

## CHAPTER XVIII.

### RECTIFICATION (III). MISCELLANEOUS THEOREMS.

#### 621. Arc of an Inverse Curve.

Let  $s$  and  $s'$  be the corresponding arcs of a curve and of its inverse with regard to a fixed point  $O$ , the constant of inversion being  $k$ .



Fig. 167.

Then if  $P, Q$  be points on the curve and  $P', Q'$  the inverse points, we have

$$P'Q' = k^2 \frac{PQ}{OP \cdot OQ}. \quad (\text{See } \textit{Diff. Calc.}, \text{ p. 174.})$$

And ultimately, when  $Q$  and  $Q'$  are made to travel along their respective paths to ultimate coincidence with  $P$  and  $P'$ ,

$$ds' = k^2 \frac{ds}{r^2},$$

*i.e.*

$$s' = k^2 \int \frac{ds}{r^2}, \dots\dots\dots(1)$$

giving the arc of the inverse in terms of elements of the original curve.

#### 622. Modifications for Various Coordinate Systems.

This formula may be modified as required for different systems of coordinates, and with the usual notation, we have for polars, the inversion being with regard to the pole,

$$s' = k^2 \int \frac{\sqrt{dr^2 + r^2 d\theta^2}}{r^2} = k^2 \int \sqrt{\frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta}\right)^2} d\theta. \dots\dots\dots(2)$$

$$= k^2 \int \frac{d\theta}{p} \dots\dots\dots(3)$$

Again, we may write

$$s' = k^2 \int \frac{1}{r^2} \frac{ds}{d\psi} d\psi = k^2 \int \frac{\rho}{r^2} d\psi, \dots\dots\dots(4)$$

*i.e.* as a formula suitable for tangential polars,

$$s' = k^2 \int \frac{p + \frac{d^2p}{d\psi^2}}{p^2 + \left(\frac{dp}{d\psi}\right)^2} d\psi, \dots\dots\dots(5)$$

or for pedal equations,

$$s' = k^2 \int \frac{1}{r^2} \frac{ds}{dr} dr = k^2 \int \frac{dr}{r^2 \cos \phi} = k^2 \int \frac{dr}{r\sqrt{r^2 - p^2}}, \dots\dots\dots(6)$$

and for Cartesians,

$$s' = \int \frac{\sqrt{dx^2 + dy^2}}{x^2 + y^2} = k^2 \int \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{x^2 + y^2} dx, \dots\dots\dots(7)$$

the inversion being with regard to the origin ;

or 
$$s' = k^2 \int \frac{\sqrt{dx^2 + dy^2}}{(x - a)^2 + (y - b)^2}, \dots\dots\dots(8)$$

if the inversion is with regard to the point  $(a, b)$ .

**623. Illustrative Examples.**

1. Consider the arc of the inverse of the parabola

$$p^2 = ar \quad \left( \text{or } \frac{2a}{r} = 1 + \cos \theta \right)$$

with regard to the focus; *i.e.* a cardioide.

Here

$$\begin{aligned} s' &= k^2 \int \frac{dr}{r\sqrt{r^2 - p^2}} = k^2 \int \frac{dr}{r\sqrt{r^2 - ar}} = -k^2 \int \frac{du}{\sqrt{1 - au}}, \quad \text{if } r = \frac{1}{u}, \\ &= \frac{2k^2}{a} \sqrt{1 - au} = \frac{2k^2}{a} \sin \frac{\theta}{2}. \end{aligned}$$

2. Rectification of the inverse with regard to the centre of the first negative pedal of an ellipse with regard to the centre.

The ellipse being  $x^2/a^2 + y^2/b^2 = 1$ , the first negative pedal is the envelope of

$$x \cos \psi + y \sin \psi = p, \quad \text{where } \frac{1}{p^2} = \frac{\cos^2 \psi}{a^2} + \frac{\sin^2 \psi}{b^2}.$$

Hence the tangential polar equation is

$$p = \frac{ab}{\sqrt{a^2 \sin^2 \psi + b^2 \cos^2 \psi}}.$$

Differentiating we have

$$\frac{dp}{d\psi} = -ab \frac{(a^2 - b^2) \sin \psi \cos \psi}{(a^2 \sin^2 \psi + b^2 \cos^2 \psi)^{\frac{3}{2}}},$$

$$\frac{d^2 p}{d\psi^2} = -ab(a^2 - b^2) \frac{b^2 \cos^4 \psi - a^2 \sin^4 \psi - 2(a^2 - b^2) \sin^2 \psi \cos^2 \psi}{(a^2 \sin^2 \psi + b^2 \cos^2 \psi)^{\frac{5}{2}}};$$

whence

$$p + \frac{d^2 p}{d\psi^2} = ab \frac{a^2(2a^2 - b^2) \sin^2 \psi + b^2(2b^2 - a^2) \cos^2 \psi}{(a^2 \sin^2 \psi + b^2 \cos^2 \psi)^{\frac{5}{2}}}$$

and

$$p^2 + \left(\frac{dp}{d\psi}\right)^2 = a^2 b^2 \frac{a^4 \sin^2 \psi + b^4 \cos^2 \psi}{(a^2 \sin^2 \psi + b^2 \cos^2 \psi)^3}.$$

Hence

$$\begin{aligned} s' \cdot \frac{ab}{k^2} &= \int \frac{a^2(2a^2 - b^2) \sin^2 \psi + b^2(2b^2 - a^2) \cos^2 \psi}{a^4 \sin^2 \psi + b^4 \cos^2 \psi} \sqrt{a^2 \sin^2 \psi + b^2 \cos^2 \psi} d\psi \\ &= \int \left( 2 - \frac{a^2 b^2}{a^4 \sin^2 \psi + b^4 \cos^2 \psi} \right) \cdot \sqrt{a^2 \sin^2 \psi + b^2 \cos^2 \psi} d\psi \\ &= \int \left[ 2 \sqrt{a^2 \sin^2 \psi + b^2 \cos^2 \psi} \right. \\ &\quad \left. - \frac{a^2 b^2}{a^2 + b^2} \frac{(a^4 \sin^2 \psi + b^4 \cos^2 \psi + a^2 b^2)}{(a^4 \sin^2 \psi + b^4 \cos^2 \psi) \sqrt{a^2 \sin^2 \psi + b^2 \cos^2 \psi}} \right] d\psi \end{aligned}$$

Hence if  $e$  be the eccentricity of the ellipse, and the integration be taken from  $\psi$  to  $\frac{\pi}{2}$  and if  $\chi$  be the complement of  $\psi$ , we have

$$s' = \frac{k^2}{b} \left[ 2E(\chi, e) - \frac{1 - e^2}{2 - e^2} F(\chi, e) - \frac{(1 - e^2)^2}{2 - e^2} \Pi(\chi, e, e^4 - 2e^2) \right].$$

This curve therefore requires all three kinds of the Legendrian integrals for its rectification.

Note for the first negative central pedal of an ellipse that we have incidentally

$$(1) \quad \rho = ab \frac{a^2(2a^2 - b^2) \sin^2 \psi + b^2(2b^2 - a^2) \cos^2 \psi}{(a^2 \sin^2 \psi + b^2 \cos^2 \psi)^{\frac{3}{2}}};$$

$$(2) \quad r^2 = a^2 b^2 \frac{a^4 \sin^2 \psi + b^4 \cos^2 \psi}{(a^2 \sin^2 \psi + b^2 \cos^2 \psi)^3};$$

$$(3) \quad \int p^2 d\theta = \int \frac{p^3 \rho}{r^2} d\psi = 3ab \tan^{-1} \left( \frac{a}{b} \tan \psi \right) - (a^2 + b^2) \tan^{-1} \left( \frac{a^2}{b^2} \tan \psi \right);$$

$$(4) \quad \int_{\psi}^{\frac{\pi}{2}} p d\psi = b \int_{\psi}^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1 - e^2 \cos^2 \psi}} = b \int_0^{\chi} \frac{d\chi}{\sqrt{1 - e^2 \sin^2 \chi}} = b F(\chi, e).$$

## 3. Central inversion of epi- or hypo-cycloids.

Here  $p = A \sin B\psi$ , where  $A = a + 2b$ ,  $B = \frac{a}{a + 2b}$ . } See *Diff. Calc.*, Art. 410

$$s' = k^2 \int_0^\psi \frac{p + \frac{d^2 p}{d\psi^2}}{p^2 + \left(\frac{dp}{d\psi}\right)^2} d\psi,$$

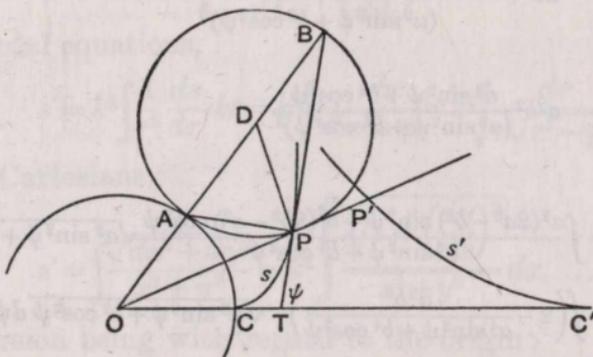


Fig. 168.

$$s' = k^2 \int_0^\psi \frac{A(1-B^2) \sin B\psi d\psi}{A^2\{1-(1-B^2)\cos^2 B\psi\}}, \text{ or } -k^2 \int_0^\psi \frac{A(B^2-1) \sin B\psi d\psi}{A^2\{1+(B^2-1)\cos^2 B\psi\}}$$

=  $\frac{k^2}{AB} \sqrt{1-B^2} \left[ \tanh^{-1}(\sqrt{1-B^2} \cos B\psi) \right]_\psi^0$  } for the inverses of epi-  
cycloids,

or =  $\frac{k^2}{AB} \sqrt{B^2-1} \left[ \tan^{-1}(\sqrt{B^2-1} \cos B\psi) \right]_0^\psi$  } for the inverses of hypo-  
cycloids.

*E.g.* in the case of the inverse of the cardioide with regard to the centre of the fixed circle  $a=b$ ,  $A=3a$ ,  $B=\frac{1}{3}$ ,

$$s' = \frac{2k^2\sqrt{2}}{3a} \left[ \tanh^{-1}\left(\frac{2\sqrt{2}}{3} \cos \frac{\psi}{3}\right) \right]_0^\psi.$$

In the case of the inverse of the three-cusped hypocycloid

$$b = -\frac{1}{3}a, \quad A = \frac{a}{3}, \quad B = 3,$$

$$s' = \frac{2k^2\sqrt{2}}{a} \left[ \tan^{-1}(2\sqrt{2} \cos 3\psi) \right]_0^\psi.$$

Note that these inverses are such that their arcs are expressible logarithmically if derived from epicycloids, or by means of circular functions if derived from hypocycloids.

4. Inverse of the parabola  $y^2 = 4ax$  with regard to the point  $x = -3a$ ,  $y = 0$ .

The general problem for any point on the axis is discussed by Mr. R. A. Roberts, in the *Proceedings of the London Mathematical Society*, vol. xviii., p. 202.

Taking  $am^2, 2am$  as the current coordinates of a point  $P$  on the curve  $y^2 = 4ax$ , an element of arc is given by

$$ds = \sqrt{dx^2 + dy^2} = 2a\sqrt{1+m^2} dm.$$

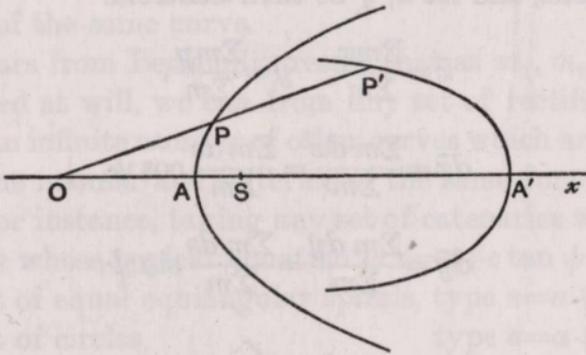


Fig. 169.

$$\begin{aligned} \text{Also } OP^2 &= (am^2 + 3a)^2 + 4a^2m^2 = a^2m^4 + 10a^2m^2 + 9a^2 \\ &= a^2(m^2 + 1)(m^2 + 9), \end{aligned}$$

and the element  $ds'$  of the inverse is  $ds' = k^2 \frac{ds}{OP^2}$ ;

$$\begin{aligned} \therefore s' &= k^2 \cdot 2a \int_0^m \frac{\sqrt{1+m^2} dm}{a^2(m^2+1)(m^2+9)} \\ &= \frac{2k^2}{a} \int_0^m \frac{dm}{(m^2+9)\sqrt{m^2+1}} \quad (\text{Let } m = \tan \phi.) \\ &= \frac{2k^2}{a} \int_0^\phi \frac{\cos \phi d\phi}{\sin^2 \phi + 9 \cos^2 \phi} \\ &= \frac{2k^2}{a} \int_0^\phi \frac{d \sin \phi}{9 - 8 \sin^2 \phi} = \frac{k^2}{4a} \int_{\frac{\pi}{2}}^{\phi} \frac{d \sin \phi}{\frac{9}{8} - \sin^2 \phi} \\ &= \frac{k^2}{6a\sqrt{2}} \log \frac{3 + 2\sqrt{2} \sin \phi}{3 - 2\sqrt{2} \sin \phi}. \end{aligned}$$

**Example.** Mr. Roberts shows in the article above cited that for points between  $-\infty$  and  $-3a$  on the  $x$ -axis the arc of the inverse curve can be expressed as a pure logarithm. For points from  $-3a$  to  $a$  such arcs are partly logarithmic, partly inverse circular. For points from  $a$  to  $+\infty$  the arcs are inverse circular expressions. Examine the truth of this.

#### 624. John Bernoulli's Theorem.

Let a number of points  $P_1(x_1, y_1), P_2(x_2, y_2),$  etc., be moving in a plane, and let  $ds_1, ds_2, ds_3,$  etc., be elements of the paths described. Let us impose upon their motion the condition that they are all moving at every instant in parallel directions

in the same sense. Let  $\psi$  be the angle the tangents to their respective paths make with the  $x$ -axis.

Suppose heavy particles of masses  $m_1, m_2$ , etc., to be placed at  $P_1, P_2$ , etc., and let  $\bar{x}, \bar{y}$  be their centroid.

Then

$$\bar{x} = \frac{\Sigma mx}{\Sigma m}, \quad \bar{y} = \frac{\Sigma my}{\Sigma m},$$

$$d\bar{x} = \frac{\Sigma m dx}{\Sigma m} = \frac{\Sigma m ds}{\Sigma m} \cos \psi,$$

$$d\bar{y} = \frac{\Sigma m dy}{\Sigma m} = \frac{\Sigma m ds}{\Sigma m} \sin \psi.$$

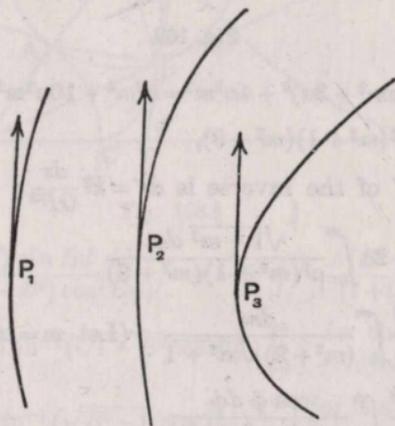


Fig. 170.

Hence  $\frac{d\bar{x}}{\cos \psi} = \frac{d\bar{y}}{\sin \psi}$ , and therefore the motion of the centroid is always parallel to the motion of the several particles; moreover, if  $d\bar{s}$  be the corresponding element of the path of the centroid,

$$d\bar{s} = \frac{\Sigma m ds}{\Sigma m}$$

and

$$\bar{s} = \frac{\Sigma ms}{\Sigma m}.$$

625. This result is ascribed by Mr. R. A. Roberts, in the paper before cited, as due to John Bernoulli, the intention being to give a method for the generation of new rectifiable curves from any system of curves whose rectification has already been effected.

It is to be remarked that the same theorem obviously holds for any system of particles moving in the manner prescribed upon twisted or tortuous curves in space.

Again, several of the points may be moving on different branches of the same curve.

It appears from Bernoulli's result that as  $m_1, m_2, m_3, \dots$  can be arranged at will, we can from any set of rectifiable curves generate an infinite number of other curves which are rectifiable in the same manner and in terms of the same functions.

Thus, for instance, taking any set of catenaries with parallel directrices whose typical equation is  $s = a + c \tan \psi$ ;

or any set of equal equiangular spirals, type  $s = a + be^{m(\psi+a)}$ ;

or any set of circles, type  $s = a + b\psi$ ;

or any set of involutes of circles, type  $s = a + b(\psi + a)^2$ ;

or any set of similar epi- or hypocycloids, type

$$s = a + b \sin(n\psi + a);$$

or any set of semi-cubical parabolas with parallel axes, type

$$s = a + b \sec^3 \psi;$$

or, in fact, any of the cases in which  $\frac{\sum ms}{\sum m}$  reduces to an expression of the same form, the locus of the centroid

$$\bar{x} = \frac{\sum mx}{\sum m}, \quad \bar{y} = \frac{\sum my}{\sum m},$$

is another curve of the same kind, and the length of any portion of its arc is to be found from the formula

$$\bar{s} = \frac{\sum ms}{\sum m}.$$

And further, when curves of different nature are taken as the original curves, though the derived locus be not of the same nature as that of any one of the original curves, yet it is still rectifiable in terms of the same functions as those in terms of which the original curves are rectifiable.

#### 626. Extension of Bernoulli's Theorem.

When the forward-drawn tangents at the several points are not all in the same sense, we may still apply the theorem, but with the precaution of reckoning all those elementary arcs which are traversed in the same sense as positive, and the remaining ones as negative.

Thus, if  $P_1(x_1, y_1)$ ,  $P_2(x_2, y_2)$  be at opposite extremities of a diameter of an ellipse, or centric oval, and if  $\cos \psi$ ,  $\sin \psi$  be the direction ratios of the tangent at  $P_1$ ,  $-\cos \psi$ ,  $-\sin \psi$  will be the direction ratios of the forward drawn tangent at  $P_2$ , and

$$\begin{aligned} d\bar{x} &= \frac{m_1 dx_1 + m_2 dx_2}{m_1 + m_2} = \frac{m_1(ds_1 \cos \psi) + m_2(-ds_2 \cos \psi)}{m_1 + m_2} \\ &= \frac{m_1 ds_1 - m_2 ds_2}{m_1 + m_2} \cos \psi, \end{aligned}$$

and  $d\bar{y} = \frac{m_1 ds_1 - m_2 ds_2}{m_1 + m_2} \sin \psi,$

and  $d\bar{s} = \frac{m_1 ds_1 - m_2 ds_2}{m_1 + m_2},$  giving  $\bar{s} = \frac{m_1 s_1 - m_2 s_2}{m_1 + m_2}.$

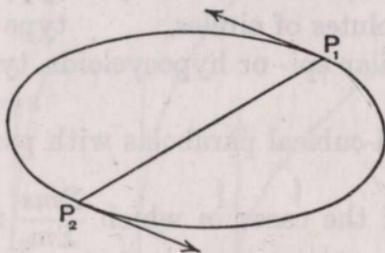


Fig. 171.

Moreover, for an ellipse, or centric oval, obviously  $ds_1 = ds_2$  and  $s_1 = s_2$ , and if we make  $m_1 = m_2$ ,  $\bar{s} = 0$ , as it should be, since all diameters are bisected at the centre, and the centroid locus degenerates into a point.

In the case when one of the curves degenerates to a point and one other point describes a given curve, Bernoulli's Theorem states that the similar and similarly situated centroid-locus is such that corresponding arcs on this locus and on the original curve are proportional, which is *a priori* obvious.

### 627. Ovoid with One Axis of Symmetry.

Let us consider the case of any ovoid with one axis of symmetry, and discuss the locus of the mid-points of chords which are such that the tangents at their extremities are parallel. Let  $P_1P_2$  be such a chord and  $G$  its mid-point. If we take the direction ratios at  $P_1$  as  $\cos \psi$ ,  $\sin \psi$ , then at  $P_2$ , where the forward-drawn tangent is parallel, but in the opposite direction, they must be taken as  $-\cos \psi$ ,  $-\sin \psi$ . If

it be a question of applying the theorem to the locus of the mid-point  $G$  of the chord  $P_1P_2$ , we have

$$d\sigma = \frac{ds_1 - ds_2}{2},$$

where  $ds_1, ds_2, d\sigma$  are the elementary arcs traced by  $P_1, P_2, G$  respectively, and as the inclination of all three tangents to the  $x$ -axis is the same,

$$\rho = \frac{\rho_1 - \rho_2}{2},$$

where  $\rho_1, \rho_2, \rho$  are the corresponding radii of curvature.

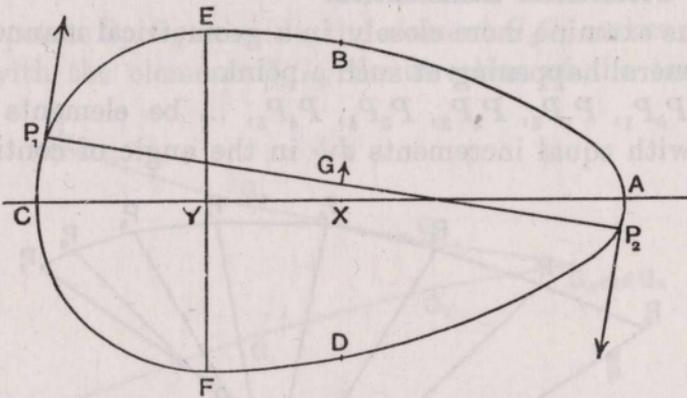


Fig. 172.

Now, in integrating to find  $\sigma$  for the whole length of the path of  $G$ , considerable care is necessary, for when the points  $P_1, P_2$  pass through positions at which the radii of curvature become equal,  $ds_1 - ds_2$  in general changes sign. So that in

estimating  $\int d\sigma$  for the whole  $G$ -locus, for some parts we must

take  $\sigma = \int \frac{ds_1 - ds_2}{2}$  and for others  $\int \frac{ds_2 - ds_1}{2}$ ; *i.e.* we must take

care that the difference of the elementary arcs at the ends of the chord is reckoned positively.

Hence we shall write the result

$$\sigma = \int \frac{ds_1 \sim ds_2}{2}.$$

In such an ovoid there will in general be points  $A, B, C, D$ , of which the first and third are the extremities of the axis of symmetry, where the radii of curvature are respectively

minimum, maximum, minimum, maximum;

and there may be a pair of points, one between  $D$  and  $A$  and one between  $B$  and  $C$ , at which the tangents are parallel, and such that the radii of curvature at those points are equal; and the same is true of the portions  $AB, CD$  of the ovoid. In such case, on the  $G$ -locus there is therefore a point at which  $\rho=0$ , with a change of sign of  $\rho$ . Hence there is at such a point a singularity on the  $G$ -locus, in general a cusp at which the tangent is parallel to the tangents at the corresponding points on the ovoid.

628. Geometrical Examination.

Let us examine more closely, in a geometrical manner, what is in general happening at such a point.

Let  $P_0P_1, P_1P_2, P_2P_3, P_3P_4, P_4P_5, \dots$  be elements of the ovoid, with equal increments  $d\psi$  in the angle of contingence,

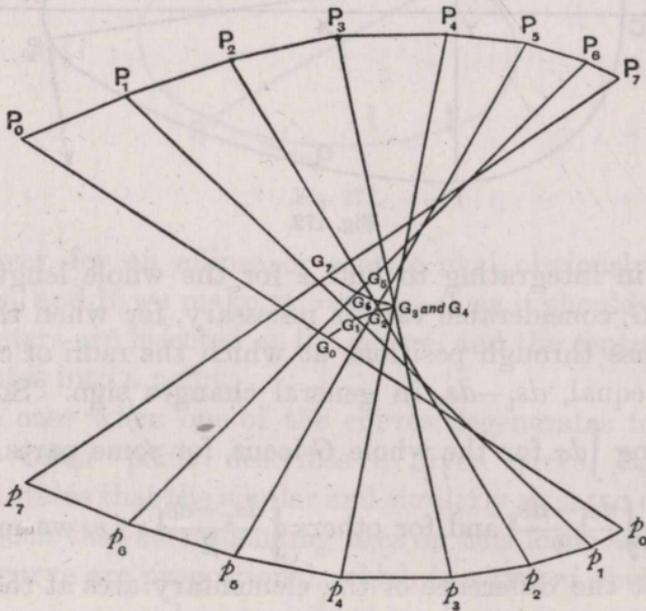


Fig. 173.

and drawn in the neighbourhood of a point on the ovoid, which has the peculiarity under consideration, viz. that the radius of curvature at that point is equal to that at the opposite extremity of the chord.

And let  $p_0p_1, p_1p_2, p_2p_3, p_3p_4, p_4p_5, \dots$  be the opposite parallel elements, the angles between consecutive pairs of either system being therefore  $d\psi$ , and let  $P_3P_4 = p_3p_4$ .

Let  $G_0, G_1, G_2, G_3, G_4, \dots$  be the mid-points of the chords  $P_0p_0, P_1p_1, P_2p_2, P_3p_3, \dots$  respectively; then it will be obvious that

$$\begin{aligned} G_0G_1 &= \frac{1}{2}(P_0P_1 - p_0p_1), \\ G_1G_2 &= \frac{1}{2}(P_1P_2 - p_1p_2), \\ G_2G_3 &= \frac{1}{2}(P_2P_3 - p_2p_3), \\ G_3G_4 &= \frac{1}{2}(P_3P_4 - p_3p_4) = 0, \\ G_4G_5 &= \frac{1}{2}(p_4p_5 - P_4P_5), \\ G_5G_6 &= \frac{1}{2}(p_5p_6 - P_5P_6), \\ &\text{etc.} \end{aligned}$$

The points  $G_3, G_4$  coincide, the element  $G_4G_5$  makes an angle  $2d\psi$  with the element  $G_2G_3$ , the direction of the tangent to

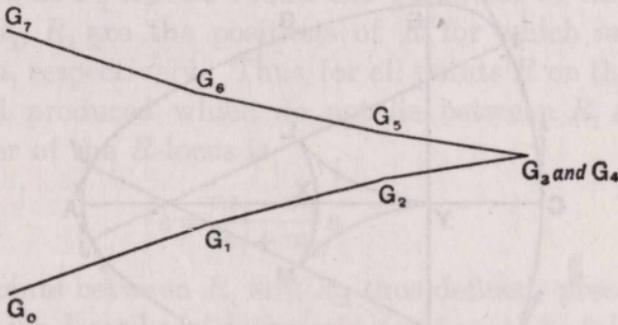


Fig. 174.

the path having turned through an angle  $\pi + 2d\psi$ . Ultimately then we have at  $G_3$  two coincident tangents to the  $G$ -locus, *i.e.* there is a cusp on the  $G$ -locus at such a point, and this cusp lies upon the envelope of the chord, for  $G_3$  is the point of intersection of two consecutive positions of the chord.

629. Again, at the points  $E, F$  on the double ordinate at the widest part of the ovoid the radii of curvature are obviously equal, and at the mid-point  $Y$  of  $EF$  there will be a cusp on the  $G$ -locus, whilst at  $X$ , the mid-point of the axis of symmetry  $AC$ , the tangent to the  $G$ -locus will be perpendicular to  $AC$ .

Let  $IJ$  be that chord of the ovoid for which the tangents at  $I$  and  $J$  are parallel and for which the radii of curvature at the ends are equal, and whose mid-point is situated at the cusp  $L$  of the  $G$ -locus, and let  $I'J'$  be the corresponding chord through the cusp  $M$ , symmetrically situated with regard to the axis of symmetry.

Then, integrating along corresponding arcs,

$$\text{arc } MXL = \frac{\text{arc } I'CI - \text{arc } J'AJ}{2},$$

$$\text{arc } LY = \frac{\text{arc } JDF - \text{arc } IE}{2},$$

$$\text{arc } YM = \frac{\text{arc } EBJ' - \text{arc } FI'}{2}.$$

Thus the whole perimeter of the tricuspidal  $G$ -locus

$$= \frac{1}{2}(\text{arc } I'CI - \text{arc } IE + \text{arc } EJ' - \text{arc } J'J + \text{arc } JF - \text{arc } FI'),$$

*i.e.* in short, half the difference of the two sums of alternate arcs of the original ovoid, the points of division being those

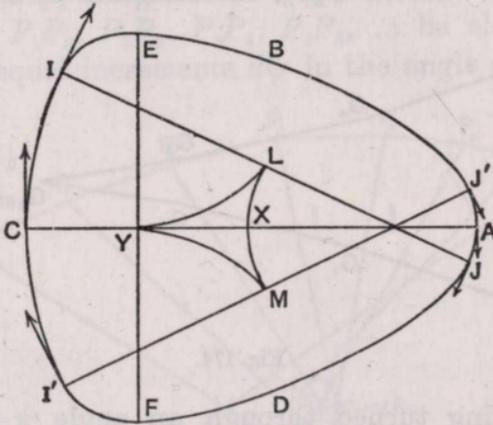


Fig. 175.

at which, whilst the opposite tangents are parallel, the radii of curvature are equal.

630. Of course, in the case of any closed oval symmetrical about two perpendicular axes, such as an ellipse, the diameters are all bisected at the intersection of the axes of symmetry, and the tricusp is evanescent, the radii of curvature at all opposite points being equal and the tangents parallel.

631. Note (i) that if lines be drawn through the points  $G$  parallel to the tangents at the extremities of the chords through  $G$ , then the points  $G$  are the points of contact of such lines with their envelope;

(ii) that the cuspidal tangents to the  $G$ -locus are parallel to those parallel tangents to the ovoid at whose points of contact the opposite radii of curvature are equal;

(iii) if  $R$  be a point on such a chord  $P_1P_2$  as has been described, and dividing it in the ratio  $m_2 : m_1$ , then the theorem

$$\sigma = \frac{m_1 s_1 - m_2 s_2}{m_1 + m_2}$$

is true for the whole perimeter  $s$  of the ovoid,

*i.e.* 
$$\sigma = \frac{m_1 - m_2}{m_1 + m_2} s$$

(for in integrating round the curve  $s_1 = s_2 = s$ ), provided that  $R$  does not lie intermediate between a certain pair of points  $R_1, R_2$  on the chord, for which  $m_1 \rho_1 - m_2 \rho_2$  can vanish, *i.e.* if  $\lambda$  and  $\lambda^{-1}$  be the greatest and least values of the ratio  $\rho_1/\rho_2$  attained as  $P_1$  travels round the perimeter of the ovoid, the points  $R_1, R_2$  are the positions of  $R$  for which  $m_1 = \lambda m_2$  and  $m_2 = \lambda m_1$  respectively. Thus, for all points  $R$  on the chord or the chord produced which do not lie between  $R_1$  and  $R_2$ , the perimeter of the  $R$ -locus is

$$\sigma = \frac{m_1 - m_2}{m_1 + m_2} s.$$

But for points between  $R_1$  and  $R_2$  thus defined, precautions similar to those described for the mid-point must be taken.

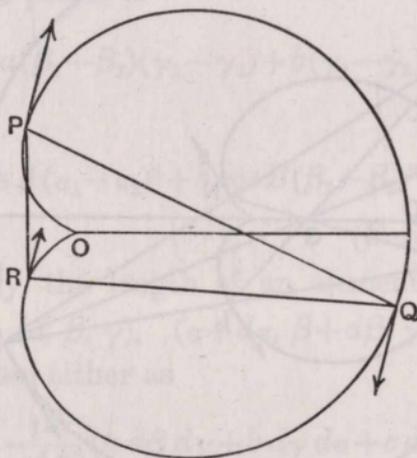


Fig. 176.

**632. An Instructive Problem.**

Let us discuss the locus of the centroid of the triangle  $PQR$  when these points lie upon a cardioid and are such that the tangents at  $P, Q, R$  are always parallel.

The equation of a normal to the curve  $r = a(1 + \cos \theta)$  at the point  $\theta = 2\alpha$  is

$$(3t - t^3)x - (1 - 3t^2)y = \frac{a}{2} \{(3t - t^3) + t(1 + t^2)\},$$

where  $t \equiv \tan \alpha$  (*Diff. Calc.*, p. 158).

The three normals will be parallel at points such that

$$\frac{3t - t^3}{1 - 3t^2} = k, \text{ say, i.e. } \tan 3\alpha = k.$$

Let  $\tan 3\chi = \tan 3\alpha$ .

Then  $3\chi = n\pi + 3\alpha$ ,

$$\chi = \alpha, \quad \alpha + \frac{\pi}{3}, \quad \alpha + \frac{2\pi}{3}.$$

Hence  $2\alpha, 2\alpha + \frac{2\pi}{3}, 2\alpha + \frac{4\pi}{3}$  are points at which the normals, and therefore also the tangents, are parallel.

Let these be called  $2\alpha, 2\beta, 2\gamma$ .

If  $(x_1, y_1), (x_2, y_2), (x_3, y_3)$  be the coordinates of  $P, Q, R$ ,

$$x_1 = 2a \cos^2 \frac{\theta}{2} \cos \theta = 2a \cos^2 \alpha \cos 2\alpha = \frac{a}{2} (1 + 2 \cos 2\alpha + \cos 4\alpha), \text{ etc.,}$$

$$y_1 = 2a \cos^2 \frac{\theta}{2} \sin \theta = 2a \cos^2 \alpha \sin 2\alpha = \frac{a}{2} (2 \sin 2\alpha + \sin 4\alpha), \text{ etc. ;}$$

$$\therefore 3\bar{x} = \sum x_1 = \frac{3a}{2}, \quad 3\bar{y} = \sum y_1 = 0.$$

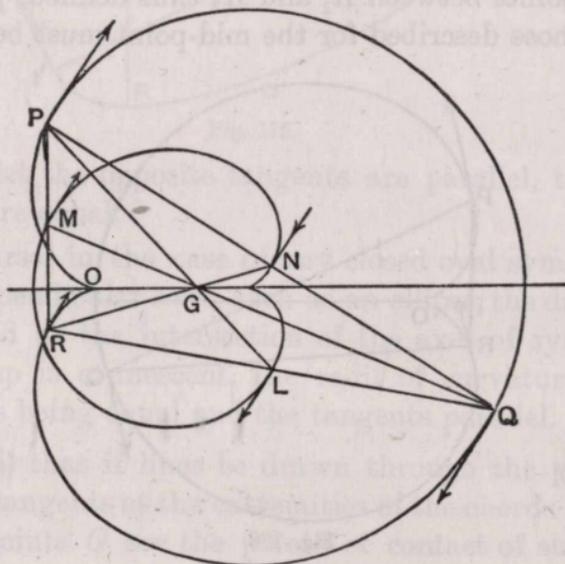


Fig. 177.

(i)  $\therefore \bar{x} = \frac{a}{2}, \bar{y} = 0$ , and the centroid is therefore at a fixed point  $G$  on the axis.

(ii) Let  $PG, QG, RG$  cut the sides of the triangle at  $L, M, N$ . Then, since  $GP=2GL$ , etc., the points  $L, M, N$ , i.e. the mid-points of the sides lie on another cardioide of half the linear dimensions of the former.

(iii) The tangents at  $L, M, N$  to this cardioide are parallel to the tangents to the original cardioide at  $P, Q, R$ .

(iv) The triangle  $PQR$  might have been described as one in which each of the sides subtends an angle  $120^\circ$  at the pole  $O$ .

(v) All other points which divide the sides, or the medians, in a constant ratio, or any points connected with the triangle  $PQR$  by the formulae

$$\xi = \frac{\sum lx}{\sum l}, \quad \eta = \frac{\sum ly}{\sum l},$$

where  $l, m, n$  are either numerical or not dependent upon the magnitude, shape and position of the triangle, also trace cardioides ; and lines through such points parallel to the tangents at  $P, Q, R$ , envelope cardioides.

### 633. Areal and Trilinears.

It has already been explained that such systems are not well adapted for metrical purposes (Art. 460).

We can, however, readily obtain suitable formulae for such cases if necessary.

Denoting the trilinear coordinates of any two points by  $(\alpha_1, \beta_1, \gamma_1), (\alpha_2, \beta_2, \gamma_2)$ , the triangle of reference being some given triangle  $ABC$  of sides  $a, b, c$ , and area  $\Delta$ , the distance between these points is

$$r^2 = -\frac{abc}{4\Delta^2} [a(\beta_1 - \beta_2)(\gamma_1 - \gamma_2) + b(\gamma_1 - \gamma_2)(\alpha_1 - \alpha_2) + c(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)]$$

or  $\frac{abc}{4\Delta^2} [a \cos A (\alpha_1 - \alpha_2)^2 + b \cos B (\beta_1 - \beta_2)^2 + c \cos C (\gamma_1 - \gamma_2)^2]$

(Ferrers' *Trilinears*, p. 6).

Accordingly, the length of an elementary arc  $ds$  between two points  $(\alpha, \beta, \gamma), (\alpha + d\alpha, \beta + d\beta, \gamma + d\gamma)$

may be written either as

$$ds^2 = -\frac{abc}{4\Delta^2} (a d\beta d\gamma + b d\gamma da + c da d\beta)$$

or as  $ds^2 = \frac{abc}{4\Delta^2} (a \cos A da^2 + b \cos B d\beta^2 + c \cos C d\gamma^2)$ ,

where  $aa + b\beta + c\gamma = 2\Delta$ ,

and therefore  $a da + b d\beta + c d\gamma = 0$ .

The corresponding expressions in Areal will obviously be

$$ds^2 = -(a^2 dy dz + b^2 dz dx + c^2 dx dy)$$

or  $ds^2 = bc \cos A dx^2 + ca \cos B dy^2 + ab \cos C dz^2,$

with the identical relations

$$x + y + z = 1, \quad dx + dy + dz = 0.$$

The Areal results are a little the simpler.

### 634. Unicursal Curves.

In the case of a curve being unicursal, *i.e.* such that the coordinates of a point upon it can be expressed as rational functions of some parameter  $t$ , then if we have taken areal coordinates  $x, y, z$ , so that their sum is unity, we may write

$$\frac{x}{f_1(t)} = \frac{y}{f_2(t)} = \frac{z}{f_3(t)} = \frac{1}{f(t)},$$

where

$$f(t) = f_1(t) + f_2(t) + f_3(t).$$

Let these functions be made homogeneous and of the same degree, say the  $n^{\text{th}}$ , by the insertion of a proper power of another letter  $\tau$ , where  $\tau = 1$ .

Then 
$$\frac{dx}{dt} = \frac{f(t)f_1'(t) - f'(t)f_1(t)}{\{f(t)\}^2}.$$

Now, by Euler's Theorem,

$$\begin{aligned} \begin{vmatrix} f(t), & f_1(t) \\ f'(t), & f_1'(t) \end{vmatrix} &= \frac{1}{n} \begin{vmatrix} t \frac{\partial f}{\partial t} + \tau \frac{\partial f}{\partial \tau}, & t \frac{\partial f_1}{\partial t} + \tau \frac{\partial f_1}{\partial \tau} \\ \frac{\partial f}{\partial t}, & \frac{\partial f_1}{\partial t} \end{vmatrix} \\ &= \frac{1}{n} \begin{vmatrix} \frac{\partial f}{\partial \tau}, & \frac{\partial f_1}{\partial \tau} \\ \frac{\partial f}{\partial t}, & \frac{\partial f_1}{\partial t} \end{vmatrix} \\ &= \frac{1}{n} J_1, \end{aligned}$$

where  $J_1$  is the Jacobian of  $f_1$  and  $f$  with regard to  $t$  and  $\tau$ , *i.e.*

$$= \frac{1}{n} \frac{\partial(f_1, f)}{\partial(t, \tau)},$$

and  $\tau$  is to be put  $= 1$  after the differentiations are performed.

Thus 
$$dx = \frac{1}{n} \frac{J_1}{f^2} dt.$$

Similarly 
$$dy = \frac{1}{n} \frac{J_2}{f^2} dt,$$

$$dz = \frac{1}{n} \frac{J_3}{f^2} dt,$$

where  $J_2$  and  $J_3$  are respectively

$$\frac{\partial(f_2, f)}{\partial(t, \tau)}, \quad \frac{\partial(f_3, f)}{\partial(t, \tau)}.$$

Thus the areal formulae for rectification in the case of a unicursal curve become

$$s = \frac{1}{n} \int \sqrt{-\frac{1}{f^4} [a^2 J_2 J_3 + b^2 J_3 J_1 + c^2 J_1 J_2]} dt$$

or 
$$s = \frac{1}{n} \int \sqrt{\frac{1}{f^4} [bc \cos A J_1^2 + ca \cos B J_2^2 + ab \cos C J_3^2]} dt.$$

These simplify a little further in the case where it is possible to take the reference triangle equilateral.

635. Ex. 1. For example, if it be required to apply this method to rectify a circle referred to a pair of tangents inclined at  $60^\circ$  and the chord of contact, the equation is

$$x^2 = yz,$$

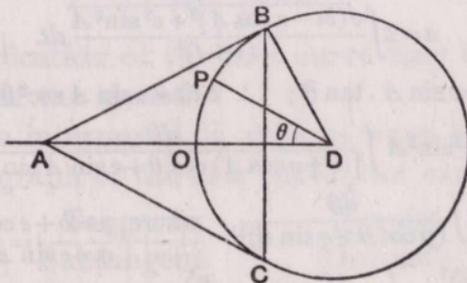


Fig. 178.

and we may put 
$$\frac{x}{t} = \frac{y}{1} = \frac{z}{t^2} = \frac{1}{1+t+t^2},$$

$$\frac{dx}{dt} = \frac{1-t^2}{(1+t+t^2)^2}, \quad \frac{dy}{dt} = -\frac{1+2t}{(1+t+t^2)^2}, \quad \frac{dz}{dt} = \frac{2t+t^2}{(1+t+t^2)^2},$$

$$ds^2 = +\frac{a^2}{2} (dx^2 + dy^2 + dz^2) = a^2 \frac{dt^2}{1+t+t^2};$$

$$\therefore ds = -a \frac{dt}{1+t+t^2}.$$

We take the negative sign, because we measure the arc from  $O$ , where  $t=1$ , the nearest point to  $A$ , and as the current point  $P$  moves from  $O$  towards  $B$  (Fig. 178),  $t$  decreases, i.e.  $s$  increases as  $t$  decreases, i.e.

$$\begin{aligned} s &= \left[ -\frac{2a}{\sqrt{3}} \tan^{-1} \frac{2t+1}{\sqrt{3}} \right]_1^t \\ &= \frac{2a}{\sqrt{3}} \left( \tan^{-1} \sqrt{3} - \tan^{-1} \frac{2t+1}{\sqrt{3}} \right) \\ &= \frac{2a}{\sqrt{3}} \tan^{-1} \left( \frac{1}{\sqrt{3}} \frac{1-t}{1+t} \right). \end{aligned}$$

[Clearly the radius  $= \frac{a}{\sqrt{3}}$ ; hence we can determine the geometrical meaning of the parameter  $t$ , viz.  $t = \frac{1 - \sqrt{3} \tan \frac{1}{2} \widehat{ODP}}{1 + \sqrt{3} \tan \frac{1}{2} \widehat{ODP}}$ .]

Ex. 2. Take as triangle of reference any pair of tangents to a parabola and the chord of contact. The equation of the curve then is

$$x^2 = 4yz,$$

and we may write

$$\frac{x}{2t} = \frac{y}{1} = \frac{z}{t^2} = \frac{1}{(1+t)^2};$$

$$\therefore \frac{dx}{dt} = 2 \frac{1-t}{(1+t)^3}, \quad \frac{dy}{dt} = -\frac{2}{(1+t)^3}, \quad \frac{dz}{dt} = \frac{2t}{(1+t)^3};$$

$$\therefore ds^2 = \frac{4}{(1+t)^6} [bc \cos A (1-t)^2 + ca \cos B + ab \cos C t^2] dt^2$$

$$= \frac{4}{(1+t)^6} (c^2 - 2bc \cos A t + b^2 t^2) dt^2,$$

$$s = 2 \int \frac{\sqrt{(bt - c \cos A)^2 + c^2 \sin^2 A}}{(1+t)^3} dt.$$

Put  $bt - c \cos A = c \sin A \cdot \tan \theta$ ;  $\therefore b dt = c \sin A \sec^2 \theta d\theta$ ;

$$\therefore s = 2b^2 c^2 \sin^2 A \int \frac{d\theta}{[(b + c \cos A) \cos \theta + c \sin A \sin \theta]^3}$$

$$= 8\Delta^2 \int \frac{d\theta}{(p \cos \theta + q \sin \theta)^3}, \quad \text{where } p = b + c \cos A,$$

$$q = c \sin A,$$

$$= \frac{8\Delta^2}{(p^2 + q^2)^{\frac{3}{2}}} \int \sec^3 \left( \theta - \tan^{-1} \frac{q}{p} \right) d\theta$$

$$= \frac{4\Delta^2}{(b^2 + 2bc \cos A + c^2)^{\frac{3}{2}}} \left[ \tan \left( \theta - \tan^{-1} \frac{q}{p} \right) \sec \left( \theta - \tan^{-1} \frac{q}{p} \right) \right.$$

$$\left. + \log \tan \left( \frac{\pi}{4} + \frac{\theta}{2} - \frac{1}{2} \tan^{-1} \frac{q}{p} \right) \right],$$

where  $\frac{q}{p} = \frac{c \sin A}{b + c \cos A}$  and  $\tan \theta = \frac{bt - c \cos A}{c \sin A}$ ,

which, when taken between limits  $t_1, t_2$ , determines the length of the intercepted arc in terms of  $t_1, t_2$  and the elements of the triangle of reference.

636. Connexion between Quadrature and Rectification.

It is perhaps of historical rather than mathematical importance to point out the connexion between the problems of rectification and of quadrature.

If  $y=f(x)$  be the Cartesian equation of the curve to be considered, we shall suppose a new curve to be constructed from

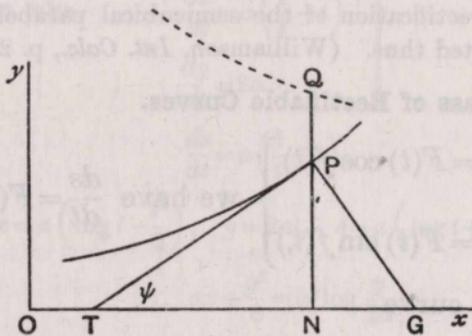


Fig. 179.

it, taking the same abscissa and an ordinate  $\eta = a \sec \psi$ , where  $\psi$  is the slope of the tangent to the original curve and  $a$  is any constant. Then

$$ds = dx \sec \psi = \frac{\eta}{a} dx;$$

$$\therefore as = \int \eta dx.$$

Hence the rectification of the first curve may be regarded as the quadrature of the second.

$\sec \psi$  may be interpreted in various ways to facilitate the drawing of the graph of the new curve; for example,

$$\sec \psi = \frac{\text{Tangent}}{\text{Subtangent}} \quad \text{or} \quad = \frac{\text{Normal}}{\text{Ordinate}}, \text{ etc.}$$

Accordingly, if the ordinates of the original curve be all increased to a length  $\eta$  so that

$$\eta : a = \frac{\text{Tangent}}{\text{Subtangent}} \quad \text{or} \quad \frac{\text{Normal}}{\text{Ordinate}},$$

a new curve will be found for which the area bounded by the new curve, the  $x$ -axis and the terminal ordinates is equal to a rectangle, one side of which is  $a$  and the other side is the corresponding arc of the given curve. Also  $a$ , being at our choice, may be taken as unit length.

637. Ex. If the ordinate of the semicubical parabola  $ay^2 = x^3$  be produced to a length  $\eta$  so that  $\eta = a \frac{\text{Normal}}{\text{Ordinate}}$ , show that the path of the new point thus found is the parabola

$$4\eta^2 = 4a^2 + 9ax.$$

Find the area of a portion of this parabola bounded by two given ordinates, and deduce the result of Ex. 1, Art. 516, for the length of the corresponding arc of the semicubical parabola.

Van Huraet's rectification of the semicubical parabola referred to in Art. 516 was effected thus. (Williamson, *Int. Calc.*, p. 249.)

638. On a Class of Rectifiable Curves.

If  $\left. \begin{aligned} \frac{dx}{dt} &= F(t) \cos f(t) \\ \text{and } \frac{dy}{dt} &= F(t) \sin f(t) \end{aligned} \right\}$  we have  $\frac{ds}{dt} = F(t).$

Hence in the curve

$$\left. \begin{aligned} x &= \int F(t) \cos f(t) dt, \\ y &= \int F(t) \sin f(t) dt, \end{aligned} \right\} \text{ we have } s = \int F(t) dt.$$

The functions  $F(t)$  and  $f(t)$  being at our choice a large number of rectifiable curves arise.

In constructing a rectifiable curve, a common method is to make  $f(t) = n \tan^{-1}t$  and make use of the formulae

$$\cos(n \tan^{-1}t) = \frac{1}{(1+t^2)^{\frac{n}{2}}} (1 - {}^nC_2 t^2 + {}^nC_4 t^4 - \dots),$$

$$\sin(n \tan^{-1}t) = \frac{1}{(1+t^2)^{\frac{n}{2}}} ({}^nC_1 t - {}^nC_3 t^3 + {}^nC_5 t^5 - \dots),$$

and either to choose an even value for  $n$ , or to take  $(1+t^2)^{\frac{n}{2}}$  as one of the factors of  $F(t)$ , if  $n$  be odd, to facilitate integration.

639. Ex. 1. Thus, taking

$$\left. \begin{aligned} \frac{dx}{dt} &= 2t, \\ \frac{dy}{dt} &= 1 - t^2 \end{aligned} \right\} \text{ here } n=2 \text{ and } F(t) = 1+t^2,$$

we have

$$\left. \begin{aligned} x &= t^2, \\ y &= t - \frac{t^3}{3} \end{aligned} \right\}$$

and

$$\frac{ds}{dt} = 1+t^2; \text{ whence } s = t + \frac{t^3}{3}.$$

The curve in question is then

$$y^2 = x \left(1 - \frac{x}{3}\right)^2, \text{ a cubic,}$$

and we have in this curve

$$s^2 = x \left(1 + \frac{x}{3}\right)^2$$

or

$$s^2 - y^2 = \frac{4}{3}x^2.$$

Ex. 2. Let us take

$$\left. \begin{aligned} \frac{dx}{dt} &= a \left(\frac{1}{t} - t\right), \\ \frac{dy}{dt} &= 2a. \end{aligned} \right\}$$

Then

$$\frac{ds}{dt} = a \left(\frac{1}{t} + t\right),$$

$$x = a \left(\log t - \frac{t^2}{2}\right), \quad y = 2at, \quad s = a \left(\log t + \frac{t^2}{2}\right);$$

$$\therefore ax + \frac{y^2}{8} = a^2 \log \frac{y}{2a}$$

is the Cartesian equation of the curve.

Also

$$s - x = \frac{y^2}{4a},$$

and the intrinsic equation is

$$s = \frac{a}{2} \tan^2 \frac{\psi}{2} + a \log \tan \frac{\psi}{2}.$$

Ex. 3. Take

$$\frac{dx}{dt} = \frac{2a}{\sqrt{t}} (1 - t^2)$$

and

$$\frac{dy}{dt} = \frac{2a}{\sqrt{t}} \cdot 2t.$$

Then

$$\left. \begin{aligned} x &= 4a \left(t^{\frac{1}{2}} - \frac{t^{\frac{5}{2}}}{5}\right), \\ y &= \frac{8}{3}at^{\frac{3}{2}}, \end{aligned} \right\}$$

and

$$\frac{ds}{dt} = \frac{2a}{\sqrt{t}} (1 + t^2),$$

$$s = 4a \left(t^{\frac{1}{2}} + \frac{t^{\frac{5}{2}}}{5}\right).$$

Hence  $s^2 - x^2 = \frac{8}{5}y^2$ , and the intrinsic equation is

$$s = 4a \sqrt{\tan \frac{\psi}{2} \left(1 + \frac{1}{5} \tan^2 \frac{\psi}{2}\right)}.$$

Ex. 4. In the curve for which

$$\frac{dx}{dt} = 1 - 6t^2 + t^4,$$

$$\frac{dy}{dt} = 4t(1 - t^2),$$

we have  $\frac{ds}{dt} = 1 + 2t^2 + t^4,$

i.e.  $s = t + \frac{2t^3}{3} + \frac{t^5}{5},$

where  $x = t - 2t^3 + \frac{t^5}{5},$   
 $y = 2t^2 - t^4,$

$$\tan \psi = \frac{dy}{dx} = \tan 4\theta, \text{ if } t \equiv \tan \theta;$$

$$\therefore \theta = \frac{\psi}{4},$$

and the intrinsic equation is

$$s = \tan \frac{\psi}{4} \left( 1 + \frac{2}{3} \tan^2 \frac{\psi}{4} + \frac{1}{5} \tan^4 \frac{\psi}{4} \right),$$

the Cartesian equation being the  $t$ -eliminant from the values of  $x$  and  $y$ .

Several examples of this class of curve will be found in Wolstenholme's *Problems* (No. 1800 onwards).

640. Since  $(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$  we may construct a curve such that

$$x = \int \phi(t) [f_1^2(t) - f_2^2(t)] dt,$$

$$y = 2 \int \phi(t) f_1(t) f_2(t) dt,$$

and then we shall obviously have

$$s = \int \phi(t) [f_1^2(t) + f_2^2(t)] dt,$$

where  $\phi(t), f_1(t), f_2(t)$  are all at our choice. This artifice amounts to a form of the last method.

641. Ex. Let  $\frac{dx}{dt} = at(t^{2p} - t^{2q}),$

$$\frac{dy}{dt} = 2at \cdot t^p t^q.$$

Then  $\frac{ds}{dt} = at(t^{2p} + t^{2q}).$

Hence, for the curve 
$$\left. \begin{aligned} x &= a \left( \frac{t^{2p+2}}{2p+2} - \frac{t^{2q+2}}{2q+2} \right), \\ y &= 2a \frac{t^{p+q+2}}{p+q+2}, \end{aligned} \right\}$$

we have 
$$s = a \left( \frac{t^{2p+2}}{2p+2} + \frac{t^{2q+2}}{2q+2} \right),$$

i.e. 
$$s^2 - x^2 = \frac{(p+q+2)^2}{(p+1)(q+1)} \frac{y^2}{4}.$$

642. **A Theorem by Mr. R. A. Roberts.**

An important transformation may be used in some cases to derive one rectifiable curve from another, as follows :

$$\text{Put } \left. \begin{array}{l} x + iy = u, \\ x - iy = v, \end{array} \right\} \text{ where } i = \sqrt{-1}.$$

$$\text{Then clearly } ds^2 = dx^2 + dy^2 = (dx + i dy)(dx - i dy) \\ = du dv.$$

In cases where the equation of the original curve takes the form

$$\phi(u) \phi(v) = \text{const.}, \text{ say unity,}$$

if another curve be derived from this one by taking

$$u' = \int [\phi(u)]^n du,$$

$$v' = \int [\phi(v)]^n dv,$$

it is plain that

$$du' dv' = [\phi(u)]^n [\phi(v)]^n du dv = du dv,$$

and therefore  $ds'^2 = ds^2$  and  $ds' = ds$ ,

and corresponding arcs will be equal.

The theorem is given by Mr. R. A. Roberts [*Proc. L.M.S.*, vol. xviii.].

643. **Precautions.**

Some circumspection is necessary in the inference to be made as to the whole perimeter of the derived curve. For instance when the point  $P(x, y)$  of the curve, supposed closed, traces out the complete path  $\phi(u) \phi(v) = 1$ , the corresponding point  $P'$  on the derived curve may not trace out the *whole* of the derived curve, or it may trace the derived curve several times. This point must be examined in all cases of application of the theorem.

644. In illustration it will be instructive to consider the most elementary case, viz. that in which the primary curve is the circle  $x^2 + y^2 = a^2$ .

With the proposed transformation, viz.  $x + iy = u$ ,  $x - iy = v$ , we have

$$uv = a^2.$$

Taking the derived curve as

$$u' = \int \frac{u^2}{a^2} du, \quad v' = \int \frac{v^2}{a^2} dv,$$

we get  $ds' = ds$ , and corresponding arcs are equal.

$$\text{Now } u' = \frac{u^3}{3a^2} \text{ gives } x' + iy' = \frac{1}{3a^2} (x + iy)^3.$$

Therefore  $3a^2x' = x^3 - 3xy^2, \dots\dots\dots(1)$

$3a^2y' = 3x^2y - y^3, \dots\dots\dots(2)$

And upon squaring and adding,

$$9a^4(x'^2 + y'^2) = (x^2 + y^2)^3 = a^6.$$

Hence the corresponding locus is the circle

$$x'^2 + y'^2 = \frac{a^2}{9},$$

viz. one of radius  $\frac{a}{3}$ .

The whole perimeters are obviously not equal.

But noticing that if we put  $\frac{y'}{x'} = \tan \theta'$  and  $\frac{y}{x} = \tan \theta$ , we get

$$\tan \theta' = \tan 3\theta, \text{ or } \theta' = 3\theta,$$

and it appears that the derived circle is traced out at *three times* the angular rate of the primary circle, and whilst the point  $P(x, y)$  traces the whole of the primary circle, the derived point  $P'(x', y')$  traces the derived circle thrice, and the circumference of the first, viz.  $2\pi a$ , is thrice the circumference of the second, i.e.  $3 \times \left(\frac{2\pi a}{3}\right)$ .

645. As an illustration of the derivation of a new rectifiable curve by this method, take as primary curve the lemniscate

$$r^2 = a^2 \cos 2\theta,$$

i.e.  $(x^2 + y^2)^2 = a^2(x^2 - y^2),$

i.e.  $u^2v^2 = \frac{a^2}{2}(u^2 + v^2),$

or  $\left(u^2 - \frac{a^2}{2}\right)\left(v^2 - \frac{a^2}{2}\right) = \frac{a^4}{4}.$

Let us derive a new curve from this by putting

$$u' = \frac{2}{a^2} \int \left(u^2 - \frac{a^2}{2}\right) du,$$

$$v' = \frac{2}{a^2} \int \left(v^2 - \frac{a^2}{2}\right) dv,$$

and therefore  $du' dv' = \frac{4}{a^4} \left(u^2 - \frac{a^2}{2}\right) \left(v^2 - \frac{a^2}{2}\right) du dv = du dv;$

whence  $ds' = ds$ , and corresponding arcs are equal.

Now  $u' = \frac{2}{a^2} \left(\frac{u^3}{3} - \frac{a^2}{2}u\right), \quad v' = \frac{2}{a^2} \left(\frac{v^3}{3} - \frac{a^2}{2}v\right),$

i.e.  $\frac{a^2}{2}(x' + iy') = \frac{(x + iy)^3}{3} - \frac{a^2}{2}(x + iy),$

$$\left. \begin{aligned} 3a^2x' &= 2(x^3 - 3xy^2) - 3a^2x, \\ 3a^2y' &= 2(3x^2y - y^3) - 3a^2y, \end{aligned} \right\} \text{ where } (x^2 + y^2)^2 = a^2(x^2 - y^2),$$

which may be written as

$$\left. \begin{aligned} \frac{3x'}{a} &= \sqrt{\cos 2\theta} [\cos 5\theta - 2 \cos \theta], \\ \frac{3y'}{a} &= \sqrt{\cos 2\theta} [\sin 5\theta - 2 \sin \theta], \end{aligned} \right\} \theta \text{ being an arbitrary parameter.}$$

Hence as arcs of a lemniscate can be expressed as elliptic integrals of the first kind, the same is true of this derived curve.

The elimination of  $u$  and  $v$  from the equations

$$u' = \frac{2}{3} \frac{u^3}{a^2} - u, \quad v' = \frac{2}{3} \frac{v^3}{a^2} - v, \quad \left( u^2 - \frac{a^2}{2} \right) \left( v^2 - \frac{a^2}{2} \right) = \frac{a^4}{4}$$

in this example may be performed as follows :

$$\text{Let} \quad u^2 = \frac{a^2}{2} (1 + t^2), \quad v^2 = \frac{a^2}{2} \left( 1 + \frac{1}{t^2} \right).$$

$$\text{Then} \quad 3u' = u(t^2 - 2), \quad 3v' = v \left( \frac{1}{t^2} - 2 \right);$$

$$\therefore 9u'^2 = \frac{a^2}{2} (1 + t^2) (t^2 - 2)^2, \quad 9v'^2 = \frac{a^2}{2} \left( 1 + \frac{1}{t^2} \right) \left( \frac{1}{t^2} - 2 \right)^2;$$

$$\therefore \frac{18u'^2}{a^2} - 4 = t^6 - 3t^4, \quad \frac{18v'^2}{a^2} - 4 = \frac{1}{t^6} - \frac{3}{t^4}$$

$$= A, \text{ say}, \quad = B, \text{ say.}$$

$$\text{Then} \quad t^2 + \frac{1}{t^2} = \frac{10 - AB}{3} = p, \text{ say};$$

$$\therefore t^4 + \frac{1}{t^4} = p^2 - 2, \quad t^6 + \frac{1}{t^6} = p^3 - 3p;$$

$$\therefore A + B = (p^3 - 3p) - 3(p^2 - 2);$$

$$\therefore A + B - 5 = (p + 1)(p^2 - 4p + 1),$$

$$27(A + B - 5) = (13 - AB)(A^2 B^2 - 8AB - 11);$$

$\therefore A^3 B^3 - 21A^2 B^2 + 93AB + 27(A + B) + 8 = 0$  is the locus required, where

$$A + B = \frac{36}{a^2} (x'^2 - y'^2) - 8,$$

$$AB = \frac{324}{a^4} (x'^2 + y'^2)^2 - \frac{144}{a^2} (x'^2 - y'^2) + 16.$$

The desired curve is therefore one of the 12th degree, and its arcs are of the same length as corresponding arcs on Bernoulli's lemniscate.

#### 646. Serret's Mode of Derivation of Rectifiable Curves.

M. Serret (*Calcul Intégral*, p. 252) indicates a process by means of which algebraic curves can be produced which are rectifiable in terms of arcs of a circle, *i.e.* without the aid of the elliptic functions. Let  $\iota \equiv \sqrt{-1}$ .

Taking  $\iota$  and  $-\iota$ ,  $a$  and  $\alpha$ ,  $b$  and  $\beta$ ,  $c$  and  $\gamma$ , etc., to be  $k$  pairs of conjugate constant complex quantities,  $C$  any real

constant quantity, and  $\omega$  a real constant angle, and  $m, n, p, q$ , etc., positive integers, and putting

$$t = (z-a)^{n+1}(z-b)^{p+1}(z-c)^{q+1}\dots,$$

$$T = (z-a)^{n+1}(z-\beta)^{p+1}(z-\gamma)^{q+1}\dots,$$

he states that the proposed problem is answered by the formula

$$x + iy = Ce^{\omega} \int \frac{t}{T} \frac{(z-i)^m}{(z+i)^{m+2}} dz, \dots\dots\dots(1)$$

provided the  $k-1$  pairs of constants  $(a, \alpha), (b, \beta)$ , etc., be chosen so as to make the result of integration algebraic. As there are  $k$  repeated factors in the denominator of the integrand, this will entail the satisfying of  $k-1$  independent conditions (Art. 149), for the degree of the denominator is greater by 2 than the degree of the numerator.

To see the truth of M. Serret's assertion, we observe that

$$dx + i dy = Ce^{\omega} \frac{t}{T} \frac{(z-i)^m}{(z+i)^{m+2}} dz;$$

$$\therefore dx - i dy = Ce^{-\omega} \frac{T}{t} \frac{(z+i)^m}{(z-i)^{m+2}} dz.$$

Hence 
$$ds^2 = dx^2 + dy^2 = C^2 \frac{dz^2}{(1+z^2)^2}$$

and 
$$ds = C \frac{dz}{1+z^2},$$

giving 
$$s = C \tan^{-1}z. \dots\dots\dots(2)$$

647. M. Serret discusses a slightly different form in Liouville's *Journal*, vol. x.,\* viz.

$$x + iy = Ce^{\omega} \int \frac{(z-a)^m(z+a)^n}{(z-a)^{m+1}(z+a)^{n+1}} dz. \dots\dots(3)$$

Here 
$$dx + i dy = Ce^{\omega} \frac{(z-a)^m(z+a)^n}{(z-a)^{m+1}(z+a)^{n+1}} dz,$$

$$dx - i dy = Ce^{-\omega} \frac{(z-a)^m(z+a)^n}{(z-a)^{m+1}(z+a)^{n+1}} dz;$$

whence 
$$ds^2 = dx^2 + dy^2 = C^2 \frac{dz^2}{(z^2-a^2)(z^2-\alpha^2)}$$

and 
$$s = C \int \frac{dz}{\sqrt{(z^2-a^2)(z^2-\alpha^2)}},$$

a form readily made to depend upon an elliptic integral.

\*See also *Lond. Math. Soc. Proc.*, vol. xviii.; Mr. R. A. Roberts; and Cayley, *Ell. Funct.*, Art. 448 (where the  $Ce^{\omega}$  is omitted).

In the equation (3), the denominator is still in degree higher by 2 than the degree of the numerator, and there are two repeated factors in the denominator; hence one condition only is necessary that the resulting rectifiable curve should be purely algebraic (Art. 149). The integral (3) is not in all cases obtainable, but if one of the indices, say  $m$ , be a positive integer and if the equation of condition be satisfied, the integration can be effected in terms of  $z$ , involving complex constants. Then, equating real and imaginary parts,  $x$  and  $y$  can be found, and when  $z$  has been eliminated the Cartesian form of the equation of the derived curve will result.

648. The Equation of Condition.

The form of the conditional equation is very remarkable, viz. taking

$$\xi = \frac{(a + \iota)^2}{4a\iota},$$

it is 
$$\frac{1}{\xi^{n-m}} \left( \frac{d}{d\xi} \right)^m \xi^n (\xi - 1)^m = 0.$$

This is discussed at length by Cayley, chap. xv., *Ell. Funct.*, to which we must refer the advanced student for the work.

MISCELLANEOUS PROBLEMS.

1. Show that any point on the Lemniscate  $r^2 = a^2 \cos 2\theta$  may be represented by

$$x = a \frac{z + z^3}{1 + z^4}, \quad y = a \frac{z - z^3}{1 + z^4},$$

and hence obtain the rectification of the curve. [SERRET.]

Show that the integral obtained for  $s$  reduces to the standard Legendrian form by the further substitution

$$\cos \phi = \frac{z\sqrt{2}}{\sqrt{1+z^4}}.$$

[CAYLEY, *Eu. Functions*, Art. 63.]

2. By the transformation  $\frac{z - \iota}{z + \iota} = \frac{a - \iota}{a + \iota} u$ ,

show that the equation

$$x + iy = Ce^{\omega} \int \frac{(z - a)^{n+1} (z - \iota)^m}{(z - a)^{n+1} (z + \iota)^{m+2}} dz$$

takes the form  $x + iy = A \int \frac{u^m (u - 1)^{n+1}}{(u - \xi)^{n+1}} du$ ,

where  $\xi = \frac{(a + \iota)(a - \iota)}{(a - \iota)(a + \iota)}$ ,  $A = \frac{C}{2\iota} e^{\omega} \left( \frac{a + \iota}{a + \iota} \right)^{n+1} \left( \frac{a - \iota}{a + \iota} \right)^{m+1}$

Hence show that the condition that  $x + iy$  should be purely algebraic is

$$\frac{d^n}{d\xi^n} \xi^m (\xi - 1)^{n+1} = 0,$$

$a$  and  $a$  being supposed conjugate, and  $m, n$  positive integers.

Discuss the roots of this equation. [SERRET, *Calc. Intég.*, p. 254.]

3. In Bernoulli's Lemniscate  $r^2 = 2a^2 \cos 2\theta$ ,

show that if  $x + iy = u$  and  $x - iy = v$ ,

the equation of the curve may be written

$$(u^2 - a^2)(v^2 - a^2) = a^4.$$

Further, expressing  $u^2$  and  $v^2$  as  $a^2(1 + t^2)$  and  $a^2\left(1 + \frac{1}{t^2}\right)$  respectively, show that the tangent of the angle which the tangent at any point makes with the  $x$ -axis is

$$\frac{1 + t^3}{1 - t^3}.$$

Hence, putting the coordinates of two points at which the tangents are parallel, as  $\omega\mu, \omega^2\mu$  where  $\omega^3 = 1$ , show that the locus of the mid-points of chords joining such points is

$$\begin{aligned} & [16u^2v^2 - 8a^2(u^2 + v^2) + 3a^4]^2 \\ & = 4a^4[16\{u^4 + v^4 - u^2v^2\} - 12a^2(u^2 + v^2) + 9a^4], \end{aligned}$$

i.e. a curve of the eighth degree.

[R. A. ROBERTS, *Proc. L.M. Soc.*, vol. xviii.]

4. Obtain an integral for the rectification of the inverse of the parabola  $y^2 = 4ax$ , with regard to a point on the axis whose coordinates are  $(h, 0)$ .

If  $h = -3a$ , show that

$$s = \frac{1}{6a\sqrt{2}} \log \frac{3 + 2\sqrt{2} \sin \omega}{3 - 2\sqrt{2} \sin \omega},$$

where  $a \tan^2 \omega, 2a \tan \omega$  are taken as the current coordinates of a point on the parabola, and the arc of the inverse is measured from the point corresponding to the vertex of the parabola.

[MR. ROBERTS, *loc. cit.*]

Show that the semiperimeter is bisected at the point  $\omega = \sin^{-1} \frac{3}{4}$ .

5. Show that the tangents to the parabola  $y^2 = 4a(x + a)$  at the points

$$\{a \sinh^2(u \pm v) - a, 2a \sinh(u \pm v)\},$$

where  $u$  is variable but  $v$  is a constant, intersect on a confocal parabola; and that if  $T$  be a point on this second parabola, and  $TP_1, TP_2$  the tangents to the first, then

$$TP_1 + TP_2 - \text{arc } P_1P_2 = a(\sinh 2v - 2v),$$

and is constant.

[OXFORD I. P., 1911.]

6. Show that  $\iint \frac{dA}{r}$ , taken over the area cut from a parabola of latus rectum  $4a$  by an ordinate distant  $c$  from the vertex ( $c < a$ ), where  $r$  denotes the distance from the focus, is equal to

$$4\sqrt{ac} - 2(a - c) \log \frac{\sqrt{a} + \sqrt{c}}{\sqrt{a} - \sqrt{c}}. \quad [\text{OXFORD I. P., 1911.}]$$

7. Show that  $\int_0^{\frac{\pi}{4}} \frac{\sin 15\theta}{\sin \theta} d\theta = \frac{\pi}{4} + \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$ .

8. If

$$u = e^{-\int \phi dx} \int e^{\int \phi dx} (c_0 + c_1x + c_2x^2 + \dots + c_nx^n) f dx + Ce^{-\int \phi dx},$$

where  $c_0, c_1, c_2, \dots, c_n, C$  are  $(n + 2)$  arbitrary constants, and

$$\frac{\phi}{f} = a_0 + a_1x + a_2x^2 + \dots + a_mx^m,$$

where  $a_0, a_1, \dots, a_m$  are  $(m + 1)$  given constants, show that if  $m$  be not greater than  $n$ ,  $\frac{du}{dx}$ , obtained by the direct differentiation of  $u$  with regard to  $x$ , contains only  $(n + 1)$  arbitrary constants.

[MATH. TRIPOS, 1878.]

9. If  $f(m, n) = \int_0^\infty x^m (\cosh x)^{-n} dx$ , where  $m$  and  $n$  are positive integers, each greater than 2, prove that

$$(n - 1)(n - 2)f(m, n) = (n - 2)^2 f(m, n - 2) - m(m - 1)f(m - 2, n - 2).$$

[OXFORD I. P., 1914.]

10. Given that  $a$  and  $c$  are positive, show that the limit when  $m \rightarrow \infty$  and  $n \rightarrow \infty$  of

$$\frac{1}{n} \left[ \frac{1}{a^r} + \frac{1}{\left(a + \frac{c}{n}\right)^r} + \frac{1}{\left(a + \frac{2c}{n}\right)^r} + \frac{1}{\left(a + \frac{3c}{n}\right)^r} + \dots + \frac{1}{\left(a + mn \frac{c}{n}\right)^r} \right]$$

is finite when  $r > 1$ ; and find this limit.

[OXF. I. P., 1914.]

11. The increase  $dS$  in a man's satisfaction  $S$  by an increased expenditure  $dx$  on a certain commodity, is expressed by the law

$$dS = \frac{\lambda}{x - a} dx. \quad \text{Similar laws, viz.}$$

$$dS = \frac{\mu}{y - b} dy, \quad dS = \frac{\nu}{z - c} dz,$$

hold for two other commodities, where  $\lambda, \mu, \nu, a, b, c$  are all positive. Find how the man should expend a given sum  $E$  ( $> a + b + c$ ) so that his total satisfaction is greatest.

[OXFORD I. P., 1914.]

Show that the maximum satisfaction is measured by

$$S = \log \frac{\lambda^\lambda \mu^\mu \nu^\nu (E - a - b - c)^{\lambda + \mu + \nu}}{(\lambda + \mu + \nu)^{\lambda + \mu + \nu}}.$$

12. Evaluate

$$\int_0^\infty \frac{x-1}{x^2 - 2x \cos v + 1} \frac{dx}{\sqrt{x^2 - 2x \cosh u + 1}}.$$

[OXFORD II. P., 1914.]

13. Show that the tangent to the curve

$$3a^2(y-x) + x^3 = 0,$$

at the point whose abscissa is  $h$ , cuts the curve again at the point whose abscissa is  $-2h$ , and that the area included between the curve and the tangent is  $9h^4/4a^2$ .

[OXF. I. P., 1918.]

14. If  $f_1(x)$  and  $f_2(x)$  are both polynomials in  $x$ , show that the integral of  $f_1(x)/f_2(x)$  with respect to  $x$  can always be written in the form

$$\phi_1(x)/\phi_2(x) + \log \phi_3(x)/\phi_4(x),$$

where  $\phi_1, \phi_2, \phi_3, \phi_4$  also denote polynomials, not necessarily real.

Find the general form of the integral with respect to  $x$  of

$$f_1(x + \sqrt{x^2 - 1})/f_2(x - \sqrt{x^2 - 1}). \quad [\text{OXF. I. P., 1918.}]$$

15. Show that the area bounded by the curve

$$x = \frac{3at^2}{1+t^3}, \quad y = \frac{3at}{1+t^3},$$

its real asymptote  $x + y + a = 0$ , and by two lines at right angles to this asymptote through the points  $t = -a$ ,  $t = 0$  of the curve, is

$$\frac{3a^2}{4} \left\{ 1 + \frac{u^4 - 1}{(u^2 + u + 1)^2} \right\},$$

and find the whole area between the curve and its real asymptote.

[OXF. I. P., 1917.]

16. If  $\phi(z)$  be a rational function of  $z$  without singularities in the range  $0 \leq z \leq 1$ , prove that

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \phi(\sin 2x) \cos^2 x \cos 2x \, dx &= \int_0^{\frac{\pi}{2}} \phi(\sin 2x) \cos^2 x \cos^2 2x \, dx \\ &= \int_0^{\frac{\pi}{2}} \phi(\sin 2x) \cos^4 x \cos 2x \, dx. \end{aligned}$$

[OXFORD I. P., 1907.]

17. Integrate (i)  $\int \frac{x(x-a)^{b-1} dx}{(x-b)\{(x-b)^{2b} - (x-a)^{2a}\}^{\frac{1}{2}}}$ ,

(ii)  $\int \frac{x^{q-1}\{px^{p+q} - qa^{p+q}\} dx}{(x^{p+q} + a^{p+q})^2 + x^{2q}a^{2q}}.$

18. In the curve  $\frac{15x}{a} = \left(\frac{2y}{a}\right)^{\frac{3}{4}} - \left(\frac{2y}{a}\right)^{\frac{5}{4}}$ , show that  $s^2 = x^2 + \frac{1}{15}y^2$ ,  $s$  being measured from the origin.

Show that the curve is a quintic of which the  $y$ -axis is an axis of symmetry, and that the area of the loop  $= \left(\frac{8}{189}\right) \left(\frac{5}{3}\right)^{\frac{7}{2}} a^2$ .

19. If  $2\phi$  be the eccentric angle of the point  $r, \theta$  on the ellipse  $c = r(1 - e \cos \theta)$ , prove that

$$\{(1+e)^2 - 4e \sin^2 \phi\} \left(\frac{d\theta}{d\phi}\right)^2 = 4(1 - e^2 \cos^2 \theta).$$

Use the fact that

$$\int_0^\pi F(\cos^2 \theta) d\theta = 2 \int_0^{\frac{\pi}{2}} F(\cos^2 \theta) d\theta$$

and the above to obtain a value of  $a$ , such that

$$\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{(1+e)^2 - 4e \sin^2 \phi}} = 2 \int_0^a \frac{d\phi}{\sqrt{(1+e)^2 - 4e \sin^2 \phi}}.$$

[OXFORD I. P., 1917.]

20. A uniform rod of mass  $M$  has its extremities at the points  $x_1, y_1; x_2, y_2$ . Show that the product of inertia of the rod with respect to the axes is given by

$$M \int_0^1 \{tx_2 + (1-t)x_1\} \{ty_2 + (1-t)y_1\} dt.$$

Hence show that the product of inertia of the rod is the same as that of three particles of masses

$$\frac{M}{6}, \quad \frac{M}{6}, \quad \frac{2M}{3},$$

placed at the extremities and the middle point of the rod respectively.

[OXFORD I. P., 1913.]

21. Show that the coordinates of any point on the curve whose intrinsic equation is  $s = a(\sec^n \psi - 1)$ ,

where  $n$  is an odd integer greater than unity, can be expressed rationally in terms of  $\tan \psi$ , and show that when  $x=0$  the curve is a cubic with a cusp. [OXF. I. P., 1911.]

22. Show how to evaluate the integral  $\int f(x, y) dx$ , where

$$y^2 = ax^2 + 2bx + c$$

and  $f(x, y)$  is a rational function of  $x$  and  $y$ .

Prove that

$$(i) \int_0^a \frac{dx}{x + \sqrt{a^2 - x^2}} = \frac{1}{2}\pi,$$

$$(ii) \int_0^a \frac{a dx}{(x + \sqrt{a^2 - x^2})^2} = \frac{1}{\sqrt{2}} \log(1 + \sqrt{2}),$$

the positive sign being taken for the radical in each of the subjects of integration. [MATH. TRIP., PART II., 1913.]

23. Show by means of the transformation  $y = \frac{(x^2 + 1)^{\frac{1}{2}}}{x + 1}$  that

$$\int_0^{\infty} \frac{dx}{(x+1)(x^2+1)^{\frac{1}{2}}} = 2 \int_{1/\sqrt{2}}^1 \frac{dy}{(2y^2-1)^{\frac{1}{2}}} = \sqrt{2} \log(\sqrt{2} + 1),$$

and verify the result in an independent manner.

[MATH. TRIP., PART II., 1914.]

24. Integrate  $\int \frac{\sin x}{\sin(x-a)} dx$ .

[MATH. TRIP., PART II., 1914.]

25. Evaluate

$$\int \frac{x+2}{(x+1)^2(x^2+4)}, \quad \int \frac{dx}{(x^2+1)^4}, \quad \int \frac{dx}{(5-3\cos x)^2},$$

and the corresponding definite integrals taken between the limits  $(0, \infty)$ ,  $(0, \infty)$  and  $(0, \pi)$  respectively. [MATH. TRIP., PART II., 1914.]

26. Show that

$$(i) \int \frac{\sin 4x}{\sin 6x} dx = \frac{\sqrt{3}}{6} \tanh^{-1} \left( \frac{\sin 2x}{\cos \frac{\pi}{6}} \right).$$

$$(ii) \int \frac{\sin 3x}{\sin 5x} dx = \frac{1}{5} \left[ \sin \frac{\pi}{5} \log \frac{\sin \left( \frac{2\pi}{5} + x \right)}{\sin \left( \frac{2\pi}{5} - x \right)} + \sin \frac{2\pi}{5} \log \frac{\sin \left( \frac{\pi}{5} + x \right)}{\sin \left( \frac{\pi}{5} - x \right)} \right].$$

27. Prove that

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 + \sin^2 \theta)(2 + \sin^2 \theta)} = \frac{\pi}{\sqrt{3}} \sin \frac{\pi}{12}.$$

28. Prove that

$$\int \frac{2 \cos \theta + \sin \theta}{(1 + \sin \theta \cos \theta)^{\frac{3}{2}}} d\theta = \frac{2 \sin \theta}{(1 + \sin \theta \cos \theta)^{\frac{1}{2}}}.$$