha},$$

if the integer $2h - 1$ be < 0 . The initial conditions are therefore satisfied.

39. The same analysis shows that, if the integer h be > 0 ,

$$\begin{aligned} \frac{\pi}{2} \cos 2h\alpha &= \int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos 2\alpha - \cos 2\theta} \{2 \cos \alpha^2 \sin \theta \sin (2h\theta - \theta) + 2 \sin \alpha^2 \cos \theta \cos (2h\theta - \theta)\} \\ &= \int_0^{\frac{\pi}{2}} d\theta \frac{\cos (2h\theta - 2\theta) - \cos 2\alpha \cos 2h\theta}{\cos 2\alpha - \cos 2\theta} = \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta \sin 2h\theta}{\cos 2\alpha - \cos 2\theta} d\theta; \end{aligned}$$

and accordingly, if we denote this last integral by f_h , we have

$$f_{h+1} + f_{h-1} - 2f_h \cos 2\alpha = \int_0^{\frac{\pi}{2}} d\theta \{\cos (2h\theta + 2\theta) - \cos (2h\theta - 2\theta)\} = 0,$$

if $h > 1$, and $= -\frac{\pi}{2}$, if $h = 1$; also $f_0 = 0$, and

$$f_1 = \int_0^{\frac{\pi}{2}} (\cos 2\alpha + \cos 2\theta) d\theta = \frac{\pi}{2} \cos 2\alpha;$$

therefore

$$f_2 = \pi \{(\cos 2\alpha)^2 - \frac{1}{2}\} = \frac{\pi}{2} \cos 4\alpha, \text{ and } f_h = \frac{\pi}{2} \cos 2h\alpha,$$

if $h > 0$. The same integral vanishes (as we have just remarked) when $h = 0$, and since it changes sign with h , it must become $= -\frac{\pi}{2} \cos 2h\alpha$, if $h < 0$.

We have therefore the discontinuous equation

$$\frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta \sin 2h\theta d\theta}{\cos 2\alpha - \cos 2\theta} = \pm \cos 2h\alpha, \text{ or } = 0,$$

according as the integer h is ≥ 0 , or $= 0$; & we found, in the last article, that

$$\frac{2}{\pi} \sin 2\alpha \int_0^{\frac{\pi}{2}} \frac{\cos 2h\theta d\theta}{\cos 2\alpha - \cos 2\theta} = \mp \sin 2h\alpha, \text{ or } = 0,$$

according as the integer h is ≥ 0 , or $= 0$. Indeed, we may consider both the two last equations as included in either of the two which occur at the beginning of article 38; & as conducting reciprocally to those two, by easy combinations.

40. We see then that if we assume

$$y_{h,t} = -\frac{2}{\pi} \cos \alpha \int_0^{\frac{\pi}{2}} \frac{d\theta \sin (2h\theta - \theta)}{\cos 2\alpha - \cos 2\theta} \{b \sin \theta \cos (2at \sin \theta) + c \sin \alpha \sin (2at \sin \theta)\},$$

b and c being any constants, & h being any integer number, we shall satisfy the indefinite equation in mixed differences

$$\alpha^2 (y_{h+1,t} + y_{h-1,t} - 2y_{h,t}) = y''_{h,t},$$

and also the initial conditions

$$\frac{y_{h,0}}{b} = \frac{y'_{h,0}}{2ac \sin \alpha} = \mp \frac{1}{2} \cos (2h\alpha - \alpha),$$

according as $2h - 1$ is $>$ or $<$ 0. In like manner, if we assume

$$y_{h,t} = -\frac{2}{\pi} \sin \alpha \int_0^{\frac{\pi}{2}} \frac{d\theta \cos \theta \cos (2h\theta - \theta)}{\cos 2\alpha - \cos 2\theta} \left\{ b \cos (2at \sin \theta) + \frac{c \sin \alpha}{\sin \theta} \sin (2at \sin \theta) \right\},$$

we shall satisfy the same indefinite equation in mixed differences, and the conditions

$$\frac{y_{h,0}}{b} = \frac{y'_{h,0}}{2ac \sin \alpha} = \pm \frac{1}{2} \sin (2h\alpha - \alpha),$$

according as $2h - 1$ is ≥ 0 . If we assume, in the third place,

$$y_{h,t} = -\frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\theta \cos \theta \sin 2h\theta}{\cos 2\alpha - \cos 2\theta} \{ b \sin \theta \cos (2at \sin \theta) + c \sin \alpha \sin (2at \sin \theta) \},$$

we shall satisfy the equation in differences, & the conditions

$$\frac{y_{h,0}}{b} = \frac{y'_{h,0}}{2ac \sin \alpha} = \mp \frac{1}{2} \cos 2h\alpha, \quad \text{or} = 0,$$

according as h is ≥ 0 , or $= 0$. And if we assume, in the fourth place,

$$y_{h,t} = -\frac{1}{\pi} \sin 2\alpha \int_0^{\frac{\pi}{2}} \frac{d\theta \cos 2h\theta}{\cos 2\alpha - \cos 2\theta} \left\{ b \cos (2at \sin \theta) + \frac{c \sin \alpha}{\sin \theta} \sin (2at \sin \theta) \right\},$$

we shall satisfy the same equation in differences, and the conditions

$$\frac{y_{h,0}}{b} = \frac{y'_{h,0}}{2ac \sin \alpha} = \pm \frac{1}{2} \sin 2h\alpha, \quad \text{or} = 0,$$

according as h is ≥ 0 , or $= 0$.

41. It follows that the first expression of article 40 corresponds to the effect, at the time t , of an initial state represented by

$$y_{h,at} = \mp \frac{1}{2} \{ b \cos (2adt \sin \alpha) + c \sin (2adt \sin \alpha) \} \cos (2h\alpha - \alpha),$$

and the second expression of the same article to the effect of an initial state represented by

$$y_{h,at} = \pm \frac{1}{2} \{ b \cos (2adt \sin \alpha) + c \sin (2adt \sin \alpha) \} \sin (2h\alpha - \alpha),$$

the upper or the lower signs being taken according as $2h >$ or $<$ 1.

It follows also that the third expression of the same article corresponds to the effect of an initial state represented by

$$y_{h,at} = \mp \frac{1}{2} \{ b \cos (2adt \sin \alpha) + c \sin (2adt \sin \alpha) \} \cos 2h\alpha, \quad \text{or} = 0,$$

and the fourth expression to the effect of the initial state

$$y_{h,at} = \pm \frac{1}{2} \{ b \cos (2adt \sin \alpha) + c \sin (2adt \sin \alpha) \} \sin 2h\alpha, \quad \text{or} = 0,$$

according as h is ≥ 0 , or $= 0$. The system of particles is here supposed to extend indefinitely in two opposite directions from the particle P_0 , so that no account is taken of any fixity of the extreme particles.

42. Resuming then the consideration of the case where half only of the system is agitated at the time 0, we see that if this system be indefinite in both directions, and if its initial state be represented by the formula

$$y_{h,at} = \cos (2h\alpha - \alpha) \cdot (b \cos + c \sin) (2a \sin \alpha dt), \quad \text{or} = 0,$$

according as h is not greater than 0 or is greater than 0, its state at the time t is represented by the formula

$$y_{h,t} = \frac{1}{2} (b \cos + c \sin) (2at \sin \alpha) \cos (2h\alpha - \alpha) - \frac{2}{\pi} \cos \alpha \int_0^{\frac{\pi}{2}} d\theta \frac{\sin (2h\theta - \theta)}{\cos 2\alpha - \cos 2\theta} (b \sin \theta \cos + c \sin \alpha \sin) (2at \sin \theta).$$

And if the initial state be

$$y_{h,at} = - (b \cos + c \sin) (2a \sin \alpha dt) \sin (2h\alpha - \alpha), \quad \text{or } = 0,$$

according as h is not greater than 0, or greater than 0, then the state at the time t will be

$$y_{h,t} = -\frac{1}{2} (b \cos + c \sin) (2at \sin \alpha) \sin (2h\alpha - \alpha) - \frac{2}{\pi} \sin \alpha \int_0^{\frac{\pi}{2}} d\theta \frac{\cos \theta \cos (2h\theta - \theta)}{\sin \theta \cos 2\alpha - \cos 2\theta} (b \sin \theta \cos + c \sin \alpha \sin) (2at \sin \theta),$$

for all (integer) values of h . And hence, by an easy combination, we find that if the initial state be

$$y_{h,at} = (b \cos + c \sin) (2a \sin \alpha dt) \sin 2h\alpha, \quad \text{or } = 0,$$

according as h is ≥ 0 or > 0 , the state at the time t is

$$y_{h,t} = \frac{1}{2} (b \cos + c \sin) (2at \sin \alpha) \sin 2h\alpha + \frac{\sin 2\alpha}{\pi} \int_0^{\frac{\pi}{2}} d\theta \frac{\cos 2h\theta}{\sin \theta \cos 2\alpha - \cos 2\theta} (b \sin \theta \cos + c \sin \alpha \sin) (2at \sin \theta).$$

In like manner, if the initial state be

$$y_{h,at} = (b \cos + c \sin) (2a \sin \alpha dt) \cos 2h\alpha, \quad \text{or } = 0, \text{ as } h \geq 0 \text{ or } > 0,$$

then the state at the time t is

$$\begin{aligned} y_{h,t} &= \frac{1}{2} (b \cos + c \sin) (2at \sin \alpha) \cos 2h\alpha \\ &+ \frac{\cos 2\alpha}{\pi} \int_0^{\frac{\pi}{2}} d\theta \frac{\cos 2h\theta}{\sin \theta \cos 2\alpha - \cos 2\theta} (b \sin \theta \cos + c \sin \alpha \sin) (2at \sin \theta) \\ &- \frac{1}{\pi} \int_0^{\frac{\pi}{2}} d\theta \frac{\cos (2h\theta - 2\theta)}{\sin \theta \cos 2\alpha - \cos 2\theta} (b \sin \theta \cos + c \sin \alpha \sin) (2at \sin \theta) \\ &= \frac{1}{2} (b \cos + c \sin) (2at \sin \alpha) \cos 2h\alpha \\ &- \frac{2}{\pi} \int_0^{\frac{\pi}{2}} d\theta \frac{\cos \theta \sin 2h\theta}{\cos 2\alpha - \cos 2\theta} (b \sin \theta \cos + c \sin \alpha \sin) (2at \sin \theta) \\ &+ \frac{1}{\pi} \int_0^{\frac{\pi}{2}} d\theta \frac{\cos 2h\theta}{\sin \theta} (b \sin \theta \cos + c \sin \alpha \sin) (2at \sin \theta). \end{aligned}$$

43. The third conclusion of article 42 might also easily have been deduced from the fourth conclusion of 41. And the fourth conclusion of article 42 might have been deduced from the third conclusion of 41, namely from the theorem that if the initial state be

$$y_{h,at} = \mp \frac{1}{2} (b \cos + c \sin) (2a \sin \alpha dt) \cos 2h\alpha, \quad \text{or } = 0,$$

according as h is ≥ 0 or $= 0$, then the state at the time t is

$$y_{h,t} = -\frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\theta \cos \theta \sin 2h\theta}{\cos 2\alpha - \cos 2\theta} (b \sin \theta \cos + c \sin \alpha \sin) (2at \sin \theta).$$

For we have only to add to this the term $\frac{1}{2}(b \cos + c \sin)(2at \sin \alpha) \cos 2h\alpha$, to allow for the additional parts of the initial states of all the particles except P_0 ; and then to allow for the remaining part of the initial state of that one particle, namely the term

$$y_{0,at} = \frac{1}{2}(b \cos + c \sin)(2a \sin \alpha dt) = \frac{1}{2}b + ac \sin \alpha dt,$$

by means of the formula (12) of article 18; which shows that, in the indefinite system here considered, the effect of this initial state $y_{0,at}$ of the particle P_0 on any other particle P_h at any time t is

$$\begin{aligned} \frac{1}{\pi} \left(b + 2ac \sin \alpha \int_0^t dt \right) \int_0^{\frac{\pi}{2}} d\theta \cos 2h\theta \cos (2at \sin \theta) \\ = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} d\theta \cos 2h\theta \left(b \cos + \frac{\sin \alpha}{\sin \theta} c \sin \right) (2at \sin \theta). \end{aligned}$$

44. To treat now the question proposed in article 32, we are to suppose, in passing to the limit there required, that for *all* integer values of $h > 0$, we have

$$y_{j+h,0} = 0, \quad y'_{j+h,0} = 0,$$

j being some very large integer number which however is to be treated as given; (but as infinite; that is, in one part of the calculation we are not to consider it as varying, but in another part of the same calculation we are to treat it as increasing without limit;) and for *all* integer values of $h > 0$,

$$\begin{aligned} y_{j+h,0} &= B_k' \cos \beta_k' \sin 2(j+h)\alpha, \\ y'_{j+h,0} &= 2a \sin \alpha B_k' \sin \beta_k' \sin 2(j+h)\alpha, \end{aligned}$$

B_k' and β_k' being arbitrary constants, and $\alpha = \frac{\frac{1}{2}k\pi}{n+1}$ = some given and finite arc. And the problem may be considered as being to find a function $y_{j+h,t}$ of h and t , which shall satisfy the initial conditions just now mentioned, & also the indefinite equation in mixed differences

$$y''_{j+h,t} = a^2 (y_{j+h+1,t} + y_{j+h-1,t} - 2y_{j+h,t}).$$

Now, the initial state $y_{j+h,at}$ here proposed may be considered as the sum of two others, of which one is expressed by the formula

$$B_k' \cos 2j\alpha (\cos \beta_k' \cos + \sin \beta_k' \sin) (2adt \sin \alpha) \sin 2h\alpha, \quad \text{or } 0,$$

and the other by the formula

$$B_k' \sin 2j\alpha (\cos \beta_k' \cos + \sin \beta_k' \sin) (2adt \sin \alpha) \cos 2h\alpha, \quad \text{or } 0,$$

according as the integer h is \succ or > 0 . The first part of the initial state gives, for its own effect at the time t , by article 42,

$$\begin{aligned} \frac{1}{2} B_k' \cos 2j\alpha (\cos \beta_k' \cos + \sin \beta_k' \sin) (2at \sin \alpha) \sin 2h\alpha \\ + \frac{1}{\pi} B_k' \cos 2j\alpha \sin 2\alpha \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin \theta} \frac{\cos 2h\theta}{\cos 2\alpha - \cos 2\theta} (\cos \beta_k' \sin \theta \cos + \sin \beta_k' \sin \alpha \sin) (2at \sin \theta); \end{aligned}$$

and the second part of the initial state gives, by the same article,

$$\begin{aligned} & \frac{1}{2} B_k \sin 2j\alpha (\cos \beta_k \cos + \sin \beta_k \sin) (2at \sin \alpha) \cos 2h\alpha \\ & - \frac{2}{\pi} B_k \sin 2j\alpha \int_0^{\frac{\pi}{2}} d\theta \frac{\cos \theta \sin 2h\theta}{\cos 2\alpha - \cos 2\theta} (\cos \beta_k \sin \theta \cos + \sin \beta_k \sin \alpha \sin) (2at \sin \theta) \\ & + \frac{1}{\pi} B_k \sin 2j\alpha \int_0^{\frac{\pi}{2}} d\theta \cos 2h\theta \left(\cos \beta_k \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta_k \sin \right) (2at \sin \theta); \end{aligned}$$

the last line of which last expression would have disappeared if the initial values $y_{j,0}$ and $y'_{j,0}$ had been only half as great as they are here supposed to be. The whole effect at the time t , or the expression for $y_{j+h,t}$, is, therefore, in the present question,

$$\begin{aligned} y_{j+h,t} = & \frac{1}{2} B_k \sin 2(j+h)\alpha \cdot (\cos \beta_k \cos + \sin \beta_k \sin) (2at \sin \alpha) \\ & + \frac{1}{\pi} B_k \int_0^{\frac{\pi}{2}} d\theta \frac{\cos 2h\theta \sin 2(j+1)\alpha - \sin 2j\alpha \cos 2(h-1)\theta}{\cos 2\alpha - \cos 2\theta} \\ & \times \left(\cos \beta_k \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta_k \sin \right) (2at \sin \theta). \quad (26) \end{aligned}$$

Accordingly it is evident that this expression satisfies the indefinite equation in mixed differences; & it satisfies also the initial conditions, because the theorem of article 38,

$$\int_0^{\frac{\pi}{2}} \frac{\cos 2h\theta d\theta}{\cos 2\alpha - \cos 2\theta} = \mp \frac{\pi \sin 2h\alpha}{2 \sin 2\alpha}, \quad \text{or } = 0,$$

according as the integer h is ≥ 0 , or $= 0$, gives

$$\begin{aligned} & \frac{1}{\pi} B_k \int_0^{\frac{\pi}{2}} d\theta \frac{\cos 2h\theta \sin 2(j+1)\alpha - \sin 2j\alpha \cos 2(h-1)\theta}{\cos 2\alpha - \cos 2\theta} \\ & = \mp \frac{1}{2} B_k \frac{\sin 2h\alpha \sin 2(j+1)\alpha - \sin 2j\alpha \sin 2(h-1)\alpha}{\sin 2\alpha} \\ & = \mp \frac{1}{2} B_k (\sin 2h\alpha \cos 2j\alpha + \sin 2j\alpha \cos 2h\alpha) \\ & = \mp \frac{1}{2} B_k \sin 2(j+h)\alpha, \end{aligned}$$

according as h is $>$ or $\nabla 0$.

45. We ought also to be able to verify the expression obtained in the last article, by deducing from it those of article 37. Suppose then $k = 2\kappa - 1$ and $n = 2j$; we shall have $\alpha = \frac{(\kappa - \frac{1}{2})\pi}{2j+1}$; therefore

$$\cos (2j\alpha + \alpha) = 0, \quad \sin (2j\alpha + \alpha) = (-1)^{\kappa+1};$$

therefore

$$\begin{aligned} \sin 2j\alpha &= \sin (2j\alpha + 2\alpha) = (-1)^{\kappa+1} \cos \alpha; \quad \cos 2j\alpha = (-1)^{\kappa+1} \sin \alpha; \\ \sin 2(j+h)\alpha &= (-1)^{\kappa+1} \cos (2h\alpha - \alpha); \end{aligned}$$

and

$$\begin{aligned} y_{j+h,t} = & \frac{1}{2} (-1)^{\kappa+1} B_{2\kappa-1} \cos (2h\alpha - \alpha) \cdot (\cos \beta_{2\kappa-1} \cos + \sin \beta_{2\kappa-1} \sin) (2at \sin \alpha) \\ & - \frac{2}{\pi} B_{2\kappa-1} (-1)^{\kappa+1} \cos \alpha \int_0^{\frac{\pi}{2}} d\theta \frac{\sin (2h\theta - \theta)}{\cos 2\alpha - \cos 2\theta} \\ & \times (\cos \beta_{2\kappa-1} \sin \theta \cos + \sin \beta_{2\kappa-1} \sin \alpha \sin) (2at \sin \theta); \end{aligned}$$

an expression which accordingly differs from the corresponding one (24) of article 37 only by the addition of the first line, which was there purposely suppressed. Again, if $k = 2\kappa$, $\alpha = \frac{\kappa\pi}{2j+1}$,

$$\text{then } \sin(2j\alpha + \alpha) = 0, \quad \cos(2j\alpha + \alpha) = (-1)^\kappa, \quad \sin(2j\alpha + 2\alpha) = (-1)^\kappa \sin \alpha = -\sin 2j\alpha,$$

$$\cos 2j\alpha = (-1)^\kappa \cos \alpha, \quad \sin 2(j+h)\alpha = (-1)^\kappa \sin(2h\alpha - \alpha),$$

$$\text{and } y_{j+h,t} = \frac{1}{2} (-1)^\kappa B_{2\kappa} \sin(2h\alpha - \alpha) \cdot (\cos \beta_{2\kappa} \cos + \sin \beta_{2\kappa} \sin) (2at \sin \alpha)$$

$$+ \frac{2}{\pi} B_{2\kappa} (-1)^\kappa \sin \alpha \int_0^{\frac{\pi}{2}} d\theta \frac{\cos \theta \cos(2h\theta - \theta)}{\cos 2\alpha - \cos 2\theta} \left(\cos \beta_{2\kappa} \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta_{2\kappa} \sin \right) (2at \sin \theta);$$

an expression agreeing, as closely as it ought, with (25) of article 37.

46. The results obtained in recent articles may be used so as to throw light upon the analysis begun in article 32. In fact we may now easily perceive that, by admitting the transformation in article 31 of sums into integrals, an expression for $y_{l,t}$ or for $y_{j+h,t}$ is deduced, involving functions of $l \pm j$, namely an expression consisting of the two following parts*:

$$\frac{B_k}{\pi} \int_0^{\frac{\pi}{2}} d\theta \frac{\cos 2(j-l)\theta \cdot \sin 2(j+1)\alpha - \cos 2(j-l+1)\theta \cdot \sin 2j\alpha}{\cos 2\alpha - \cos 2\theta} \times \left(\cos \beta_k \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta_k \sin \right) (2at \sin \theta);$$

and

$$- \frac{B_k}{\pi} \int_0^{\frac{\pi}{2}} d\theta \frac{\cos 2(j+l)\theta \cdot \sin 2(j+1)\alpha - \cos 2(j+l+1)\theta \cdot \sin 2j\alpha}{\cos 2\alpha - \cos 2\theta} \times \left(\cos \beta_k \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta_k \sin \right) (2at \sin \theta).$$

The first of these two parts coincides with the second part of the expression for $y_{j+h,t}$ in article 44, when we change l to $j+h$. With respect to the second of the two parts assigned in the present article, it may be remarked that (see article 32)

$$- \cos 2(j+l)\theta \cdot \sin 2(j+1)\alpha + \cos 2(j+l+1)\theta \cdot \sin 2j\alpha \\ = -\sin(\theta + \alpha) \cdot \cos \{(2j+2l+1)(\theta - \alpha) + 2l\alpha\} + \sin(\theta - \alpha) \cdot \cos \{(2j+2l+1)(\theta + \alpha) - 2l\alpha\};$$

and that $\cos 2\alpha - \cos 2\theta = 2 \sin(\theta - \alpha) \sin(\theta + \alpha)$; therefore, dividing by this latter function, and neglecting the terms which have no small divisors and those which change sign with $\theta - \alpha$, we find, for the part still to be considered, the expression

$$\frac{1}{2\pi} B_k \int_0^{\frac{\pi}{2}} d\theta \frac{\sin(2j+2l+1)(\theta - \alpha)}{\sin(\theta - \alpha)} \sin 2l\alpha (\cos \beta_k \cos + \sin \beta_k \sin) (2at \sin \alpha) \\ = \frac{1}{2} B_k \sin 2(j+h)\alpha \cdot \cos(2at \sin \alpha - \beta_k),$$

coinciding with the first part of the expression in article 44, because $j+l$ increases without limit.

* [If these two parts are joined together under the integral sign, the resulting integrand does not become infinite when $\theta = \alpha$ and can be evaluated by ordinary methods. The process here consists of taking each part separately and interpreting each integral by Cauchy's method and the method of fluctuation.]

47. Resuming now the analysis of articles 30 and 32, we see, in the first place, that the expression (18) may be decomposed into two parts, namely the two values of the expression

$$\pm \frac{\phi}{\pi} B_k \sum_{(i)1}^n \frac{\cos 2(j \mp l) i \phi \cdot \sin 2(j+1) k \phi - \cos 2(j \mp l + 1) i \phi \cdot \sin 2jk \phi}{\cos 2k \phi - \cos 2i \phi} \times \left(\cos \beta_k \cos + \frac{\sin k \phi}{\sin i \phi} \sin \beta_k \sin \right) (2at \sin i \phi);$$

in which $\phi = \frac{\frac{1}{2}\pi}{n+1}$, and j, k, l, n are finite. We may also change

$$y_{l,t} = \sum_{(i)1}^n C_i \quad \text{to} \quad C_k + \sum_{(i)1}^{k-1} C_i + \sum_{(i)k+1}^n C_i,$$

in which

$$C_k = \frac{\frac{1}{2} B_k}{n+1} \cos(2at \sin k \phi - \beta_k) \sin 2kl \phi \left\{ 2j+1 - \frac{\sin 2(2j+1) k \phi}{\sin 2k \phi} \right\};$$

this last expression being rigorously that part of $y_{l,t}$ which has rigorously the periodic time $\frac{\pi}{a} \operatorname{cosec} k \phi$. Let j, k, l, n be very large, but such that the ratios $\frac{j}{n}, \frac{k}{n}, \frac{l}{n}$ are sensibly > 0 and < 1 ,

and that $\frac{j-l}{n}$ is sensibly $= 0$. Then ϕ is extremely small, and so is even $2(j-l)\phi = \frac{j-l}{n+1} \pi$,

although the number $j-l$ may be considerable. Thus, the part C_i of C_i which involves $j-i$ alters very little when i is changed to $i+1$; unless the denominator becomes small by i being nearly $= k$, or at least by $i\phi - k\phi$ being small, which may be while $i-k$ is considerable; and therefore, with respect to this part C_i , the conversion of summation into integration is permitted unless it shall be found that this conversion is invalid near the critical value $i\phi = k\phi$. To examine what happens near this value, let $i = k + g, g$ being an integer > 0 or < 0 which may be considerable itself but is to be so chosen that the product $g\phi$ may be moderately small: & let us calculate $C_{k+g} + C_{k-g}$. This sum is found to involve $\frac{\sin 2(l-j)g\phi}{\sin g\phi}$ and $\frac{\sin(2at \cos k\phi \sin g\phi)}{\sin g\phi}$; that is, we

have to sum expressions of the form

$$\phi \times f(g\phi), \quad \phi \times \frac{\sin 2(l-j)g\phi}{\sin g\phi} f(g\phi), \quad \phi \times \frac{\sin(2at \cos k\phi \sin g\phi)}{\sin g\phi} f(g\phi),$$

from $g = 1$ to $g = a$ large integer, and the functions $f(g\phi)$ not varying rapidly near the lower limit of this summation, while ϕ is still extremely small and tends to 0. But such summations $\sum_{(g)1}^a \phi \times (\&c.)$ may be replaced by the definite integrations $\int_0^{g\phi} d(g\phi) (\&c.)$; & therefore, (on

account chiefly of $\sin \frac{\pi(l-j)}{n+1}$ and $\sin \left(2at \cos \alpha \sin \frac{\frac{1}{2}\pi}{n+1} \right)$ bearing determined limits to $\sin \frac{\frac{1}{2}\pi}{n+1}$ when n tends to ∞ .) it is permitted to change the summation $(\sum_{(i)1}^{k-1} + \sum_{(i)k+1}^n) \phi$ into a definite

integration $\int_{\frac{\pi}{2}}^{\pi} d\theta$, for that part C_i of C_i which depends on $j-l$. With respect to the other part C_i of C_i which involves $j+l$, we see that this part involves cosines of arcs which receive a finite increment

$$= 2(j+l)\phi = \frac{(j+l)\pi}{n+1},$$

when i is changed to $i+1$; while these cosines are multiplied by functions which receive, by

the same change of i , only infinitely small alterations, except near the critical value $i\phi = k\phi$. It is therefore permitted to reject all values of i which do not render $g\phi = i\phi - k\phi$ small; and the limit of this smallness is 0. We may therefore, after employing the transformation indicated in article 32, change i into k , or θ into α , except in

$$\frac{\cos \{2l\alpha + (2j + 2l + 1) g\phi\} + \cos \{2l\alpha - (2j + 2l + 1) g\phi\}}{\sin g\phi}$$

which may be reduced to

$$\frac{2 \sin 2l\alpha \sin (2j + 2l + 1) g\phi}{g\phi}$$

It remains therefore to calculate the sum

$$\sum_{(g)1}^{\infty} \frac{1}{g} \sin \frac{(j + l + \frac{1}{2}) g\pi}{n + 1};$$

for we shall have

$$(\sum_{(i)1}^{k-1} + \sum_{(i)k+1}^n) C_i^n = \frac{B_k'}{\pi} \sin 2l\alpha \cos (2at \sin \alpha - \beta_k') \sum_{(g)1}^{\infty} \frac{1}{g} \sin \frac{g(j + l + \frac{1}{2}) \pi}{n + 1}.$$

But $\frac{(j + l + \frac{1}{2}) \pi}{n + 1}$ is $> 0, < 2\pi$; therefore, by a known theorem,

$$\sum_{(g)1}^{\infty} \frac{1}{g} \sin \frac{g(j + l + \frac{1}{2}) \pi}{n + 1} = \frac{\pi}{2} \left(1 - \frac{j + l + \frac{1}{2}}{n + 1} \right)$$

accurately; we may therefore write, for this sum, $\frac{\pi}{2} \left(1 - \frac{j}{n} \right)$, & we have for that part of $\sum_{(i)1}^{(k-1 + n_{k+1})} C_i$ (observe this notation) which depends on $j + l$, the expression

$$\left(\frac{1}{2} - \frac{j}{n} \right) B_k' \sin 2l\alpha \cos (2at \sin \alpha - \beta_k').$$

And since $C_k = \frac{j}{n} B_k' \sin 2l\alpha \cos (2at \sin \alpha - \beta_k')$, we have, upon the whole,

$$y_{l,i} = \frac{1}{2} B_k' \sin 2l\alpha \cos (2at \sin \alpha - \beta_k') + \frac{1}{\pi} B_k' \int_0^{\frac{\pi}{2}} d\theta \frac{\cos 2(l-j)\theta \cdot \sin 2(j+1)\alpha - \cos 2(l-j-1)\theta \cdot \sin 2j\alpha}{\cos 2\alpha - \cos 2\theta} \times \left(\cos \beta_k' \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta_k' \sin \right) (2at \sin \theta); \quad (27)$$

an expression which coincides with that marked (26) in article 44.

48. The analysis of the foregoing article shows, at the same time, by what steps we may pass back from this expression (27) or (26) to that marked (18) in article 30; that is, from the supposition of n infinite to that of n finite. In this return, we are 1st to restore for α its value $\frac{1}{2}k\pi$; 2nd to change $\frac{1}{\pi} \int_0^{\frac{\pi}{2}} d\theta F(\theta)$ to $\frac{1}{2(n+1)} \sum_{(i)1}^n F\left(\frac{1}{2}i\pi\right)$; 3rd in the part free from the sign of integration, namely $\frac{1}{2} B_k' \sin 2l\alpha \cos (2at \sin \alpha - \beta_k')$, (α retaining for abridgement its meaning just recited,) to resolve the factor $\frac{1}{2}$ into two parts which are nearly $\frac{j}{n}$ and $\frac{1}{2} - \frac{j}{n}$; or, more precisely, to substitute for this part free of the sign \int the sum of the two following functions:

$$C_k = \frac{B_k'}{n+1} \cos (2at \sin \alpha - \beta_k') \sin 2l\alpha \left\{ \frac{2j+1}{2} - \frac{\sin 2(2j+1)\alpha}{2 \sin 2\alpha} \right\};$$

and

$$\frac{B_k'}{4(n+1)} \sum_{(i)} \binom{k-1+n}{i} \left(\frac{\cos \{(2j+2l+1)(\theta+\alpha) - 2l\alpha\}}{\sin(\theta+\alpha)} - \frac{\cos \{(2j+2l+1)(\theta-\alpha) + 2l\alpha\}}{\sin(\theta-\alpha)} \right) \times \left(\cos \beta_k' \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta_k' \sin \right) (2at \sin \theta);$$

in which $\theta = \frac{\frac{1}{2}i\pi}{n+1}$.

49. With respect to the physical meaning of this last resolution of the factor $\frac{1}{2}$ into the two parts $\frac{j}{n}$ and $\frac{1}{2} - \frac{j}{n}$, the foregoing analysis shows that the part $\frac{j}{n}$ corresponds to the immediate effect of the initial state, namely

$$y_{l,at} = B_k' \sin 2l\alpha \cos (2a \sin \alpha t - \beta_k'), \quad \text{or } = 0,$$

according as $l >$ or $> j$, in producing the part

$$\frac{j}{n} B_k' \sin 2l\alpha \cos (2at \sin \alpha - \beta_k'),$$

for all values of l (from 1 to n) or, more precisely (when n is finite) the part

$$\frac{2j+1 - \frac{\sin 2(2j+1)\alpha}{\sin 2\alpha}}{2(n+1)} B_k' \sin 2l\alpha \cos (2at \sin \alpha - \beta_k'),$$

with the same periodic time $\frac{\pi}{a} \operatorname{cosec} \alpha$, & the same number of venters $k = \frac{2\alpha}{\pi} (n+1)$, as there would be in the initial state, if that were extended to all values of l and t . In such a manner that if one third part only of the system (supposed numerous) be originally agitated so as to correspond with a given simple mode of vibration, or with a given value of k , then the whole system becomes agitated with all possible simple modes superposed upon each other, corresponding to all possible values of i (from 1 to n), but the amplitude of the k^{th} mode is $\frac{1}{2}$ of the initial amplitude. And the modes for which i is nearly equal to k , or more precisely for which θ is nearly equal to α , so that their periodic times $\frac{\pi}{a} \operatorname{cosec} \theta$ are only a little less or a little greater than $\frac{\pi}{a} \operatorname{cosec} \alpha$, besides producing the effect expressed by the definite integral in the formula (27) or (26), produce also a resultant mode which (if n be large) coincides nearly with the simple initial mode k & has an amplitude which bears to the initial amplitude the ratio of $\frac{1}{2} - \frac{j}{n}$ to 1. Thus, when (as in the case just now mentioned) the initial agitation occupied only the third part of the (numerous) system, so that $\frac{j}{n} = \frac{1}{3}$, we have $\frac{1}{2} - \frac{j}{n} = \frac{1}{6}$, and the indirect effect (extending to the whole system) increases by $\frac{1}{6}$ of the initial amplitude the immediate or direct effect which had been found to amount to $\frac{1}{3}$. This indirect effect is produced in an indefinitely short time t , and then is permanent; so that if n be very large, there is at once produced for the whole system a permanent mode of vibration which coincides with the initial simple mode in all respects except that of having an amplitude only half as great; which ratio does not require for its establishment that the part originally agitated should be exactly or nearly half of the whole system. The remaining effect, expressed by the definite integral, corresponds to a complex mode of vibration formed by the superposition of infinitely many simple modes; but when the time elapsed is very

small, it reduces itself sensibly to a single mode, namely the initial mode k ; or rather to two coexisting movements with the initial period & epoch, but with amplitudes which bear to the initial amplitude the ratios of $\frac{1}{2}$ and $-\frac{1}{2}$, according as l is $<$ or $>$ j , and so reproduce the given initial discontinuity. We shall soon consider whether any and what reduction of the same sort takes place when the time elapsed is large.

50. The definite integral in (26) or (27) may be transformed by observing that of the two parts $\sin(\theta + \alpha) \cos\{(2j - 2l + 1)\theta - (2j + 1)\alpha\}$ and $-\sin(\theta - \alpha) \cos\{(2j - 2l + 1)\theta + (2j + 1)\alpha\}$, into which (as was remarked in article 32) the numerator

$$\cos 2(l - j)\theta \cdot \sin 2(j + 1)\alpha - \cos 2(l - j - 1)\theta \cdot \sin 2j\alpha$$

may be decomposed, the second results from the first by changing θ to $\pi - \theta$; while $\cos 2\theta$ in the denominator & $\sin \theta$ do not alter by such change. In this manner we find that the formula (27) may be thus written:

$$y_{l,t} = \frac{1}{2} B_k \sin 2l\alpha \cos(2at \sin \alpha - \beta_k) + \frac{1}{2\pi} B_k \int_0^\pi d\theta \frac{\cos\{(2l - 2j - 1)(\theta - \alpha) + 2l\alpha\}}{\sin(\theta - \alpha)} \left(\cos \beta_k \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta_k \sin \right) (2at \sin \theta). \quad (28)$$

(As this is a decided simplification of the integral (27), it will be interesting to inquire whether we cannot find a similar simplification of the sum (18).)

51. Under this last form, as under those found before, we see clearly that the function $y_{l,t}$ satisfies the indefinite equation in mixed differences; and to show that it satisfies also the initial conditions, we ought to be able to show that

$$\frac{1}{\pi} \int_0^\pi d\theta \frac{\cos\{(2h - 1)(\theta - \alpha) + 2l\alpha\}}{\sin(\theta - \alpha)} = \mp \sin 2l\alpha,$$

according as $2h - 1$ is ≥ 0 . This discontinuous equation appears to resolve itself into the two following:

$$\int_0^\pi d\theta \frac{\cos(2h - 1)(\theta - \alpha)}{\sin(\theta - \alpha)} = 0,$$

for all integer values of h ; and

$$\int_0^\pi d\theta \frac{\sin(2h - 1)(\theta - \alpha)}{\sin(\theta - \alpha)} = \pm \pi,$$

according as the integer h is $>$ or ≥ 0 . Accordingly

$$\Delta_h \int_0^\pi d\theta \frac{\cos(2h - 1)(\theta - \alpha)}{\sin(\theta - \alpha)} = -2 \int_0^\pi d\theta \sin 2h(\theta - \alpha) = 0,$$

$$\Delta_h \int_0^\pi d\theta \frac{\sin(2h - 1)(\theta - \alpha)}{\sin(\theta - \alpha)} = 2 \int_0^\pi d\theta \cos 2h(\theta - \alpha) = 0, \quad \text{or } 2\pi,$$

according as h is \geq or $= 0$; so that it only remains to prove that $\int_0^\pi \frac{d\theta}{\tan(\theta - \alpha)} = 0$, and this integral

$$= \int_0^{\frac{\pi}{2}} d\theta \{\cotan(\theta - \alpha) - \cotan(\theta + \alpha)\} = \int_0^{\frac{\pi}{2}} \frac{d\theta \sin 2\alpha}{\sin(\theta + \alpha) \sin(\theta - \alpha)} = 2 \sin 2\alpha \int_0^{\frac{\pi}{2}} \frac{d\theta}{\cos 2\alpha - \cos 2\theta} = 0$$

by article 38.*

* [See note, p. 469.]

52. It is evident also from inspection of the integral in (28) that this integral reduces itself to $-\frac{1}{2}B_k \sin 2l\alpha \cos (2at \sin \alpha - \beta_k)$, that is to the part free from the sign \int , taken negatively, (so that the one part of the formula destroys the other,) if we take $j = -\infty$, or $h = l - j = \infty$. A result which might have been expected, because, by throwing indefinitely far back in the system the origin of the disturbance we must render the effect of that disturbance insensible for any finite values of l and t . And we have thus a new explanation of the term independent of integration; namely that it is the negative of the value of the integral term for $j = -\infty$. We may therefore write the formula (28) as follows:

$$y_{l,t} = \frac{B_k}{2\pi} \int_0^\pi \frac{d\theta}{\sin(\theta - \alpha)} [\cos \{(2l - 2j - 1)(\theta - \alpha) + 2l\alpha\} - \cos \{(2l - 2j + 1)(\theta - \alpha) + 2l\alpha\}] \\ \times \left(\cos \beta_k \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta_k \sin \right) (2at \sin \theta); \quad (29)$$

in which $j = -\infty$. And if we treat j , as finite, we may then consider this last formula as expressing the solution of the question:

To find a function $y_{l,t}$ which shall satisfy the indefinite equation in mixed differences (1), and also the initial conditions

$$y_{l,at} = B_k \sin 2l\alpha \cos (2a \sin \alpha dt - \beta_k), \quad \text{or } = 0,$$

according as l is, or is not, one of the $j - j + 1$ successive integers $j, j + 1, \dots, j - 1, j$.

53. This question might have been resolved by the help of the formula (12), which, when applied to it, becomes

$$y_{l,t} = \frac{2}{\pi} B_k \int_0^{\frac{\pi}{2}} d\theta \{ \sum_{(j)j}^j \sin 2j\alpha \cos 2(j-l)\theta \} \left(\cos \beta_k \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta_k \sin \right) (2at \sin \theta); \quad (30)$$

in which

$$2 \sum_{(j)j}^j \sin 2j\alpha \cos 2(j-l)\theta = \sum_{(j)j}^j \{ \sin 2(j\theta + j\alpha - l\theta) - \sin 2(j\theta - j\alpha - l\theta) \} \\ = \frac{\cos \{(2j - 1)(\theta + \alpha) - 2l\theta\} - \cos \{(2j + 1)(\theta + \alpha) - 2l\theta\}}{2 \sin(\theta + \alpha)} \\ - \frac{\cos \{(2j - 1)(\theta - \alpha) - 2l\theta\} - \cos \{(2j + 1)(\theta - \alpha) - 2l\theta\}}{2 \sin(\theta - \alpha)};$$

so that the formula (30) reduces itself to (29). And because the formula (12) admits of a very easy proof, and may almost be said to be obviously true, it might have been a better or at least a more elementary mode of proceeding to have begun by deducing (30) from it & to have then transformed (30) into (29) in the manner just now indicated; after which it would have been easy to pass to (28) as the limit corresponding to the supposition $j = -\infty$.

54. The formula (29) may be thus written:

$$y_{l,t} = \frac{1}{\pi} B_k \int_0^\pi d\theta \frac{\sin(j - j + 1)(\theta - \alpha)}{\sin(\theta - \alpha)} \sin \{(2l - j - j)(\theta - \alpha) + 2l\alpha\} \\ \times \left(\cos \beta_k \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta_k \sin \right) (2at \sin \theta); \quad (31)$$

in which we may remember that $j - j + 1$ is the number of particles P_j, \dots, P_j originally agitated;

and $l - \frac{j+j_1}{2}$ is the distance of the particle P_l from the middle of the initial agitation. If $j_1 = j$, that is, if there be but one particle originally agitated, this last expression becomes

$$y_{l,t} = \frac{1}{\pi} B_k \int_0^\pi d\theta \sin \{2(l-j)\theta + 2j\alpha\} \left(\cos \beta_k \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta_k \sin \right) (2at \sin \theta) \\ = \frac{2}{\pi} B_k \sin 2j\alpha \int_0^{\frac{\pi}{2}} d\theta \cos \{2(l-j)\theta\} \left(\cos \beta_k \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta_k \sin \right) (2at \sin \theta),$$

as it ought to do.

55. Article 32 and all the subsequent articles have had reference chiefly to the case of a system or series of particles extending indefinitely in *both* directions from the particle P_l , of which the motion is to be examined; but it is easy to deduce analogous results for the case considered in article 31, in which the system is indefinite in *one* direction only, the particle P_0 being fixed. The formula (19) for this last mentioned case may be thus written:

$$y_{l,t} = \frac{1}{\pi} B_k \int_0^\pi d\theta \frac{\sin (2j+1)(\theta-\alpha)}{\sin (\theta-\alpha)} \sin 2l\theta \left(\cos \beta_k \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta_k \sin \right) (2at \sin \theta); \quad (32)$$

and might have been obtained from (10) under the form

$$y_{l,t} = \frac{4}{\pi} B_k \int_0^{\frac{\pi}{2}} d\theta (\Sigma_{(j)1}^j \sin 2j\alpha \sin 2j\theta) \sin 2l\theta \left(\cos \beta_k \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta_k \sin \right) (2at \sin \theta). \quad (33)$$

And if, instead of $\Sigma_{(j)1}^j$, we take $\Sigma_{(j)j}^j$, that is, if we suppose only the $j-j_1+1$ consecutive particles P_{j_1}, \dots, P_j to be originally agitated, we have then

$$y_{l,t} = \frac{2}{\pi} B_k \int_0^\pi d\theta \frac{\sin (j-j_1+1)(\theta-\alpha)}{\sin (\theta-\alpha)} \cos (j+j_1)(\theta-\alpha) \sin 2l\theta \\ \times \left(\cos \beta_k \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta_k \sin \right) (2at \sin \theta). \quad (34)$$

If only one particle P_j be originally agitated, then $j_1 = j$, and the last formula becomes

$$y_{l,t} = \frac{2}{\pi} B_k \int_0^\pi d\theta \cos 2j(\theta-\alpha) \sin 2l\theta \left(\cos \beta_k \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta_k \sin \right) (2at \sin \theta) \\ = \frac{4}{\pi} B_k \sin 2j\alpha \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \left(\cos \beta_k \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta_k \sin \right) (2at \sin \theta),$$

agreeing evidently with (10).

56. It is worth observing that the formula (32) may be obtained from (31) by changing j , to $-j$. And it is easy to explain this circumstance. In fact, instead of supposing P_0 fixed by any external cause, we may suppose it to be originally at rest, and to remain so because $y_{-l,0} = -y_{l,0}$ and $y'_{-l,0} = -y'_{l,0}$, as was remarked in article 19. But in this view we must suppose, in the question of articles 31 and 55, that the $2j+1$ particles P_{-j}, \dots, P_j are all originally agitated according to the law $B_k \sin 2l\alpha \cos (2a \sin \alpha t - \beta_k)$; except the particle P_0 , which fulfils this law by being undisturbed, & any others which in like manner have $\sin 2l\alpha = 0$.

57. An analogous reasoning may be employed to deduce the solution of the 1st Problem from the formula (12), or the laws of vibration of a finite from those of an infinite system. To

illustrate this transition, let us begin by considering the case where, for all values of j , positive & negative,

$$y_{j,0} = B_1 \sin \frac{j\pi}{2}, \quad y'_{j,0} = 0;$$

so that

$$y_{2j,0} = 0 \quad \text{and} \quad y_{4j+1,0} = -y_{4j-1,0} = B_1.$$

The formula (12) may in general be thus written:

$$y_{i,t} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} d\theta \sum_{(j)-\infty}^{\infty} \cos(2l\theta - 2j\theta) \left(y_{j,0} + y'_{j,0} \int_0^t dt \right) \cos(2at \sin \theta); \quad (35)$$

& in the present case it becomes

$$y_{i,t} = \frac{4B_1}{\pi} \int_0^{\frac{\pi}{2}} d\theta \left(\sum_{(j)1}^{\infty} \sin 2j\theta \sin \frac{j\pi}{2} \right) \sin 2l\theta \cos(2at \sin \theta);$$

in which

$$2 \sum_{(j)1}^{\infty} \sin 2j\theta \sin \frac{j\pi}{2} = 2(\sin 2\theta - \sin 6\theta + \sin 10\theta - \&c.)$$

$$= \lim_{j=\infty} 2 \sum_{(j)1}^j \sin 2j\theta \sin \frac{j\pi}{2} = \lim_{\lambda=\infty} \frac{(-1)^{\lambda+1} \sin 4\lambda\theta}{\cos 2\theta} = \lim_{\lambda=\infty} \frac{\sin 4\lambda \left(\theta - \frac{\pi}{4} \right)}{\sin 2 \left(\theta - \frac{\pi}{4} \right)};$$

therefore

$$\begin{aligned} y_{i,t} &= \frac{2B_1}{\pi} \lim_{\lambda=\infty} \int_0^{\frac{\pi}{2}} d\theta \frac{\sin 4\lambda \left(\theta - \frac{\pi}{4} \right)}{\sin 2 \left(\theta - \frac{\pi}{4} \right)} \sin 2l\theta \cos(2at \sin \theta) \\ &= B_1 \sin \frac{l\pi}{2} \cos(at\sqrt{2}) = y_{1,0} \sin \frac{l\pi}{2} \cos(at\sqrt{2}), \end{aligned}$$

as found in the 1st example, article 11, for the case of a single moveable particle. Indeed, we there considered, on the one hand, only the value $l = 1$; &, on the other hand, supposed $y'_{1,0}$ not to vanish. But with respect to this last part of the conditions of article 11, if we now suppose $y'_{j,0} = y'_{1,0} \sin \frac{j\pi}{2}$, we get, by the analysis of the present article, the additional term

$$\begin{aligned} \frac{2y'_{1,0}}{\pi} \lim_{\lambda=\infty} \int_0^{\frac{\pi}{2}} d\theta \frac{\sin 4\lambda \left(\theta - \frac{\pi}{4} \right)}{\sin 2 \left(\theta - \frac{\pi}{4} \right)} \sin 2l\theta \int_0^t dt \cos(2at \sin \theta) \\ = y'_{1,0} \sin \frac{l\pi}{2} \int_0^t dt \cos(at\sqrt{2}) = y'_{1,0} \sin \frac{l\pi}{2} \frac{\sin(at\sqrt{2})}{a\sqrt{2}}, \end{aligned}$$

which completes the agreement with the results of the 1st example. In fact

$$\lim_{\lambda=\infty} \int_0^{\frac{\pi}{2}} d\theta \frac{\sin 4\lambda \left(\theta - \frac{\pi}{4} \right)}{\sin 2 \left(\theta - \frac{\pi}{4} \right)} F(\theta) = \frac{\pi}{2} F\left(\frac{\pi}{4}\right),$$

if the function $F(\theta)$ remain finite for the whole extent of the integral.

58. We might even have considered it as evident à priori that the indefinite integration of the equation in mixed differences

$$y''_{l,t} = \frac{a^2 \Delta_l^2}{1 + \Delta_l} y_{l,t}, \tag{2}$$

combined with the initial conditions

$$y_{l,0} = y_{1,0} \sin \frac{l\pi}{2}, \quad y'_{l,0} = y'_{1,0} \sin \frac{l\pi}{2},$$

if these be supposed to hold good for all values of the integer l , from a large negative to a large positive value, must conduct nearly, and more and more nearly as these initial conditions hold good for a greater extent of l , to an expression of the form

$$y_{l,t} = y_{1,t} \sin \frac{l\pi}{2},$$

in which $y_{1,t}$ is a function determined by the differential equation

$$y''_{1,t} = -2a^2 y_{1,t};$$

and therefore that the integral of this equation, namely

$$y_{1,t} = y_{1,0} \cos (at\sqrt{2}) + y'_{1,0} \frac{\sin (at\sqrt{2})}{a\sqrt{2}},$$

when multiplied by $\sin \frac{l\pi}{2}$, must express the limit to which the expression

$$y_{l,t} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} d\theta \left(\sum_{(j)-h}^h \cos (2l\theta - 2j\theta) \sin \frac{j\pi}{2} \right) \left(y_{1,0} \cos (2at \sin \theta) + y'_{1,0} \frac{\sin (2at \sin \theta)}{2a \sin \theta} \right)$$

tends as h increases without limit. And since the sum

$$\sum_{(j)-h}^h \cos (2l\theta - 2j\theta) \sin \frac{j\pi}{2} = \sin 2l\theta \frac{\sin 4\lambda \left(\theta - \frac{\pi}{4} \right)}{\sin 2 \left(\theta - \frac{\pi}{4} \right)},$$

in which $\lambda = \frac{h}{2}$ or $\frac{h+1}{2}$ according as h is even or odd, we might thus be led by the consideration of the differential equations to discover the following limiting values of definite integrals:

$$\lim_{\lambda=\infty} \int_0^{\frac{\pi}{2}} d\theta \frac{\sin 4\lambda \left(\theta - \frac{\pi}{4} \right)}{\sin 2 \left(\theta - \frac{\pi}{4} \right)} \sin 2l\theta \cos (2at \sin \theta) = \frac{\pi}{2} \sin \frac{l\pi}{2} \cos (at\sqrt{2});$$

$$\lim_{\lambda=\infty} \int_0^{\frac{\pi}{2}} d\theta \frac{\sin 4\lambda \left(\theta - \frac{\pi}{4} \right)}{\sin 2 \left(\theta - \frac{\pi}{4} \right)} \frac{\sin 2l\theta}{\sin \theta} \sin (2at \sin \theta) = \frac{\pi}{\sqrt{2}} \sin \frac{l\pi}{2} \sin (at\sqrt{2}).$$

59. In like manner, if the initial conditions be

$$y_{j,0} = \eta_j \sin \frac{ij\pi}{n+1}, \quad y'_{j,0} = \eta'_j \sin \frac{ij\pi}{n+1},$$

and if we use the formula (35) of article 57, we have the expression

$$y_{l,t} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} d\theta \left(\sum_{(j)-\infty}^{\infty} \sin 2j\theta \sin \frac{ij\pi}{n+1} \right) \sin 2l\theta \left(\eta_i + \eta'_i \int_0^t dt \right) \cos (2at \sin \theta);$$

in which $\sum_{(j)-\infty}^{\infty} = 2 \lim_{j=\infty} \sum_{(j)1}^j$; and, by article 24,

$$4 \sum_{(j)1}^j \sin 2j\theta \sin 2j\alpha = \frac{\sin (2j+1)(\theta-\alpha)}{\sin(\theta-\alpha)} - \frac{\sin (2j+1)(\theta+\alpha)}{\sin(\theta+\alpha)};$$

therefore

$$y_{l,t} = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} d\theta \lim_{j=\infty} \left(\frac{\sin (2j+1)(\theta-\alpha)}{\sin(\theta-\alpha)} - \frac{\sin (2j+1)(\theta+\alpha)}{\sin(\theta+\alpha)} \right) \sin 2l\theta \left(\eta_i + \eta'_i \int_0^t dt \right) \cos (2at \sin \theta),$$

in which $\alpha = \frac{i\pi}{2(n+1)}$, so that α is > 0 & $< \frac{\pi}{2}$, if $i > 0, < n+1$. We may therefore neglect the part depending on the rapidly fluctuating term $\sin (2j+1)(\theta+\alpha)$ as being rigorously null at the limit $j = \infty$; and in the part depending on $\sin (2j+1)(\theta-\alpha)$ may confine ourselves to the consideration of infinitely small values, positive or negative, of $\theta - \alpha$. We find therefore, as the limit sought,

$$y_{l,t} = \sin 2l\alpha \left(\eta_i + \eta'_i \int_0^t dt \right) \cos (2at \sin \alpha),$$

the initial conditions being, for all integer values of j ,

$$y_{j,0} = \eta_i \sin 2j\alpha, \quad y'_{j,0} = \eta'_i \sin 2j\alpha;$$

and thus the formula of the 4th corollary, article 6, for the case of a finite number of particles is deduced from that of an infinite number. And hence by reasoning similar to that of article 7, we may infer, for an infinite system, that if the initial conditions be, for all integer values of j ,

$$y_{j,0} = \sum_{(i)1}^n \eta_i \sin \frac{ij\pi}{n+1}, \quad y'_{j,0} = \sum_{(i)1}^n \eta'_i \sin \frac{ij\pi}{n+1},$$

which require only that (if i be integer) we should have the kind & degree of initial periodicity expressed by the formula

$$y_{j,0} = -y_{2n+2-j,0} = y_{2n+2+j,0}, \quad y'_{j,0} = -y'_{2n+2-j,0} = y'_{2n+2+j,0},$$

we shall then have, for all integer values of l & for all values of t , the same kind & degree of periodicity, which may be expressed as follows:

$$y_{l,t} = \sum_{(i)1}^n \sin \frac{il\pi}{n+1} \left(\eta_i + \eta'_i \int_0^t dt \right) \cos \left(2at \sin \frac{i\pi}{2n+2} \right); \tag{8}$$

$$y_{l,t} = -y_{2n+2-l,t} = y_{2n+2+l,t}, \quad y'_{l,t} = -y'_{2n+2-l,t} = y'_{2n+2+l,t}.$$

Thus the theory of a finite system is included in that of an infinite system, since the formula (8) has been deduced from the formula (12).

60. The reasoning of the foregoing article shows also that if the initial conditions be, for the whole extent of an infinite system, or for all integer values of j ,

$$y_{j,0} = \eta_\alpha \sin 2j\alpha, \quad y'_{j,0} = \eta'_\alpha \sin 2j\alpha,$$

α being any real arc between 0 and $\frac{\pi}{2}$, (and therefore also if α be any real arc), we shall have, for the whole extent of the same system at any time t ,

$$y_{l,t} = \sin 2l\alpha \left(\eta_\alpha + \eta'_\alpha \int_0^t dt \right) \cos (2at \sin \alpha).$$

In fact it has been shown, & is evident, that this expression satisfies the indefinite equation in mixed differences, whatever α may be. And here we might commence, from a new point of view, reasonings analogous to those of article 23; but it seems desirable to pass on to other things.

61. By a transformation analogous to that of article 50, we may simplify the formula (18) of article 30. For if, in that formula, we extend the summation relatively to i as far as the value $i = 2n + 1$, we merely double the whole expression, because $(2n + 2 - i)\phi = \pi - i\phi$, and the value $i = n + 1$ gives $\sin 2i\phi = \sin l\pi = 0$; & the two lines of the last expression in the 1st sentence of article 32 are changed each into the other by changing θ to $\pi - \theta$; so that by confining ourselves to one alone we again halve the expression. In this manner we find that the formula (18) may be thus written:

$$y_{l,t} = \frac{B_k}{2(n+1)} \sum_{(i)1}^{2n+1} \sin 2l\theta \frac{\sin (2j+1)(\theta-\alpha)}{\sin(\theta-\alpha)} \left(\cos \beta_k \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta_k \sin \right) (2at \sin \theta); \quad (36)$$

in which $\theta = \frac{i\pi}{2(n+1)}$, $\alpha = \frac{k\pi}{2(n+1)}$. It is evident that this expression for $y_{l,t}$ satisfies the equation in differences, & gives $y_{0,t} = 0$, $y_{n+1,t} = 0$; it ought also to give

$$y_{l,0} = B_k \cos \beta_k \sin 2l\alpha, \quad y'_{l,0} = 2a \sin \alpha B_k \sin \beta_k \sin 2l\alpha,$$

if l be 1, 2, ... j ; but $y_{l,0} = 0$, $y'_{l,0} = 0$, if $l = j + 1, j + 2, \dots n$. We ought therefore to find that

$$\sum_{(i)1}^{2n+1} \frac{\sin 2l\theta \sin (2j+1)(\theta-\alpha)}{2n+2 \sin(\theta-\alpha)} = \sin 2l\alpha, \quad \text{or } = 0,$$

according as l is not greater, or greater, than j ; l and j being integers which are each > 0 and $< n + 1$; and θ, α having their recent values. In fact

$$\frac{\sin (2j+1)(\theta-\alpha)}{\sin(\theta-\alpha)} = \sum_{(j)-j}^j \cos 2j(\theta-\alpha) = 1 + 2 \sum_{(j)1}^j \cos 2j(\theta-\alpha);$$

& making $\theta = i\phi$, we have

$$2 \sin 2l\theta \cos 2j(\theta-\alpha) = \sin \{2(j+l)i\phi - 2j\alpha\} - \sin \{2(j-l)i\phi - 2j\alpha\};$$

$$\begin{aligned} \sum_{(i)1}^{2n+1} \sin \{2(j \pm l)i\phi - 2j\alpha\} &= \frac{\cos \{(j \pm l)\phi - 2j\alpha\} - \cos \{(4n+3)(j \pm l)\phi - 2j\alpha\}}{2 \sin (j \pm l)\phi} \\ &= \frac{\sin \{(2n+1)(j \pm l)\phi\}}{\sin (j \pm l)\phi} \sin \{(2n+2)(j \pm l)\phi - 2j\alpha\} = \sin 2j\alpha, \end{aligned}$$

unless denominator = 0; therefore

$$\sum_{(i)1}^{2n+1} \sin 2li\phi \cos 2j(i\phi - \alpha) = 0$$

unless $j = l$, if j & l be each > 0 , $< n + 1$; but when $j = l$, this sum becomes $(2n + 2) \sin 2l\alpha$.

62. Another mode of verifying, and indeed proving the formula (36), is to show that the expression of which it is (relatively to j) the sum, namely

$$y_{l,t} = \frac{B_k}{n+1} \sum_{(i)1}^{2n+1} \sin 2l\theta \cos 2j(\theta - \alpha) \left(\cos \beta_k' \cos + \frac{\sin \alpha}{\sin \theta} \sin \beta_k' \sin \right) (2at \sin \theta),$$

corresponds to the case of only one particle P_j initially disturbed,

$$y_{j,0} = B_k \cos \beta_k' \sin 2j\alpha, \quad y'_{j,0} = 2a \sin \alpha B_k \sin \beta_k' \sin 2j\alpha.$$

Accordingly the last expression for $y_{l,t}$ may be thus written:

$$y_{l,t} = \frac{2}{n+1} \sum_{(i)1}^n \sin \frac{il\pi}{n+1} \sin \frac{ij\pi}{n+1} \left(y_{j,0} + y'_{j,0} \int_0^t dt \right) \cos \left(2at \sin \frac{\frac{1}{2}i\pi}{n+1} \right);$$

and under this form it agrees with (7). Reciprocally, in (7), if we suppress the sign of summation $\sum_{(j)1}^n$ so as to attend only to the effect of the initial state of a single particle P_j ; & if we represent this state by the formulae

$$y_{j,0} = \eta_i \sin \frac{ij\pi}{n+1}; \quad y'_{j,0} = \eta'_i \sin \frac{ij\pi}{n+1};$$

we may write

$$\begin{aligned} y_{l,t} &= \frac{1}{n+1} \sum_{(k)1}^{2n+1} \sin \frac{lk\pi}{n+1} \cos \frac{j(k-i)\pi}{n+1} \left(\eta_i + \eta'_i \int_0^t dt \right) \cos \left(2at \sin \frac{\frac{1}{2}k\pi}{n+1} \right) \\ &= \frac{1}{n+1} \left(\eta_i + \eta'_i \int_0^t dt \right) \sum_{(i)1}^{2n+1} \sin \frac{lk\pi}{n+1} \cos \frac{j(k-i)\pi}{n+1} \cos \left(2at \sin \frac{\frac{1}{2}k\pi}{n+1} \right). \end{aligned}$$

Interchanging i & k to conform more closely to the notation of article 28, and summing relatively to j from 1 to j , we get this other formula, equivalent to (18) or (36):

$$y_{l,t} = \frac{1}{2n+2} \left(\eta_k + \eta'_k \int_0^t dt \right) \sum_{(i)1}^{2n+1} \frac{\sin \frac{il\pi}{n+1} \sin \frac{(2j+1)(i-k)\pi}{2n+2}}{\sin \frac{(i-k)\pi}{2n+2}} \cos \left(2at \sin \frac{i\pi}{2n+2} \right). \quad (37)$$

And if we sum, instead, from j , to j , we get this transformation of (17) & therefore this other form of the solution of Problem II:

$$\begin{aligned} y_{l,t} &= \frac{1}{n+1} \left(\eta_k + \eta'_k \int_0^t dt \right) \sum_{(i)1}^{2n+1} \frac{\sin(j-j,+1)(\theta - \alpha)}{\sin(\theta - \alpha)} \\ &\quad \times \sin 2l\theta \cos(j+j,)(\theta - \alpha) \cos(2at \sin \theta); \end{aligned} \quad (38)$$

in which, as in many former equations, $\theta = \frac{i\pi}{2n+2}$, $\alpha = \frac{k\pi}{2n+2}$. It is evident that this includes the formula (34).

Problem III.

63. It is proposed to determine the consequences of the supposition that the initial states of some number of successive particles correspond to one of the two conjugate components of a simple movement, (considered in article 20,) that is to the uniform transmission of phase in one direction.

64. Before passing to this determination, it will be convenient to review & recapitulate the chief results already obtained.

(I). The number n of moveable particles being *finite*, so that the differential equations to be satisfied are n in number, namely

$$y''_{1,t} = a^2(0 - 2y_{1,t} + y_{2,t}), \quad y''_{2,t} = a^2(y_{1,t} - 2y_{2,t} + y_{3,t}),$$

$$y''_{3,t} = a^2(y_{2,t} - 2y_{3,t} + y_{4,t}), \quad \dots, \quad y''_{n-1,t} = a^2(y_{n-2,t} - 2y_{n-1,t} + y_{n,t}), \quad y''_{n,t} = a^2(y_{n-1,t} - 2y_{n,t}),$$

we found that we might satisfy these equations, & therefore also the dynamical conditions of the question, by supposing all the displacements to correspond to that simple mode of vibration, which is expressed by the formula

$$y_{i,t} = \sin 2l\alpha_i \left(\eta_i + \eta'_i \int_0^t dt \right) \cos (2at \sin \alpha_i);$$

in which $\alpha_i = \frac{i\pi}{2n+2}$, and η_i, η'_i are constants. In this mode, the n moveable or intermediate and the two fixed or extreme particles are, at any moment t , arranged all upon the i alternate branches of a *sinusoid*, which has 2 extreme & $i-1$ intermediate nodes and i venters. This sinusoid varies with the time, & oscillates between two extreme positions determined by those of the first venter. The sinusoidal form is expressed by the factor $\sin 2l\alpha_i$, and the oscillation of the first venter by the factor

$$\begin{aligned} & \left(\eta_i + \eta'_i \int_0^t dt \right) \cos (2at \sin \alpha_i) \\ &= \left(\eta_i \cos + \frac{\eta'_i \sin}{2a \sin \alpha_i} \right) (2at \sin \alpha_i) = (\eta_i \cos + r_i^{-1} \eta'_i \sin) (tr_i), \quad \text{if } r_i = 2a \sin \alpha_i. \end{aligned}$$

The greatest positive excursion of the venter is attained at those moments, (succeeding each other after equal intervals or periods of time, each period T_i being

$$= \frac{2\pi}{r_i} = \frac{\pi}{a} \operatorname{cosec} \alpha_i = \frac{\pi}{a} \operatorname{cosec} \frac{i\pi}{2n+2},$$

when

$$\cos tr_i = \frac{\eta_i}{\sqrt{\eta_i^2 + r_i^{-2} \eta_i'^2}}, \quad \sin tr_i = \frac{r_i^{-1} \eta'_i}{\sqrt{\eta_i^2 + r_i^{-2} \eta_i'^2}};$$

and this greatest positive excursion = $B_i = \sqrt{\eta_i^2 + r_i^{-2} \eta_i'^2}$. The greatest negative excursion = $-B_i$, & is attained at moments which follow or precede, by exactly half the periodic time T_i , the moments of greatest positive excursion; so that if these last be of the form $\epsilon_i + \nu T_i$, in which ν is any integer, positive, negative or null, while ϵ_i is such that

$$\cos r_i \epsilon_i = \frac{\eta_i}{\sqrt{(\eta_i^2 + r_i^{-2} \eta_i'^2)}}, \quad \sin r_i \epsilon_i = \frac{r_i^{-1} \eta'_i}{\sqrt{(\eta_i^2 + r_i^{-2} \eta_i'^2)}}$$

the moments of greatest negative excursion are expressed by the formula $\epsilon_i + (\nu + \frac{1}{2}) T_i$. The intermediate moments $\epsilon_i + (\nu \pm \frac{1}{4}) T_i$ are such that in them the sinusoid reduces itself to a straight line, the displacements of the particles all vanishing; in such a manner that $y_{i,t} = 0$, if $t = \epsilon_i + (\nu \pm \frac{1}{4}) T_i$. In fact we have then $tr_i = r_i \epsilon_i + 2\nu\pi \pm \frac{\pi}{2}$; therefore

$$\cos tr_i = \mp \sin r_i \epsilon_i, \quad \sin tr_i = \pm \cos r_i \epsilon_i, \quad (\eta_i \cos + r_i^{-1} \eta'_i \sin) (tr_i) = 0.$$

The variable velocity $y'_{i,t}$ is expressed as follows, in this simple mode of vibration, for any particle P_i :

$$y'_{i,t} = \sin 2l\alpha_i \left(\eta'_i + \eta_i \frac{d}{dt} \right) \cos tr_i;$$

in which the first factor $\sin 2l\alpha_i$ corresponds to the sinusoidal relation between the several particles, & the other factor

$$\left(\eta'_i + \eta_i \frac{d}{dt}\right) \cos tr_i = (\eta'_i \cos - r_i \eta_i \sin) tr_i$$

expresses the velocity of the first venter. This velocity vanishes when $t = \epsilon_i + \nu T_i$ and when $t = \epsilon_i + (\nu + \frac{1}{2}) T_i$, that is, at the moments of greatest positive or negative excursion; but at the moments when $t = \epsilon_i + (\nu \pm \frac{1}{4}) T_i$, & when therefore $tr_i = r_i \epsilon_i + 2\nu\pi \pm \frac{\pi}{2}$, the velocity of the venter becomes $= \mp r_i \sqrt{\eta_i^2 + r_i^{-2} \eta_i'^2}$; it attains therefore at these moments a negative or positive maximum of amount & this greatest velocity is equal to $\sqrt{\eta_i'^2 + r_i^2 \eta_i^2} = \sqrt{\eta_i^2 + r_i^{-2} \eta_i'^2}$ multiplied by the coefficient r_i which multiplies the time t under the signs of periodicity. (In former articles $r_i \epsilon_i$ has been called β_i .)

(II). The foregoing being a *possible permanent mode* of vibration of the system, it follows that if at any one moment, such as the moment $t = 0$, the displacements $y_{l,0}$ & the velocities $y'_{l,0}$ are all such as to agree with it, then, at all subsequent moments t , the displacements & velocities $y_{l,t}$ & $y'_{l,t}$ will still agree with the same simple mode. In other words, if the particles are all arranged on a sinusoidal curve of the form $y_{x,0} = Y_0 \sin 2x\alpha_i$ at the moment 0 & also on another such curve $y_{x,dt} = Y_{dt} \sin 2x\alpha_i$ at the infinitely near moment dt , the coefficient α_i being still $= \frac{i\pi}{2n+2}$ and the coefficients Y_0 and Y_{dt} representing for these two near moments 0 and dt the displacements of the first venter, (for which $x = \frac{\pi}{4\alpha_i} = \frac{n+1}{2i}$), then at any subsequent moment t the particles will all be arranged on a curve of the same kind, namely $y_{x,t} = Y_t \sin 2x\alpha_i$; in which the coefficient Y_t represents the displacement of the venter and satisfies the differential equation of the second order $Y''_t + r_i^2 Y_t = 0$, so that it may be deduced from Y_0 & from $Y_{dt} = Y_0 + Y'_0 dt$ by the formula

$$Y_t = \left(Y_0 + Y'_0 \int_0^t dt \right) \cos tr_i.$$

(III). By the linear form of the differential equations of the question, it is permitted to add together any number of particular integrals or to superpose any number of small motions of which each is separately possible. On the other hand, any single initial displacement $y_{j,0}$, of any one particle P_j , may be considered as the sum or resultant of n different initial sinusoidal displacements of the form $y_{l,0} = \eta_i \sin 2l\alpha_i$, of which each separately extends to all the particles P_l , but which destroy each other by interference or superposition for all the particles except P_j . For we may write $y_{j,0} = \sum_{(i)1}^n \eta_i \sin 2j\alpha_i$, if we so choose the n coefficients η_i as to have $\eta_i = \frac{2}{n+1} y_{j,0} \sin 2j\alpha_i$; because $\sum_{(i)1}^n (\sin 2j\alpha_i)^2 = \sum_{(i)1}^n \left(\sin \frac{ij\pi}{n+1} \right)^2 = \frac{n+1}{2}$. And with the same choice of the coefficients η_i we shall have as the resultant initial displacement of any other particle P_l the null expression

$$y_{l,0} = \sum_{(i)1}^n \eta_i \sin 2l\alpha_i = \frac{2}{n+1} y_{j,0} \sum_{(i)1}^n \sin 2j\alpha_i \sin 2l\alpha_i = 0,$$

l being different from j . The effect of a single initial displacement $y_{j,0}$ of any single particle P_j

is to produce, at the time t , a system of displacements, or a complex mode of vibration, represented by the formula $y_{l,t} = \frac{2}{n+1} y_{j,0} \sum_{(i)1}^n \sin 2j\alpha_i \sin 2l\alpha_i \cos tr_i$. In fact this complex mode is a possible permanent mode, because it is the sum of n simple possible & permanent modes; & it reproduces the initial conditions, giving $y_{l,0} = 0$ or $= y_{j,0}$ according as l (being integer) is different from or equal to j ; & giving $y'_{l,0} = 0$ for all values of l . In like manner the effect of any single initial velocity $y'_{j,0}$ is to produce the complex mode of vibration represented as follows:

$$y_{l,t} = \frac{2}{n+1} y'_{j,0} \int_0^t dt \sum_{(i)1}^n \sin 2j\alpha_i \sin 2l\alpha_i \cos tr_i.$$

And therefore the effect of any arbitrary initial state, or the complete solution of Problem I, may be expressed thus:

$$y_{l,t} = \frac{2}{n+1} \sum_{(i)1}^n \left(y_{j,0} + y'_{j,0} \int_0^t dt \right) \sum_{(i)1}^n \sin 2j\alpha_i \sin 2l\alpha_i \cos tr_i;$$

in which it is important to observe that the part of the state of any particle P_i at the time t , which corresponds to a given value of i and to any given initial displacement or velocity of any other particle P_j , is equal to that part of the state of the latter particle P_j at the same time t , which corresponds to the same value of i (or to the same mode of component & simple vibration), and to an equal initial displacement or velocity of the former particle P_i ; because the product $\sin 2j\alpha_i \sin 2l\alpha_i$ is symmetric relatively to j and l .*

(IV). To pass to Problem II, we are to suppose that the initial states of some one or more successive particles correspond to the k^{th} mode of simple vibration, so that for one or more successive values of j we have

$$y_{j,0} = \eta_k \sin 2j\alpha_k, \quad y'_{j,0} = \eta'_k \sin 2j\alpha_k, \quad \left(\alpha_k = \frac{k\pi}{2n+2} \right);$$

but that, for all the other values of j , $y_{j,0}$ and $y'_{j,0}$ vanish. And it now is necessary to sum the product $\sin 2j\alpha_i \sin 2j\alpha_k$ between some given limits of j ; or at least this is the operation which first presents itself. But because we have afterwards to multiply by $\sin 2l\alpha_i$ and to sum relatively to i , and because $\alpha_{2n+2-i} = \pi - \alpha_i$, $\alpha_{n+1} = \frac{\pi}{2}$, we may substitute $\cos 2j(\alpha_i - \alpha_k)$ for $\sin 2j\alpha_i \sin 2j\alpha_k$, if we afterwards change $2 \sum_{(i)1}^n$ to $\sum_{(i)1}^{2n+1}$. In this manner we find, if j be confined to one value, the expression

$$y_{l,t} = \frac{1}{n+1} \left(\eta_k + \eta'_k \int_0^t dt \right) \sum_{(i)1}^{2n+1} \sin 2l\alpha_i \cos 2j(\alpha_i - \alpha_k) \cos tr_i;$$

and if we are to sum relatively to j from j , to j , then we find

$$\begin{aligned} y_{l,t} &= \frac{1}{2n+2} \left(\eta_k + \eta'_k \int_0^t dt \right) \sum_{(i)1}^{2n+1} \sin 2l\alpha_i \left\{ \frac{\sin(2j+1)(\alpha_i - \alpha_k) - \sin(2j-1)(\alpha_i - \alpha_k)}{\sin(\alpha_i - \alpha_k)} \right\} \cos tr_i \\ &= \frac{1}{n+1} \left(\eta_k + \eta'_k \int_0^t dt \right) \sum_{(i)1}^{2n+1} \sin 2l\alpha_i \frac{\sin(j-j, +1)(\alpha_i - \alpha_k)}{\sin(\alpha_i - \alpha_k)} \cos(j+j,)(\alpha_i - \alpha_k) \cos tr_i; \end{aligned}$$

an expression which is the complete solution of the IInd Problem.

* [Rayleigh, *Theory of Sound*, I, pp. 150-157.]

When all the j first particles are originally in the k^{th} mode of vibration and all the others are originally without displacement or velocity, so that $j, = 1$, we may write more simply

$$y_{l,t} = \frac{1}{2n+2} \left(\eta_k + \eta'_k \int_0^t dt \right) \sum_{(i)1}^{2n+1} \frac{\sin(2j+1)(\alpha_i - \alpha_k)}{\sin(\alpha_i - \alpha_k)} \sin 2l\alpha_i \cos tr_i.$$

(V). The number n of moveable particles being still supposed finite, the most general mode of motion of the system may be considered (as we have seen) as the resultant of n simple modes, of the kind lately described: so that we may write generally

$$\begin{aligned} y_{l,t} &= \sum_{(i)1}^n \sin 2l\alpha_i \left(\eta_i + \eta'_i \int_0^t dt \right) \cos tr_i \\ &= \sum_{(i)1}^n B_i \sin 2l\alpha_i \cos r_i(t - \epsilon_i). \end{aligned}$$

And because $\alpha_{2n+2-i} = \pi - \alpha_i$, $r_{2n+2-i} = r_i$, we may write

$$y_{l,t} = \frac{1}{2} \sum_{(i)1}^{2n+1} B_i \sin 2l\alpha_i \cos r_i(t - \epsilon_i),$$

if we assume $B_{2n+2-i} = -B_i$, but $\epsilon_{2n+2-i} = \epsilon_i$.

With these last assumptions, we may therefore write also

$$y_{l,t} = \frac{1}{2} \sum_{(i)1}^{2n+1} B_i \sin(2l\alpha_i - r_i t + r_i \epsilon_i);$$

and consequently, (B_{n+1} being $= 0$), may consider the general mode of complex vibration $y_{l,t}$ as the sum of n pairs of component vibrations, of which each pair might separately continue to exist, but not (in general) each component semi-mode of vibration itself, if taken without its conjugate semi-mode, which has the same periodic time for the vibration of any single particle.

Those n component semi-modes for which $i < n+1$ would correspond, if the system were indefinite, to a continual transmission of phase in the forward or positive direction with a velocity (for the i^{th} mode) $= \frac{r_i}{2\alpha_i} = a \frac{\sin \alpha_i}{\alpha_i}$; and those n other component semi-modes for which $i > n+1$ would correspond, if the system were indefinite, to a continual transmission of phase in the backward or negative direction with a velocity which, for the semi-mode conjugate to the i^{th} , is $= -a \frac{\sin \alpha_i}{\alpha_i}$, and is therefore equal in amount (though different in sign) to that just now determined for the i^{th} semi-mode itself. In fact, the parts of $y_{l,t}$ corresponding to these two conjugate semi-modes are

$$\frac{1}{2} B_i \sin(2l\alpha_i - r_i t + r_i \epsilon_i) \quad \text{and} \quad \frac{1}{2} B_i \sin(2l\alpha_i + r_i t - r_i \epsilon_i).$$

Their resultant vanishes for the extreme particles P_0 and P_{n+1} , whatever t may be; & for any intermediate particle P_l , it is, as before,

$$B_i \sin 2l\alpha_i \cos(r_i t - r_i \epsilon_i).$$

In general, whatever may be the arbitrary initial state of the system, we may represent its state at the time t by the formula

$$y_{l,t} = \frac{1}{2} \sum_{(i)1}^{2n+1} (\eta_i \sin + r_i^{-1} \eta'_i \cos)(2l\alpha_i - 2at \sin \alpha_i),$$

if we assume that $\eta_{2n+2-i} = -\eta_i$, $\eta'_{2n+2-i} = -\eta'_i$, and therefore that $\eta_{n+1} = 0$, $\eta'_{n+1} = 0$; α_i being still $= \frac{i\pi}{2n+2}$. And we shall still have, in this last formula for $y_{l,t}$, as in others,

$$\eta_i = \frac{2}{n+1} \sum_{(j)1}^n y_{j,0} \sin 2j\alpha_i, \quad \eta'_i = \frac{2}{n+1} \sum_{(j)1}^n y'_{j,0} \sin 2j\alpha_i.$$

Thus, generally, we have the expression

$$y_{l,t} = \frac{1}{n+1} \sum_{(j)1}^n \sum_{(i)1}^{2n+1} \sin 2j\alpha_i (y_{j,0} \sin + r_i^{-1} y'_{j,0} \cos) (2l\alpha_i - tr_i),$$

in which r_i is still $= 2a \sin \alpha_i$, and α_i is still $= \frac{i\pi}{2n+2}$.

Accordingly it is easy to prove à posteriori the truth of this last expression for $y_{l,t}$. And if in it we make $y_{j,0} = C_{i,j} \cos \gamma_{i,j}$, $r_i^{-1} y'_{j,0} = C_{i,j} \sin \gamma_{i,j}$, so that $C_{2n+2-i,j} = C_{i,j}$ and $\gamma_{2n+2-i,j} = \gamma_{i,j}$, we shall have

$$y_{l,t} = \frac{1}{n+1} \sum_{(j)1}^n \sum_{(i)1}^{2n+1} C_{i,j} \sin 2j\alpha_i \sin (2l\alpha_i + \gamma_{i,j} - tr_i).$$

But this last transformation does not seem to be attended with any advantage.

65. The foregoing article contains a recapitulation of the chief results obtained already in this manuscript for the case of a finite system. If the system be unlimited in one direction, so that only the condition $y_{0,t} = 0$ but not the condition $y_{n+1,t} = 0$ is to be attended to, we have then the following results:

(I)'. The differential equations to be satisfied are now infinite in number; they need involve only positive values of l , but l may be taken as great as we please; they may be written thus:

$$y''_{1,t} = a^2 (0 - 2y_{1,t} + y_{2,t}), \quad y''_{2,t} = a^2 (y_{1,t} - 2y_{2,t} + y_{3,t}), \dots$$

$$y''_{l,t} = a^2 (y_{l-1,t} - 2y_{l,t} + y_{l+1,t}), \text{ \&c. ad infinitum.}$$

A particular integral, or possible permanent mode of motion of the system, which may also be considered as a simple mode, is expressed by the formula

$$y_{l,t} = \sin 2l\alpha \left(\eta_\alpha + \eta'_\alpha \int_0^t dt \right) \cos (2at \sin \alpha);$$

in which α is any real arc & η_α , η'_α are any arbitrary real functions thereof. This formula indicates an arrangement of all the particles on a sinusoidal curve, containing indefinitely many alternate branches and varying with the time, but so that a first node is always at the fixed particle P_0 & a first venter at a distance, as measured on the axis of the system, $= \frac{\pi}{4\alpha}$. The

whole space-period, or interval between two similar modes, is $S_\alpha = \frac{\pi}{\alpha}$; the whole time-period, or

periodic time of vibration of any one particle, is connected therewith, being $T_\alpha = \frac{\pi}{a} \operatorname{cosec} \alpha$.

The positive maximum of excursion of the first venter is $B_\alpha = \sqrt{\eta_\alpha^2 + r_\alpha^{-2} \eta'_\alpha^2}$, in which $r_\alpha = 2a \sin \alpha$;

and is attained when $t = \epsilon_\alpha + \nu T_\alpha$, ν being any integer & ϵ_α being such that $\cos r_\alpha \epsilon_\alpha = \frac{\eta_\alpha}{B_\alpha}$,

$\sin r_\alpha \epsilon_\alpha = \frac{r_\alpha^{-1} \eta'_\alpha}{B_\alpha}$. The negative maximum of excursion of the same venter is $= -B_\alpha$, & is attained

when $t = \epsilon_\alpha + (\nu + \frac{1}{2}) T_\alpha$. At both these two sets of moments, the velocity of the venter vanishes (& so do therefore the velocities of all the particles); while, on the contrary, the velocity of the venter attains the negative or positive maximum $\mp r_\alpha B_\alpha$ at the intermediate moments when $t = \epsilon_\alpha + (\nu \pm \frac{1}{4}) T_\alpha$; & at these last mentioned moments the displacements all vanish, or the particles are all in the axis.

(II)'. A mode of vibration, such as has been just now described, if once established, will persist; if then, at each of any two near moments 0 & dt , we have

$$y_{l,0} = \eta_\alpha \sin 2l\alpha \quad \text{and} \quad y_{l,dt} = (\eta_\alpha + \eta'_\alpha dt) \sin 2l\alpha,$$

we shall have, for all subsequent moments,

$$y_{l,t} = \sin 2l\alpha \left(\eta_\alpha + \eta'_\alpha \int_0^t dt \right) \cos tr_\alpha,$$

that is,

$$y_{l,t} = B_\alpha \sin 2l\alpha \cos \{r_\alpha (t - \epsilon_\alpha)\}.$$

(III)'. The sum or integral of any number of such simple vibrations, that is the resultant of the superposition of any finite or infinite number of them, if once established, will persist; but any single initial displacement $y_{j,0}$ may be expressed by such an integral as follows:

$$y_{l,0} = \frac{4}{\pi} y_{j,0} \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \left(= \frac{2}{\pi} y_{j,0} \int_0^\pi d\alpha \sin 2j\alpha \sin 2l\alpha \right),$$

because this integral becomes $= y_{j,0}$ or $= 0$, according as the positive integer l is equal to or different from j ; the effect of any single initial displacement $y_{j,0}$ is therefore to produce, at the time t , the system of displacements represented by the formula

$$y_{l,t} = \frac{4}{\pi} y_{j,0} \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos tr_\theta.$$

In like manner the effect of any single initial velocity $y'_{j,0}$ is to produce, at the time t , the system of displacements

$$y_{l,t} = \frac{4}{\pi} y'_{j,0} \int_0^t dt \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos tr_\theta;$$

and therefore the effect of any arbitrary initial state of the indefinite system of particles P_1, P_2, \dots is to produce, at the time t , a state which may be thus expressed:

$$y_{l,t} = \frac{4}{\pi} \sum_{(j)1}^\infty \left(y_{j,0} + y'_{j,0} \int_0^t dt \right) \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos tr_\theta.$$

This result may be connected with the corresponding one in the subdivision (III) of article 64 for the case of a finite system; & the same remark respecting the symmetry of $\sin 2j\alpha \sin 2l\alpha$ applies.

(IV)'. If, for some set of successive values of j , from $j=j_1$ to $j=j_2$, we have the initial conditions $y_{j,0} = \eta_\alpha \sin 2j\alpha$, $y'_{j,0} = \eta'_\alpha \sin 2j\alpha$, while $y_{j,0}$ and $y'_{j,0}$ vanish for all other (positive) values

of j , we have then, by changing $2 \int_0^{\frac{\pi}{2}} d\theta$ to $\int_0^\pi d\theta$ & summing $\cos 2j(\theta - \alpha)$ relatively to j ,

$$\begin{aligned} y_{l,t} &= \frac{1}{\pi} \left(\eta_\alpha + \eta'_\alpha \int_0^t dt \right) \int_0^\pi d\theta \left\{ \frac{\sin (2j_2 + 1)(\theta - \alpha) - \sin (2j_1 - 1)(\theta - \alpha)}{\sin (\theta - \alpha)} \right\} \sin 2l\theta \cos tr_\theta \\ &= \frac{2}{\pi} \left(\eta_\alpha + \eta'_\alpha \int_0^t dt \right) \int_0^\pi d\theta \frac{\sin (j_2 - j_1 + 1)(\theta - \alpha)}{\sin (\theta - \alpha)} \cos (j_2 + j_1)(\theta - \alpha) \sin 2l\theta \cos tr_\theta; \end{aligned}$$

and if in particular $j, = 1$, so that the first j particles all satisfy the foregoing initial conditions, then

$$y_{i,t} = \frac{1}{\pi} \left(\eta_\alpha + \eta'_\alpha \int_0^t dt \right) \int_0^\pi d\theta \frac{\sin(2j+1)(\theta-\alpha)}{\sin(\theta-\alpha)} \sin 2l\theta \cos tr_\theta.$$

(V)'. The most general mode of vibration of the present system may be expressed as follows:

$$y_{i,t} = \int_0^{\frac{\pi}{2}} d\theta \sin 2l\theta \left(\eta_\theta + \eta'_\theta \int_0^t dt \right) \cos tr_\theta,$$

in which η_θ and η'_θ are connected with the initial state of the system by the relations

$$y_{i,0} = \int_0^{\frac{\pi}{2}} \eta_\theta \sin 2l\theta d\theta, \quad y'_{i,0} = \int_0^{\frac{\pi}{2}} \eta'_\theta \sin 2l\theta d\theta,$$

so that, by what was lately shown (section (III)'), we have

$$\eta_\theta = \frac{4}{\pi} \sum_{(j)1}^\infty y_{j,0} \sin 2j\theta, \quad \eta'_\theta = \frac{4}{\pi} \sum_{(j)1}^\infty y'_{j,0} \sin 2j\theta.$$

If we extend these last expressions to all values of θ from 0 to π , we shall have

$$y_{i,0} = \frac{1}{2} \int_0^\pi \eta_\theta \sin 2l\theta d\theta, \quad y'_{i,0} = \frac{1}{2} \int_0^\pi \eta'_\theta \sin 2l\theta d\theta,$$

and

$$y_{i,t} = \frac{1}{2} \int_0^\pi d\theta \sin 2l\theta \left(\eta_\theta + \eta'_\theta \int_0^t dt \right) \cos tr_\theta.$$

And this expression again may be put under the form

$$y_{i,t} = \frac{1}{2} \int_0^\pi d\theta (\eta_\theta \sin + r_\theta^{-1} \eta'_\theta \cos) (2l\theta - tr_\theta),$$

or, substituting for η_θ and η'_θ their values,

$$y_{i,t} = \frac{2}{\pi} \sum_{(j)1}^\infty \int_0^\pi d\theta \sin 2j\theta (y_{j,0} \sin + r_\theta^{-1} y'_{j,0} \cos) (2l\theta - tr_\theta).$$

The general expression for the mode of vibration or complex motion of a system indefinite in one direction may therefore be considered as the sum of an infinite number of pairs of conjugate component motions, in each of which there is a continual & uniform transmission of phase in one of two opposite directions. In any one such component motion, corresponding to $\theta = \alpha$, if $\alpha > 0$, $< \frac{\pi}{2}$, there is as above a space-period $S_\alpha = \frac{\pi}{\alpha}$, and a time-period $T_\alpha = \frac{2\pi}{r_\alpha} = \frac{\pi}{a} \operatorname{cosec} \alpha$; and the velocity of transmission is $\frac{S_\alpha}{T_\alpha} = a \frac{\sin \alpha}{\alpha}$. In the conjugate component motion, corresponding to $\theta = \pi - \alpha$, we have the same length of space-period & of time-period, but the velocity of transmission is negative and may be represented by $-a \frac{\sin \alpha}{\alpha}$. The combination of the two is necessary in order to preserve the fixity of P_0 .

66. To recapitulate in like manner the results already obtained relative to a system which extends indefinitely in two opposite directions, without any condition of fixity, we may observe that:

(I)''. The differential equations are now all those included in the formula

$$y''_{l,t} = \alpha^2 (y_{l-1,t} - 2y_{l,t} + y_{l+1,t}),$$

l receiving all integer values. A particular integral is

$$y_{l,t} = \cos(2l\alpha - \gamma_\alpha) \left(\eta_\alpha + \eta'_\alpha \int_0^t dt \right) \cos tr_\alpha,$$

in which α is an arbitrary real quantity, $r_\alpha = 2a \sin \alpha$, and $\eta_\alpha, \eta'_\alpha, \gamma_\alpha$ are arbitrary real functions of α . The function γ_α is introduced, instead of the constant $\frac{\pi}{2}$ which occupied its place in (I) and (I)', because we do not now suppose $y_{0,t}$ to vanish, & therefore retain the cosines as well as the sines of $2l\alpha$. This simple mode of vibration is still sinusoidal, but the particle P_0 is not now necessarily a node. With this exception the remarks of (I)' apply to it.

(II)''. In this indefinite system also a sinusoidal mode of vibration, if once established, will be permanent. A node and venter may be assumed at pleasure, but when the space-period is thus determined, the time-period is so too. By making $\gamma_\alpha = \frac{\pi}{2} + 2\alpha\lambda_\alpha$, in which λ_α is a new arbitrary but real function of α , namely the abscissa of a node, we may write for any one mode of this sort

$$y_{l,t} = B_\alpha \sin \{2(l - \lambda_\alpha)\alpha\} \cos \{r_\alpha(t - \epsilon_\alpha)\}.$$

(III)''. The sum or integral of any number of such vibrations will be permanent; therefore the effect of any single initial displacement $y_{j,0}$ is

$$y_{l,t} = \frac{2}{\pi} y_{j,0} \int_0^{\frac{\pi}{2}} d\theta \cos 2(l-j)\theta \cos tr_\theta,$$

and the effect of any single initial velocity $y'_{j,0}$ is

$$y_{l,t} = \frac{2}{\pi} y'_{j,0} \int_0^t dt \int_0^{\frac{\pi}{2}} d\theta \cos 2(l-j)\theta \cos tr_\theta.$$

The effect therefore of an initial arbitrary state of the system is, at the time t , expressed by the formula

$$y_{l,t} = \frac{2}{\pi} \sum_{(j)=-\infty}^{\infty} \left(y_{j,0} + y'_{j,0} \int_0^t dt \right) \int_0^{\frac{\pi}{2}} d\theta \cos 2(l-j)\theta \cos tr_\theta.$$

By supposing $y_{-j,0} = -y_{j,0}$ and $y'_{-j,0} = -y'_{j,0}$, we can reduce this general expression for a system indefinite in *both* directions to the corresponding expression in (III)' for a system which is indefinite in one direction only, the particle P_0 being fixed. It is also possible, by consideration of limits, to connect the expression for a doubly infinite with that for a doubly finite system, so as to deduce each from the other. In deducing the infinite from the finite, we suppose j, l, n to increase indefinitely together, preserving finite ratios; in deducing the finite from the infinite, we suppose a certain periodicity of initial state, for greater and greater distances from the origin P_0 , in each of two opposite directions.

(IV)''. In the recent expression for $y_{l,t}$ we may change $2 \int_0^{\frac{\pi}{2}}$ to \int_0^π ; & then, if

$$y_{j,0} = \eta_\alpha \cos(2j\alpha - \gamma_\alpha), \quad y'_{j,0} = \eta'_\alpha \cos(2j\alpha - \gamma_\alpha),$$

we may change

$$\cos \{2(l-j)\theta\} \cos (2j\alpha - \gamma_\alpha) \text{ to } \cos \{2l\theta \mp \gamma_\alpha - 2j(\theta \mp \alpha)\};$$

in which it is remarkable that we may take at pleasure the upper or the lower signs. Summing relatively to j , we find

$$\begin{aligned} y_{l,t} &= \frac{1}{2\pi} \left(\eta_\alpha + \eta'_\alpha \int_0^t dt \right) \int_0^\pi d\theta [\sin \{(2j+1)(\theta \mp \alpha) - (2l\theta \mp \gamma_\alpha)\} \\ &\quad - \sin \{(2j-1)(\theta \mp \alpha) - (2l\theta \mp \gamma_\alpha)\}] \frac{\cos tr_\theta}{\sin(\theta \mp \alpha)} \\ &= \frac{1}{\pi} \left(\eta_\alpha + \eta'_\alpha \int_0^t dt \right) \int_0^\pi d\theta \frac{\sin(j-j, +1)(\theta - \alpha)}{\sin(\theta - \alpha)} \cos \{(j+j,)(\theta - \alpha) - (2l\theta - \gamma_\alpha)\} \cos tr_\theta, \end{aligned}$$

as the effect, at the time t , of an initial state in which all the particles from P_j , to P_j inclusive are disturbed according to the simple mode (I)'', so as to have

$$y_{l,0} = \eta_\alpha \cos(2l\alpha - \gamma_\alpha), \quad y'_{l,0} = \eta'_\alpha \cos(2l\alpha - \gamma_\alpha),$$

and all the other particles are originally undisturbed.

In the particular case when $\gamma_\alpha = \frac{\pi}{2}$, the recent expression for $y_{l,t}$ reduces itself to (29) or (31), in articles 52, 54; the initial conditions being then

$$y_{l,0} = \eta_\alpha \sin 2l\alpha, \quad y'_{l,0} = \eta'_\alpha \sin 2l\alpha,$$

if l be $> j, -1$ but $< j+1$, & $y_{l,0} = 0, y'_{l,0} = 0$, for all other integer values of l . By assuming also the relations $y_{-l,0} = -y_{l,0}, y'_{-l,0} = -y'_{l,0}$, we can pass from the case of a doubly infinite to that of a singly infinite system. And by consideration of limits, the cases of a doubly infinite and of a doubly finite system may be connected so as to deduce each from the other. The consideration of limits shows also that if $j = -\infty$, so that, in the doubly infinite system, the particle P_j & all behind it are initially disturbed according to the law $\frac{y_{l,0}}{\eta_\alpha} = \frac{y'_{l,0}}{\eta'_\alpha} = \cos(2l\alpha - \gamma_\alpha)$, while all beyond it have neither initial displacement nor velocity, the state of the system at the time t is expressed as follows (at least if α be between 0 & π , or more generally if $\sin \alpha$ be different from 0):

$$\begin{aligned} y_{l,t} &= \frac{1}{2} \cos(2l\alpha - \gamma_\alpha) \left(\eta_\alpha + \eta'_\alpha \int_0^t dt \right) \cos tr_\alpha \\ &\quad - \frac{1}{2\pi} \left(\eta_\alpha + \eta'_\alpha \int_0^t dt \right) \int_0^\pi d\theta \sin \{2l\theta - \gamma_\alpha - (2j+1)(\theta - \alpha)\} \frac{\cos tr_\theta}{\sin(\theta - \alpha)}. \end{aligned}$$

By making $\gamma_\alpha = \frac{\pi}{2}$, this reduces itself to the formula (28) of the 50th article.

(V)''. The general formula for the doubly indefinite system may be thus written:

$$y_{l,t} = \frac{1}{\pi} \sum_{(j)-\infty}^{\infty} \left(y_{j,0} + y'_{j,0} \int_0^t dt \right) \int_0^\pi d\theta \cos(2l\theta - 2j\theta - tr_\theta); \quad (r_\theta = 2a \sin \theta).$$

This most general mode of motion of this system may therefore be considered as the resultant of an infinite number of component motions, of which each separately corresponds to the continual and uniform transmission of phase in one of two opposite directions. Nor is it necessary now to compound or conjugate two opposite transmissions of this sort, in order to obtain a particular integral; we may employ either singly, & shall still obtain thereby a possible permanent mode of motion of the system.

67. The three preceding articles, 64, 65, 66, contain a recapitulation of the chief results obtained in the earlier articles of this manuscript. A few remarks may however be usefully made here, before passing to the solution of Problem III. In particular, it seems useful to observe that the particular integrals hitherto considered correspond either to *oscillating* or to *travelling sinusoids*. The *oscillating* are those which have fixed nodes, but oscillating venters; the *travelling* are those which have neither nodes nor venters fixed, but which, instead of oscillating & thereby *changing form, change place* by moving uniformly & continually either in the positive or in the negative direction. When any particle is fixed, this condition of fixity obliges us either to suppose a node to be fixed thereat, & therefore the sinusoid to oscillate, or else two oppositely travelling but otherwise similar sinusoids to be always conjugated together. But when the system is doubly infinite, this conjugation is not necessary & we may suppose a doubly indefinite sinusoid to travel continually in one direction, without being accompanied by any other travelling in the direction opposite. Even if the system be finite, we may suppose the sinusoids, whether oscillating or travelling, to be infinite. Finally, in the case of a finite system & finite sinusoid, we may suppose the number of venters to exceed the number of particles; but this will lead to no essentially new law of arrangement or vibration of the particles themselves. Thus, in the case of a single vibrating particle, we may treat that particle as a 3rd, 5th, ... venter; but its motion will be the same as when it was treated as the 1st.

68. Returning now to Problem III, article 63, we are to suppose that the initial states of some finite number of successive particles correspond to some one *travelling sinusoid*, while the other particles are initially undisturbed; & are to investigate the consequences of this supposition.

69. For the case of a finite system, we have found (see page 492)

$$y_{l,t} = \frac{1}{n+1} \sum_{(j)1}^n \sum_{(i)1}^{2n+1} \sin 2j\alpha_i (y_{j,0} \sin + r_i^{-1} y'_{j,0} \cos) (2l\alpha_i - tr_i),$$

in which $\alpha_i = \frac{i\pi}{2n+2}$ and $r_i = 2a \sin \alpha_i$. And we are now to suppose that for certain successive values of j , namely from j , to j , we have

$$y_{j,0} = B_k \sin (2j\alpha_k + \beta_k), \quad y'_{j,0} = -r_k B_k \cos (2j\alpha_k + \beta_k),$$

k being an integer which is less or greater than $n+1$, according as the initial sinusoid is travelling forward or backward.

Instead of $\sum_{(i)1}^{2n+1} \sin 2j\alpha_i \sin (2l\alpha_i - tr_i)$ we may write

$$\sum_{(i)1}^{2n+1} \sin 2j\alpha_i \sin 2l\alpha_i \cos tr_i, \quad \& \text{ we have } 0 = \sum_{(i)1}^{2n+1} \cos 2j\alpha_i \sin 2l\alpha_i \cos tr_i;$$

we may therefore change, under the signs of summation, the product

$$\sin 2j\alpha_i \sin (2j\alpha_k + \beta_k) \quad \text{to} \quad \cos \{2j(\alpha_i - \alpha_k) - \beta_k\},$$

& in like manner

$$-\sin 2j\alpha_i \cos (2j\alpha_k + \beta_k) \quad \text{to} \quad -\sin \{2j(\alpha_i - \alpha_k) - \beta_k\} = \cos \left(2j(\alpha_i - \alpha_k) - \beta_k + \frac{\pi}{2} \right);$$

therefore

$$y_{l,t} = \frac{B_k}{n+1} \sum_{(i)1}^{2n+1} \sin 2l\alpha_i \frac{\sin (j-j, +1) (\alpha_i - \alpha_k)}{\sin (\alpha_i - \alpha_k)} \\ \times \left\{ \cos \{(j+j,) (\alpha_i - \alpha_k) - \beta_k\} \cos tr_i - \frac{r_k}{r_i} \sin \{(j+j,) (\alpha_i - \alpha_k) - \beta_k\} \sin tr_i \right\}.$$

70. This is the general expression for $y_{l,t}$ in the present question; if $j = 1$, that is, if all the first j particles are originally disturbed in the way supposed, we have

$$y_{l,t} = \frac{B_k}{n+1} \sum_{(i)1}^{2n+1} \frac{\sin 2l\alpha_i \left(\sin \{(2j+1)(\alpha_i - \alpha_k) - \beta_k\} \cos tr_i + \frac{r_k}{r_i} \cos \{(2j+1)(\alpha_i - \alpha_k) - \beta_k\} \sin tr_i \right)}{2 \sin (\alpha_i - \alpha_k)} + \frac{B_k}{n+1} \sum_{(i)1}^{2n+1} \frac{\sin 2l\alpha_i \left(\sin \beta_k \cos tr_i - \frac{r_k}{r_i} \cos \beta_k \sin tr_i \right)}{2 \tan (\alpha_i - \alpha_k)}.$$

Making both $j = 1$ and $j = n$, that is, supposing all the n particles to be originally disturbed in the way already mentioned, we find

$$\frac{\sin (2j+1)(\alpha_i - \alpha_k)}{\sin (\alpha_i - \alpha_k)} = \frac{\sin \{(i-k)\pi - (\alpha_i - \alpha_k)\}}{\sin (\alpha_i - \alpha_k)} = \frac{\sin (i-k)\pi}{\tan (\alpha_i - \alpha_k)} - \cos (i-k)\pi;$$

the first part vanishes or is equal to $2n + 2$, according as i is different from or equal to k ; & the second part disappears in the summation; in like manner

$$\frac{\cos (2j+1)(\alpha_i - \alpha_k)}{\sin (\alpha_i - \alpha_k)} = \frac{\cos (i-k)\pi}{\tan (\alpha_i - \alpha_k)} + \sin (i-k)\pi = \frac{\cos (i-k)\pi}{\tan (\alpha_i - \alpha_k)};$$

therefore the expression for the state of the system at the time t is

$$y_{l,t} = B_k \sin 2l\alpha_k \cos (tr_k - \beta_k) + \frac{B_k}{2n+2} \sum_{(i)1}^{2n+1} \sin 2l\alpha_i \cotan (\alpha_i - \alpha_k) \text{vers } (i-k)\pi \left(\sin \beta_k \cos tr_i - \frac{r_k}{r_i} \cos \beta_k \sin tr_i \right).$$

In this expression the first part corresponds to a possible and permanent mode of simple vibration; & the second part must correspond to an initial state in which the displacements and velocities of all the particles are represented by the formulae

$$y_{j,0} = B_k \cos 2j\alpha_k \sin \beta_k, \quad y'_{j,0} = -r_k B_k \cos 2j\alpha_k \cos \beta_k.$$

Accordingly, the effect of such an initial state of any single particle P_j is

$$y_{l,t} = \frac{B_k}{n+1} \sum_{(i)1}^{2n+1} \sin 2l\alpha_i \sin 2j(\alpha_i - \alpha_k) \left(\sin \beta_k \cos tr_i - r_k r_i^{-1} \cos \beta_k \sin tr_i \right);$$

and

$$\sum_{(j)1}^n \sin 2j(\alpha_i - \alpha_k) = \frac{\cos (\alpha_i - \alpha_k) - \cos (2n+1)(\alpha_i - \alpha_k)}{2 \sin (\alpha_i - \alpha_k)} = \frac{\text{vers } (i-k)\pi}{2 \tan (\alpha_i - \alpha_k)}.$$

This last expression becomes = 0 when $i - k$ is any even integer (zero included); if therefore k be odd, of the form $2\kappa - 1$, we must take i even, of the form 2ι , & we have, as the corresponding value of the formula of the last article,

$$y_{l,t} = B_{2\kappa-1} \sin 2l\alpha_{2\kappa-1} \cos (tr_{2\kappa-1} - \beta_{2\kappa-1}) + \frac{B_{2\kappa-1}}{n+1} \sum_{(i)1}^n \sin 2l\alpha_{2\iota} \cotan (\alpha_{2\iota} - \alpha_{2\kappa-1}) \left(\sin \beta_{2\kappa-1} \cos tr_{2\iota} - \frac{r_{2\kappa-1}}{r_{2\iota}} \cos \beta_{2\kappa-1} \sin tr_{2\iota} \right);$$

but if k be of the form 2κ , we must take i of the form $2\iota - 1$, and

$$y_{l,t} = B_{2\kappa} \sin 2l\alpha_{2\kappa} \cos (tr_{2\kappa} - \beta_{2\kappa}) + \frac{B_{2\kappa}}{n+1} \sum_{(i)1}^{n+1} \sin 2l\alpha_{2\iota-1} \cotan (\alpha_{2\iota-1} - \alpha_{2\kappa}) \times \left(\sin \beta_{2\kappa} \cos tr_{2\iota-1} - \frac{r_{2\kappa}}{r_{2\iota-1}} \cos \beta_{2\kappa} \sin tr_{2\iota-1} \right).$$

71. For example, if $n = 2$ and $k = 1$, so that the system contains only two moveable particles and that these are originally in the state of the travelling sinusoid

$$y_{j,at} = B_1 \sin(2j\alpha_1 + \beta_1 - r_1 dt),$$

in which $\alpha_1 = \frac{\pi}{6}$ and $r_1 = a$; so that

$$y_{j,0} = B_1 \sin\left(\frac{j\pi}{3} + \beta_1\right), \quad y'_{j,0} = -r_1 B_1 \cos\left(\frac{j\pi}{3} + \beta_1\right);$$

then the parts proportional to $\sin \frac{j\pi}{3}$, namely the initial partial displacements

$$y_{j,0} = B_1 \cos \beta_1 \sin \frac{j\pi}{3}$$

& the initial partial velocities

$$y'_{j,0} = a B_1 \sin \beta_1 \sin \frac{j\pi}{3},$$

will produce the permanent partial vibration corresponding to a fixed sinusoid with one venter, and represented by the partial formula

$$y_{l,t} = B_1 \sin \frac{l\pi}{3} \cos(at - \beta_1);$$

and the parts proportional to $\cos \frac{j\pi}{3}$, namely the initial partial displacements

$$y_{j,0} = B_1 \sin \beta_1 \cos \frac{j\pi}{3},$$

or more fully

$$y_{1,0} = \frac{1}{2} B_1 \sin \beta_1, \quad y_{2,0} = -\frac{1}{2} B_1 \sin \beta_1,$$

& the initial partial velocities

$$y'_{j,0} = -a B_1 \cos \beta_1 \cos \frac{j\pi}{3},$$

or more fully

$$y'_{1,0} = -\frac{a}{2} B_1 \cos \beta_1, \quad y'_{2,0} = \frac{a}{2} B_1 \cos \beta_1,$$

will produce another partial vibration at the time t , which may be thus represented,

$$y_{l,t} = \frac{1}{3} B_1 \sum_{(l)1}^2 \sin \frac{2l\pi}{3} \cotan\left(\frac{l\pi}{3} - \frac{\pi}{6}\right) (\sin \beta_1 \cos tr_{2l} - ar_{2l}^{-1} \cos \beta_1 \sin tr_{2l}),$$

in which

$$r_{2l} = 2a \sin \frac{l\pi}{3} = a\sqrt{3};$$

also

$$\cotan\left(\frac{\pi}{3} - \frac{\pi}{6}\right) - \cotan\left(\frac{2\pi}{3} - \frac{\pi}{6}\right) = \cotan \frac{\pi}{6} - \cotan \frac{\pi}{2} = \sqrt{3};$$

$$\therefore y_{l,t} = \frac{B_1}{\sqrt{3}} \sin \frac{2l\pi}{3} \left(\sin \beta_1 \cos at \sqrt{3} - \frac{1}{\sqrt{3}} \cos \beta_1 \sin at \sqrt{3} \right).$$

Accordingly this also is a possible permanent mode of vibration of the system of two particles, & satisfies the initial conditions.

72. Again, if $n=2$ & if the initial conditions be

$$y_{j,0} = B_2 \sin \beta_2 \cos \frac{2j\pi}{3}, \quad y'_{j,0} = -r_2 B_2 \cos \beta_2 \cos \frac{2j\pi}{3}, \quad (r_2 = a\sqrt{3}),$$

we find by the formula for k even

$$y_{l,t} = \frac{B_2}{3} \sum_{(i)1}^3 \sin \frac{l(2i-1)\pi}{3} \cotan \frac{(2i-3)\pi}{6} \left(\sin \beta_2 \cos tr_{2i-1} - \frac{a\sqrt{3}}{r_{2i-1}} \cos \beta_2 \sin tr_{2i-1} \right).$$

Also

$$\cotan \left(-\frac{\pi}{6} \right) - \cotan \frac{\pi}{2} = -\sqrt{3}; \quad \therefore y_{l,t} = -\frac{B_2}{\sqrt{3}} \sin \frac{l\pi}{3} (\sin \beta_2 \cos at - \sqrt{3} \cos \beta_2 \sin at).$$

Accordingly this gives

$$y_{1,0} = -\frac{B_2}{2} \sin \beta_2 = y_{2,0}; \quad y'_{1,0} = \frac{a\sqrt{3}}{2} B_2 \cos \beta_2 = y'_{2,0}.$$

73. Next let the system be infinite in one direction, so that the condition $y_{0,t} = 0$ but not the condition $y_{n+1,t} = 0$ is to be attended to. We have now to suppose that for all values of j , from j , to j , the initial state is represented by the formulae

$$y_{j,0} = B_\alpha \sin (2j\alpha + \beta_\alpha), \quad y'_{j,0} = -r_\alpha B_\alpha \cos (2j\alpha + \beta_\alpha);$$

but, because we already know the effect of the initial state

$$y_{j,0} = B_\alpha \cos \beta_\alpha \sin 2j\alpha, \quad y'_{j,0} = r_\alpha B_\alpha \sin \beta_\alpha \sin 2j\alpha, \quad (\text{see page 493}),$$

it is sufficient now to calculate the effect of

$$y_{j,0} = B_\alpha \sin \beta_\alpha \cos 2j\alpha, \quad y'_{j,0} = -r_\alpha B_\alpha \cos \beta_\alpha \cos 2j\alpha.$$

Using for this purpose the formula

$$y_{l,t} = \frac{2}{\pi} \sum_{(j)1}^{\infty} \int_0^\pi d\theta \sin 2j\theta \sin 2l\theta (y_{j,0} \cos tr_\theta + r_\theta^{-1} y'_{j,0} \sin tr_\theta),$$

& changing under the sign of summation $\sin 2j\theta \cos 2j\alpha$ to $\sin 2j(\theta - \alpha)$, we get, as the new part of the final state of the system,

$$y_{l,t} = \frac{2B_\alpha}{\pi} \int_0^\pi d\theta \frac{\sin(j-j, +1)(\theta - \alpha)}{\sin(\theta - \alpha)} \sin(j+j,)(\theta - \alpha) \sin 2l\theta \left(\sin \beta_\alpha \cos tr_\theta - \frac{r_\alpha}{r_\theta} \cos \beta_\alpha \sin tr_\theta \right).$$

The old part was (see page 493)

$$y_{l,t} = \frac{2B_\alpha}{\pi} \int_0^\pi d\theta \frac{\sin(j-j, +1)(\theta - \alpha)}{\sin(\theta - \alpha)} \cos(j+j,)(\theta - \alpha) \sin 2l\theta \left(\cos \beta_\alpha \cos tr_\theta + \frac{r_\alpha}{r_\theta} \sin \beta_\alpha \sin tr_\theta \right);$$

therefore the sum of these two parts, or the solution of the question proposed in the present article, is

$$y_{l,t} = \frac{2B_\alpha}{\pi} \int_0^\pi d\theta \frac{\sin(j-j, +1)(\theta - \alpha)}{\sin(\theta - \alpha)} \sin 2l\theta \\ \times [\cos\{(j+j,)(\theta - \alpha) - \beta_\alpha\} \cos tr_\theta - r_\alpha r_\theta^{-1} \sin\{(j+j,)(\theta - \alpha) - \beta_\alpha\} \sin tr_\theta].$$

It might have been deduced from that of article 69 by changing B_i to B_α , α_i to θ , α_k to α , β_k to β_α , r_i to r_θ , r_k to r_α , and $\frac{1}{n+1} \sum_{(i)1}^{2n+1}$ to $\frac{2}{\pi} \int_0^\pi d\theta$.

It may also be thus written:

$$y_{l,t} = \frac{B_\alpha}{\pi} \int_0^\pi d\theta \frac{\sin 2l\theta}{\sin(\theta - \alpha)} \left\{ [\sin \{(2j+1)(\theta - \alpha) - \beta_\alpha\} - \sin \{(2j-1)(\theta - \alpha) - \beta_\alpha\}] \cos tr_\theta \right. \\ \left. + \frac{r_\alpha}{r_\theta} [\cos \{(2j+1)(\theta - \alpha) - \beta_\alpha\} - \cos \{(2j-1)(\theta - \alpha) - \beta_\alpha\}] \sin tr_\theta \right\} = \Phi(2j+1) - \Phi(2j-1).$$

74. The function Φ introduced at the end of the last article is such that

$$\Phi(1) = \frac{B_\alpha}{\pi} \int_0^\pi d\theta \frac{\sin 2l\theta}{\tan(\theta - \alpha)} \left\{ -\sin \beta_\alpha \cos tr_\theta + r_\alpha r_\theta^{-1} \cos \beta_\alpha \sin tr_\theta \right\};$$

this, therefore, with its sign changed, is to be added to $\Phi(2j+1)$, in order to obtain the effect of the initial disturbance of the first j particles of this singly indefinite system. On the other hand, if we seek the effect of the initial disturbance of the j^{th} and all following particles, we are to suppose $j = \infty$, & to calculate $\Phi(\infty)$; which is

$$\Phi(\infty) = B_\alpha \sin 2l\alpha (\cos \beta_\alpha \cos tr_\alpha + \sin \beta_\alpha \sin tr_\alpha) = B_\alpha \sin 2l\alpha \cos (tr_\alpha - \beta_\alpha).$$

We have therefore, for the effect of an initial disturbance of the kind supposed, but extending to the whole system, the expression

$$y_{l,t} = \Phi(\infty) - \Phi(1) = B_\alpha \sin 2l\alpha \cos (tr_\alpha - \beta_\alpha) + \frac{B_\alpha}{\pi} \int_0^\pi d\theta \frac{\sin 2l\theta}{\tan(\theta - \alpha)} \\ \times (\sin \beta_\alpha \cos tr_\theta - r_\alpha r_\theta^{-1} \cos \beta_\alpha \sin tr_\theta);$$

& the second part of this expression, namely $-\Phi(1)$, must be the effect of that part of the initial state of the whole system which is represented by the formula $y_{j,at} = B_\alpha \cos 2j\alpha \sin(\beta_\alpha - r_\alpha dt)$, or by $y_{j,0} = B_\alpha \cos 2j\alpha \sin \beta_\alpha$, $y'_{j,0} = -r_\alpha B_\alpha \cos 2j\alpha \cos \beta_\alpha$. Accordingly it is easy to verify and reduce this result by making $j=1$ & $j=\infty$ in the formula given near the middle of the preceding page. We may also easily deduce the present result, as the limit of either of those given at the end of article 70 for the case of a finite system.

75. As an example we may take the case $\alpha = \frac{\pi}{2}$, in which the initial state of the system corresponds to a travelling sinusoid of the form $y_{j,at} = B \sin(j\pi + \beta - 2adt)$, so that

$$y_{j,0} = B \sin \beta (-1)^j, \quad y'_{j,0} = 2aB \cos \beta (-1)^{j+1}.$$

In this case, the permanent part, free from the sign of integration, disappears, & we find

$$y_{l,t} = -\frac{B}{\pi} \int_0^\pi d\theta \sin 2l\theta \tan \theta \left(\sin \beta \cos tr_\theta - \frac{2a}{r_\theta} \cos \beta \sin tr_\theta \right).$$

Accordingly

$$\int_0^\pi d\theta \sin 2l\theta \tan \theta = \pi, \quad \text{and} \quad \int_0^\pi d\theta \{ \sin 2(l+1)\theta + \sin 2l\theta \} \tan \theta = 0,$$

if l , being integer, is > 0 ; so that

$$-\frac{1}{\pi} \int_0^\pi d\theta \sin 2l\theta \tan \theta = (-1)^l,$$

if l is > 0 , and the initial conditions are satisfied. These initial conditions correspond to an

alternate arrangement of the particles both in displacement and in velocity; & as long as l is much greater than t , the arrangement ought to remain nearly alternate; that is, the above expression ought to give, nearly,

$$y_{l,t} = B \sin(l\pi + \beta - 2at) = B \cos l\pi \sin(\beta - 2at),$$

if l be much greater than t . Accordingly this result is obtained by integrating from $\frac{\pi}{2} - \delta\theta$ to $\frac{\pi}{2} + \delta\theta$. On the contrary, if t be much greater than l , we have nearly $y_{l,t} = 0$. But the consequences of supposing t large, or the state of a system after a very long time, shall be the object of a full examination hereafter.

76. Finally if the system be indefinite in both directions, and if for some set of successive particles, $P_j, \dots P_j$, the initial state is represented by the formula

$$y_{j,at} = B \sin(\beta + 2j\alpha - r_\alpha dt),$$

that is, more fully, by

$$y_{j,0} = B \sin(\beta + 2j\alpha), \quad y'_{j,0} = -r_\alpha B \cos(\beta + 2j\alpha),$$

we may resolve this travelling sinusoid into two fixed sinusoids, namely,

$$1^{\text{st}}, \quad y_{j,0} = B \sin \beta \cos 2j\alpha, \quad y'_{j,0} = -r_\alpha B \cos \beta \cos 2j\alpha,$$

and

$$2^{\text{nd}}, \quad y_{j,0} = B \cos \beta \sin 2j\alpha, \quad y'_{j,0} = r_\alpha B \sin \beta \sin 2j\alpha.$$

The effect of the first is, by page 496, (making $\gamma_\alpha = 0$),

$$y_{l,t} = \frac{B}{\pi} \int_0^\pi d\theta \frac{\sin(j-j, +1)(\theta - \alpha)}{\sin(\theta - \alpha)} \cos\{(j+j,)(\theta - \alpha) - 2l\theta\} (\sin \beta \cos tr_\theta - r_\alpha r_\theta^{-1} \cos \beta \sin tr_\theta);$$

and the effect of the 2nd part is, by the same page, (making $\gamma_\alpha = \frac{\pi}{2}$),

$$y_{l,t} = -\frac{B}{\pi} \int_0^\pi d\theta \frac{\sin(j-j, +1)(\theta - \alpha)}{\sin(\theta - \alpha)} \sin\{(j+j,)(\theta - \alpha) - 2l\theta\} (\cos \beta \cos tr_\theta + r_\alpha r_\theta^{-1} \sin \beta \sin tr_\theta);$$

therefore the whole effect, or the solution of the present problem, is

$$y_{l,t} = \frac{B}{\pi} \int_0^\pi d\theta \frac{\sin(j-j, +1)(\theta - \alpha)}{\sin(\theta - \alpha)} \\ \times [\sin\{\beta + 2l\theta - (j+j,)(\theta - \alpha)\} \cos tr_\theta - r_\alpha r_\theta^{-1} \cos\{\beta + 2l\theta - (j+j,)(\theta - \alpha)\} \sin tr_\theta].$$

It may also be thus written,

$$y_{l,t} = \frac{B}{2\pi} \int_0^\pi \frac{d\theta}{\sin(\theta - \alpha)} [\{\cos((2j+1)(\theta - \alpha) - (2l\theta + \beta)) - \cos((2j, -1)(\theta - \alpha) - (2l\theta + \beta))\} \cos tr_\theta \\ - r_\alpha r_\theta^{-1} \{\sin((2j+1)(\theta - \alpha) - (2l\theta + \beta)) - \sin((2j, -1)(\theta - \alpha) - (2l\theta + \beta))\} \sin tr_\theta] \\ = \Psi(2j+1) - \Psi(2j, -1).$$

77. In this expression the function Ψ is such that

$$-\Psi(-\infty) = \frac{B}{2} \{\sin(2l\alpha + \beta) \cos tr_\alpha - \cos(2l\alpha + \beta) \sin tr_\alpha\} = \frac{1}{2} B \sin(2l\alpha + \beta - tr_\alpha);$$

if therefore all the particles as far as P_j inclusive are originally disturbed in the way supposed in the last article, we have, at the time t , a state which is thus expressed,

$$y_{l,t} = \frac{1}{2}B \sin(\beta + 2l\alpha - tr_\alpha) + \frac{B}{2\pi} \int_0^\pi \frac{d\theta}{\sin(\theta - \alpha)} [\cos\{(2j+1)(\theta - \alpha) - (2l\theta + \beta)\} \cos tr_\theta - r_\alpha r_\theta^{-1} \sin\{(2j+1)(\theta - \alpha) - (2l\theta + \beta)\} \sin tr_\theta].$$

The part involving the sign \int must therefore express the effect of an initial state in which all the particles as far as P_j inclusive are agitated according to the formula

$$y_{l,at} = +\frac{1}{2}B \sin(\beta + 2l\alpha - r_\alpha dt),$$

& all the particles beyond P_j according to

$$y_{l,at} = -\frac{1}{2}B \sin(\beta + 2l\alpha - r_\alpha dt).$$

We ought therefore to have

$$\begin{aligned} \pm \pi \sin(\beta + 2l\alpha) &= \int_0^\pi \frac{d\theta}{\sin(\theta - \alpha)} \cos\{(2j+1)(\theta - \alpha) - (2l\theta + \beta)\}, \\ \pm \pi \cos(\beta + 2l\alpha) &= \int_0^\pi \frac{d\theta}{\sin(\theta - \alpha)} \sin\{(2j+1)(\theta - \alpha) - (2l\theta + \beta)\}, \end{aligned}$$

the upper signs to be taken if l be not greater than j : that is, we ought to have

$$\pm \pi = \int_0^\pi \frac{d\theta}{\sin(\theta - \alpha)} \sin(2j - 2l + 1)(\theta - \alpha),$$

according as l is \geq or $>$ j ; and

$$0 = \int_0^\pi \frac{d\theta}{\sin(\theta - \alpha)} \cos(2j - 2l + 1)(\theta - \alpha).$$

And it is easy to prove, in fact, that these equations are true. (See article 51.)

Problem IV.

78. It is now required to determine the approximate or limiting forms to which the solution of the foregoing problem tends, when the system is numerous & the time elapsed is large.

79. Beginning with the case when the system extends indefinitely in both directions, and when all the particles as far as P_j inclusive are originally agitated according to the formula

$$y_{j,at} = B \sin(\beta + 2j\alpha - r_\alpha dt), \quad r_\alpha = 2a \sin \alpha, \quad \alpha > 0, < \pi,$$

but all beyond P_j are originally undisturbed, we have to discuss the formula of article 77, on the supposition that t is very great. In this manner we obtain, approximately, attending only to values of θ nearly equal to α ,

$$\begin{aligned} y_{l,t} &= \frac{1}{2}B \sin(\beta + 2l\alpha - tr_\alpha) + \frac{1}{2\pi} B \int_{\alpha - \delta\alpha}^{\alpha + \delta\alpha} \frac{d\theta}{\theta - \alpha} \cos\{(2j+1)(\theta - \alpha) - (2l\theta + \beta) + tr_\theta\} \\ &= \frac{1}{2}B \sin(\beta + 2l\alpha - tr_\alpha) \left\{ 1 + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\theta}{\theta - \alpha} \sin(2j + 1 - 2l + tr'_\alpha)(\theta - \alpha) \right\}, \end{aligned}$$

in which $r'_\alpha = 2a \cos \alpha$. If therefore l be considerably less than $j + \frac{1}{2} + at \cos \alpha$, then we have nearly $y_{l,t} = B \sin(\beta + 2l\alpha - tr_\alpha)$; but if l be considerably greater than $j + \frac{1}{2} + at \cos \alpha$, then, nearly, $y_{l,t} = 0$. And these conclusions hold good, whether t be large or small, & even for negative values of t ; we may therefore consider the initial state of the system as having been and as continuing to be

dynamically propagated, forwards or backwards according as $\cos \alpha$ is $>$ or $<$ 0, and with a velocity $= a \cos \alpha$.

80. Let us consider particularly the case $\alpha = \frac{\pi}{2}$, for which this velocity vanishes. The rigorous formula of article 77 becomes in this case

$$y_{l,t} = \frac{1}{2} B (-1)^j \sin(\beta - 2at) - \frac{1}{2\pi} B (-1)^j \int_0^\pi d\theta \sec \theta \\ \times \left\{ \sin \{(2j+1-2l)\theta - \beta\} \cos(2at \sin \theta) + \cos \{(2j+1-2l)\theta - \beta\} \frac{\sin(2at \sin \theta)}{\sin \theta} \right\};$$

which may also be rigorously thus expressed:

$$y_{l,t} = \frac{1}{2} (-1)^j B \sin(\beta - 2at) - \frac{1}{2\pi} (-1)^j B \int_0^\pi d\theta \frac{1 + \sin \theta}{\sin 2\theta} \sin \{(2j+1-2l)\theta - \beta + 2at \sin \theta\} \\ + \frac{1}{2\pi} (-1)^j B \int_0^\pi d\theta \frac{1 - \sin \theta}{\sin 2\theta} \sin \{(2j+1-2l)\theta - \beta - 2at \sin \theta\}.$$

Now, while $\Theta_\theta = (2j+1-2l)\theta - \beta - 2at \sin \theta$ receives a small but finite increment, θ in general receives, nearly, the increment $\frac{\Delta \Theta_\theta}{2j+1-2l-2at \cos \theta} = \Delta \theta$; if then l be considerably different from $j + \frac{1}{2} - at \cos \theta$, the factor $\sin \{(2j+1-2l)\theta - \beta - 2at \sin \theta\}$ will fluctuate often between its extreme values, ± 1 , while the other factor $\frac{1 - \sin \theta}{\sin 2\theta}$ will vary little, unless θ be nearly $= 0$ or π ; thus, in calculating the 2nd definite integral, we may in general attend only to these particular values of θ . But for these values, we must combine the corresponding parts of the 1st definite integral, and to do this we may write the 2nd integral as follows:

$$-\frac{1}{2\pi} (-1)^j B \int_0^\pi d\theta \frac{1 - \sin \theta}{\sin 2\theta} \sin \{(2j+1-2l)\theta + \beta + 2at \sin \theta\};$$

the whole expression for $y_{l,t}$ may therefore rigorously be thus written,*

$$y_{l,t} = \frac{1}{2} (-1)^j B \sin(\beta - 2at) - \frac{1}{\pi} (-1)^j B \cos \beta \int_0^\pi \frac{d\theta}{\sin 2\theta} \sin \{(2j+1-2l)\theta + 2at \sin \theta\} \\ + \frac{1}{2\pi} (-1)^j B \sin \beta \int_0^\pi \frac{d\theta}{\cos \theta} \cos \{(2j+1-2l)\theta + 2at \sin \theta\};$$

so that if l be considerably greater or less than both $j + \frac{1}{2} + at$ and $j + \frac{1}{2} - at$, the sum of the parts corresponding to θ nearly $= 0$ and θ nearly $= \pi$ is insensible; but if l be considerably greater than $j + \frac{1}{2} - at$ and at the same time considerably less than $j + \frac{1}{2} + at$, (t being large and positive,) then

* [Accepting Hamilton's method of treating the integrals, it is a question of finding the value of the integral

$$-\frac{1}{\pi} (-1)^j B \cos \beta \int \frac{d\theta}{\sin 2\theta} \sin \{(2j+1-2l)\theta + 2at \sin \theta\}$$

between the limits $0, \epsilon$ and $\pi - \eta, \pi$, where ϵ, η are small and $2j+1-2l+2at, 2j+1-2l-2at$ are large positive or negative numbers. We get then, easily,

$$-\frac{1}{2\pi} (-1)^j B \cos \beta \int_0^\epsilon \frac{d\theta}{\theta} \sin(2j+1-2l+2at)\theta, \\ + \frac{1}{2\pi} (-1)^j B \cos \beta \int_0^\eta \frac{d\theta}{\theta} \sin(2j+1-2l-2at)\theta.$$

If ϵ, η are such that $|(2j+1-2l+2at)\epsilon|, |(2j+1-2l-2at)\eta|$ are large, the results follow as above.]

this sum of parts is sensibly $= -\frac{1}{2}(-1)^j B \cos \beta$; & consequently, a disturbance or displacement, represented thus and due to the initial velocities, spreads with two equal and opposite velocities $\pm a$ on the two sides of the particle P_j which terminated the initial disturbance, or rather on both sides of the point $j + \frac{1}{2}$; and this constant amount of resultant displacement is $= \frac{1}{4}ay'_{j,0}$. If we have exactly $l = j + \frac{1}{2} + at$, & if t be large, we have to consider

$$\int_0^\pi \frac{d\theta}{\sin 2\theta} \sin \{2at(\theta - \sin \theta)\}, \quad \int_0^\pi \frac{d\theta}{\cos \theta} \cos \{2at(\theta - \sin \theta)\};$$

of which two integrals the second may be neglected, so far as depends on values of θ near to 0 or π , but the first gives, for the parts depending on those values,

$$\frac{1}{2} \int_0^\infty \frac{d\theta}{\theta} \sin \left(\frac{at\theta^3}{3} \right) = \frac{\pi}{12}, \quad \text{and} \quad -\frac{1}{2} \int_{-\infty}^0 \frac{d\theta}{\theta} \sin (4at\theta) = -\frac{\pi}{4},$$

$2at$ being here an odd integer; so that the sum is $-\frac{\pi}{6}$, and the resultant displacement is

$$-\frac{1}{6}(-1)^j B \cos \beta = \frac{1}{12}ay'_{j,0}.$$

If l exactly equals $j + \frac{1}{2} - at$, $t > 0$, then $2at$ is still a large odd integer and the parts considered are

$$-\frac{1}{\pi}(-1)^j B \cos \beta \left(\int_0^{\delta\theta} + \int_{\pi-\delta\theta}^\pi \right) \frac{d\theta}{\sin 2\theta} \sin \{2at(\theta + \sin \theta)\} = -\frac{1}{\pi}(-1)^j B \cos \beta \left(\frac{\pi}{4} - \frac{\pi}{12} \right),$$

giving still the same sum

$$-\frac{1}{6}(-1)^j B \cos \beta = \frac{1}{12}ay'_{j,0}.$$

And it seems likely that if l be the nearest integer either to $j + \frac{1}{2} + at$ or to $j + \frac{1}{2} - at$, when t is large, we shall still have nearly this same displacement $\frac{a}{12}y'_{j,0}$ as the part of the general expression which corresponds to values of θ near to 0 and π . As to the values of θ near $\frac{\pi}{2}$, we may use the 1st definite integral in the second formula of the present article, which, for this purpose, may be put under the form

$$\begin{aligned} & + \frac{1}{2\pi}(-1)^j B \int_{\frac{\pi}{2}-\delta\theta}^{\frac{\pi}{2}+\delta\theta} d\theta \frac{\sin \left\{ (2j+1-2l) \left(\frac{\pi}{2} + \theta - \frac{\pi}{2} \right) - \beta + 2at - 2at \text{vers } \theta - \frac{\pi}{2} \right\}}{\theta - \frac{\pi}{2}} \\ & = \frac{1}{2\pi}(-1)^j B \int_{\frac{\pi}{2}-\delta\theta}^{\frac{\pi}{2}+\delta\theta} \frac{d\theta}{\theta - \frac{\pi}{2}} \cos \left\{ 2at - \beta + (2j+1-2l) \left(\theta - \frac{\pi}{2} \right) - 2at \text{vers} \left(\theta - \frac{\pi}{2} \right) \right\} \\ & = \frac{1}{\pi}(-1)^j B \int_0^\infty \frac{d\theta'}{\theta'} \sin (\beta - 2at \cos \theta') \sin (2j+1-2l)\theta', \end{aligned}$$

If $2j+1-2l$ be > 0 and large, and if we put $(2j+1-2l)\theta' = \Theta$, we may suppose Θ , large enough to allow of our changing, with a sufficient approximation, this integral to the form

$$\frac{1}{\pi}(-1)^j B \int_0^{\Theta'} \frac{d\Theta'}{\Theta'} \sin \Theta' \sin \left(\beta - 2at \cos \frac{\Theta'}{2j+1-2l} \right),$$

while $2at \text{vers } \theta' = at \left(2 \sin \frac{\theta'}{2} \right)^2$ is very small, being nearly $= \frac{at \Theta'^2}{(2j+1-2l)^2}$; and then the integral

becomes $\frac{1}{2}(-1)^l B \sin(\beta - 2at)$. In like manner if $2j + 1 - 2l$ be large but negative, so that $\frac{at}{(2j + 1 - 2l)^2}$ is still extremely small, this integral becomes $-\frac{1}{2}(-1)^l B \sin(\beta - 2at)$.

Hence, adding the term free from the sign \int at the beginning of page 504, we find the following results, as consequences of the initial state expressed by the formula

$$y_{l,t} = (-1)^l B \sin(\beta - 2at) \quad \text{for } l \gg j;$$

(a)... If l be much greater than $j + \frac{1}{2} + at$, $y_{l,t} = 0$;

(b)... If l be much less (algebraically) than $j + \frac{1}{2} - at$,

$$y_{l,t} = (-1)^l B \sin(\beta - 2at);$$

(c)... If l be much less than $j + \frac{1}{2} + at$ but much greater than $j + \frac{1}{2}$, and if $\frac{2l - 2j - 1}{\sqrt{at}}$ be much greater than a certain large number Θ , then

$$y_{l,t} = -\frac{1}{2}(-1)^l B \cos \beta;$$

(d)... If l be much greater than $j + \frac{1}{2} - at$ but much less than $j + \frac{1}{2}$, and if $\frac{2j + 1 - 2l}{\sqrt{at}}$ be much greater than the same large number Θ , then

$$y_{l,t} = -\frac{1}{2}(-1)^l B \cos \beta + (-1)^l B \sin(\beta - 2at);$$

but peculiar calculations are required near the critical values $l = j + \frac{1}{2}$, $l = j + \frac{1}{2} \pm at$. Thus, if $l = j$, & if we wish to calculate the part depending on values of θ near $\frac{\pi}{2}$, we have, by the last page, for this part, the expression (if t be large)

$$\frac{1}{\pi}(-1)^j B \int_0^\infty d\theta, \frac{\sin \theta}{\theta}, \sin(\beta - 2at \cos \theta);$$

which is insensible.

And generally if $2l - 2j - 1$ be small (whether positive or negative) in comparison with \sqrt{at} , so that $2at \text{ vers } \theta$, may attain a considerable value while $(2l - 2j - 1)\theta$, is very small, we have then, by the last page, to consider the part*

$$\frac{B}{\pi}(-1)^j \frac{2j + 1 - 2l}{\sqrt{at}} \int_0^\infty d\theta_n \sin(\beta - 2at + \theta_n^2) = \frac{1}{2}B(-1)^j \frac{2j + 1 - 2l}{\sqrt{a\pi t}} \sin\left(\beta - 2at + \frac{\pi}{4}\right);$$

which corresponds to a vibration, but with diminished amplitude, and with a *change of phase*; to which is to be added the constant displacement,

$$-\frac{1}{2}(-1)^j B \cos \beta, \quad \& \text{ also } \frac{1}{2}(-1)^l B \sin(\beta - 2at).$$

* [The integral is of the form

$$-\frac{1}{\pi}(-1)^j B \int_0^\Theta \frac{d\theta}{\theta} \sin \theta \sin\left(\beta - 2at + \frac{at\theta^2}{(2j + 1 - 2l)^2}\right);$$

where since Θ , is small we may put $\theta = \sin \theta$. Putting $\theta_n^2 = \frac{at\theta^2}{2j + 1 - 2l}$, we get the required form.]

81. The corresponding constant part has not yet been calculated in the problem of article 79. To do so, we must resume the formula of article 77, attending to the factor r_θ^{-1} & to the values of θ which are near to 0 & π .

The part depending on these values of θ is

$$\begin{aligned} & \frac{B}{4\pi} \int_0^\infty \frac{d\theta}{\theta} [\cos \{(2j+1)(\theta-\alpha) - (2l\theta+\beta) - 2at \sin \theta\} - \cos \{(2j+1)(\theta-\alpha) - (2l\theta+\beta) + 2at \sin \theta\}] \\ & - \frac{B}{4\pi} \int_0^\infty \frac{d\theta}{\theta} [\cos \{(2j+1)(\theta+\alpha) - (2l\theta-\beta) - 2at \sin \theta\} - \cos \{(2j+1)(\theta+\alpha) - (2l\theta-\beta) + 2at \sin \theta\}] \\ & = \frac{B}{2\pi} \sin \{(2j+1)\alpha + \beta\} \int_0^\infty \frac{d\theta}{\theta} [\sin \{(2j+1-2l)\theta - 2at \sin \theta\} - \sin \{(2j+1-2l)\theta + 2at \sin \theta\}]; \end{aligned}$$

it vanishes or is insensible if $j + \frac{1}{2} - l - at$ and $j + \frac{1}{2} - l + at$ are both large and have the same sign, that is, if l be much greater than $j + \frac{1}{2} + at$ or much less than $j + \frac{1}{2} - at$; but if l be much less than $j + \frac{1}{2} + at$ and yet much greater than $j + \frac{1}{2} - at$, so that $2j + 1 - 2l - 2at$ is large and negative while $2j + 1 - 2l + 2at$ is large and positive, then the above part becomes, nearly,

$$-\frac{1}{2} B \sin \{(2j+1)\alpha + \beta\}.$$

When $\alpha = \frac{\pi}{2}$, this reduces itself to the value found in article 80, namely, $-\frac{1}{2}(-1)^j B \cos \beta$. In general it may be represented by $-\frac{1}{2} y_{j+\frac{1}{2},0}$, if the initial formula $y_{j,0} = B \sin (2j\alpha + \beta)$ be conceived to extend as far as the point $j + \frac{1}{2}$, or to the middle point between the particles P_j and P_{j+1} . It may also be thus written, $\frac{a}{4} y'_{j,0} - \frac{\cos \alpha}{2} y_{j,0}$.

82. If it happen that $j + \frac{1}{2} - at$ is an integer and if we take l equal hereto, so that $2at = 2j + 1 - 2l = a$ large positive odd integer number, the formula of the last article conducts us to calculate the integrals

$$\int_0^\infty \frac{d\theta}{\theta} \sin \{2at(\theta - \sin \theta)\}, \quad - \int_0^\infty \frac{d\theta}{\theta} \sin \{2at(\theta + \sin \theta)\};$$

which are $\frac{\pi}{6}$, $-\frac{\pi}{2}$; their sum is therefore $-\frac{\pi}{3}$, and the corresponding displacement is

$$-\frac{B}{6} \sin \{(2j+1)\alpha + \beta\}.$$

And the same result is obtained by supposing $l = j + \frac{1}{2} + at$. At these critical positions, the constant displacement is therefore only one third part of the value which it has for particles nearer to P_j .

83. If $2j + 1 - 2l - 2at$ is only small in comparison with $\sqrt[3]{at}$ but not exactly = 0, we may still reduce the integral

$$\int_0^\infty \frac{d\theta}{\theta} \sin \{(2j+1-2l)\theta + 2at \sin \theta\}$$

to $\frac{\pi}{2}$, but the integral

$$\int_0^\infty \frac{d\theta}{\theta} \sin \{(2j+1-2l)\theta - 2at \sin \theta\}$$

assumes the form

$$\int_0^\infty \frac{d\theta}{\theta} \sin \{(2j+1-2l-2at)\theta + 2at(\theta - \sin \theta)\},$$

in which the second part within the brackets now predominates; $2at(\theta - \sin \theta)$ being able now to attain a considerable magnitude, while not only θ is small but also $(2j + 1 - 2l - 2at)\theta$, this latter being nearly equal to

$$\frac{2j + 1 - 2l - 2at}{\sqrt[3]{\left(\frac{at}{3}\right)}} \sqrt[3]{2at(\theta - \sin \theta)}.$$

Also
$$\int_0^\infty \frac{d\theta}{\theta} \sin \{2at(\theta - \sin \theta)\} \text{ is } = \frac{\pi}{6},$$

at being very large; we have therefore only to calculate $\frac{2j + 1 - 2l - 2at}{\sqrt[3]{\left(\frac{1}{3}at\right)}} \int_0^\infty d\theta \cos(\theta^3)$. Now (see 1st Blank Book of the present year, 1839, page 65, left hand & the references there made)*,

$$\begin{aligned} \Gamma(n) &= \int_0^\infty x^{n-1} e^{-x} dx = \int_0^\infty (ax + \sqrt{-1}bx)^{n-1} e^{-(ax + \sqrt{-1}bx)} d(ax + \sqrt{-1}bx), \quad (\text{if } a > 0), \\ &= (a + \sqrt{-1}b)^n \int_0^\infty x^{n-1} e^{-ax} (\cos bx - \sqrt{-1} \sin bx) dx; \end{aligned}$$

therefore

$$\int_0^\infty e^{-ax} x^{n-1} \cos bx dx = r^{-n} \cos nv \Gamma(n), \quad \text{and} \quad \int_0^\infty e^{-ax} x^{n-1} \sin bx dx = r^{-n} \sin nv \Gamma(n),$$

Γ being the celebrated function tabulated by Legendre, and r, v being connected with a, b by the relations $a = r \cos v (> 0)$, $b = r \sin v$. These theorems hold however near a may be to 0; they hold even at that limit, and thereby give (making $v = \frac{\pi}{2}$, $a = 0$, $r = b$)

$$\int_0^\infty x^{n-1} \cos bxdx = b^{-n} \cos \frac{n\pi}{2} \Gamma(n), \quad \int_0^\infty x^{n-1} \sin bxdx = b^{-n} \sin \frac{n\pi}{2} \Gamma(n).$$

Making $b = 1$ and $x = \theta^{\frac{1}{n}}$, these become

$$\int_0^\infty d\theta \cos(\theta^{\frac{1}{n}}) = n \cos \frac{n\pi}{2} \Gamma(n), \quad \int_0^\infty d\theta \sin(\theta^{\frac{1}{n}}) = n \sin \frac{n\pi}{2} \Gamma(n);$$

or, changing n to $\frac{1}{m}$, and observing that $n \Gamma(n) = \Gamma(n + 1)$,

$$\int_0^\infty d\theta \cos(\theta^m) = \cos \frac{\pi}{2m} \Gamma\left(1 + \frac{1}{m}\right), \quad \int_0^\infty d\theta \sin(\theta^m) = \sin \frac{\pi}{2m} \Gamma\left(1 + \frac{1}{m}\right).$$

For example, $\int_0^\infty d\theta \cos \theta = 0$, $\int_0^\infty d\theta \sin \theta = 1$; (to be integrated as limiting results;)

$$\int_0^\infty d\theta \cos(\theta^2) = \sqrt{\frac{1}{2}} \Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \sqrt{\frac{\pi}{2}}, \quad \int_0^\infty d\theta \sin(\theta^2) = \sqrt{\frac{1}{2}} \Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \sqrt{\frac{\pi}{2}};$$

and finally, returning to the present question, $\int_0^\infty d\theta \cos(\theta^3) = \frac{\sqrt{3}}{2} \Gamma\left(\frac{4}{3}\right)$; in which, by Legendre's Table, at the end of the 2nd Part of his Exercises, we have $\Gamma\left(\frac{4}{3}\right) = 10^{1.9508414} = 0,892979$. Thus, in the question of the present article, we have, because $\Gamma\left(\frac{4}{3}\right) = \frac{1}{3} \Gamma\left(\frac{1}{3}\right)$,

$$\int_0^\infty \frac{d\theta}{\theta} \sin \{(2j + 1 - 2l)\theta - 2at \sin \theta\} = \frac{\pi}{6} + \frac{2j + 1 - 2l - 2at}{2 \sqrt[3]{at} \sqrt{3}} \Gamma\left(\frac{1}{3}\right);$$

* [There is no trace of this book among the manuscripts.]

and the expression at the beginning of article 81 becomes

$$-B \sin \{(2j+1)\alpha + \beta\} \left\{ \frac{1}{6} - \frac{2j+1-2l-2at}{4\pi \sqrt[3]{at} \sqrt{3}} \Gamma\left(\frac{1}{3}\right) \right\}.$$

84. In like manner, if $2j+1-2l+2at$ be small in comparison with $\sqrt[3]{at}$, the same expression becomes

$$-B \sin \{(2j+1)\alpha + \beta\} \left\{ \frac{1}{6} + \frac{2j+1-2l+2at}{4\pi \sqrt[3]{at} \sqrt{3}} \Gamma\left(\frac{1}{3}\right) \right\};$$

so that if l be as much less than $j + \frac{1}{2} + at$, in this last result, as it was greater than $j + \frac{1}{2} - at$, in the result immediately preceding, or vice versa, these two results (of the present & the former articles) will coincide; or in other words the amount of disturbance, as distinct from vibration, increases very nearly according to the same law as we advance inward from both extremities towards the middle of its extent = $2at$, for the greater part of which extent it is nearly constant, but is reduced to one third of this constant amount at each extremity.

85. If l be nearly $= j + \frac{1}{2} + at \cos \alpha$, the integral of article 79 will take another form. In this case, because

$$(2j+1)(\theta - \alpha) - (2l\theta + \beta) + 2at \sin \theta = 2at \sin \alpha - (2l\alpha + \beta) \\ + (2j+1-2l+2at \cos \alpha)(\theta - \alpha) + 2at \{\sin \theta - \sin \alpha - (\theta - \alpha) \cos \alpha\},$$

and

$$2at \{\sin \theta - \sin \alpha - (\theta - \alpha) \cos \alpha\} = -at(\theta - \alpha)^2 \sin \alpha,$$

nearly ($\alpha > 0$, $< \pi$), we may write

$$\frac{B}{2\pi} \int_{\alpha - \delta\alpha}^{\alpha + \delta\alpha} \frac{d\theta}{\theta - \alpha} \cos \{(2j+1)(\theta - \alpha) - (2l\theta + \beta) + 2at \sin \theta\} \\ = \frac{B}{\pi} \frac{2j+1-2l+2at \cos \alpha}{\sqrt{at \sin \alpha}} \int_0^\infty d\theta \sin (2l\alpha + \beta - 2at \sin \alpha + \theta^2) \\ = \frac{B}{2} \frac{2j+1-2l+2at \cos \alpha}{\sqrt{\pi at \sin \alpha}} \sin \left(2l\alpha + \beta + \frac{\pi}{4} - 2at \sin \alpha \right);$$

the *change of phase* presenting itself still, as at the end of article 80. But we must add the constant part and also the part free from the sign \int ; & thus we find that if l be nearly $= j + \frac{1}{2} + at \cos \alpha$, we have, nearly,*

$$y_{l,t} = \frac{1}{2} B \sin (\beta + 2l\alpha - 2at \sin \alpha) - \frac{1}{2} B \sin (\beta + 2j\alpha + \alpha) \\ + \frac{1}{2} B \frac{2j+1-2l+2at \cos \alpha}{\sqrt{\pi at \sin \alpha}} \sin \left(\beta + 2l\alpha + \frac{\pi}{4} - 2at \sin \alpha \right).$$

86. We see, then, that although a disturbance, distinct from vibration, spreads, with two equal but opposite velocities, $\pm a$, and with a certain constant amount $= \frac{a}{4} y'_{j,0} - \frac{\cos \alpha}{2} y_{j,0}$, in both directions, from the point intermediate between the particles P_j and P_{j+1} , accompanied by two terminal diffusions, which are similar to each other, and are nearly proportional in longitudinal extent to the cube-root of the time t elapsed from the original state of the system;

* [The general term in the asymptotic expansion of the Bessel Function was first given, without proof, by Hamilton, *R.I.A. Trans.* Vol. XIX (1843), p. 313. See Watson, *Theory of Bessel Functions*, p. 12.]

there is also a real spreading or *dynamical propagation* of the initial mode of vibration, preserving the same constants of amplitude and phase, and accompanied by a terminal diffusion which is nearly proportional, in longitudinal extent, to the square-root of the time t ; and the velocity of this forward spreading of the vibration is represented by $a \cos \alpha$, if $\alpha > 0$, $< \frac{\pi}{2}$. If $\alpha = \frac{\pi}{2}$, there is only terminal diffusion and spreading of a constant disturbance, but no proper propagation of vibration. And if $\alpha > \frac{\pi}{2}$, $< \pi$, there is a backward propagation of vibration, or an uniform rate of abandonment of particles originally occupied by that vibration, the negative velocity of this propagation being still represented by $a \cos \alpha$.

87. It is remarkable that this *velocity of propagation*, $a \cos \alpha$, is the algebraical *sum of all the velocities of transmission of phase*,

$$\frac{a \sin \alpha}{\alpha} + \frac{a \sin \alpha}{\alpha - \pi} + \frac{a \sin \alpha}{\alpha + \pi} + \frac{a \sin \alpha}{\alpha - 2\pi} + \frac{a \sin \alpha}{\alpha + 2\pi} + \&c. = a \cos \alpha,$$

these several velocities corresponding to the several ways in which the phase may be expressed, namely

$$\beta + 2l\alpha - 2at \sin \alpha, \quad \beta + 2l(\alpha \mp \pi) - 2at \sin \alpha, \quad \beta + 2l(\alpha \mp 2\pi) - 2at \sin \alpha, \quad \&c.$$

In fact

$$\sin \alpha = \alpha \left(1 - \frac{\alpha}{\pi}\right) \left(1 + \frac{\alpha}{\pi}\right) \left(1 - \frac{\alpha}{2\pi}\right) \left(1 + \frac{\alpha}{2\pi}\right) \dots;$$

therefore

$$\cos \alpha = \frac{d \sin \alpha}{d \alpha} = \frac{\sin \alpha}{\alpha} - \frac{1 \sin \alpha}{\pi} + \frac{1 \sin \alpha}{\pi} - \frac{1 \sin \alpha}{2\pi} + \frac{1 \sin \alpha}{2\pi} - \frac{1 \sin \alpha}{2\pi} + \frac{1 \sin \alpha}{2\pi} - \&c.$$

For example, if $\alpha = \frac{\pi}{3}$, the series $\frac{a \sin \alpha}{\alpha} + \&c.$ becomes

$$\frac{3a\sqrt{3}}{2\pi} \left\{1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \&c.\right\},$$

which may also be thus written

$$\frac{a}{\alpha} \left(\frac{\sin 2\alpha}{1} + \frac{\sin 4\alpha}{2} + \frac{\sin 6\alpha}{3} + \frac{\sin 8\alpha}{4} + \&c. \right) = \frac{a(\pi - 2\alpha)}{2\alpha} = \frac{a}{2} = a \cos \frac{\pi}{3}.$$

But it is not obvious what dynamical interpretation ought to be put upon this theorem of summation, as applied to the present question; or in other words, it is not clear, *a priori*, *why* the actual velocity of propagation of vibration *ought* to be the sum of all the possible velocities of transmission of phase.

88. It is evident that the solution of the case of Problem IV, proposed for consideration in article 79, includes the solution of that other case of the same Problem, in which the initial disturbance is confined to a limited number of successive particles; since this finite number may be regarded as the difference of two infinities. Thus, for this latter case also, analogous results hold good; & we have still a propagation of vibration in one direction & with one velocity expressed still by $a \cos \alpha$. It seems then that even a single undulation tends to propagate itself with this velocity.

To illustrate this subject, let us consider the following Problem.

Problem V.

89. A single particle P_0 of a doubly indefinite system being compelled to vibrate according to a known law, it is required to determine the motion of the system which will ensue in consequence of this vibration of this particle.

90. Suppose, first, that the particle P_0 is obliged to remain fixed from the time 0 to the time t , so that $y_{0,t}$ is, during this interval of time, constant & $= y_{0,0}$. We may now consider ourselves as falling back on the case of a singly indefinite system, and may employ a modification of the formula (10) of article 17, namely the following, in which $l > 0$,

$$y_{l,t} = \frac{4}{\pi} \sum_{(j)1}^{\infty} \left(y_{j,0} - y_{0,0} + y'_{j,0} \int_0^t dt \right) \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos (2at \sin \theta) + y_{0,0}.$$

In fact this reduces itself to $y_{l,0}$ when $t = 0$; it gives also an expression for $y'_{l,t}$ which reduces itself to $y'_{l,0}$ when $t = 0$; & it gives $y''_{l,t} = a^2 (y_{l+1,t} - 2y_{l,t} + y_{l-1,t})$; if l be any integer > 0 . Had we suppressed the terms proportional to $y_{0,0}$, we should have had $y''_{1,t} = a^2 (y_{2,t} - 2y_{1,t})$, instead of having $y''_{1,t} = a^2 (y_{2,t} - 2y_{1,t} + y_{0,0})$. The part proportional to $y_{0,0}$ is

$$y_{0,0} \left\{ 1 - \frac{4}{\pi} \sum_{(j)1}^{\infty} \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos (2at \sin \theta) \right\};$$

it is the effect of the displacement $y_{0,0}$ of P_0 , continued forcibly constant throughout the interval of time t , the system being supposed to extend only in the positive direction.

Instead of $\sum_{(j)1}^{\infty} \sin 2j\theta$, we may write $\lim_{j \rightarrow \infty} \sum_{(j)1}^j \sin 2j\theta$, that is,

$$\lim_{j \rightarrow \infty} \frac{\cos \theta - \cos (2j+1)\theta}{2 \sin \theta} \left(= \lim_{j \rightarrow \infty} \frac{\sin j\theta \sin (j\theta + \theta)}{\sin \theta} \right);$$

& at the limit we may reduce this to the term $\frac{\cos \theta}{2 \sin \theta} = \frac{1}{2} \cotan \theta$, on account of the integration \int .*

Thus, the solution of the question of the present article may be expressed as follows,

$$y_{l,t} = \frac{4}{\pi} \sum_{(j)1}^{\infty} \left(y_{j,0} + y'_{j,0} \int_0^t dt \right) \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos (2at \sin \theta) + y_{0,0} \left\{ 1 - \frac{2}{\pi} \int_0^{\frac{\pi}{2}} d\theta \frac{\sin 2l\theta}{\tan \theta} \cos (2at \sin \theta) \right\};$$

the first line being only the old expression (10) for the effect of an arbitrary initial state of a singly indefinite system, the particle P_0 being fixed at the origin of coordinates; and the second line being that new part which results from the fixing of that particle P_0 in the displaced position $y_{0,0}$ during the time t .

91. Imagine next, that after being thus displaced during an interval of time $= \tau$, the particle P_0 is suddenly removed to a new position $y_{0,\tau}$ & is retained there during some subsequent

* [The terms which vanish are $\lim_{j \rightarrow \infty} \sum_{(j)1}^j J_{2j+1+2s} (2at)$.]

interval $\Delta\tau$; & let us inquire what will be the state of the system at the end of the time $\tau + \Delta\tau$. The immediate effect of the displacement $y_{0,\tau}$, continued during the time $\Delta\tau$, is

$$y_{0,\tau} \left\{ 1 - \frac{2}{\pi} \int_0^{\frac{\pi}{2}} d\theta \frac{\sin 2l\theta}{\tan \theta} \cos (2a \Delta\tau \sin \theta) \right\};$$

but we have to consider also the effect of the initial displacements & velocities $y_{i,0}$, $y'_{i,0}$, expressed still by the first line of the formula at the end of the previous page, or by the old expression (10), in which $t = \tau + \Delta\tau$; we have also to consider the effect of the new displacements and velocities,

$$y_{j,\tau} = y_{0,0} \left\{ 1 - \frac{2}{\pi} \int_0^{\frac{\pi}{2}} d\theta \frac{\sin 2j\theta}{\tan \theta} \cos (2a\tau \sin \theta) \right\},$$

$$y'_{j,\tau} = \frac{4a}{\pi} y_{0,0} \int_0^{\frac{\pi}{2}} d\theta \cos \theta \sin 2j\theta \sin (2a\tau \sin \theta),$$

which result, at the end of the interval τ , from the first fixed displacement $y_{0,0}$. The effect of these is

$$\frac{4}{\pi} \sum_{(j)1}^{\infty} \left(y_{j,\tau} + y'_{j,\tau} \int_0^{\Delta\tau} d\Delta\tau \right) \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos (2a \Delta\tau \sin \theta);$$

in which the part of $y_{j,\tau}$ independent of j , namely the term $y_{0,0}$, gives, as its part of the effect,

$$\frac{2}{\pi} y_{0,0} \int_0^{\frac{\pi}{2}} d\theta \frac{\sin 2l\theta}{\tan \theta} \cos (2a \Delta\tau \sin \theta);$$

the element $-\frac{2d\alpha}{\pi} y_{0,0} \frac{\sin 2j\alpha}{\tan \alpha} \cos (2a\tau \sin \alpha)$, in the expression for $y_{j,\tau}$, produces the effect

$$-\frac{2d\alpha}{\pi} y_{0,0} \frac{\sin 2l\alpha}{\tan \alpha} \cos (2a\tau \sin \alpha) \cos (2a \Delta\tau \sin \alpha);$$

and the element $\frac{4ad\alpha}{\pi} y_{0,0} \cos \alpha \sin 2j\alpha \sin (2a\tau \sin \alpha)$, in the expression for $y'_{j,\tau}$, produces the effect

$$\frac{2d\alpha}{\pi} y_{0,0} \frac{\sin 2l\alpha}{\tan \alpha} \sin (2a\tau \sin \alpha) \sin (2a \Delta\tau \sin \alpha);$$

so that the joint effect of these two elements is $-\frac{2d\alpha}{\pi} y_{0,0} \frac{\sin 2l\alpha}{\tan \alpha} \cos (2at \sin \alpha)$, in which $t = \tau + \Delta\tau$.

The state of the system at the end of the time t is therefore expressed by the formula*

$$y_{i,t} = \frac{4}{\pi} \sum_{(j)1}^{\infty} \left(y_{j,0} + y'_{j,0} \int_0^t dt \right) \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos (2at \sin \theta)$$

$$+ y_{0,\tau} - \frac{2}{\pi} (y_{0,\tau} - y_{0,0}) \int_0^{\frac{\pi}{2}} d\theta \cotan \theta \sin 2l\theta \cos \{2a(t - \tau) \sin \theta\}$$

$$- \frac{2}{\pi} y_{0,0} \int_0^{\frac{\pi}{2}} d\theta \cotan \theta \sin 2l\theta \cos (2at \sin \theta).$$

* [This could, of course, have been inferred from article 90 by taking the displacement $y_{0,0}$ from 0 to t and $y_{0,\tau} - y_{0,0}$ from τ to t .]

Accordingly this expression gives, if $l > 0$,

$$y_{l,\tau} = \frac{4}{\pi} \sum_{(j)1}^{\infty} \left(y_{j,0} + y'_{j,0} \int_0^{\tau} dt \right) \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos (2at \sin \theta) \\ + y_{0,0} \left\{ 1 - \frac{2}{\pi} \int_0^{\frac{\pi}{2}} d\theta \cotan \theta \sin 2l\theta \cos (2a\tau \sin \theta) \right\}, \\ y'_{l,\tau} = \frac{4}{\pi} \sum_{(j)1}^{\infty} \left(y_{j,0} \frac{d}{d\tau} + y'_{j,0} \right) \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos (2a\tau \sin \theta) \\ + \frac{4a}{\pi} y_{0,0} \int_0^{\frac{\pi}{2}} d\theta \cos \theta \sin 2l\theta \sin (2a\tau \sin \theta),$$

so that it reproduces the known state of the system at the end of the first interval τ ; it gives also, for any moment of the 2nd interval $\Delta\tau$, that is, for any value of t from τ to $\tau + \Delta\tau$, (l being still > 0),

$$y''_{l,t} = a^2 (y_{l+1,t} - 2y_{l,t} + y_{l-1,t}),$$

including the equation

$$y''_{1,t} = a^2 (y_{2,t} - 2y_{1,t} + y_{0,\tau}),$$

& therefore it satisfies the differential equations of vibration, on the hypothesis of $y_{0,t}$ being, throughout the whole of this second interval, constant & $= y_{0,\tau}$.

92. In the next place, if there be three successive displacements $y_{0,0}$, y_{0,t_1} , y_{0,t_2} lasting for the successive intervals τ_1 , τ_2 , τ_3 , we must suppose that at the moment $t_2 = \tau_1 + \tau_2$ the displacements and velocities are represented thus:

$$y_{l,t_2} = \frac{4}{\pi} \sum_{(j)1}^{\infty} \left(y_{j,0} + y'_{j,0} \int_0^{t_2} dt_2 \right) \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos (2at_2 \sin \theta) \\ + y_{0,t_1} - \frac{2}{\pi} (y_{0,t_1} - y_{0,0}) \int_0^{\frac{\pi}{2}} d\theta \frac{\sin 2l\theta}{\tan \theta} \cos (2a\tau_2 \sin \theta) - \frac{2}{\pi} y_{0,0} \int_0^{\frac{\pi}{2}} d\theta \frac{\sin 2l\theta}{\tan \theta} \cos (2at_2 \sin \theta); \\ y'_{l,t_2} = \frac{4}{\pi} \sum_{(j)1}^{\infty} \left(y_{j,0} \frac{d}{dt_2} + y'_{j,0} \right) \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos (2at_2 \sin \theta) \\ + \frac{4a}{\pi} (y_{0,t_1} - y_{0,0}) \int_0^{\frac{\pi}{2}} d\theta \cos \theta \sin 2l\theta \sin (2a\tau_2 \sin \theta) \\ + \frac{4a}{\pi} y_{0,0} \int_0^{\frac{\pi}{2}} d\theta \cos \theta \sin 2l\theta \sin (2at_2 \sin \theta);$$

and the function $y_{l,t}$, in which $l > 0$ and $t = t_2 + \tau_3$, is to satisfy the differential equations of the form $y''_{l,t} = a^2 (y_{l+1,t} - 2y_{l,t} + y_{l-1,t})$, including the equation $y''_{1,t} = a^2 (y_{2,t} - 2y_{1,t} + y_{0,t_2})$; the final effect is therefore expressed by the following formula:

$$\begin{aligned}
y_{i,t} = & \frac{4}{\pi} \sum_{(j)1}^{\infty} \left(y_{j,0} + y'_{j,0} \int_0^t dt \right) \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos (2at \sin \theta) \\
& + y_{0,t_2} - \frac{2}{\pi} (y_{0,t_2} - y_{0,t_1}) \int_0^{\frac{\pi}{2}} d\theta \frac{\sin 2l\theta}{\tan \theta} \cos (2a\tau_3 \sin \theta) \\
& - \frac{2}{\pi} (y_{0,t_1} - y_{0,0}) \int_0^{\frac{\pi}{2}} d\theta \frac{\sin 2l\theta}{\tan \theta} \cos (2a\tau_3 + \tau_2 \sin \theta) \\
& - \frac{2}{\pi} y_{0,0} \int_0^{\frac{\pi}{2}} d\theta \frac{\sin 2l\theta}{\tan \theta} \cos (2a\tau_3 + \tau_2 + \tau_1 \sin \theta);
\end{aligned}$$

which may also be thus written,

$$\begin{aligned}
y_{i,t} = & \frac{4}{\pi} \sum_{(j)1}^{\infty} \left(y_{j,0} + y'_{j,0} \int_0^t dt \right) \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos (2at \sin \theta) \\
& + \frac{2}{\pi} y_{0,t_2} \int_0^{\frac{\pi}{2}} d\theta \frac{\sin 2l\theta}{\tan \theta} \{1 - \cos (2a\tau_3 \sin \theta)\} \\
& + \frac{2}{\pi} y_{0,t_1} \int_0^{\frac{\pi}{2}} d\theta \frac{\sin 2l\theta}{\tan \theta} \{\cos (2a\tau_3 \sin \theta) - \cos (2a\tau_3 + \tau_2 \sin \theta)\} \\
& + \frac{2}{\pi} y_{0,0} \int_0^{\frac{\pi}{2}} d\theta \frac{\sin 2l\theta}{\tan \theta} \{\cos (2a\tau_3 + \tau_2 \sin \theta) - \cos (2a\tau_3 + \tau_2 + \tau_1 \sin \theta)\}.
\end{aligned}$$

93. It is easy to see that this law continues; and that it gives, as the solution of Problem V in article 89, the expression

$$\begin{aligned}
y_{i,t} = & \frac{4}{\pi} \sum_{(j)1}^{\infty} \left(y_{j,0} + y'_{j,0} \int_0^t dt \right) \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos (2at \sin \theta) \\
& + \frac{4a}{\pi} \int_0^t d\tau y_{0,\tau} \int_0^{\frac{\pi}{2}} d\theta \cos \theta \sin 2l\theta \sin \{2a(t-\tau) \sin \theta\}.
\end{aligned}$$

Accordingly this expression gives

$$\begin{aligned}
y'_{i,t} = & \frac{4}{\pi} \sum_{(j)1}^{\infty} \left(y_{j,0} \frac{d}{dt} + y'_{j,0} \right) \int_0^{\frac{\pi}{2}} d\theta \sin 2j\theta \sin 2l\theta \cos (2at \sin \theta) \\
& + \frac{8a^2}{\pi} \int_0^t d\tau y_{0,\tau} \int_0^{\frac{\pi}{2}} d\theta \sin \theta \cos \theta \sin 2l\theta \cos \{2a(t-\tau) \sin \theta\}, \\
y''_{i,t} = & - \frac{16a^2}{\pi} \sum_{(j)1}^{\infty} \left(y_{j,0} + y'_{j,0} \int_0^t dt \right) \int_0^{\frac{\pi}{2}} d\theta \sin \theta^2 \sin 2j\theta \sin 2l\theta \cos (2at \sin \theta) \\
& - \frac{16a^2}{\pi} \int_0^t d\tau y_{0,\tau} \int_0^{\frac{\pi}{2}} d\theta \sin \theta^2 \cos \theta \sin 2l\theta \sin \{2a(t-\tau) \sin \theta\} \\
& + \frac{4a^2}{\pi} y_{0,t} \int_0^{\frac{\pi}{2}} d\theta \sin 2\theta \sin 2l\theta,
\end{aligned}$$

so that $y_{l,t}$ and $y'_{l,t}$ reduce themselves when $t=0$ to the given initial values, and also the differential equations of the form $y''_{l,t} = a^2 (y_{l+1,t} - 2y_{l,t} + y_{l-1,t})$ are satisfied if $l > 0$, including the equation $y''_{1,t} = a^2 (y_{2,t} - 2y_{1,t} + y_{0,t})$, in which the function $y_{0,t}$ is arbitrary. As a verification we may observe that if this function reduce itself to the constant $y_{0,0}$, we have

$$\int_0^t d\tau y_{0,\tau} \sin \{2a(t-\tau) \sin \theta\} = y_{0,0} \frac{1 - \cos(2at \sin \theta)}{2a \sin \theta},$$

so that if the initial displacements and velocities $y_{l,0}$ and $y'_{l,0}$ vanish, we have

$$y_{l,t} = \frac{2}{\pi} y_{0,0} \int_0^{\frac{\pi}{2}} d\theta \frac{\sin 2l\theta}{\tan \theta} \text{vers}(2at \sin \theta) = y_{0,0} \left\{ 1 - \frac{2}{\pi} \int_0^{\frac{\pi}{2}} d\theta \frac{\sin 2l\theta}{\tan \theta} \cos(2at \sin \theta) \right\},$$

as found in article 90.

94. The effect of the vibration $y_{0,t}$ of the particle P_0 being thus found to be

$$\frac{4a}{\pi} \int_0^t d\tau y_{0,\tau} \int_0^{\frac{\pi}{2}} d\theta \cos \theta \sin 2l\theta \sin \{2a(t-\tau) \sin \theta\},$$

let us suppose that $y_{0,\tau} = -\eta \sin(2a\tau \sin \alpha - \beta)$, α being > 0 , $< \frac{\pi}{2}$.

Multiplying this by $2 \sin \{2a(t-\tau) \sin \theta\}$, we get

$$\eta \cos \{2at \sin \theta - \beta - 2a\tau(\sin \theta - \sin \alpha)\} - \eta \cos \{2at \sin \theta + \beta - 2a\tau(\sin \theta + \sin \alpha)\};$$

and multiplying again by $2 \sin 2l\theta$, we get

$$\eta \sin \{2at \sin \theta - \beta + 2l\theta - 2a\tau(\sin \theta - \sin \alpha)\} - \eta \sin \{2at \sin \theta + \beta + 2l\theta - 2a\tau(\sin \theta + \sin \alpha)\} \\ - \eta \sin \{2at \sin \theta - \beta - 2l\theta - 2a\tau(\sin \theta - \sin \alpha)\} + \eta \sin \{2at \sin \theta + \beta - 2l\theta - 2a\tau(\sin \theta + \sin \alpha)\},$$

which is to be multiplied by $\frac{a}{\pi} \cos \theta$, & integrated relatively to τ from 0 to t and relatively to θ from 0 to $\frac{\pi}{2}$. In this manner we obtain*

* [The expression for $y_{l,t}$ may be written

$$\frac{\eta}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta \{ \cos \theta \sin 2l\theta \sin(2at \sin \theta - \beta) - \cos \alpha \sin 2l\alpha \sin(2at \sin \alpha - \beta) \}}{\sin \theta - \sin \alpha} \\ - \frac{\eta}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta (\cos \theta \sin 2l\theta - \cos \alpha \sin 2l\alpha) \sin(2at \sin \alpha - \beta)}{\sin \theta - \sin \alpha}.$$

The first integral may, obviously, be written

$$\text{Lt}_{\epsilon \rightarrow 0} \frac{\eta}{\pi} \left(\int_{-\frac{\pi}{2}}^{\alpha - \epsilon} + \int_{\alpha + \epsilon}^{\frac{\pi}{2}} \right) \frac{d\theta}{\sin \theta - \sin \alpha} \{ \quad \}$$

or
$$\text{Lt}_{\rightarrow 0} \frac{\eta}{\pi} \left(\int_{-\frac{\pi}{2}}^{\alpha - \epsilon} + \int_{\alpha + \epsilon}^{\frac{\pi}{2}} \right) \frac{d\theta}{\sin \theta - \sin \alpha} \cos \theta \sin 2l\theta \sin(2at \sin \theta - \beta),$$

$$\begin{aligned}
& \frac{\eta}{2\pi} \int_0^{\frac{\pi}{2}} \frac{d\theta \cos \theta}{\sin \theta - \sin \alpha} \{ \cos (2at \sin \alpha - \beta + 2l\theta) - \cos (2at \sin \theta - \beta + 2l\theta) \\
& \quad - \cos (2at \sin \alpha - \beta - 2l\theta) + \cos (2at \sin \theta - \beta - 2l\theta) \} \\
& - \frac{\eta}{2\pi} \int_0^{\frac{\pi}{2}} \frac{d\theta \cos \theta}{\sin \theta + \sin \alpha} \{ \cos (-2at \sin \alpha + \beta + 2l\theta) - \cos (2at \sin \theta + \beta + 2l\theta) \\
& \quad - \cos (-2at \sin \alpha + \beta - 2l\theta) + \cos (2at \sin \theta + \beta - 2l\theta) \} \\
& = \frac{\eta}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta \cos \theta}{\sin \theta - \sin \alpha} \{ \cos (2at \sin \alpha - \beta + 2l\theta) - \cos (2at \sin \theta - \beta + 2l\theta) \\
& \quad - \cos (2at \sin \alpha - \beta - 2l\theta) + \cos (2at \sin \theta - \beta - 2l\theta) \} \\
& = \frac{\eta}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta \cos \theta \sin 2l\theta}{\sin \theta - \sin \alpha} \{ \sin (2at \sin \theta - \beta) - \sin (2at \sin \alpha - \beta) \} \\
& = \frac{\eta}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta \cos \theta \sin 2l\theta \sin (2at \sin \theta - \beta)}{\sin \theta - \sin \alpha} - \eta \cos 2l\alpha \sin (2at \sin \alpha - \beta),
\end{aligned}$$

observing that by article 39 we have, if l be any integer > 0 ,

$$\begin{aligned}
\cos 2l\alpha &= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\theta \sin 2\theta \sin 2l\theta}{\cos 2\alpha - \cos 2\theta} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\theta \sin \theta \cos \theta \sin 2l\theta}{\sin \theta^2 - \sin \alpha^2} \\
&= \frac{1}{\pi} \int_0^{\frac{\pi}{2}} d\theta \cos \theta \sin 2l\theta \left\{ \frac{1}{\sin (\theta - \alpha)} + \frac{1}{\sin (\theta + \alpha)} \right\} \\
&= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta \cos \theta \sin 2l\theta}{\sin (\theta - \alpha)}.
\end{aligned}$$

As a verification, the effect calculated in the present article ought to vanish when l becomes infinite, t remaining finite. Now

$$\sin 2l\theta \sin (2at \sin \theta - \beta) = \frac{1}{2} \cos (2l\theta + \beta - 2at \sin \theta) - \frac{1}{2} \cos (2l\theta - \beta + 2at \sin \theta);$$

& the first of these two terms gives, at the limit, $-\frac{\eta}{2} \sin (2l\alpha + \beta - 2at \sin \alpha)$, while the second gives $+\frac{\eta}{2} \sin (2l\alpha - \beta + 2at \sin \alpha)$; their sum gives therefore $+\eta \cos 2l\alpha \sin (2at \sin \alpha - \beta)$, as it ought to do.

95. We find therefore that in the singly indefinite system, extending only in the positive direction, the effect of the vibration $y_{0,t} = \eta \sin (-2at \sin \alpha + \beta)$ (in which $\alpha > 0$, $< \frac{\pi}{2}$) of the first

because

$$\text{Lt}_{\epsilon \rightarrow 0} \left(\int_{-\frac{\pi}{2}}^{\alpha - \epsilon} + \int_{\alpha + \epsilon}^{\frac{\pi}{2}} \right) \frac{d\theta}{\sin \theta - \sin \alpha} = 0.$$

The second line gives

$$-\eta \cos 2l\alpha \sin (2at \sin \alpha - \beta).$$

We thus arrive at the final form given for $y_{i,t}$, the integral to be interpreted as this special Cauchy value. The case $\sin 2l\alpha = 0$ is exceptional. Here the Cauchy value is unnecessary.]

particle P_0 is represented by the formula

$$\begin{aligned}
 y_{l,t} &= \frac{\eta}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta \cos \theta \sin 2l\theta \sin (2at \sin \theta - \beta)}{\sin \theta - \sin \alpha} - \eta \cos 2l\alpha \sin (2at \sin \alpha - \beta) \\
 &= \frac{\eta}{2\pi} \int_{-1}^{+1} \frac{d(\sin \theta)}{\sin \theta - \sin \alpha} \{ \cos (2l\theta + \beta - 2at \sin \theta) - \cos (2l\theta - \beta + 2at \sin \theta) \} \\
 &\qquad\qquad\qquad - \eta \cos 2l\alpha \sin (2at \sin \alpha - \beta).
 \end{aligned}$$

If l be much larger than $at \cos \alpha$, this effect is insensible; but if, on the contrary, $at \cos \alpha$ be much larger than l , the effect is nearly represented thus,

$$\begin{aligned}
 y_{l,t} &= \eta \left\{ \frac{1}{2} \sin (2l\alpha + \beta - 2at \sin \alpha) + \frac{1}{2} \sin (2l\alpha - \beta + 2at \sin \alpha) - \cos 2l\alpha \sin (2at \sin \alpha - \beta) \right\} \\
 &= \eta \{ \sin 2l\alpha \cos (2at \sin \alpha - \beta) - \cos 2l\alpha \sin (2at \sin \alpha - \beta) \} \\
 &= \eta \sin (2l\alpha + \beta - 2at \sin \alpha).
 \end{aligned}$$

Thus it is true, in a certain sense, that even the vibration of a single particle P_0 , with a periodic time $= \frac{\pi}{\alpha} \operatorname{cosec} \alpha$, (in which α is any real arc > 0 , but $< \frac{\pi}{2}$), produces vibration, with the same periodic time, in all the other particles; the transmission of phase having a velocity $= \frac{a \sin \alpha}{\alpha}$, but the propagation of vibration having a somewhat less velocity, namely $a \cos \alpha$.

96. If $\alpha = \frac{\pi}{2}$, so that the vibration of P_0 is

$$y_{0,t} = \eta \sin (\beta - 2at),$$

then the same analysis shows that the effect of this continued vibration of this single particle is

$$y_{l,t} = \frac{\eta}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta \cos \theta \sin 2l\theta}{1 - \sin \theta} \{ \sin (\beta - 2at \sin \theta) - \sin (\beta - 2at) \};$$

now

$$- \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta \cos \theta \sin 2l\theta}{1 - \sin \theta} = -2 \int_0^{\frac{\pi}{2}} d\theta \tan \theta \sin 2l\theta = \pi (-1)^l;$$

therefore in the present case,

$$y_{l,t} = \eta \left\{ (-1)^l \sin (\beta - 2at) + \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta \cos \theta \sin 2l\theta \sin (\beta - 2at \sin \theta)}{1 - \sin \theta} \right\}.$$

Changing θ to $\frac{\pi}{2} - \theta$, this becomes

$$\begin{aligned}
 y_{l,t} &= \eta (-1)^l \left\{ \sin (\beta - 2at) - \frac{1}{\pi} \int_0^{\pi} \frac{d\theta \sin \theta \sin 2l\theta}{1 - \cos \theta} \sin (\beta - 2at \cos \theta) \right\} \\
 &= \eta (-1)^l \left\{ \sin (\beta - 2at) - \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\theta \sin 4l\theta}{\tan \theta} \sin (\beta - 2at \cos 2\theta) \right\}.
 \end{aligned}$$

This is insensible, if l be much greater than \sqrt{at} ; but if, on the contrary, l be much less than \sqrt{at} , it becomes, making $2\sqrt{at}\sin\theta = \theta$,

$$\begin{aligned} \eta(-1)^l & \left\{ \sin(\beta - 2at) - \frac{4l}{\pi\sqrt{at}} \int_0^\infty d\theta, \sin(\beta - 2at + \theta^2) \right\} \\ & = \eta(-1)^l \left\{ \sin(\beta - 2at) - \frac{2l}{\sqrt{\pi at}} \sin\left(\beta - 2at + \frac{\pi}{4}\right) \right\}. \end{aligned}$$

97. We might suppose $y_{0,t} = \eta \sin(\beta - 2aAt)$, $A > 1$; & the analysis of article 94 would then give

$$y_{l,t} = \frac{\eta}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta \cos\theta \sin 2l\theta}{A - \sin\theta} \{ \sin(\beta - 2at \sin\theta) - \sin(\beta - 2aAt) \};$$

but whether l be large or small in comparison with t , if both be very great, this function will become insensible, because the denominator $A - \sin\theta$ cannot now vanish. Thus a vibration of shorter periodic time than the minimum $\frac{\pi}{a}$ cannot sensibly propagate itself far.

However, when t is very great, we have nearly

$$y_{l,t} = \frac{\eta}{\pi} \sin(\beta - 2aAt) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta \cos\theta \sin 2l\theta}{\sin\theta - A};$$

in which*

$$\frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta \cos\theta \sin 2l\theta}{\sin\theta - A} = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\theta \sin 2\theta \sin 2l\theta}{\sin^2\theta - A^2} = -f_l$$

= a function such that

$$f_{l+1} + f_{l-1} - 2f_l + 4A^2 f_l = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} d\theta \sin 2\theta \sin 2l\theta = 0 \text{ or } 1,$$

according as l , being > 0 , is > 1 or $= 1$. Also

$$f_0 = 1; \quad f_1 = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \frac{(\sin 2\theta)^2 d\theta}{A^2 - \sin^2\theta} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\theta \cos\theta \sin 2\theta}{\pi(A - \sin\theta)},$$

in which

$$\sin 2\theta \cos\theta = 2(\sin\theta - A + A)(1 - A^2 + A^2 - \sin^2\theta),$$

therefore

$$f_1 = \frac{4A^2(1 - A^2)}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\theta}{A^2 - \sin^2\theta} - 1 + 2A^2;$$

also

$$A^2(A^2 - \sin^2\theta)^{-1} = 1 + A^{-2}\sin^2\theta + A^{-4}\sin^4\theta + \&c.,$$

* [If $A = \cosh \alpha$, $\cos\theta/(A - \sin\theta) = \sum_{n=1}^\infty e^{-n\alpha} \sin n\left(\frac{\pi}{2} - \theta\right).]$

and

$$\frac{2}{\pi} \int_0^{\frac{\pi}{2}} \sin \theta^{2n} d\theta = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos \theta^{2n} d\theta = 2^{-2n} [2n]^n [0]^{-n} = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots (2n)} = [-\frac{1}{2}]^n [0]^{-n} (-1)^n;$$

therefore

$$\frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\theta}{1 - A^{-2} \sin^2 \theta} = (1 - A^{-2})^{-\frac{1}{2}},$$

and

$$f_1 = \frac{2(1 - A^2)}{\sqrt{1 - A^2}} - 1 + 2A^2 = -2A\sqrt{A^2 - 1} - 1 + 2A^2 = (A - \sqrt{A^2 - 1})^2;$$

$$f_2 = 1 + 2(1 - 2A^2)f_1 = -(A - \sqrt{A^2 - 1})^4;$$

and generally

$$f_l = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\theta \sin 2\theta \sin 2l\theta}{A^2 - \sin^2 \theta} = (-1)^{l+1} (A - \sqrt{A^2 - 1})^{2l},$$

l being any integer > 0 .

Hence in the present question, if t be great, we have

$$y_{l,t} = \eta (-1)^l (A + \sqrt{A^2 - 1})^{-2l} \sin(\beta - 2\alpha At).$$

The amplitudes therefore, in this case, decrease in geometrical proportion, being proportional to the power of the fraction $\frac{1}{(A + \sqrt{A^2 - 1})^2}$.

98. This result, on the analytic side, bears some analogy to the well-known theorems

$$\int_0^\infty \frac{\cos qx dq}{1 + q^2} = \frac{\pi}{2} e^{-x}, \quad \int_0^\infty \frac{q \sin qx dq}{1 + q^2} = \frac{\pi}{2} e^{-x},$$

in which x is any real quantity > 0 , & of which the former includes the latter and may be proved by observing that if we put

$$X_x = \int_0^\infty \frac{\cos qx dq}{1 + q^2}, \quad \text{we have } X_x'' = - \int_0^\infty \frac{q^2 \cos qx dq}{1 + q^2} = X_x - \int_0^\infty \cos qx dq,$$

which gives $X_x'' = X_x$ if x be > 0 ; therefore $X_x = ae^x + be^{-x} = be^{-x}$ because $X_\infty = 0$; and although the differential equation $X_x'' = X_x$ does not hold good for the particular value $x = 0$, yet the coefficient b must be = the limit to which $e^x X_x$ tends as x decreases to 0, & must therefore be equal to $\frac{\pi}{2}$.

Laplace, in the Analytic Theory of Probabilities, Article 26, deduces the theorem from the consideration of a double definite integral, as follows:

$$\begin{aligned} X_r &= \int_0^\infty \frac{dx \cos rx}{1 + x^2} = \int_0^\infty dx \int_0^\infty dy 2ye^{-y^2(1+x^2)} \cos rx = \sqrt{\pi} \int_0^\infty dy e^{-y^2 - \frac{r^2}{4y^2}} \\ &= \sqrt{\pi} e^{-r} \int_0^\infty dy e^{-\left(y - \frac{r}{2y}\right)^2} = \frac{\sqrt{\pi}}{2} e^{-r} \int_{-\infty}^\infty dz e^{-z^2} \left(1 + \frac{z}{\sqrt{z^2 + 2r}}\right) = \frac{\pi}{2e^r}. \end{aligned}$$

And he remarks that by pursuing a similar analysis, the following theorems may be deduced:

$$\int_{-\infty}^{\infty} dx \frac{(a+bx) \cos rx}{m+2nx+x^2} = \left(\frac{a-bn}{\sqrt{m-n^2}} \cos rn + b \sin rn \right) \pi e^{-r\sqrt{m-n^2}};$$

$$\int_{-\infty}^{\infty} dx \frac{(a+bx) \sin rx}{m+2nx+x^2} = \left(b \cos rn - \frac{a-bn}{\sqrt{m-n^2}} \sin rn \right) \pi e^{-r\sqrt{m-n^2}};$$

and ultimately the values of the definite integrals

$$\int_{-\infty}^{\infty} dx \frac{M}{N} \cos rx, \quad \int_{-\infty}^{\infty} dx \frac{M}{N} \sin rx,$$

in which M and N are rational and integral polynomials, such that the degree of M (relatively to x) is less than that of N , and the roots of $N=0$ are all imaginary. The two last theorems may thus be written,

$$\int_{-\infty}^{\infty} dx \frac{(a+x) \cos rx}{m^2+x^2} = \frac{\pi a}{m} e^{-rm}, \quad \int_{-\infty}^{\infty} dx \frac{(a+x) \sin rx}{m^2+x^2} = \pi e^{-rm};$$

& under this form they follow easily from those first cited in this article.

99. In the foregoing investigations, we have, expressly or tacitly, employed often the principle that if any finite function of a real variable be multiplied by the sine or cosine of an infinitely great multiple of that variable, & integrated within any finite limits, (or even, in most cases, between negative and positive infinity,) the result is evanescent; & therefore that if the function be not constantly finite, we need attend only to those values of the variable which differ infinitely little from the values which make the function infinite.* And in some cases of a constantly finite function, such as $F(\theta) = \frac{\cos \theta}{\sin \theta - A}$, ($A > 1$), in article 97, or $F(q) = \frac{1}{1+q^2}$ in article 98, we have been able to assign the law according to which the integral tends to become infinitely small as the multiplier of the variable under the sign sine or cosine in the rapidly fluctuating factor tends to become infinitely great; namely, in these cases, the exponential law expressed by the formulæ

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \frac{\cos \theta}{\sin \theta - A} \sin 2l\theta = \pi (-1)^l (A + \sqrt{A^2 - 1})^{-2l},$$

A being any real quantity > 1 , and l being any integer number > 1 ; and

$$\int_{-\infty}^{\infty} dq \frac{1}{1+q^2} \cos qx = \pi e^{-ax},$$

x being any real quantity < 0 .

It seems that in most cases of this last kind, that is, in most cases of a constantly finite function $F(q)$ multiplied by the cosine of a large multiple qx of the variable q & integrated from $-\infty$ to $+\infty$, ($F(\mp\infty)$ being supposed $= 0$), we must have, nearly, the expression†

$$\int_{-\infty}^{\infty} dq F(q) \cos qx = \frac{2}{x} \sum_{(n)-\infty}^{\infty} (-1)^n F\left(\frac{n\pi}{x}\right);$$

x being large.

* [This is Hamilton's principle of fluctuation, which he later developed at length in a memoir on fluctuating functions. *Trans. R.I.A.* (1843) XIX, pp. 264-321.]

† $\left[\sum_{(n)} \int_{(n-\frac{1}{2})\frac{\pi}{x}}^{(n+\frac{1}{2})\frac{\pi}{x}} F(q) \cos qx dq. \right]$

It will however be probably in general more exact to substitute, in this expression, instead of the middle ordinate $F\left(\frac{n\pi}{x}\right)$ of the curve in which q is abscissa and $F(q)$ ordinate, taken between the limits $q = \frac{(n \mp \frac{1}{2})\pi}{x}$, the average ordinate of this curve between these limits, namely

$$\frac{x}{\pi} \int_{\frac{(n-\frac{1}{2})\pi}{x}}^{\frac{(n+\frac{1}{2})\pi}{x}} dq F(q); \text{ \& thus we find}$$

$$\int_{-\infty}^{\infty} dq F(q) \cos qx = \frac{2}{\pi} \sum_{(n)-\infty}^{\infty} (-1)^n \int_{\frac{(n-\frac{1}{2})\pi}{x}}^{\frac{(n+\frac{1}{2})\pi}{x}} dq F(q);$$

x being large. It will be useful to test these formulae by some examples.

100. Let $F(q) = \cos q$; then $2 \cos q \cos qx = \cos(qx + q) + \cos(qx - q)$; & since this function F does not vanish when $q = \infty$, we shall take finite limits of integration, such as 0 and π , & employ the approximate formula

$$\int_0^{\pi} dq F(q) \cos qx = \frac{1}{x} \left\{ F(0) + 2 \sum_{(n)1}^{x-1} (-1)^n F\left(\frac{n\pi}{x}\right) + F(\pi) \cos(x\pi) \right\},$$

x being now some large and positive integer. Applying this to the case $F(q) = \cos q$, the first member vanishes, & the 2nd member ought to be found to be nearly = 0. We ought therefore to have, nearly, $\frac{2}{x} \sum_{(n)1}^{x-1} (-1)^n \cos \frac{n\pi}{x} = \frac{\cos x\pi - 1}{x}$, if x be a large positive integer. Now

$$2 \cos \frac{\pi}{2x} \left(-\cos \frac{\pi}{x} + \cos \frac{2\pi}{x} - \cos \frac{3\pi}{x} + \dots + \cos m\pi \cos \frac{m\pi}{x} \right) = -\cos \frac{\pi}{2x} + \cos m\pi \cos \left(\frac{m\pi}{x} + \frac{\pi}{2x} \right);$$

making therefore $m = x - 1$, we have

$$-\cos \frac{\pi}{2x} + \cos(x\pi - \pi) \cos \left(\pi - \frac{\pi}{2x} \right) = (\cos x\pi - 1) \cos \frac{\pi}{2x};$$

in this case therefore the theorem is rigorously true. If we had employed *average* instead of *middle* ordinates, we should have had

$$\int_0^{\pi} dq F(q) \cos qx = \frac{2}{\pi} \left\{ \int_0^{\frac{\pi}{2x}} + \sum_{(n)1}^{x-1} (-1)^n \int_{\frac{(n-\frac{1}{2})\pi}{x}}^{\frac{(n+\frac{1}{2})\pi}{x}} + \cos x\pi \int_{\pi-\frac{\pi}{2x}}^{\pi} \right\} dq F(q);$$

x being still a large and positive integer. Making $F(q) = \cos q$, we ought to find nearly

$$0 = \frac{2}{\pi} \sin \frac{\pi}{2x} \left\{ 1 + 2 \sum_{(n)1}^{x-1} (-1)^n \cos \frac{n\pi}{x} - \cos x\pi \right\};$$

& in fact this also is rigorously true. But we cannot expect to find so perfect an agreement, in general, between the equated expressions.

101. Let $F(q) = e^{-q^2}$; then

$$\int_{-\infty}^{\infty} dq F(q) \cos qx = \sqrt{\pi} e^{-\frac{x^2}{4}};$$

is this then nearly $= \frac{2}{x} \sum_{(n)-\infty}^{\infty} (-1)^n e^{-\frac{n^2\pi^2}{x^2}}$? In general, we have, *rigorously*,

$$\begin{aligned} \sum_{(n)-\infty}^{\infty} (-1)^n F(n\alpha) &= \text{Lt}_{s=\infty} \int_{-\infty}^{\infty} dt F(t\alpha) \frac{\sin 2st\pi}{\sin t\pi} \\ &= 2 \int_{-\infty}^{\infty} dt F(t\alpha) (\cos t\pi + \cos 3t\pi + \cos 5t\pi + \&c.). \end{aligned}$$

We ought therefore to have, nearly, in general, if x be large,

$$\begin{aligned} \int_{-\infty}^{\infty} dq F(q) \cos qx &= \frac{4}{x} \int_{-\infty}^{\infty} dt F\left(\frac{t\pi}{x}\right) (\cos t\pi + \cos 3t\pi + \cos 5t\pi + \&c.) \\ &= \frac{4}{\pi} \int_{-\infty}^{\infty} dq F(q) (\cos qx + \cos 3qx + \cos 5qx + \&c.). \end{aligned}$$

But in the case $F(q) = e^{-q^2}$, this would give

$$\sqrt{\pi} e^{-\frac{x^2}{4}} = \frac{4}{\sqrt{\pi}} \left(e^{-\frac{x^2}{4}} + e^{-\frac{9x^2}{4}} + e^{-\frac{25x^2}{4}} + \&c. \right);$$

& the second member is greater than the first, if x be large, in the ratio nearly of 4 to π . There can be no doubt but that this arises chiefly from the circumstance that the definite integral

$$\int_{-\frac{\pi}{2x}}^{\frac{\pi}{2x}} dq \cos qx = \frac{2}{x} \text{ is greater than } \int_{-\frac{\pi}{2x}}^{\frac{\pi}{2x}} dq e^{-q^2} \cos qx;$$

and generally from the inequality of

$$\int_{(n-\frac{1}{2})\frac{\pi}{x}}^{(n+\frac{1}{2})\frac{\pi}{x}} dq e^{-q^2} \cos qx \text{ and } e^{-\frac{n^2\pi^2}{x^2}} \int_{(n-\frac{1}{2})\frac{\pi}{x}}^{(n+\frac{1}{2})\frac{\pi}{x}} dq \cos qx = (-1)^n \frac{2}{x} e^{-\frac{n^2\pi^2}{x^2}};$$

such inequalities though small being numerous & giving an accumulated result, which bears a sensible ratio ($4 - \pi$ to π) to the small value of the integral in question, when x is very great. To allow for these inequalities, at least nearly, we might make

$$e^{-q^2} = e^{-\frac{n^2\pi^2}{x^2}} e^{-\left(q^2 - \frac{n^2\pi^2}{x^2}\right)} = e^{-\frac{n^2\pi^2}{x^2}}.$$

.....

103. Returning to the investigation of article 97, & to the first expression there given for $y_{l,t}$, that is for the transversal displacement of the particle P_l at the time t , l being any integer > 0 , and the particle P_0 being obliged to vibrate according to the law $y_{0,t} = \eta \sin(\beta - 2aAt)$, in which $A > 1$, while $y_{l,0} = 0$, $y'_{l,0} = 0$; we see that, in order to developpe this expression according to the powers of t , it is sufficient to calculate generally

$$f_{l,t} = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \frac{\sin \theta^i \cos \theta \sin 2l\theta}{A - \sin \theta} = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \frac{(A + \sin \theta) \sin \theta^i \cos \theta \sin 2l\theta}{A^2 - \sin^2 \theta},$$

or simply

$$f_{l,2i} = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} d\theta \frac{\sin \theta^{2i} \sin 2\theta \sin 2l\theta}{A^2 - \sin^2 \theta},$$

since $f_{l, 2i+1} = Af_{l, 2i}$. And we have already found that

$$f_l = f_{l, 0} = (-1)^{l+1} (A - \sqrt{A^2 - 1})^{2l},$$

if $l > 0; f_{0, 0} = 0$. Now

$$A^{2i} f_l - f_{l, 2i} = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} d\theta \{A^{2i-2} + A^{2i-4} \sin^2 \theta + \dots + \sin^{2i-2} \theta\} \sin 2\theta \sin 2l\theta;$$

we are therefore to calculate

$$g_{l, k} = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} d\theta \sin \theta^{2k} \sin 2\theta \sin 2l\theta;$$

and it is convenient to suppose at first $l > 1$, reserving the case $l = 1$ for separate study.

$$\begin{aligned} \sin \theta^{2k} &= \left(-\frac{1}{4}\right)^k (e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}})^{2k} = \left(-\frac{1}{4}\right)^k [2k]^{2k} \sum_{(n)_0}^{2k} e^{2(k-n)\theta\sqrt{-1}} [0]^{-(2k-n)} [0]^{-n} (-1)^{n*} \\ &= \left(\frac{1}{4}\right)^k [2k]^{2k} \sum_{(n)_k}^k [0]^{-(k+n)} [0]^{-(k-n)} (-1)^n \cos 2n\theta; \end{aligned}$$

hence, coefficient of $\cos(2l\theta - 2\theta)$, if $l > 1$, is $2\left(\frac{1}{4}\right)^k [2k]^{2k} (-1)^{l-1} [0]^{-(k+l-1)} [0]^{-(k-l+1)}$; and coefficient of $\cos(2l\theta + 2\theta)$ is $2\left(\frac{1}{4}\right)^k [2k]^{2k} (-1)^{l+1} [0]^{-(k+l+1)} [0]^{-(k-l-1)}$; therefore $g_{l, k} = \frac{1}{8}$ th of (former - latter) coefficient, if $l > 1$,

$$\begin{aligned} &= \left(\frac{1}{4}\right)^{k+1} [2k]^{2k} (-1)^l [0]^{-(k+l+1)} [0]^{-(k-l+1)} \{[k-l+1]^2 - [k+l+1]^2\} \\ &= (-1)^{l+1} [2k+1]^{2k+1} \left(\frac{1}{2}\right)^{2k+1} l [0]^{-(k+l+1)} [0]^{-(k-l+1)} = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} d\theta \sin \theta^{2k} \sin 2\theta \sin 2l\theta, \end{aligned}$$

if $l > 1$. And

$$\begin{aligned} g_{1, k} &= \left(\frac{1}{4}\right)^{k+1} [2k]^{2k} \{([0]^{-k})^2 - [0]^{-(k+2)} [0]^{-(k-2)}\} \\ &= [2k+1]^{2k+1} \left(\frac{1}{2}\right)^{2k+1} [0]^{-k} [0]^{-(k+2)} = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} d\theta \sin \theta^{2k} \sin 2\theta^2; \end{aligned}$$

so that the formula just now found for $g_{l, k}$ holds even when $l = 1$.

104. Hence in the expression of article 97,

$$\begin{aligned} y_{l, t} &= \frac{\eta}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \frac{\cos \theta \sin 2l\theta}{A - \sin \theta} \{ \sin(\beta - 2at \sin \theta) - \sin(\beta - 2aAt) \} \\ &= \frac{\eta}{\pi} \int_0^{\frac{\pi}{2}} d\theta \frac{\sin 2\theta \sin 2l\theta}{A^2 - \sin^2 \theta} \left\{ \sin \beta \{ \cos(2at \sin \theta) - \cos(2aAt) \} \right. \\ &\quad \left. + \cos \beta \left(\sin(2aAt) - \frac{A}{\sin \theta} \sin(2at \sin \theta) \right) \right\} \\ &= \frac{\eta}{\pi} \int_0^{\frac{\pi}{2}} d\theta \frac{\sin 2\theta \sin 2l\theta}{A^2 - \sin^2 \theta} \sum_{(i)_1}^{\infty} (A^{2i} - \sin^2 \theta) (2at)^{2i} [0]^{-2i} (-1)^i \left\{ -\sin \beta + \frac{2aAt}{2i+1} \cos \beta \right\} \\ &= \eta \sum_{(i)_1}^{\infty} (-4a^2 t^2)^i [0]^{-2i} \left\{ \frac{2aAt}{2i+1} \cos \beta - \sin \beta \right\} \sum_{(k)_0}^{i-1} A^{2i-2k-2} g_{l, k} \\ &= \eta \sum_{(i)_1}^{\infty} (-4a^2 A^2 t^2)^i [0]^{-2i} \left\{ \frac{2aAt}{2i+1} \cos \beta - \sin \beta \right\} A^{-2} \sum_{(k)_0}^{i-1} A^{-2k} g_{l, k}, \end{aligned}$$

* $[[n]^m = n!/(n-m)!; [0]^{-m} = 1/(m!);$ this is Vandermonde's notation. See Vol. I, p. 468.]

we have $g_{l,0} = (-1)^{l+1} \frac{l}{2} [0]^{-(1+l)} [0]^{-(1-l)}$, $\therefore g_{1,0} = \frac{1}{2}$, and $g_{l,0} = 0$ if $l > 1$; so that the part of $y_{l,t}$ which involves t^2 and t^3 vanishes for all the particles beyond P_1 , but becomes, for that particle,

$$\frac{\eta a^2 t^2}{2} \left(\sin \beta - \frac{2aAt}{3} \cos \beta \right).$$

In fact, the differential equations are, in the present question,

$$y''_{1,t} = a^2 \{ \eta \sin (\beta - 2aAt) - 2y_{1,t} + y_{2,t} \} \quad \text{and} \quad y''_{l,t} = a^2 (y_{l-1,t} - 2y_{l,t} + y_{l+1,t}),$$

if $l > 1$; also $y_{l,0} = y'_{l,0} = 0$, if $l > 0$; and these conditions are satisfied if we neglect t^4 in $y_{l,t}$, or t^2 in $y''_{l,t}$, & suppose $y_{1,t} = \eta a^2 \left(\frac{t^2}{2} \sin \beta - \frac{aAt^3}{3} \cos \beta \right)$, $y_{2,t} = \&c. = 0$.

For the parts involving t^4 and t^5 , we are to suppose $i = 2$, & to calculate

$$\sum_{(k)0}^1 A^{-2k} g_{l,k} = g_{l,0} + A^{-2} g_{l,1},$$

in which we already know $g_{l,0}$ and in which $g_{l,1} = (-1)^{l+1} \frac{3l}{4} [0]^{-(2+l)} [0]^{-(2-l)}$, so that

$$g_{3,1} = g_{4,1} = \&c. = 0, \quad \text{and} \quad g_{1,1} = \frac{3}{4} [0]^{-3} = \frac{1}{8}, \quad g_{2,1} = -\frac{3}{2} [0]^{-4} = -\frac{1}{16};$$

these parts therefore vanish for all particles beyond P_2 ; they are, for P_1 ,

$$\frac{2}{3} \eta a^4 A^2 t^4 \left(\frac{2aAt}{5} \cos \beta - \sin \beta \right) \left(\frac{1}{4} + \frac{A^{-2}}{8} \right) = -\frac{\eta a^4 t^4}{12} \left(\sin \beta - \frac{2aAt}{5} \cos \beta \right) (1 + 2A^2),$$

and, for P_2 ,

$$\frac{\eta a^4 t^4}{24} \left(\sin \beta - \frac{2aAt}{5} \cos \beta \right).$$

Accordingly, if we suppose

$$y_{1,t} = \eta \sin \beta \left\{ \frac{a^2 t^2}{2} - \frac{a^4 t^4}{12} (1 + 2A^2) \right\} + \eta A \cos \beta \left\{ -\frac{a^3 t^3}{3} + \frac{a^5 t^5}{30} (1 + 2A^2) \right\},$$

$$y_{2,t} = \eta \sin \beta \frac{a^4 t^4}{24} - \eta A \cos \beta \frac{a^5 t^5}{60}, \quad y_{3,t} = \&c. = 0,$$

and neglect t^4 in $y''_{l,t}$, we shall have

$$y''_{3,t} = \&c. = 0; \quad y''_{2,t} = a^2 y_{1,t};$$

$$y''_{1,t} + 2a^2 y_{1,t} = a^2 \eta \{ \sin \beta (1 - 2a^2 A^2 t^2) + A \cos \beta (-2at + \frac{4}{3} a^3 A^2 t^3) \} = a^2 \eta \sin (\beta - 2aAt);$$

so that the differential equations are as well satisfied as they ought to be.

105. No power of t being neglected, the displacement of the particle P_1 is

$$y_{1,t} = \eta A^{-2} \sum_{(i)1}^{\infty} [0]^{-2i} (-4a^2 A^2 t^2)^i \left\{ \frac{2aAt}{2i+1} \cos \beta - \sin \beta \right\} \sum_{(k)0}^{i-1} A^{-2k} g_{1,k};$$

in which

$$g_{1,k} = 2^{-(2k+1)} [2k+1]^{2k+1} [0]^{-k} [0]^{-(k+2)}.$$

We may order $y_{1,t}$ according to the ascending powers of A ; & the coefficient of A^0 is

$$-\sin \beta \cdot \eta \sum_{(i)1}^{\infty} [0]^{-2i} (-4a^2 t^2)^i g_{1,i-1} = \eta \sin \beta \sum_{(i)1}^{\infty} [0]^{-i} [0]^{-(i+1)} (at)^{2i} (-1)^{i+1};$$

which latter series can be summed, if we can sum $1 - \left(\frac{x}{1}\right)^2 + \left(\frac{x^2}{1 \cdot 2}\right)^2 - \left(\frac{x^3}{1 \cdot 2 \cdot 3}\right)^2 + \&c.$ Now this

last sum = $\frac{1}{\pi} \int_0^\pi dr \cos(2x \sin r)$, because $\frac{2^{2i}}{\pi} [0]^{-2i} \int_0^\pi dr \sin r^{2i} = ([0]^{-i})^2$, (see page 523); therefore, differentiating relatively to x ,

$$\frac{1}{\pi} \int_0^\pi dr \sin r \sin(2x \sin r) = x - \frac{x^3}{1 \cdot 1 \cdot 2} + \frac{x^5}{1 \cdot 2 \cdot 1 \cdot 2 \cdot 3} - \&c.;$$

\therefore coefficient of A^0 in $y_{1,t}$ is $\left\{ 1 - \frac{1}{\pi x} \int_0^\pi dr \sin r \sin(2x \sin r) \right\} \eta \sin \beta$, if $x = at$. In fact

$$\int_0^\pi dr \sin r \sin(2x \sin r) = -\Delta \{ \cos r \sin(2x \sin r) \} + 2x \int_0^\pi dr \cos r^2 \cos(2x \sin r);$$

the coefficient of A^0 in $y_{1,t}$ is therefore

$$\eta \sin \beta \left\{ 1 - \frac{2}{\pi} \int_0^\pi dr \cos r^2 \cos(2at \sin r) \right\};$$

& accordingly if we make $A = 0$, $l = 1$, in the expression of 97 or 104 for $y_{l,t}$, we get

$$y_{1,t} = \frac{2\eta \sin \beta}{\pi} \int_0^{\frac{\pi}{2}} d\theta \, 2 \cos \theta^2 \{ 1 - \cos(2at \sin \theta) \} = \frac{\eta \sin \beta}{\pi} \int_0^\pi d\theta \, \&c.$$

But it is probably improper, or at least disadvantageous, to develop according to ascending powers of A , when A is > 1 .

106. It may be convenient to make $A - \sqrt{A^2 - 1} = B$, and therefore

$$A + \sqrt{A^2 - 1} = \frac{1}{B}, \quad A = \frac{1}{2} (B + B^{-1}) = \frac{1 + B^2}{2B},$$

$$\begin{aligned} y_{l,t} &= \frac{(-1)^l \eta}{\pi} \int_0^\pi d\theta \frac{2B \sin \theta \sin 2l\theta}{1 - 2B \cos \theta + B^2} \{ \sin(\beta - 2aAt) - \sin(\beta - 2at \cos \theta) \} \\ &= \frac{2(-1)^l \eta}{\pi} \sum_{(n)1}^\infty \int_0^\pi d\theta \, B^n \sin n\theta \sin 2l\theta \{ \sin(\beta - 2aAt) - \sin(\beta - 2at \cos \theta) \} \\ &= \eta (-B^2)^l \sin(\beta - 2aAt) + \frac{\eta (-1)^l}{\pi} \sum_{(n)1}^\infty B^n \int_0^\pi \{ \cos(2l\theta + n\theta) - \cos(2l\theta - n\theta) \} \\ &\quad \times \sin(\beta - 2at \cos \theta) d\theta. \end{aligned}$$

Accordingly it is evident that this last expression for $y_{l,t}$ satisfies the equation in mixed differences $y''_{l,t} = a^2 (y_{l+1,t} - 2y_{l,t} + y_{l-1,t})$, because $B^2 + B^{-2} + 2 = 4A^2$; also $y_{0,t} = \eta \sin(\beta - 2atA)$, & $y_{l,0} = y'_{l,0} = 0$ if $l > 0$.

We may therefore write

$$\begin{aligned} y_{l,t} &= \eta (-B^2)^l \sin(\beta - 2aAt) \\ &\quad + \frac{2}{\pi} \eta (-1)^l \sin \beta \sum_{(n)1}^\infty B^{2n} \int_0^{\frac{\pi}{2}} d\theta \{ \cos(2l\theta + 2n\theta) - \cos(2l\theta - 2n\theta) \} \cos(2at \cos \theta) \\ &\quad - \frac{2}{\pi} \eta (-1)^l \cos \beta \sum_{(n)1}^\infty B^{2n-1} \int_0^{\frac{\pi}{2}} d\theta \{ \cos(2l\theta + 2n\theta - \theta) - \cos(2l\theta - 2n\theta + \theta) \} \sin(2at \cos \theta). \end{aligned}$$

And we see that it remains to calculate the values of the definite integrals

$$\frac{2}{\pi} \int_0^1 dx \frac{x^{2m} \cos(2atx)}{\sqrt{1-x^2}}, \quad \frac{2}{\pi} \int_0^1 dx \frac{x^{2m+1} \sin(2atx)}{\sqrt{1-x^2}},$$

or simply of the former. We may even deduce all, by differentiation, from the function

$$\begin{aligned} \frac{2}{\pi} \int_0^1 dx \frac{\cos(2atx)}{\sqrt{1-x^2}} &= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} d\theta \cos(2at \cos \theta) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} d\theta \cos(2at \sin \theta) \\ &= 1 - \left(\frac{at}{1}\right)^2 + \left(\frac{a^2t^2}{1 \cdot 2}\right)^2 - \left(\frac{a^3t^3}{1 \cdot 2 \cdot 3}\right)^2 + \&c. \end{aligned}$$

Or we may consider the question as being now to determine, at least approximately & for integer values of m much larger than t , the definite integrals

$$\begin{aligned} \int_0^{\frac{\pi}{2}} d\theta \cos 2m\theta \cos(2at \cos \theta) &= M_m, \\ \int_0^{\frac{\pi}{2}} d\theta \cos(2m\theta - \theta) \sin(2at \cos \theta) &= N_m. \end{aligned}$$

107. Adopting this last view, & integrating by parts, so as to develop according to descending powers of m , we find

$$\begin{aligned} \int_0^{\theta} d\theta \cos 2m\theta \cos(2at \cos \theta) &= \frac{\sin 2m\theta}{2m} \cos(2at \cos \theta) - \frac{at}{m} \int_0^{\theta} d\theta \sin \theta \sin 2m\theta \sin(2at \cos \theta), \\ \int_0^{\theta} d\theta \cos(2m\theta - \theta) \cos(2at \cos \theta) &= \frac{\sin(2m\theta - \theta)}{2m-1} \sin(2at \cos \theta) + \frac{2at}{2m-1} \int_0^{\theta} d\theta \sin \theta \sin(2m\theta - \theta) \cos(2at \cos \theta). \end{aligned}$$

If then we employ the symbols M_m and N_m as defined in the last article, we have*

$$\begin{aligned} M_m &= \frac{at}{2m} (N_{m+1} - N_m), \quad N_m = \frac{at}{2m-1} (M_{m-1} - M_m); \\ \therefore N_{m+1} &= \frac{at}{2m+1} (M_m - M_{m+1}), \quad M_m = \frac{a^2t^2}{2m(4m^2-1)} \{(2m-1)(M_m - M_{m+1}) \\ &\quad - (2m+1)(M_{m-1} - M_m)\}; \\ \therefore 0 &= M_m \left\{ 1 - \frac{2a^2t^2}{4m^2-1} \right\} + \frac{a^2t^2}{2m} \left(\frac{M_{m+1}}{2m+1} + \frac{M_{m-1}}{2m-1} \right); \\ \therefore \text{nearly, } \frac{2m}{at} \text{ being large, } M_{m+1} - 2M_m + M_{m-1} &= -\frac{4m^2}{a^2t^2} M_m. \end{aligned}$$

[Manuscript ends.]

* [$M_m = (-)^m J_{2m}(2at)$; $N_m = (-)^m J_{2m-1}(2at)$.]