

## 719.

SUGGESTION OF A MECHANICAL INTEGRATOR FOR THE  
CALCULATION OF  $\int (Xdx + Ydy)$  ALONG AN ARBITRARY  
PATH\*.

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I CONSIDER an integral  $\int (Xdx + Ydy)$ , where  $X$ ,  $Y$  are each of them a given function of the variables  $(x, y)$ ;  $Xdx + Ydy$  is thus not in general an exact differential; but assuming a relation between  $(x, y)$ , that is, a path of the integral, there is in effect one variable only, and the integral becomes calculable. I wish to show how for any given values of the functions  $X$ ,  $Y$ , but for an arbitrary path, it is possible to construct a mechanism for the calculation of the integral: viz. a mechanism such that, a point  $D$  thereof being moved in a plane along a path chosen at pleasure, the corresponding value of the integral shall be exhibited on a dial.

The mechanism (for convenience I speak of it as actually existing) consists of a square block or inverted box, the upper horizontal face whereof is taken as the plane of  $xy$ , the equations of its edges being  $y=0$ ,  $y=1$ ,  $x=0$ ,  $x=1$  respectively. In the wall faces represented by these equations, we have the endless bands  $A$ ,  $A'$ ,  $B$ ,  $B'$  respectively; and in the plane of  $xy$ , a driving point  $D$ , the coordinates of which are  $(x, y)$ , and a regulating point  $R$ , mechanically connected with  $D$ , in suchwise that the coordinates of  $R$  are always the given functions  $X$ ,  $Y$  of the coordinates of  $D$ †; the nature of the mechanical connexion will of course depend upon the particular functions  $X$ ,  $Y$ .

This being so,  $D$  drives the bands  $A$  and  $B$  in such manner that, to the given motions  $dx$ ,  $dy$  of  $D$ , correspond a motion  $dx$  of the band  $A$  and a motion  $dy$  of

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† It might be convenient to have as the coordinates of  $R$ , not  $X$ ,  $Y$  but  $\xi$ ,  $\eta$ , determinate functions of  $X$ ,  $Y$  respectively.

the band  $B$ ;  $A$  drives  $A'$  with a velocity-ratio depending on the position of the regulator  $R$  in suchwise that, the coordinates of  $R$  being  $X, Y$ , then to the motion  $dx$  of  $A$  corresponds a motion  $Xdx$  of  $A'$ ; and, similarly,  $B$  drives  $B'$  with a velocity-ratio depending on the position of  $R$ , in suchwise that to the motion  $dy$  of  $B$  corresponds a motion  $Ydy$  of  $B'$ . Hence, to the motions  $dx, dy$  of the driver  $D$ , there correspond the motions  $Xdx$  and  $Ydy$  of the bands  $A'$  and  $B'$  respectively; the band  $A'$  drives a hand or index, and the band  $B'$  drives in the contrary sense a graduated dial, the hand and dial rotating independently of each other about a common centre; the increased reading of the hand on the dial is thus  $=Xdx + Ydy$ ; and supposing the original reading to be zero, and the driver  $D$  to be moved from its original position along an arbitrary path to any other position whatever, the reading on the dial will be the corresponding value of the integral  $\int(Xdx + Ydy)$ .

It is obvious that we might, by means of a combination of two such mechanisms, calculate the value of an integral  $\int f(u) du$  along an arbitrary path of the complex variable  $u, = x + iy$ ; in fact, writing  $f(x + iy) = P + iQ$ , the differential is

$$(P + iQ)(dx + idy), = Pdx - Qdy + i(Qdx + Pdy);$$

and we thus require the calculation of the two integrals

$$\int(Pdx - Qdy) \text{ and } \int(Qdx + Pdy),$$

each of which is an integral of the above form. Taking for the path a closed curve, it would be very curious to see the machine giving a value zero or a value different from zero, according as the path did not include or included within it a critical point; it seems to me that this discontinuity would really exhibit itself without the necessity of any change in the setting of the machine.

The ordinary modes of establishing a continuously-variable velocity-ratio between two parts of a machine depend upon friction; and, in particular, this is the case in Prof. James Thomson's mechanical integrator—there is thus of course a limitation of the driving power. It seems to me that a variable velocity-ratio, the variation of which is practically although not strictly continuous, might be established by means of toothed wheels (and so with unlimited driving power) in the following manner.

Consider a revolving wheel  $A$ , which by means of a link  $BC$ , pivoted to a point  $B$  of the wheel  $A$  and a point  $C$  of a toothed wheel or arc  $D$ , communicates a reciprocating motion to  $D$ ; the extent of this reciprocating motion depending on the distance of  $B$  from the centre of  $A$ , which distance, or say the half-throw, is assumed to be variable. Here during a half-revolution of  $A$ ,  $D$  moves in one direction, say upwards; and during the other half-revolution of  $A$ ,  $D$  moves in the other direction, say downwards; the extent of these equal and opposite motions varying with the throw. Suppose then that  $D$  works a pinion  $E$ , the centre of which is not absolutely fixed but is so connected with  $A$  that during the first half-revolution of  $A$  (or while  $D$  is moving upwards),  $E$  is in gear with  $D$ , and during the second half-revolution of  $A$ , or while

$D$  is moving downwards,  $E$  is out of gear with  $D$ ; the continuous rotation of  $A$  will communicate an intermittent rotation to  $E$ , in such manner nevertheless that, to each entire revolution of  $A$  or rotation through the angle  $2\pi$ , there will (the throw remaining constant) correspond a rotation of  $E$  through the angle  $n.2\pi$ , where the coefficient  $n$  depends upon the throw\*. And evidently if  $A$  be driven by a wheel  $A'$ , the angular velocity of which is  $\frac{1}{\lambda}$  times that of  $A$ , then to a rotation of  $A'$  through each angle  $\frac{2\pi}{\lambda}$ , there will correspond an entire revolution of  $A$ , and therefore, as before, a rotation of  $E$  through the determinate angle  $n.2\pi$ ; hence,  $\lambda$  being sufficiently large to each increment of rotation of  $A'$ , there corresponds in  $E$  an increment of rotation which is  $n\lambda$  times the first-mentioned increment; viz.  $E$  moves (intermittently and possibly also with some "loss of time" on  $E$  coming successively in gear and out of gear with  $D$ , or in beats as explained) with an angular velocity which is  $=n\lambda$  times the angular velocity of  $A'$ . And thus the throw (and therefore  $n$ ) being variable, the velocity-ratio  $n\lambda$  is also variable.

We may imagine the wheel  $A$  as carrying upon it a piece  $L$  sliding between guides, which piece  $L$  carries the pivot  $B$  of the link  $BC$ , and works by a rack on a toothed wheel  $\alpha$  concentric with  $A$ , but capable of rotating independently thereof. Then if  $\alpha$  rotates along with  $A$ , as if forming one piece therewith, it will act as a clamp upon  $L$ , keeping the distance of  $B$  from the centre of  $A$ , that is, the half-throw, constant; whereas, if  $\alpha$  has given to it an angular velocity different from that of  $A$ , the effect will be to vary the distance in question; that is, to vary the half-throw, and consequently the velocity-ratio of  $A$  and  $E$ . And, in some such manner, substituting for  $A$  and  $E$  the bands  $A$  and  $A'$  of the foregoing description, it might be possible to establish between these bands the required variable velocity-ratio.

\* If instead of the wheel or arc  $D$  with a reciprocating circular motion, we have a double rack  $D$  with a reciprocating rectilinear motion, such that the wheel  $E$  is placed between the two racks, and is in gear on the one side with one of them when the rack is moving upwards, and on the other side with the other of them when the rack is moving downwards; then the continuous circular motion of  $A$  will communicate to  $E$  a continuous circular motion, not of course uniform, but such that to each entire revolution of  $A$  or rotation through the angle  $2\pi$ , there will correspond a rotation of  $E$  through an angle  $n.2\pi$  as before. This is in fact a mechanical arrangement made use of in a mangle, the double rack being there the follower instead of the driver.