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The mathematical yield and/or fracture conditions of elastoplastic solids

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A RATE-TYPE elastoplastic constitutive equation which may be thought to be a kind of extension of the hypoelastic one is represented by the terms of the invariants and the basic products of the tensor variables involved in it, i.e. stress $\boldsymbol{\sigma}$, plastic strain $\boldsymbol{\epsilon}$, strain-rate $\mathbf{D}$ and the density tensor of internal defects $\boldsymbol{\rho}$. Then the conditions of constitutive instability are examined in detail to lead to some equations which are, according to the reference situation, interpreted to be initial or subsequent yield conditions and/or fracture conditions. Specifically, when co-axiality between the tensor variables holds, the conditions are decomposed into two parts, i.e. normal and shear types, and involve some simple forms such as Yoshimura's yield function and an extended Tresca function. The general form of the subsequent yield condition established here would offer us a powerful clue to determine the concrete forms of such condition. The fracture condition is expressed by the terms of $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$ and the direction of the stress- or strain-rate vector. The last term is a new proposition on the so-called fracture criterion and implies that an abrupt change of the loading (or straining) condition would induce fracture of the material which could otherwise continue to deform stably or could prolong the life of the material which would otherwise cease to deform stably.

Równanie konstytutywne dla ciała sprężysto-plastycznego typu prędkościowego, stanowiące np. uogólnienie równania hyposprężystości, przedstawiono w postaci wiążącej niezmienniki i podstawowe iloczyny zmiennych tesnorowych, tzn. naprężenia $\boldsymbol{\sigma}$, odkształcenia plastycznego $\boldsymbol{\epsilon}$, prędkości odkształcenia $\mathbf{D}$ i tensora gęstości defektów wewnętrznych $\boldsymbol{\rho}$. Następnie zbadano szczegółowo warunki niestateczności materiału, by wyprowadzić równania, które w zależności od konfiguracji odniesienia mogą być interpretowane jako początkowy lub kolejne warunki plastyczności i/lub warunki zniszczenia. W szczególności, gdy zachodzi współosiowość zmiennych tensorowych, warunki te rozpadają się na dwie części, tzn. warunki typu naprężenia normalnego i ścinania, przyjmując proste formy np. warunku plastyczności Yoshimury i uogólnionego warunku Treski. Ogólna postać wyprowadzonego tu warunku na kolejne powierzchnie płynięcia stwarza nam ogromną możliwość wyznaczenia konkretnych postaci takiego warunku. Warunek zniszczenia wyrażony jest przez $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$ i kierunek wektora naprężenia lub prędkości odkształcenia. Prędkość odkształcenia jest nową propozycją w tak zwanym kryterium zniszczenia i implikuje wniosek, że raptowna zmiana warunków obciążenia (lub odkształcenia) mogłaby spowodować zniszczenie materiału, który w innych warunkach mógłby dalej odkształcać się w sposób stateczny, lub też mogłaby przedłużyć istnienie materiału, który w innych warunkach przestałby się deformować statecznie.

Определяющее уравнение для упруго-пластического тела скоростного типа, составляющее, например, обобщение уравнения гипоупругости, представлено в виде связывающем инварианты и основные произведения тензорных переменных т. зн. напряжения $\boldsymbol{\sigma}$, пластической деформации $\boldsymbol{\epsilon}$, скорости деформации $\mathbf{D}$ и тензора плотности внутренних дефектов $\boldsymbol{\rho}$. Затем исследованы подробно условия неустойчивости материала, чтобы вывести уравнения, которые в зависимости от конфигурации отсчета могут быть интерпретированы как начальное или последовательные условия пластичности и/или условия разрушения. В частности, когда имеет место соосность тензорных переменных, эти условия распадаются на две части т. наз. условия типа нормального напряжения и сдвига, принимая простые формы, например условия пластичности Иошимура и обобщенного условия Треска. Общий вид выведенного здесь условия для последовательных поверхностей течения создает нам огромную возможность определения конкретного вида такого условия. Условие разрушения выражается через $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$ и направление вектора напряжения или скорости деформации. Скорость деформации является новым

предложением в так называемом критерии разрушения и вызывает вывод, что внезапное изменение условий нагружения (или деформации) могло бы вызвать разрушение материала, который в других условиях мог бы дальше деформироваться устойчивым образом, или же мог бы продлить существование материала, который в других условиях перестал бы деформироваться устойчивым образом.

Notations

$\boldsymbol{\sigma}$ Euler stress tensor,
σ^{ij} components of $\boldsymbol{\sigma}$,
σ_m principal components of $\boldsymbol{\sigma}$,
$\hat{\boldsymbol{\sigma}}$ vectorial expression of $\boldsymbol{\sigma}$,
$\mathbf{D}$ stretching tensor or Euler strain-rate tensor,
D_{ij} components of $\mathbf{D}$,
D_m principal components of $\mathbf{D}$,
$\hat{\mathbf{D}}$ vectorial expression of $\mathbf{D}$,
$\mathbf{C}^p$ elastoplastic coefficient tensor of the fourth rank,
$\hat{\mathbf{C}}^p$ (6×6)-matrix expression of $\mathbf{C}^p$,
$\mathring{*}$ co-rotational (or Jaumann) rate of $*$,
$\dot{*}$ material time derivatives of $*$,
$\boldsymbol{\epsilon}$ total plastic strain,
$\boldsymbol{\rho}$ symmetric part of density tensor of internal defect $\boldsymbol{\rho}^*$,
$\mathbf{Q}$ arbitrary orthogonal tensor,
$\mathrm{tr}*$ trace of tensor $*$, i.e. $= (*)_{ii}$,
$:$ two indices, product of tensors, e.g. $[\mathbf{C}:\mathbf{D}]^{ij} = C^{ijkl}D_{kl}$,
$\mathbf{1}$ unit tensor of rank 2,
$\mathbf{S}_1+\mathbf{S}_2$ linear decomposition of the function of $\mathring{\boldsymbol{\sigma}}$. $\mathbf{S}_1$ does not contain any terms of $\boldsymbol{\rho}$ whereas $\mathbf{S}_2$ does as well as $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$,
$\mathbf{C}_1, \mathbf{C}_2$ fourth-rank tensors in the relation $\mathbf{S}_1 = \mathbf{C}_1:\mathbf{D}$ and $\mathbf{S}_2 = \mathbf{C}_2:\mathbf{D}$, respectively,
$\hat{\mathbf{C}}_1, \hat{\mathbf{C}}_2$ (6×6)-matrix expressions of $\mathbf{C}_1$ and $\mathbf{C}_2$, respectively,
$\mathbf{C}_{1\sigma}+\mathbf{C}_{1\epsilon}$ linear decomposition of $\mathbf{C}_1 \cdot \mathbf{C}_{1\sigma}$ does not have any terms of $\boldsymbol{\rho}$ whereas $\mathbf{C}_{1\epsilon}$ does as well as $\boldsymbol{\rho}$,
A^p_{ik}, B^p_{ij} coefficients in a decomposition of $\mathbf{C}_{1p}$,
$\mathbf{G}$ metric tensor,
δ_{ij} Kronecker delta,
$\mathbf{H}^p_N, \mathbf{H}^p_S$ (3×3)-matrices occupying the upper and lower diagonal places in $\mathbf{C}_1$ or $\mathbf{C}_2$ for the case when co-axiality between variables holds,
$H^{(i)}_N$ proper numbers of the characteristic equations of $\mathbf{H}_N$,
$\mathbf{H}^{\sigma\epsilon}_N, \mathbf{H}^{\sigma\epsilon}_S$ $= \mathbf{H}^\sigma_N+\mathbf{H}^\epsilon_N$ and $\mathbf{H}^\sigma_S+\mathbf{H}^\epsilon_S$, respectively,
$\det|*|$ determinant of $*$,
s_i principal deviatoric stress,
$\alpha_i, \beta_i, \gamma_i, \delta_i, \lambda_i, \mu_i, \nu_i, \omega_i$ material constants or scalar functions of the invariants of variables involved,
$\mathbf{C}^*_2$ fourth-rank tensor in the relation $\mathbf{S}_2 = \mathbf{C}^*_2:\boldsymbol{\rho}$,
$\hat{\mathbf{C}}^*_2$ (6×6)-matrix expression of $\mathbf{C}^*_2$,
$A_{ij}, B_{ij}, C_{ij}, F_{ij}, G_{ij}, H_{ij}$ coefficients in $\mathbf{C}^*_2$ and $\mathbf{E}^*_2$,
$\mathbf{T}_1+\mathbf{T}_2$ $= \mathbf{E}^p:\mathring{\boldsymbol{\sigma}} = \mathbf{D}, \mathbf{T}_1$ does not contain any terms of $\boldsymbol{\rho}$ whereas $\mathbf{T}_2$ does,
$\mathbf{E}^*_2$ fourth-rank tensor in $\mathbf{T}_2 = \mathbf{E}^*_2:\boldsymbol{\rho}$,
$\hat{\mathbf{E}}^*_2$ (6×6)-matrix expression of $\mathbf{E}^*_2$.

1. Introduction

THE YIELD conditions or the fracture conditions of elastoplastic materials have been given intuitively, experimentally, a priori or from the view point of energy balance. The classical Tresca and Mises yield conditions and the recent PRAGER'S kinematical yield condition [1] and the like are such examples altogether. REINER'S yield condition for viscoelastic materials [2] and the like are based on energy considerations. And there are also many works on the yield condition from the crystallographic view points, such as the well-known BISHOP and HILL'S work [3]. Studies based on the dislocation theory have also been developed (e.g. KRÖNER [4]).

On the other hand, the fracture conditions have been established almost individually according to fracture type. For example, in brittle fracture the classical Griffith's theory [5] which derives a fracture criterion from kinetic and energy balance considerations on elastic materials with pre-existing cracks and the recent fracture mechanics (see e.g. SIH and KASSIER [6]) which seems to be based on the idea like Griffith's are often referred to by the workers in the field of structural mechanics. The conditions of fatigue fracture are also derived from a similar point of view. As for the ductile fracture condition, there exist many theories based on the plasticity theory and plastic kinematical instability conditions like the bifurcation phenomenon with regard to the materials with pre-existing voids or cavities (e.g. MCCLINTOCK [7] and RICE and TRACEY [8]). In atomic scale, interactions between dislocations and cracks, voids or inclusions are discussed to derive the fundamental fracture conditions (e.g. GOTOH [9] and YOKOBORI [10]).

As for the yield condition, recently TOKUOKA [11] has given a condition derived mathematically from a consideration on the constitutive instability of hypoelastic materials which includes the Tresca and Mises yield functions as special cases.

In this paper we deduce various conditions of the constitutive instability of rate-type elastoplastic materials whose constitutive equation is introduced by the author. This seems to be an extension of the idea of hypoelasticity established by TRUESDELL [12], making use of the representation theorems for isotropic functions as Tokuoka did. They may be understood to be subsequent yield conditions or fracture ones according to the reference situation. The results contain a general form of the subsequent yield condition which nowadays attracts a great deal of attention of workers in the field of the plasticity theory (see e.g. [13]). It is expected that this condition will give us a powerful clue to establish rationally the concrete form of such a condition which should be explored according to the circumferences encountered by workers. The results also contain a new proposition on the so-called fracture criterion which involves the terms of the stress- or strain-rate vector and implies that an abrupt change of the external condition may induce fracture of the material which could otherwise continue to deform stably or prolong the life of the material which would otherwise cease to deform by fracture. The motivation of this idea on the fracture criterion stems from the author's experiences with sheet metal formability in press working in which various types of fracture phenomenon of sheet metals play an important role [14], [15].

Here we should understand the word "fracture" to be that in a macroscopic scale or a state where crack-formation (i.e. a loss of material continuity) originates in a microscopic scale. The representation theorems for isotropic functions proposed by SPENCER and RIVLIN [16] and WANG [17] are adopted.

2. Fundamental statements

2.1. The elastoplastic constitutive equation of rate-type

The elastoplastic constitutive equation adopted in this paper has the following form of rate-type [18]:

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{C}^p : \mathbf{D} \quad \text{or} \quad \hat{\overset{\circ}{\boldsymbol{\sigma}}} = \hat{\mathbf{C}}^p \hat{\mathbf{D}}, \tag{1.1}$$

where $\boldsymbol{\sigma}$ is Euler stress tensor, $\mathbf{D}$ is the stretching tensor or Euler strain-rate tensor, $^\circ$ denotes the co-rotational (Jaumann) rate [19], i.e. $\overset{\circ}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \mathbf{w}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{w}$, where $\mathbf{w}$ is the spin tensor and $\cdot$ denotes material time derivatives, $\hat{\boldsymbol{\sigma}}$ is a (column) vectorial expression of $\boldsymbol{\sigma}$, i.e. $\hat{\boldsymbol{\sigma}} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}]^T$, where T means the transpose and σ_{ij} (or σ^{ij}) are the components of $\boldsymbol{\sigma}$, similarly $\hat{\mathbf{D}} = [D_{11}, D_{22}, D_{22}, 2D_{23}, 2D_{31}, 2D_{12}]^T$ and $\hat{\mathbf{C}}^p$ is a (6×6)-positive definite matrix obtained by rearrangement of the elements of the coefficient tensor $\mathbf{C}^p$ which is of rank 4. Here we consider only isothermal deformations. We call Eq. $(1.1)_1$ the constitutive equation of tensorial expression and denote it by CET and Eq. $(1.1)_2$ that of vectorial expression denoted by CEV.

The constitutive equation (1.1) may be understood as an extension of the hypoelastic one introduced by TRUESDELL [12] to elastoplasticity and is derived by the author from the view point of irreversible thermodynamics, see GOTOH [18], although we can find similar expressions in several papers based on the conventional plasticity theory (e.g. HIBBITT et al [20]).

In Eq. (1.1), $\mathbf{D}$ can be decomposed into elastic and plastic parts as follows [18]:

$$\mathbf{D} = \operatorname{sym}(\dot{\mathbf{F}}_{(t)}\mathbf{F}_{(t)}^{-1}) = \operatorname{sym}(\dot{\mathbf{F}}_{(t)}) = \mathbf{D}^e + \mathbf{D}^p = \dot{\boldsymbol{\epsilon}}^e + \dot{\boldsymbol{\epsilon}}^p, \tag{1.2}$$

where $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$, $\mathbf{x}$ — Euler coordinates, $\mathbf{X}$ — the referential coordinates and the suffix (t) means that the referential configuration is taken to be that at the current time t, where t is a positive parameter (or time) characterizing the deformation process. $\mathbf{D}^e = \dot{\boldsymbol{\epsilon}}^e$ and $\mathbf{D}^p = \dot{\boldsymbol{\epsilon}}^p$ are the elastic and plastic parts of $\mathbf{D}$, respectively, where we decompose $\dot{\mathbf{F}}_{(t)}$ as follows:

$$\dot{\mathbf{F}}_{(t)} = \partial\dot{\mathbf{x}}/\partial\mathbf{X}_{(t)} = \partial\dot{\mathbf{u}}/\partial\mathbf{X}_{(t)} = \partial\dot{\mathbf{u}}^e/\partial\mathbf{X}_{(t)} + \partial\dot{\mathbf{u}}^p/\partial\mathbf{X}_{(t)} = \dot{\boldsymbol{\epsilon}}^e + \dot{\boldsymbol{\epsilon}}^p$$

in which the temporary displacement $\mathbf{u}$ is decomposed into the elastic and plastic parts (cf. e.g. [21]). The total plastic strain up to the time t, $\boldsymbol{\epsilon}^p$, should be such an integration of $\dot{}^p$ along the whole deformation history as that $\boldsymbol{\epsilon}^p$ possesses the property of objectivity [22]. Namely, under an arbitrary observer transformation which is characterized by an arbitrary orthogonal tensor $\mathbf{Q}$, it is transformed into $\mathbf{Q}\boldsymbol{\epsilon}^p\mathbf{Q}^T$. In an actual computation, $\boldsymbol{\epsilon}^p$ may be obtained by subtracting the total elastic strain from the total strain. For convenience we denote $\boldsymbol{\epsilon}^p$ by $\boldsymbol{\epsilon}$ hereafter.

Now the coefficient tensor $\mathbf{C}^p$ is generally a function of $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$ (and temperature T, though it is omitted here), in which $\boldsymbol{\alpha}$ is a general term for the internal variables, $\boldsymbol{\epsilon}$ may be thought to be a macroscopic reflection of the internal structural changes of the material due to plastic deformation. In this sense we adopt $\boldsymbol{\epsilon}$ as an internal variable. Furthermore we consider the density of any internal defect produced by plastic deformation such as dislocations which are associated directly with plastic deformation, and for convenience we express it by a symmetric second-rank tensor $\boldsymbol{\rho}$ (cf. PERZYNA [23]). When we specifically consider $\boldsymbol{\rho}$ at the symmetric part of the dislocation density tensor $\boldsymbol{\rho}^*$, (i.e. $\boldsymbol{\rho} = (\boldsymbol{\rho}^{*T} + \boldsymbol{\rho}^*)/2$), in the recent continuously distributed dislocation theory, we should refer to the earlier theory by KRÖNER [4] and others in which $\boldsymbol{\rho}^*$ is defined by the terms of elastic distortion around the dislocation lines and not to the theory by MURA [24] in which $\boldsymbol{\rho}^*$ is connected directly to plastic distortion. This is so since we require $\boldsymbol{\rho}$ to express the internal state of the material at the current time and not all the dislocations which swept out the area under consideration in the past and we will treat it as a variable independent of $\boldsymbol{\epsilon}$. We can assume safely that $\boldsymbol{\rho}$ has objectivity because it is an attribute of the material. Thus all of the variables in Eq. $(1.1)_1$ have the property of objectivity and then under an arbitrary observer transformation Eq. $(1.1)_1$ is transformed into the following form (see e.g. [22]):

$$\mathbf{Q}\mathring{\boldsymbol{\sigma}}\mathbf{Q}^T = \mathbf{f}(\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T, \mathbf{Q}\boldsymbol{\epsilon}\mathbf{Q}^T, \mathbf{Q}\boldsymbol{\rho}\mathbf{Q}^T, \mathbf{Q}\mathbf{D}\mathbf{Q}^T),$$

in which $\mathring{\boldsymbol{\sigma}} = \mathbf{f}(\boldsymbol{\sigma}, \boldsymbol{\epsilon}, \boldsymbol{\rho}, \mathbf{D})$; this the right-hand side of Eq. $(1.1)_1$, and thus

$$\mathbf{Q}\mathbf{f}(\boldsymbol{\sigma}, \boldsymbol{\epsilon}, \boldsymbol{\rho}, \mathbf{D})\mathbf{Q}^T = \mathbf{f}(\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T, \mathbf{Q}\boldsymbol{\epsilon}\mathbf{Q}^T, \mathbf{Q}\boldsymbol{\rho}\mathbf{Q}^T, \mathbf{Q}\mathbf{D}\mathbf{Q}^T)$$

which means that $\mathbf{f}$ must be an isotropic symmetric tensor function of rank 2 with the variables $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$, $\boldsymbol{\rho}$ and $\mathbf{D}$, where $\mathbf{D}$ is linearly involved due to the form of Eq. $(1.1)_1$. We should note that no assumption of material isotropy in the engineering sense is imposed on this statement and that only the objectivity requirement is completely satisfied. As we will see later, $\mathbf{f}$ can express material anisotropy such as the Bauschinger effect, and topological distortion of the yield surface and so forth through the terms of $\boldsymbol{\epsilon}$ and $\boldsymbol{\rho}$, although we initially assume the materials to be isotropic.

2.2. Invariants and basic products

Now, according to the representation theorem for isotropic functions [25], we know that the right hand side of Eq. $(1.1)_1$ can be expressed by a function of the invariants and the basic products of $\mathbf{D}, \boldsymbol{\sigma}, \boldsymbol{\epsilon}$ and $\boldsymbol{\rho}$, where $\mathbf{D}$ should be linearly involved. Here we show these invariants and basic products by the use of the theorem established by SPENCER and RIVLIN [16].

2.2.1. Invariants

first order: $\mathrm{tr}\boldsymbol{\sigma}$, $\mathrm{tr}\boldsymbol{\epsilon}$, $\mathrm{tr}\boldsymbol{\rho}$, $\mathrm{tr}\mathbf{D}$,
second order: $\mathrm{tr}\boldsymbol{\sigma}^2$, $\mathrm{tr}\boldsymbol{\epsilon}^2, \ldots$, (9 in all),
third order: $\mathrm{tr}\boldsymbol{\sigma}^3$, $\mathrm{tr}\boldsymbol{\epsilon}^3$, $\mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{\epsilon}^2), \ldots$, (16 in all),
fourth order: $\mathrm{tr}(\boldsymbol{\sigma}^2\boldsymbol{\epsilon}^2)$, $\mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{\epsilon}^2\mathbf{D}), \ldots$, (13 in all),
fifth order: $\mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{\epsilon}^2\boldsymbol{\rho}^2)$, $\mathrm{tr}(\boldsymbol{\epsilon}\boldsymbol{\rho}^2\boldsymbol{\sigma}^2)$, $\mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{\epsilon}\boldsymbol{\rho}^2\mathbf{D}), \ldots$, (12 in all).

sixth order: $\operatorname{tr}(\boldsymbol{\sigma}^2\boldsymbol{\epsilon}^2\boldsymbol{\rho}\mathbf{D})$, $\operatorname{tr}(\boldsymbol{\sigma}^2\boldsymbol{\rho}\boldsymbol{\epsilon}^2\mathbf{D})$, ..., (9 in all), where tr means trace, i.e. $\operatorname{tr}\mathbf{A} = A_{ii}$. We call all the invariants above Eq. (2.1) *en bloc*. In the above there are in all 35 and 28 invariants with and without $\mathbf{D}$, respectively.

2.2.2. Basic products (except the corresponding transpose forms)

i) without $\mathbf{D}$
0-th order: $\mathbf{1}$ (= the unit tensor of rank 2),
first order: $\boldsymbol{\sigma}, \boldsymbol{\epsilon}, \boldsymbol{\rho}$,
second order: $\boldsymbol{\sigma}^2, \boldsymbol{\epsilon}^2, \boldsymbol{\rho}^2, \boldsymbol{\sigma\epsilon}, \boldsymbol{\epsilon\rho}, \boldsymbol{\rho\sigma}$,
third order: $\boldsymbol{\sigma}^2\boldsymbol{\epsilon}, \boldsymbol{\sigma\epsilon\rho}$, ..., (9 in all),
fourth order: $\boldsymbol{\sigma}^2\boldsymbol{\epsilon}^2, \boldsymbol{\sigma}^2\boldsymbol{\epsilon\rho}$, ..., (12 in all),
fifth order: $\boldsymbol{\sigma}^2\boldsymbol{\epsilon}^2\boldsymbol{\rho}, \boldsymbol{\sigma}^2\boldsymbol{\rho}\boldsymbol{\epsilon}^2, \boldsymbol{\sigma\epsilon\rho}\boldsymbol{\sigma}^2$, ..., (27 in all).
We call all the above basic products Eq. (2.2) *en bloc*. In the above there are 58 basic products in all total.

ii) with $\mathbf{D}$
first order: $\mathbf{D}$,
second order: $\boldsymbol{\sigma}\mathbf{D}, \boldsymbol{\epsilon}\mathbf{D}, \boldsymbol{\rho}\mathbf{D}$,
third order: $\boldsymbol{\sigma}^2\mathbf{D}, \boldsymbol{\epsilon}^2\mathbf{D}, \boldsymbol{\sigma\epsilon}\mathbf{D}$, ..., (12 in all),
fourth order: $\boldsymbol{\sigma}^2\boldsymbol{\epsilon}\mathbf{D}, \boldsymbol{\sigma}^2\mathbf{D}\boldsymbol{\epsilon}, \boldsymbol{\sigma\epsilon\rho}\mathbf{D}$, ..., (25 in all),
fifth order: $\boldsymbol{\sigma}^2\boldsymbol{\epsilon}^2\mathbf{D}, \boldsymbol{\sigma}^2\mathbf{D}\boldsymbol{\epsilon}^2, \boldsymbol{\sigma}^2\boldsymbol{\epsilon\rho}\mathbf{D}, \boldsymbol{\epsilon\sigma}^2\boldsymbol{\rho}\mathbf{D}, \boldsymbol{\epsilon\rho\sigma}^2\mathbf{D}$, ..., (39 in all),
sixth order $\boldsymbol{\sigma}^2\boldsymbol{\epsilon}^2\boldsymbol{\rho}\mathbf{D}, \boldsymbol{\epsilon}^2\boldsymbol{\sigma}^2\boldsymbol{\rho}\mathbf{D}, \boldsymbol{\epsilon}^2\boldsymbol{\rho\sigma}^2\mathbf{D}, \boldsymbol{\epsilon}^2\boldsymbol{\rho}\mathbf{D}\boldsymbol{\sigma}^2$, ..., (120 in all).
We call all the above basic products Eq. (2.3) *en bloc*. In the above there are 200 basic products in all total.

3. Representation of the constitutive equation by the terms of the invariants and the basic products

The constitutive equation $(1.1)_1$, CET, can be represented by the invariants and the basic products given above, i.e. in the terms of $\boldsymbol{\sigma}, \mathbf{D}$; $\boldsymbol{\epsilon}, \mathbf{D}$ and $\boldsymbol{\sigma}, \boldsymbol{\epsilon}, \mathbf{D}$ and in the other terms which include $\boldsymbol{\rho}$. Namely,

$$(3.1)\quad \overset{\circ}{\boldsymbol{\sigma}} = \mathbf{C}^p\!:\!\mathbf{D} = \mathbf{S}_1+\mathbf{S}_2 = [[\alpha_1\operatorname{tr}\mathbf{D}+\alpha_2\operatorname{tr}(\boldsymbol{\sigma}\mathbf{D})+\alpha_3\operatorname{tr}(\boldsymbol{\epsilon}\mathbf{D})+\alpha_4\operatorname{tr}(\boldsymbol{\sigma}^2\mathbf{D}) \\ +\alpha_5\operatorname{tr}(\boldsymbol{\epsilon}^2\mathbf{D})+\alpha_6\operatorname{tr}(\boldsymbol{\sigma\epsilon}\mathbf{D})+\alpha_7\operatorname{tr}(\boldsymbol{\sigma\epsilon}^2\mathbf{D})+\alpha_8\operatorname{tr}(\boldsymbol{\epsilon\sigma}^2\mathbf{D})+\alpha_9\operatorname{tr}(\boldsymbol{\sigma}^2\boldsymbol{\epsilon}^2\mathbf{D})]\mathbf{1} \\ +[\alpha_{10}\operatorname{tr}\mathbf{D}+\ \ldots]\boldsymbol{\sigma}+\ \ldots\ +\alpha_{82}\mathbf{D}+\alpha_{83}(\boldsymbol{\sigma}\mathbf{D}+\mathbf{D}\boldsymbol{\sigma})+\ \ldots\ +\alpha_{98}(\mathbf{D}\boldsymbol{\sigma}^2\boldsymbol{\epsilon}^2+\boldsymbol{\epsilon}^2\boldsymbol{\sigma}^2\mathbf{D})]+\mathbf{S}_2,$$

where $\mathbf{S}_1$ consists of the terms without $\boldsymbol{\rho}$ and $\mathbf{S}_2$, the terms with $\boldsymbol{\rho}$ in which $\mathbf{S}_2 = \mathbf{0}$ if $\boldsymbol{\rho} = 0$. The coefficients α_1 to α_{98} are the functions of ten invariants associated with $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$, i.e. $[\operatorname{tr}\boldsymbol{\sigma}, \operatorname{tr}\boldsymbol{\epsilon}, \operatorname{tr}\boldsymbol{\sigma}^2, \operatorname{tr}\boldsymbol{\epsilon}^2, \operatorname{tr}(\boldsymbol{\sigma\epsilon}), \operatorname{tr}\boldsymbol{\sigma}^3, \operatorname{tr}\boldsymbol{\epsilon}^3, \operatorname{tr}(\boldsymbol{\sigma\epsilon}^2), \operatorname{tr}(\boldsymbol{\sigma}^2\boldsymbol{\epsilon}), \operatorname{tr}(\boldsymbol{\sigma}^2\boldsymbol{\epsilon}^2)]$. $\mathbf{S}_2$ has 2230 invariant coefficients say α_{99} to α_{2328}. Expressing $\mathbf{S}_1$ in the component form we have

$$(3.2)\quad [\mathbf{S}_1]^{ij} = [(\alpha_1 G^{kl}+\alpha_2\sigma^{kl}+\alpha_3\varepsilon^{kl}+\alpha_4\sigma^{l}_{\cdot m}\sigma^{mk}+\ \ldots\ +\alpha_9\sigma^{lm}\sigma_m^{\cdot n}\varepsilon_n^{\cdot r}\varepsilon_r^{\cdot k})G^{ij} \\ +(\alpha_{10}G^{kl}+\ldots)\sigma^{ij}+\ \ldots\ +\alpha_{82}g^{ijkl}+\alpha_{83}(\sigma^{ik}G^{lj}+G^{il}\sigma^{kj})+\ \ldots\ +\alpha_{96}(\sigma^{im}\sigma_m^{\cdot n}\varepsilon_n^{\cdot r}\varepsilon_r^{\cdot k}G^{lj} \\ +\sigma^{jm}\sigma_m^{\cdot n}\varepsilon_n^{\cdot r}\varepsilon_r^{\cdot k}G^{li})+\ \ldots]D_{kl} = [\mathbf{C}_1]^{ijkl}\!:\![\mathbf{D}]_{kl},\quad \text{(say)},$$

where G^{ij} are the contravariant components of the metric tensor $\mathbf{G}$ and $g^{ijkl} = (G^{ik}G^{jl} + G^{il}G^{jk})/2$. In Eq. (3.2) and the following equations we use the summation convention if we do not mark the indices with under-bars $(-)$ or do not say otherwise.

Now we adopt the rectangular Cartesian coordinates which coincide with the principal axes of $\boldsymbol{\sigma}$ at the current time. Then, putting $G^{ij} = \delta_{ij}$, $\sigma^{ii} = \sigma_i$ and $\sigma^{ij} = 0$ for $i \neq j$, $(i,j = 1,2,3)$, $\mathbf{C}_1$ in Eq. (3.2) is decomposed into the following form:

$$\mathbf{C}_1 = \mathbf{C}_{1\sigma} + \mathbf{C}_{1\varepsilon}, \tag{3.3}$$

$$[\mathbf{C}_{1\sigma}]_{ijkl} = A^\sigma_{ik}\delta_{ij}\delta_{kl} + B^\sigma_{ij}g_{ijkl}, \tag{3.4}$$

$$g_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})/2,$$

$$\begin{aligned} A^\sigma_{ik} &= \alpha_1 + \alpha_2\sigma_k + \alpha_4\sigma_k^2 + \alpha_{10}\sigma_i + \alpha_{11}\sigma_i\sigma_k + \alpha_{13}\sigma_i\sigma_k^2 + \alpha_{28}\sigma_i^2 + \alpha_{29}\sigma_i^2\sigma_k + \alpha_{31}\sigma_i^2\sigma_k^2, \\ B^\sigma_{ij} &= \alpha_{82} + \alpha_{83}(\sigma_i + \sigma_j) + \alpha_{85}(\sigma_i^2 + \sigma_j^2) \quad \text{(not summed over } i,j,k,l); \end{aligned} \tag{3.5}$$

$$\begin{aligned} [\mathbf{C}_{1\varepsilon}]_{ijkl} = {} & \{\alpha_3\varepsilon_{kl} + \alpha_5\varepsilon_{lm}\varepsilon_{mk} + 0.5\alpha_6(\sigma_l + \sigma_k)\varepsilon_{kl} + 0.5\alpha_7(\sigma_l\varepsilon_{lm}\varepsilon_{mk} + \sigma_k\varepsilon_{km}\varepsilon_{ml}) \\ & + 0.5\alpha_8(\sigma_l^2 + \sigma_k^2)\varepsilon_{kl} + 0.5\alpha_9(\sigma_l^2\varepsilon_{lm}\varepsilon_{mk} + \sigma_k^2\varepsilon_{km}\varepsilon_{ml})\}\delta_{ij} + \{\alpha_{12}\varepsilon_{kl} + \ \dots\ \}\sigma_i\delta_{ij} \\ & + \{\alpha_{19}\delta_{kl} + \alpha_{20}\sigma_k\delta_{kl} + \alpha_{21}\varepsilon_{kl} + \alpha_{22}\sigma_k^2\delta_{kl} + \alpha_{23}\varepsilon_{lm}\varepsilon_{mk} + \dots\}\varepsilon_{ij} + \ \dots\ + \alpha_{84}(\varepsilon_{ir}g_{rjkl} \\ & + \varepsilon_{jr}g_{rikl}) + 0.5\alpha_{86}(\sigma_i\delta_{ik}\varepsilon_{lj} + \sigma_j\delta_{jk}\varepsilon_{li} + \sigma_i\delta_{il}\varepsilon_{kj} + \sigma_j\delta_{jl}\varepsilon_{ki}) + 0.5\alpha_{87}(\delta_{ik}\sigma_l\varepsilon_{lj} + \delta_{jk}\sigma_l\varepsilon_{li} \\ & + \delta_{il}\sigma_k\varepsilon_{kj} + \delta_{jl}\sigma_k\varepsilon_{ki}) + \ \dots\ + 0.5\alpha_{98}\{(\delta_{ik}\varepsilon_{mj} + \delta_{jk}\varepsilon_{mi})\sigma_l^2\varepsilon_{lm} + (\delta_{il}\varepsilon_{mj} + \delta_{jl}\varepsilon_{mi})\sigma_k^2\varepsilon_{km}\}, \\ & \text{not summed over } i,j,k,l. \end{aligned} \tag{3.6}$$

$\mathbf{C}_{1\sigma}$ involves only $\boldsymbol{\sigma}$ and $\mathbf{C}_{1\varepsilon}$ consists of $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$. Tokuoka is concerned only with $\mathbf{C}_{1\sigma}$.

As for CEV [Eq. $(1.1)_2$], correspondingly to Eq. (3.3) we have

$$\hat{\mathbf{C}}^p = \hat{\mathbf{C}}_1 + \hat{\mathbf{C}}_2 = (\hat{\mathbf{C}}_{1\sigma} + \hat{\mathbf{C}}_{1\varepsilon}) + \hat{\mathbf{C}}_2. \tag{3.7}$$

From Eqs. (3.4) and (3.5) we have

$$\begin{aligned} \hat{\mathbf{C}}^p_{1\sigma} &= \begin{Bmatrix} \mathbf{H}^\sigma_N & \mathbf{0} \\ \mathbf{0} & \mathbf{H}^\sigma_S \end{Bmatrix}, \\ \mathbf{H}^\sigma_N &= \begin{Bmatrix} (A^\sigma_{11} + B^\sigma_{11}) & A^\sigma_{12} & A^\sigma_{13} \\ A^\sigma_{21} & (A^\sigma_{22} + B^\sigma_{22}) & A^\sigma_{23} \\ A^\sigma_{31} & A^\sigma_{32} & (A^\sigma_{33} + B^\sigma_{33}) \end{Bmatrix}, \\ \mathbf{H}^\sigma_S &= \begin{Bmatrix} B^\sigma_{23} & 0 & 0 \\ 0 & B^\sigma_{31} & 0 \\ 0 & 0 & B^\sigma_{12} \end{Bmatrix}, \end{aligned} \tag{3.8}$$

where $[\mathbf{0}] = a\ (3 \times 3)$ — null matrix.

$\hat{\mathbf{C}}_{1\varepsilon}$ is not reduced to the form of Eq. $(3.8)_1$ except when the principal axes of $\boldsymbol{\epsilon}$ coincide with those of $\boldsymbol{\sigma}$, i.e. co-axiality between $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ holds. For this special case we can put $\varepsilon_{ii} = \varepsilon_i$ and $\varepsilon_{ij} = 0$ for $i \neq j$ $(i,j = 1,2,3)$, and obtain the following equation:

$$\begin{aligned} [\mathbf{C}_{1\varepsilon}]_{ijkl} = {} & [\varepsilon_k(\alpha_3 + \alpha_5\varepsilon_k + \alpha_6\sigma_k + \alpha_7\sigma_k\varepsilon_k + \alpha_8\sigma_k^2 + \alpha_9\sigma_k^2\varepsilon_k) + \sigma_i\varepsilon_k(\alpha_{12} + \ \dots) + \ \dots \\ & + \varepsilon_i^2\sigma_i^2(\alpha'_{73} + \alpha'_{74}\sigma_k + \alpha'_{75}\varepsilon_k + \alpha'_{76}\sigma_k^2 + \alpha'_{77}\varepsilon_k^2 + \ \dots\ + \alpha'_{81}\sigma_k^2\varepsilon_k^2)]\delta_{ij}\delta_{kl} + [\alpha_{84}(\varepsilon_i + \varepsilon_j) \\ & + \alpha_{86}(\sigma_i\varepsilon_j + \sigma_j\varepsilon_i) + (\alpha_{87} + \alpha_{88})(\sigma_i\varepsilon_i + \sigma_j\varepsilon_j) + \alpha_{89}(\varepsilon_i^2 + \varepsilon_j^2) + (\alpha_{90} + \alpha_{92})(\sigma_i^2\varepsilon_i + \sigma_j^2\varepsilon_j) \\ & + \alpha_{91}(\sigma_i^2\varepsilon_j + \sigma_j^2\varepsilon_i) + (\alpha_{93} + \alpha_{95})(\sigma_i\varepsilon_i^2 + \sigma_j\varepsilon_j^2) + \alpha_{94}(\sigma_i\varepsilon_j^2 + \sigma_j\varepsilon_i^2) + (\alpha_{96} + \alpha_{98})(\sigma_i^2\varepsilon_i^2 \\ & + \sigma_j^2\varepsilon_j^2) + \alpha_{97}(\sigma_i\varepsilon_j^2 + \sigma_j\varepsilon_i^2)]g_{klij} = A^\varepsilon_{ik}\delta_{ij}\delta_{kl} + B^\varepsilon_{ij}g_{ijkl}, \quad \text{(say)}, \quad \alpha'_n = 2\alpha_n, \\ & \text{not summed over } i,j,k,l. \end{aligned} \tag{3.9}$$

Consequently, we obtain correspondingly to Eq. (3.8) the following equations:

$$\hat{\mathbf{C}}_{1\varepsilon} = \begin{Bmatrix} \mathbf{H}_N^\varepsilon & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_S^\varepsilon \end{Bmatrix}, \tag{3.10}$$

$\mathbf{H}_N^\varepsilon, \mathbf{H}_S^\varepsilon$ = those given by the replacement of the superscript σ by ε in Eqs. $(3.8)_2$ and $(3.8)_3$, respectively.

Next, let us concern ourselves with $\mathbf{S}_2$. Here we take only the first order of $\boldsymbol{\rho}$ into consideration, for the formulation becomes too complicated if the higher orders of $\boldsymbol{\rho}$ are contained. Then $\mathbf{S}_2$ has the following expression:

$$[\mathbf{S}_2]^{ij} = C_*^{ijklmn} \varrho_{kl} D_{mn} = [\mathbf{C}_2]^{ijmn} : [\mathbf{D}]_{mn}, \tag{3.11}$$

$$\begin{aligned} \mathbf{S}_2 = {} & [\tilde{\beta}_1 \operatorname{tr}(\boldsymbol{\rho}\mathbf{D}) + \tilde{\beta}_2 \operatorname{tr}(\boldsymbol{\epsilon}\boldsymbol{\rho}\mathbf{D}) + \tilde{\beta}_3 \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{\rho}\mathbf{D}) + \tilde{\beta}_4 \operatorname{tr}(\boldsymbol{\sigma}^2\boldsymbol{\rho}\mathbf{D}) \\ & + \tilde{\beta}_5 \operatorname{tr}(\boldsymbol{\epsilon}^2\boldsymbol{\rho}\mathbf{D}) + \tilde{\beta}_6 \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{\epsilon}\boldsymbol{\rho}\mathbf{D}) + \tilde{\beta}_7 \operatorname{tr}(\boldsymbol{\epsilon}\boldsymbol{\sigma}^2\boldsymbol{\rho}\mathbf{D}) + \ldots + \tilde{\beta}_{13} \operatorname{tr}(\boldsymbol{\sigma}^2\boldsymbol{\epsilon}^2\boldsymbol{\rho}\mathbf{D})]\mathbf{1} \\ & + [\tilde{\beta}_{14} \operatorname{tr}(\boldsymbol{\rho}\mathbf{D}) + \ldots]\boldsymbol{\sigma} + \ldots + [\tilde{\beta}_{370} \operatorname{tr}\mathbf{D} + \tilde{\beta}_{371} \operatorname{tr}(\boldsymbol{\sigma}\mathbf{D}) \\ & + \tilde{\beta}_{372} \operatorname{tr}(\boldsymbol{\epsilon}\mathbf{D}) + \ldots + \tilde{\beta}_{378} \operatorname{tr}(\boldsymbol{\sigma}^2\boldsymbol{\epsilon}^2\mathbf{D})](\boldsymbol{\epsilon}\boldsymbol{\sigma}\boldsymbol{\epsilon}^2\boldsymbol{\rho} + \boldsymbol{\rho}\boldsymbol{\epsilon}^2\boldsymbol{\sigma}\boldsymbol{\epsilon}) \\ & + \tilde{\beta}_{379}(\boldsymbol{\rho}\mathbf{D} + \mathbf{D}\boldsymbol{\rho}) + \tilde{\beta}_{380}(\boldsymbol{\epsilon}\boldsymbol{\rho}\mathbf{D} + \mathbf{D}\boldsymbol{\rho}\boldsymbol{\epsilon}) + \tilde{\beta}_{381}(\boldsymbol{\epsilon}\mathbf{D}\boldsymbol{\rho} + \boldsymbol{\rho}\mathbf{D}\boldsymbol{\epsilon}) \\ & + \tilde{\beta}_{382}(\mathbf{D}\boldsymbol{\epsilon}\boldsymbol{\rho} + \boldsymbol{\rho}\boldsymbol{\epsilon}\mathbf{D}) + \tilde{\beta}_{383}(\boldsymbol{\sigma}\boldsymbol{\rho}\mathbf{D} + \mathbf{D}\boldsymbol{\rho}\boldsymbol{\sigma}) + \ldots + \tilde{\beta}_{386}(\boldsymbol{\sigma}\boldsymbol{\epsilon}\boldsymbol{\rho}\mathbf{D} + \mathbf{D}\boldsymbol{\rho}\boldsymbol{\epsilon}\boldsymbol{\sigma}) \\ & + \ldots + \tilde{\beta}_{402}(\boldsymbol{\sigma}^2\boldsymbol{\epsilon}\boldsymbol{\rho}\mathbf{D} + \mathbf{D}\boldsymbol{\rho}\boldsymbol{\epsilon}\boldsymbol{\sigma}^2) + \ldots + \tilde{\beta}_{422}(\boldsymbol{\sigma}^2\boldsymbol{\epsilon}^2\boldsymbol{\rho}\mathbf{D} + \mathbf{D}\boldsymbol{\rho}\boldsymbol{\epsilon}^2\boldsymbol{\sigma}^2) \\ & + \tilde{\beta}_{491}(\mathbf{D}\boldsymbol{\epsilon}\boldsymbol{\sigma}\boldsymbol{\epsilon}^2\boldsymbol{\rho} + \boldsymbol{\rho}\boldsymbol{\epsilon}^2\boldsymbol{\sigma}\boldsymbol{\epsilon}\mathbf{D}), \end{aligned} \tag{3.12}$$

where the coefficients $\tilde{\beta}_1$ to $\tilde{\beta}_{491}$ are the functions of the ten invariants composed of $\boldsymbol{\sigma}, \boldsymbol{\epsilon}$ mentioned earlier.

In the vectorial expression of Eq. (3.11) which corresponds to Eq. $(1.1)_2$, $\hat{\mathbf{C}}_2$ is reduced to such a concise form as Eq. (3.8) or Eq. (3.10) only when co-axiality between $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$ and $\boldsymbol{\rho}$ holds at the same time. For this special case we can put $\varrho_{ii} = \varrho_i$ and $\varrho_{ij} = 0$ for $i \neq j$, $(i, j = 1, 2, 3)$, and obtain

$$\begin{aligned} [\mathbf{C}_2]_{ijkl} = {} & [(\beta_1 \varrho_k + \beta_2 \varrho_k \varepsilon_k + \beta_3 \varrho_k \sigma_k + \beta_4 \varrho_k \sigma_k^2 + \beta_5 \varrho_k \varepsilon_k^2 \\ & + \beta_6 \varrho_k \varepsilon_k \sigma_k + \beta_7 \varrho_k \varepsilon_k \sigma_k^2 + \beta_8 \varrho_k \varepsilon_k \sigma_k^2 + \beta_9 \varrho_k \sigma_k^2 \varepsilon_k^2) + \sigma_i(\beta_{10} \varrho_k + \ldots) \\ & + \varepsilon_i(\beta_{19} \varrho_k + \ldots) + \ldots + \sigma_i^2 \varepsilon_i^2 (\beta_{73} \varrho_k + \ldots) + \varrho_i(\beta_{82} + \beta_{83} \sigma_k + \beta_{84} \varepsilon_k + \beta_{85} \sigma_k^2 + \beta_{86} \varepsilon_k^2 \\ & + \beta_{87} \sigma_k \varepsilon_k + \beta_{88} \sigma_k^2 \varepsilon_k + \beta_{89} \sigma_k \varepsilon_k^2 + \beta_{90} \sigma_k^2 \varepsilon_k^2) + \varrho_i \sigma_i(\beta_{91} + \ldots) \\ & + \varrho_i \varepsilon_i(\beta_{100} + \ldots) + \ldots + \varrho_i \varepsilon_i^2 \sigma_i^2 (\beta_{154} + \ldots)] \delta_{ij} \delta_{kl} + [\beta_{163}(\varrho_i + \varrho_j) \\ & + \beta_{164}(\varrho_i \sigma_i + \varrho_j \sigma_j) + \beta_{165}(\varrho_i \sigma_j + \varrho_j \sigma_i) + \beta_{166}(\varrho_i \varepsilon_i + \varrho_j \varepsilon_j) + \beta_{167}(\varrho_i \varepsilon_j + \varrho_j \varepsilon_i) \\ & + \beta_{168}(\varrho_i \sigma_i \varepsilon_i + \varrho_j \sigma_j \varepsilon_j) + \beta_{169}(\varrho_i \sigma_j \varepsilon_i + \varrho_j \sigma_i \varepsilon_j) + \beta_{170}(\varrho_i \sigma_j \varepsilon_j + \varrho_j \sigma_i \varepsilon_i) \\ & + \beta_{171}(\varrho_i \sigma_i \varepsilon_j + \varrho_j \sigma_j \varepsilon_i) + \ldots + \beta_{183}(\varrho_i \sigma_i^2 \varepsilon_j^2 + \varrho_j \sigma_j^2 \varepsilon_i^2)] g_{ijkl} = A_{ik}^{\varrho} \delta_{ij} \delta_{kl} \\ & + B_{ij}^{\varrho} g_{ijkl}, \quad \text{(say)}, \quad \text{not summed over } i, j, k, l, \end{aligned} \tag{3.13}$$

where β_1 to β_{183} are also the functions of the invariants of $\boldsymbol{\sigma}, \boldsymbol{\epsilon}$. β_1 to β_6 coincide with $\tilde{\beta}_1$ to $\tilde{\beta}_6$ in Eq. (3.12), respectively. Finally we have

$$\hat{\mathbf{C}}_2 = \begin{pmatrix} \mathbf{H}_N^\varrho & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_S^\varrho \end{pmatrix}, \tag{3.14}$$

$\mathbf{H}_\varrho^N, \mathbf{H}_S^\varrho$ = those given by the replacement of the superscript σ by ϱ in Eq. $(3.8)_2$ and $(3.8)_3$, respectively.

4. The conditions of constitutive instability and their interpretation

4.1. The case of $\boldsymbol{\epsilon} = \boldsymbol{\rho} = \mathbf{0}$

This is the case that Tokuoka was concerned with. $\hat{\mathbf{C}}^P = \hat{\mathbf{C}}_{1\sigma}$ and thus

$$\det|\hat{\mathbf{C}}_{1\sigma}| = 0 \tag{4.1}$$

is the condition of constitutive instability which means that $\hat{\mathbf{D}}$ becomes indefinite for a finite value of $\mathring{\boldsymbol{\sigma}}$ and thus the constitutive equation (1.1) loses its meaning. This situation may be thought to express mathematically a catastrophic phase change of the physical state of the material. Then the introduction (or appearance) of $\boldsymbol{\epsilon}$ and/or $\boldsymbol{\rho}$ would recover the material constitution and hence Eq. (4.1) may be considered to be an initial yield condition which is Tokuoka's postulate. We should note that the supposition $\boldsymbol{\rho} = \mathbf{0}$ does not mean that the material has no internal defects in the initial state but means that the defects keep their state unchanged until Eq. (4.1) holds and thus $\boldsymbol{\rho}$ needs not be thought as a variable. From Eqs. (3.8) and (4.1) we obtain two kinds of the initial yield condition:

$$\text{normal type:} \quad \det|\mathbf{H}_N^\sigma| = 0, \tag{4.2}$$

$$\text{shear type:} \quad \det|\mathbf{H}_S^\sigma| = 0, \tag{4.3}$$

and the associated flow rules are easily recognized to be the non-trivial solutions of the equations

$$\mathbf{H}_N^\sigma \hat{\mathbf{D}} = \mathbf{0} \quad \text{and} \quad \mathbf{H}_S^\sigma \hat{\mathbf{D}} = \mathbf{0}, \tag{4.4}$$

respectively. As Tokuoka shows, Eqs. (4.2) and $(4.4)_1$ include the so-called Mises yield condition and the flow rule associated with it; (4.3) and $(4.4)_2$ include the so-called Tresca yield condition and also the flow rule associated with it.

As we will see later, such a decomposition of a condition as above always holds only for the initial yield condition.

4.2. The case of $\boldsymbol{\rho} = \mathbf{0}$ and $\boldsymbol{\epsilon} \neq \mathbf{0}$

Now we consider the situations where plastic deformation takes place at least once. For this case

$$\det|\hat{\mathbf{C}}_1| = \det|\hat{\mathbf{C}}_{1\sigma} + \hat{\mathbf{C}}_{1\epsilon}| = 0 \tag{4.5}$$

is the condition of constitutive instability. If we assume that the subsequent yield condition, i.e. the yield condition after any plastic deformation, is not influenced by any intermediate unloading, then it may be thought to be the same both for the continuous loading and intermediate unloading-reloading process (see Fig. 1). Namely, the subsequent yield condition may be thought to be just the re-yielding condition. Thus, considering the meaning of the supposition $\boldsymbol{\rho} = \mathbf{0}$ in the same manner as in Sect. 4.1, we can interpret Eq. (4.5) as subsequent yield condition after arbitrary plastic-straining. In this case the stability of the material constitution may be recovered by $\boldsymbol{\epsilon}$ and $\boldsymbol{\rho}$ which are newly introduced by re-yielding. This is the reason why we do not think that the condition (4.5) is a fracture condition which means an origination of loss of material continuity. The condition (4.5) cannot be necessarily decomposed into two parts like Eqs. (4.2) and (4.3) because Eq. $(3.10)_1$ has no generality. We should understand it as a single condition, referring to Eqs. (3.6) and $(3.8)_1$. Specifically, however, when co-axiality between $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ holds, such a decomposition is again available, and then from Eqs. $(3.8)_1$, $(3.10)_1$ and

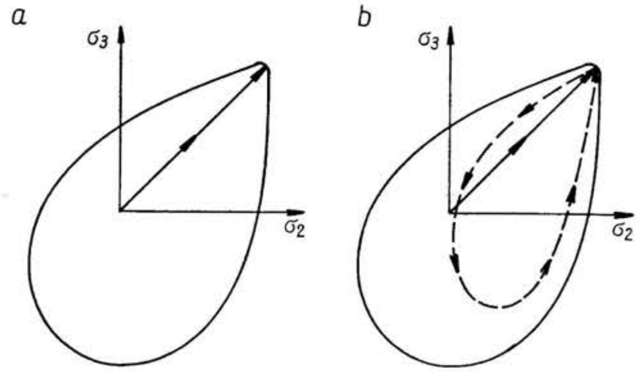

FIG. 1. A subsequent yield surface: a) a proportional loading, b) an unloading-reloading process. (Schematic illustrations in a two-dimensional stress space).

(4.5) we obtain the subsequent yield condition of normal type and the associated flow rule

$$\det|\mathbf{H}_N^{\sigma}+\mathbf{H}_N^{\varepsilon}| = \det|\mathbf{H}_N^{\sigma\varepsilon}| = 0 \tag{4.6}$$

and the non-trivial solution of the equation

$$\mathbf{H}_N^{\sigma\varepsilon}\hat{\mathbf{D}} = \mathbf{0}, \tag{4.7}$$

and the subsequent yield condition of shear type and the associated flow rule

$$\det|\mathbf{H}_S^{\sigma}+\mathbf{H}_S^{\varepsilon}| = \det|\mathbf{H}_S^{\sigma\varepsilon}| = 0 \tag{4.8}$$

and the non-trivial solution of the equation

$$\mathbf{H}_S^{\sigma\varepsilon}\hat{\mathbf{D}} = \mathbf{0}. \tag{4.9}$$

From Eqs. $(3.8)_{2,3}$, $(3.10)_{2,3}$, (4.6) and (4.8) it is easily found that Eqs. (4.6) and (4.8) involve the Mises and Tresca yield conditions for the isotropic work-hardening materials for the case when the effect of $\boldsymbol{\epsilon}$ is expressed only by scalars (e.g. $\mathrm{tr}\,\boldsymbol{\epsilon}^2$) as the usual assumption made in the classical plasticity theory.

4.3. Examples of the subsequent yield condition in the case when co-axiality between σ and ϵ holds

4.3.1. Normal type (i)

In Eqs. (3.4) and (3.9) let us put the following:

$$\begin{aligned} &A_{ik}^{\sigma}+A_{ik}^{\varepsilon} = \lambda_0+\lambda_1 s_i s_k-\frac{1}{2}\lambda_4(\varepsilon_i s_k+\varepsilon_k s_i),\\ &B_{ij}^{\sigma}+B_{ij}^{\varepsilon} = \mu,\\ &s_i = \sigma_i-p, \quad p = \mathrm{tr}\boldsymbol{\sigma}/3, \end{aligned} \tag{4.10}$$

where $\lambda_0, \ldots, \lambda_4$ and μ are the material constants and s_i is the principal deviatoric stress. Then the proper numbers $H_N^{(i)}$ of Eq. (4.6) are obtained as follows:

$$\begin{aligned} H_N^{(1)} &= \mu+(\lambda_1 s_k s_k-\lambda_4\varepsilon_k s_k), \\ H_N^{(2)} &= 3\lambda_0+\mu, \quad H_N^{(3)} = \mu. \end{aligned} \tag{4.11}$$

Putting $H_N^{(1)} = 0$, we obtain the following subsequent yield condition which shows a shift of the center of the yield surface:

$$\begin{aligned} s_k s_k-\lambda_4'\varepsilon_k s_k &= \mu', \\ \lambda_4' = \lambda_4/\lambda_1, &\quad \mu' = -\mu/\lambda_1. \end{aligned} \tag{4.12}$$

4.3.2. Normal type (ii)

In Eq. (4.11) we regard μ as a function of the invariants and put

$$\mu = \mu_0+(\lambda_2\varepsilon_k+\lambda_3\varepsilon_k^2)s_k^2, \tag{4.13}$$

then, correspondingly to Eq. (4.11), we obtain

$$\begin{aligned} H_N^{(1)} &= \mu_0+\{(\lambda_1+\lambda_2\varepsilon_k+\lambda_3\varepsilon_k^2)s_k^2-\lambda_4\varepsilon_k s_k\}, \\ H_N^{(2)} &= 3\lambda_0+\mu_0+(\lambda_2\varepsilon_k+\lambda_3\varepsilon_k^2)s_k^2, \\ H_N^{(3)} &= \mu_0+(\lambda_2\varepsilon_k+\lambda_3\varepsilon_k^2)s_k^2, \end{aligned} \tag{4.14}$$

where ε_k^2 and s_k^2 are understood to have a single index k and not to be equal to $\varepsilon_k\varepsilon_k$ and $s_k s_k$, respectively. Putting $H_N^{(1)} = 0$, we obtain another subsequent yield condition of normal type as follows:

$$(\lambda_1+\lambda_2\varepsilon_k+\lambda_3\varepsilon_k^2)s_k^2-\lambda_4\varepsilon_k s_k = -\mu_0. \tag{4.15}$$

It is just worthy to note that Eq. (4.15) is a special form of the so-called Yoshimura's yield condition [26] which can express both a shape change of the yield surface and a shift of its center due to plastic deformation.

4.3.3. Shear type

In Eqs. $(3.8)_3$ and $(3.10)_3$ we put

$$\begin{aligned} B_{ij}^{\sigma}+B_{ij}^{\varepsilon} = {} & \mu_0+(\mu_1+\mu_4\varepsilon_j+\mu_5\varepsilon_i+\mu_7\varepsilon_j^2+\mu_8\varepsilon_i^2+\mu_{17}\varepsilon_i\varepsilon_j)s_i \\ & +(\mu_1+\mu_4\varepsilon_i+\mu_5\varepsilon_j+\mu_7\varepsilon_i^2+\mu_8\varepsilon_j^2+\mu_{17}\varepsilon_i\varepsilon_j)s_j+(\mu_2+\mu_9\varepsilon_j+\mu_{10}\varepsilon_i+\mu_{11}\varepsilon_j^2+\mu_{12}\varepsilon_i^2 \\ & +\mu_{18}\varepsilon_i\varepsilon_j)s_i^2+(\mu_2+\mu_9\varepsilon_i+\mu_{10}\varepsilon_j+\mu_{11}\varepsilon_i^2+\mu_{12}\varepsilon_j^2+\mu_{18}\varepsilon_i\varepsilon_j)s_i^2+\{\mu_{13}+\mu_{14}(\varepsilon_i+\varepsilon_j) \\ & +\mu_{15}(\varepsilon_i^2+\varepsilon_j^2)+\mu_{19}\varepsilon_i\varepsilon_j\}s_i s_j+\mu_3(\varepsilon_i+\varepsilon_j)+\mu_6(\varepsilon_i^2+\varepsilon_j^2) \\ & +\mu_{16}\varepsilon_i\varepsilon_j = B_{ij}, \quad \text{(say)}, \quad \text{not summed over } i \text{ and } j. \end{aligned} \tag{4.16}$$

With regard to a combination of the fixed i and j, $B_{ij} = 0$ gives a subsequent yield condition of shear type associated with D_{ij} when σ_k is the intermediate principal stress, where $i \neq j \neq k \neq i$, (see Eqs. (4.8) and (4.9)). Specifically, when we put

$$\begin{gathered} \mu_1 = \mu_2 = \mu_9 = \mu_{10} = \mu_{13} = \mu_{14} = \mu_{17} = 0, \quad \mu_8 = -\mu_7, \quad \mu_5 = -\mu_4, \\ \mu_{11} = \mu_{12} = -0.5\mu_{18}, \quad \mu_{19} = -2\mu_{15} = -2\mu_{18}, \quad \mu_{15} = -2\mu_{11}, \quad \mu_{16} = 2\mu_6, \\ \mu_3 = \mu_4\mu_7/2\mu_{11}, \quad \mu_6 = \mu_7^2/4\mu_{11}, \quad \mu_0^* = \mu_0-(\mu_4^2/4\mu_{11}) \quad \text{and} \quad B_{23} = 0, \end{gathered}$$

Eq. (4.16) reduces to

$$(4.17)\qquad |\varepsilon_2-\varepsilon_3|\cdot|(s_2-s_3)-\{\mu_4+\mu_7(\varepsilon_2+\varepsilon_3)\}/\{2\mu_{11}(\varepsilon_2-\varepsilon_3)\}| = \sqrt{-\mu_0^{**}/\mu_{11}},$$

which is for the case when σ_1 is the intermediate stress. Or, when we put

$$\mu_1=\mu_3=\mu_7=\mu_8=\mu_9=\mu_{10}=\mu_{11}=\mu_{12}=\mu_{18}=0,$$

$$\mu_{13}=-2\mu_2,\quad \mu_{14}=\mu_{15}=\mu_{19}=0,\quad \mu_{16}=-2\mu_6,\quad \mu_6=\mu_5^2/4\mu_2,\quad B_{23}=0,$$

Eq. (4.16) reduces to

$$(4.18)\qquad |(s_2-s_3)-(\varepsilon_2-\varepsilon_3)(\mu_5/2\mu_2)| = \sqrt{-\mu_0/\mu_2}.$$

It seems natural to call the conditions (4.17) and (4.18) the extended Tresca yield conditions.

5. Representations by C.-C. Wang's theorem and their interpretation

Here we examine the condition of constitutive instability by means of Wang's representation theorem for isotropic functions [16]. Specifically, we concern ourselves with some fracture conditions. First, rewritting Eq. (3.12), we have

$$\begin{aligned}(5.1)\qquad \mathbf{S}_2 = {}& [\gamma_1\,\mathrm{tr}(\boldsymbol{\rho}\mathbf{D})+\gamma_2\,\mathrm{tr}(\boldsymbol{\epsilon}\boldsymbol{\rho}\mathbf{D})+\gamma_3\,\mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{\rho}\mathbf{D})]\mathbf{1}+[\gamma_4\,\mathrm{tr}(\boldsymbol{\rho}\mathbf{D})\\
&+\ldots]\boldsymbol{\sigma}+[\gamma_7\,\mathrm{tr}(\boldsymbol{\rho}\mathbf{D})+\ldots]\boldsymbol{\epsilon}+[\gamma_{10}\,\mathrm{tr}(\boldsymbol{\rho}\mathbf{D})+\ldots]\boldsymbol{\sigma}^2+[\gamma_{13}\,\mathrm{tr}(\boldsymbol{\rho}\mathbf{D})+\ldots]\boldsymbol{\epsilon}^2\\
&+[\gamma_{16}\,\mathrm{tr}(\boldsymbol{\rho}\mathbf{D})+\ldots](\boldsymbol{\sigma}\boldsymbol{\epsilon}+\boldsymbol{\epsilon}\boldsymbol{\sigma})+[\gamma_{19}\,\mathrm{tr}(\boldsymbol{\rho}\mathbf{D})+\ldots](\boldsymbol{\sigma}^2\boldsymbol{\epsilon}+\boldsymbol{\epsilon}\boldsymbol{\sigma}^2)+[\gamma_{22}\,\mathrm{tr}(\boldsymbol{\rho}\mathbf{D})+\ldots]\\
&\times(\boldsymbol{\epsilon}^2\boldsymbol{\sigma}+\boldsymbol{\sigma}\boldsymbol{\epsilon}^2)+[\gamma_{25}\,\mathrm{tr}(\boldsymbol{\rho}\mathbf{D})+\ldots](\boldsymbol{\sigma}^2\boldsymbol{\epsilon}^2+\boldsymbol{\epsilon}^2\boldsymbol{\sigma}^2)+[\gamma_{28}\,\mathrm{tr}\,\mathbf{D}+\gamma_{29}\,\mathrm{tr}(\boldsymbol{\sigma}\mathbf{D})+\gamma_{30}\,\mathrm{tr}(\boldsymbol{\epsilon}\mathbf{D})\\
&+\gamma_{31}\,\mathrm{tr}(\boldsymbol{\sigma}^2\mathbf{D})+\gamma_{32}\,\mathrm{tr}(\boldsymbol{\epsilon}^2\mathbf{D})\gamma_{33}\,\mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{\epsilon}\mathbf{D})]\boldsymbol{\rho}+[\gamma_{34}\,\mathrm{tr}\,\mathbf{D}+\ldots](\boldsymbol{\epsilon}\boldsymbol{\rho}+\boldsymbol{\rho}\boldsymbol{\epsilon})+[\gamma_{40}\,\mathrm{tr}\,\mathbf{D}\\
&+\ldots](\boldsymbol{\sigma}\boldsymbol{\rho}+\boldsymbol{\rho}\boldsymbol{\sigma})+[\gamma_{46}\,\mathrm{tr}\,\mathbf{D}+\ldots](\boldsymbol{\sigma}^2\boldsymbol{\rho}+\boldsymbol{\rho}\boldsymbol{\sigma}^2)+[\gamma_{52}\,\mathrm{tr}\,\mathbf{D}\\
&+\ldots](\boldsymbol{\epsilon}^2\boldsymbol{\rho}+\boldsymbol{\rho}\boldsymbol{\epsilon}^2)+\gamma_{58}(\boldsymbol{\rho}\mathbf{D}+\mathbf{D}\boldsymbol{\rho}),\end{aligned}$$

where γ_1 to γ_{58} are the functions of the ten invariants of $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$ already mentioned earlier.

5.1. Fracture condition expressed by the terms of σ, ϵ and D

Equation (5.1) can be written by the following component form:

$$\begin{aligned}(5.2)\qquad [\mathbf{S}_2]^{ij} = {}& [\mathbf{C}_*]^{ijklmn}:[\mathbf{D}]_{mn}:[\boldsymbol{\rho}]_{kl} = [\mathbf{C}_2^*]^{ijkl}:[\boldsymbol{\rho}]_{kl} = [\{\gamma_1 g^{mnkl}\\
&+\gamma_2(\varepsilon^{mk}G^{nl}+\varepsilon^{ml}G^{nk}+\varepsilon^{nk}G^{ml}+\varepsilon^{nl}G^{mk})+\gamma_3(\sigma^{mk}G^{nl}+\sigma^{ml}G^{nk}+\sigma^{nk}G^{ml}+\sigma^{nl}G^{mk})\}G^{ij}\\
&+\{\gamma_4 g^{mnkl}+\ldots\}\sigma^{ij}+\{\gamma_7 g^{mnkl}+\ldots\}\varepsilon^{ij}+\{\gamma_{10}\ldots\}\sigma^{iw}\sigma^{rj}G_{wr}+\{\gamma_{13}\ldots\}\varepsilon^{iw}\varepsilon^{rj}G_{wr}+\\
&+\{\gamma_{16}\ldots\}(\sigma^{ir}\varepsilon^{rj}+\varepsilon^{ir}\sigma^{wj})G_{wr}+\{\gamma_{19}\ldots\}(\sigma^{ir}\sigma_{rs}\varepsilon^{sj}+\varepsilon^{ir}\sigma_{rs}\sigma^{sj})+\{\gamma_{22}\ldots\}(\varepsilon^{ir}\varepsilon_{rs}\sigma^{sj}+\\
&+\sigma^{ir}\varepsilon_{rs}\varepsilon^{sj})+\{\gamma_{25}\ldots\}(\sigma^{ir}\sigma_{rs}\varepsilon^{sw}\varepsilon_{wv}+\varepsilon^{ir}\varepsilon_{rs}\sigma^{sw}\sigma_{wv})G^{vj}+[\gamma_{28}G^{mn}+\gamma_{29}\sigma^{mn}+\gamma_{30}\varepsilon^{mn}\\
&+\gamma_{31}\sigma^{mw}\sigma^{rn}G_{wr}+\gamma_{32}\varepsilon^{mw}\varepsilon^{nr}G_{wr}+\gamma_{33}\sigma^{mw}\varepsilon^{rn}G_{wr}]g^{ijkl}+[\gamma_{34}\ldots](\varepsilon^{ik}G^{jl}+\varepsilon^{il}G^{jk}\\
&+\varepsilon^{jk}G^{il}+\varepsilon^{jl}G^{ik})+[\gamma_{40}\ldots](\sigma^{ik}G^{jl}+\sigma^{il}G^{jk}+\sigma^{jk}G^{il}+\sigma^{jl}G^{ik})+[\gamma_{46}\ldots]\{(G^{ik}\sigma^{ls}\\
&+G^{il}\sigma^{ks})\sigma_s^{\cdot j}+(G^{jk}\sigma^{ls}+G^{jl}\sigma^{ks})\sigma_s^{\cdot i}\}+[\gamma_{52}\ldots]\{(G^{ik}\varepsilon^{ls}+G^{il}\varepsilon^{ks})\varepsilon_s^{\cdot j}+(G^{jk}\varepsilon^{ls}\\
&+G^{jl}\varepsilon^{ks})\varepsilon_s^{\cdot i}\}+\gamma_{58}(g^{ikmn}G^{jl}+g^{ilmn}G^{jk}+g^{jkmn}G^{il}+g^{jlmn}G^{ik})]D_{mn}\varrho_{kl}.\end{aligned}$$

In the vectorial form, Eq. (5.2) is rewritten into

$$\hat{\mathbf{S}}_2 = \hat{\mathbf{C}}_2^* \hat{\boldsymbol{\rho}}, \tag{5.3}$$

where $\hat{\boldsymbol{\rho}} = [\varrho_{11}, \varrho_{22}, \varrho_{33}, 2\varrho_{23}, 2\varrho_{31}, 2\varrho_{12}]^T$ and $\hat{\mathbf{C}}_2^* = a$ (6×6)-matrix.

In the special case when co-axiallity between $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ holds, $\mathbf{C}_2^*$ in Eq. (5.2) is simplified into the following form with respect to the rectangular Cartesian coordinates which coincide with their principal axes:

$$\begin{aligned}[\mathbf{C}_2^*]_{ijkl} = {} & [[\{\gamma_1+\gamma_2(\varepsilon_{\mathrm{m}}+\varepsilon_{\mathrm{n}})+\gamma_3(\sigma_{\mathrm{m}}+\sigma_{\mathrm{n}}\}\{\gamma_4+\ldots\}\sigma_i+\{\gamma_7+\ldots\}\varepsilon_i \\ & +\{\gamma_{10}+\ldots\}\sigma_i^2+\{\gamma_{13}+\ldots\}\varepsilon_i^2+\{\gamma_{16}+\ldots\}\sigma_i\varepsilon_i+\{\gamma_{19}+\ldots\}\sigma_i^2\varepsilon_i \\ & +\{\gamma_{22}+\ldots\}\sigma_i\varepsilon_i^2+\{\gamma_{25}+\ldots\}\sigma_i^2\varepsilon_i^2]g_{\mathrm{mn}kl}\delta_{ij}+[\{\gamma_{28}+\gamma_{29}\sigma_{\mathrm{m}}+\gamma_{30}\varepsilon_{\mathrm{m}} \\ & +\gamma_{31}\sigma_{\mathrm{m}}^2+\gamma_{32}\varepsilon_{\mathrm{m}}^2+\gamma_{33}\sigma_{\mathrm{m}}\varepsilon_{\mathrm{m}}\}+\{\gamma_{34}+\ldots\}(\varepsilon_i+\varepsilon_j)+\{\gamma_{40}+\ldots\}(\sigma_i+\sigma_j) \\ & +\{\gamma_{46}+\ldots\}(\sigma_i^2+\sigma_j^2)+\{\gamma_{52}+\ldots\}(\varepsilon_i^2+\varepsilon_j^2)]\delta_{\mathrm{mn}}g_{ijkl}+\gamma_{58}(g_{ijml}\delta_{kn}+g_{ijnl}\delta_{km} \\ & +g_{ijmk}\delta_{ln}+g_{ijnk}\delta_{lm})]D_{mn} = A_{ij'}\delta_{ij}+B_{ij}g_{ijj'}+\gamma_{58}C_{i'j} \\ = {} & \{A^*_{i'j'm'}\delta_{m'}+B^*_{i'm'}g_{m'j'}+\gamma_{58}C^*_{i'j'm'}\}D_{m'} = [\hat{\mathbf{C}}_2^*]_{i'j'},\end{aligned} \tag{5.4}$$

where no sum is taken except over m and m', and the rule on the index replacement is as follows: $ij \to i'$, $kl \to j'$, $mn \to m'$; $i' = i$ for $i = j$, $i' = 4$ for $i = 2$, $j = 3$ and $i = 3$, $j = 2$, $i' = 5$ for $i = 3$, $j = 1$ and $i = 1$, $j = 3$ and $i' = 6$ for $i = 1$, $j = 2$ and $i = 2$, $j = 1$ and so forth. In detail we have

$$\hat{\mathbf{C}}_2^* = \begin{Bmatrix} (A_{11}+B_{11}) & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & (A_{22}+B_{22}) & A_{23} & A_{24} & A_{25} & A_{26} \\ A_{31} & A_{32} & (A_{33}+B_{33}) & A_{34} & A_{35} & A_{36} \\ 0 & 0 & 0 & B_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{12} \end{Bmatrix} + \gamma_{58} \begin{Bmatrix} 4D_{11} & 0 & 0 & 0 & 2D_{31} & 2D_{12} \\ & 4D_{22} & 0 & 2D_{23} & 0 & 2D_{12} \\ & & 4D_{33} & 2D_{23} & 2D_{31} & 0 \\ & & & (D_{22}+D_{33}) & D_{12} & D_{31} \\ & & & & (D_{33}+D_{11}) & D_{23} \\ \text{(symmetric)} & & & & & (D_{11}+D_{22}) \end{Bmatrix}, \tag{5.5}$$

$$\begin{aligned} A_{ij'} = {} & [\{\gamma_1+\gamma_2(\varepsilon_k+\varepsilon_l)+\gamma_3(\sigma_k+\sigma_l)\}+\{\gamma_4+\ldots\}\sigma_i+\{\gamma_7+\ldots\}\varepsilon_i \\ & +\{\gamma_{10}+\ldots\}\sigma_i^2+\{\gamma_{13}+\ldots\}\varepsilon_i^2+\{\gamma_{16}+\ldots\}\sigma_i\varepsilon_i+\{\gamma_{19}+\ldots\}\sigma_i^2\varepsilon_i \\ & +\{\gamma_{22}+\ldots\}\sigma_i\varepsilon_i^2+\{\gamma_{25}+\ldots\}\sigma_i^2\varepsilon_i^2]D_j, \quad \text{for} \quad i = 1, 2, 3, \\ A_{ij'} = {} & 0 \quad \text{for} \quad i = 4, 5, 6. \end{aligned} \tag{5.6}$$

$$\begin{aligned} B_{ij} = {} & [\{\gamma_{28}\delta_m+\gamma_{29}\sigma_m+\gamma_{30}\varepsilon_m+\gamma_{31}\sigma_m^2+\gamma_{32}\varepsilon_m^2+\gamma_{33}\sigma_{\mathrm{m}}\varepsilon_{\mathrm{m}}\} \\ & +\{\gamma_{34}\delta_m+\ldots\}(\varepsilon_i+\varepsilon_j)+\{\gamma_{40}\delta_m+\ldots\}(\sigma_i+\sigma_j)+\{\gamma_{46}\delta_m+\ldots\}(\sigma_i^2+\sigma_j^2) \\ & +\{\gamma_{52}\delta_m+\ldots\}(\varepsilon_i^2+\varepsilon_j^2)]D_m, \end{aligned} \tag{5.7}$$

where the rule on the indices is the same as in Eq. (5.4).

According to Eq. (5.3), the equation

$$\det|\hat{\mathbf{C}}_2^*| = 0 \tag{5.8}$$

is the condition of constitutive instability for this case in which $\boldsymbol{\rho}$ becomes indefinite for definite $\mathring{\boldsymbol{\sigma}}$ and $\mathbf{D}$. However, there would be no way any more to recover the material stable constitution once Eq. (5.8) holds. In this sense, considering this state to be the onset of "loss of material continuity", it seems natural to call Eq. (5.8) a fracture condition expressed by the terms of $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$ and $\mathbf{D}$. Physically, the state at the instant when Eq. (5.8) is satisfied may correspond to the dislocation avalanche which is accompanied by a local instable deformation followed by fracture within the material. Of course, the condition (5.8) is a mathematical abstraction of such a physical phenomenon and thus includes somewhat its idealization.

For the further special case when $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$ and $\mathbf{D}$ are co-axial at the same time, we can put $D_{ii} = D_i$ and $D_{ij} = 0$ for $i \neq j$. Then $A_{ij'} = 0$ for $i = 1, 2, 3$ and $j = 4, 5, 6$ and $i = 4, 5, 6$ and $j = 1, 2, \ldots, 6$. $C_{i'j'} = 0$ for $i' \neq j'$ and $C_{i'i'} = 4D_{i'}$ for $i' = 1, 2, 3$. $C_{44} = D_2 + D_3$, $C_{55} = D_3 + D_1$ and $C_{66} = D_1 + D_2$. Putting $D'_i = \gamma_{58} D_i$, $i = 1, 2, 3$, we obtain the following equations which have the same form as Eqs. $(3.8)_1$, $(3.10)_1$ or (3.14):

$$\hat{\mathbf{C}}_2^* = \begin{Bmatrix} \mathbf{H}_N^* & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_S^* \end{Bmatrix}, \tag{5.9}$$

$$\mathbf{H}_N^* = \begin{Bmatrix} (A_{11}+B_{11}+4D'_1) & A_{12} & A_{13} \\ A_{21} & (A_{22}+B_{22}+4D'_2) & A_{23} \\ A_{31} & A_{32} & (A_{33}+B_{33}+4D'_3) \end{Bmatrix}, \tag{5.10}$$

$$\mathbf{H}_S^* = \begin{Bmatrix} (B_{23}+D'_2+D'_3) & 0 & 0 \\ 0 & (B_{31}+D'_3+D'_1) & 0 \\ 0 & 0 & (B_{12}+D'_1+D'_2) \end{Bmatrix}. \tag{5.11}$$

Consequently, the condition (5.5) is decomposed into two parts as follows:

$$\text{normal type:} \quad \det|\mathbf{H}_N^*| = 0, \tag{5.12}$$

$$\text{shear type:} \quad \det|\mathbf{H}_S^*| = 0. \tag{5.13}$$

And the flow modes of internal defects at the instant of the onset of fracture will be derived in a similar manner as the plastic flow rules associated with the subsequent yielding with which we shall not deal in detail here.

5.1.1. Examples of the fracture condition of normal type

If we put $A_{ij'} = 0$ in Eq. (5.12), we obtain a fracture condition

$$B'_{ii} + 4D'_i = 0. \tag{5.14}$$

This equation involves the following form for $D_i \neq 0$:

$$\{(\lambda_1 + \lambda_2 \varepsilon_m + \lambda_3 \varepsilon_m^2)\sigma_m^2 - \lambda_4 \varepsilon_m \sigma_m\} D_m^* = \lambda_0, \tag{5.15}$$

$D_m^* = D_m / D_i$, $\quad \lambda_0, \ldots, \lambda_4$ = material constants, where we add the terms $\varepsilon_m \sigma_m^2$ and $\varepsilon_m^2 \sigma_m^2$ to Eq. (5.8) and then the coefficients γ_ν may be different from those in Eq. (5.8). (If we adopt Spencer and Rivlin's theorem, these terms are inevitably involved).

If we put $B_{ij} = 0$ in Eq. (5.12), we obtain

(5.16) $$A_{ii} + 4D_i' = 0$$

which is also a kind of fracture condition. This involves the following form:

(5.17) $$(\lambda_1 + \lambda_2 \varepsilon_m + \lambda_3 \varepsilon_m^2)\sigma_m^2 - \lambda_4 \varepsilon_m \sigma_m = \lambda_0$$

$\lambda_0, \ldots, \lambda_4$ = material constants, which has the same form as the subsequent yield condition (4.15).

More generally we have

(5.18) $$A_{ii} + B_{ii} + 4D_i' = 0,$$

which involves the following form:

(5.19) $$f(\sigma_m, \varepsilon_m) \cdot D_m^* + g(\sigma_i, \varepsilon_i) = \lambda_0, \qquad D_m^* = D_m / D_i,$$

f, g = some quadratic functions of σ_k and ε_k, where we assume $D_i \neq 0$.

A striking aspect of the conditions (5.15) and (5.19) is that the strain rate $\mathbf{D}$ may generally affect the fracture condition by the terms of the ratios between its individual elements, that is, the direction of the strain rate vector $\hat{\mathbf{D}}$ in the strain space. (We should recall that any fracture criterion ever proposed does not include such terms as $\hat{\mathbf{D}}$ here). Of course, if $\hat{\mathbf{D}}$ never changes its direction throughout the deformation history imposed on the material, the fracture condition reduces to the same form as the subsequent yield one. Equation (5.17) may be understood to represent such an example.

5.1.2. Examples of the fracture condition of shear type

Adding the terms $\sigma_i \sigma_j$, $\varepsilon_i \varepsilon_j$ etc. to Eq. (5.8), the condition (5.13) gives the following fracture condition of shear type for the case when σ_1 is the intermediate principal stress:

(5.20) $$|(\sigma_2 - \sigma_3) - \mu_1(\varepsilon_2 - \varepsilon_3)| = -\mu_0 \pm \sqrt{-\gamma_{58}(D_2 + D_3)/\{f(\sigma_m, \varepsilon_m) D_m\}},$$

$\gamma_{58}, \mu_0, \mu_1$ = material constants, which is derived from the condition $B_{23} + D_2' + D_3' = 0$. The function f has a similar meaning as that in Eq. (5.19). Equation (5.20) gives a fracture condition of the same type as the subsequent yield condition (4.18) if $\gamma_{58} = 0$. However, for $\gamma_{58} \neq 0$, it also shows that the effect of the strain rate vector like that in Eq. (5.15) would exist.

5.2. Fracture condition expressed by the terms of $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$ and $\mathring{\boldsymbol{\sigma}}$

Equation $(1.1)_1$ can be rewritten into the following form:

(5.21) $$\mathbf{D} = \mathbf{E}^p : \mathring{\boldsymbol{\sigma}} = \mathbf{T}_1 + \mathbf{T}_2,$$

where $\mathbf{T}_1$ involves no terms of $\boldsymbol{\rho}$, whereas $\mathbf{T}_2$ does [cf. Eq. (3.1)]. $\mathbf{T}_2$ is expressed by the same form as the right hand side of Eq. (5.1), replacing $\mathbf{D}$ by $\mathring{\boldsymbol{\sigma}}$ and γ_1 to γ_{58} by δ_1 to δ_{58}, (say), which are also the functions of the invariants of $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$. Hence

(5.22) $$\mathbf{T}_2 = [\delta_1 \operatorname{tr}(\boldsymbol{\rho}\mathring{\boldsymbol{\sigma}}) + \delta_2 \operatorname{tr}(\boldsymbol{\epsilon}\boldsymbol{\rho}\mathring{\boldsymbol{\sigma}}) + \delta_3 \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{\rho}\mathring{\boldsymbol{\sigma}})]\mathbf{1} + \ldots + \delta_{58}(\boldsymbol{\rho}\mathring{\boldsymbol{\sigma}} + \mathring{\boldsymbol{\sigma}}\boldsymbol{\rho}).$$

Correspondingly to Eqs. (5.2) and (5.3), we have

$$(5.23)\qquad [\mathbf{T}_2]_{ij} = [\mathbf{E}_*]_{ij\cdot\cdot mn}^{\cdot\cdot kl\cdot\cdot} : [\mathring{\boldsymbol{\sigma}}]^{mn} : [\boldsymbol{\rho}]_{kl},$$

$$(5.24)\qquad \mathbf{T}_2 = \mathbf{E}_2^* : \boldsymbol{\rho}, \quad \hat{\mathbf{T}}_2 = \hat{\mathbf{E}}_2^* \boldsymbol{\rho}.$$

Then the equation

$$(5.25)\qquad \det|\hat{\mathbf{E}}_2^*| = 0$$

makes $\boldsymbol{\rho}$ indefinite with respect to definite $\mathbf{D}$ and $\mathring{\boldsymbol{\sigma}}$. Thus we may call Eq. (5.25) a fracture condition expressed by the terms of $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$ and $\mathring{\boldsymbol{\sigma}}$ for the case when they are given. This also shows a striking aspect that the stress rate could affect the fracture condition.

For the special case when co-axiality between $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ holds, we obtain the following equation corresponding to Eq. (5.4) with respect to the rectangular Cartesian coordinates which coincide with their principal axes:

$$(5.26)\qquad \begin{aligned}[\mathbf{E}_2^*]_{ijkl} &= [[\{\delta_1+\delta_2(\varepsilon_{\mathrm{m}}+\varepsilon_{\mathrm{n}})+\delta_3(\sigma_{\mathrm{m}}+\sigma_{\mathrm{n}})+\ldots]g_{\mathrm{mn}kl}\delta_{ij} \\ &\quad +[\{\delta_{28}+\ \ldots\}+\ \ldots]\delta_{\mathrm{mn}}g_{ijkl}+\delta_{58}(\ldots)]\dot{\sigma}_{mn} = F_{\mathrm{i}j'}\delta_{\mathrm{i}j}+G_{\mathrm{ij}}g_{\mathrm{ii}j'}+\delta_{58}H_{i'j'} \\ &= [F^*_{i'j'\mathrm{m}'}\delta_{\mathrm{m}'}+G^*_{i'\mathrm{m}'}g_{\mathrm{m}'j'}+\delta_{58}H^*_{i'j'm'}]\mathring{\sigma}_{m'} = [\hat{\mathbf{E}}_2^*]_{i'j'},\end{aligned}$$

where the rule on the indices is the same as that in Eq. (5.4). We can obtain the expressions for $\hat{\mathbf{E}}_2^*$ corresponding to Eqs. (5.5) to (5.8) just by the replacement of A, B, C, D_{ij} and γ_k in them by $F, G, H, \mathring{\sigma}_{ij}$ and δ_k, respectively.

For the further special case when $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$ and $\mathring{\boldsymbol{\sigma}}$ are co-axial at the same time, we put $\mathring{\sigma}_{\mathrm{ii}} = \mathring{\sigma}_i$ and $\mathring{\sigma}_{ij} = 0$ for $i \neq j$. Then again we obtain two kinds of fracture condition corresponding to Eqs. (5.9) to (5.13). Furthermore, correspondingly to Eqs. (5.15), (5.17) and (5.19), we have the following examples of fracture condition of normal type:

$$(5.27)\qquad \{(\nu_1+\nu_2\varepsilon_{\mathrm{m}}+\nu_3\varepsilon_{\mathrm{m}}^2)\sigma_{\mathrm{m}}^2-\nu_4\varepsilon_{\mathrm{m}}\sigma_{\mathrm{m}}\}\mathring{\sigma}_m^* = \nu_0,$$

$$(5.28)\qquad (\nu_1+\nu_2\varepsilon_m+\nu_3\varepsilon_m^2)\sigma_m^2-\nu_4\varepsilon_m\sigma_m = \nu_0,$$

$$(5.29)\qquad f(\sigma_m, \varepsilon_m)\mathring{\sigma}_m^*+g(\sigma_l, \varepsilon_l) = \nu_0,$$

where ν_0 to ν_4 are the material constants, $\mathring{\sigma}_m^* = \mathring{\sigma}_m/\mathring{\sigma}_l$, f and g = some quadratic functions of σ_k and ε_k and the assumption of $\mathring{\sigma}_l \neq 0$ is made.

As an example of a fracture condition of shear type, corresponding to Eq. (5.20), we obtain

$$|(\sigma_2-\sigma_3)-\omega_1(\varepsilon_2-\varepsilon_3)| = -\omega_0 \pm \sqrt{-\delta_{58}(\mathring{\sigma}_2+\mathring{\sigma}_3)/f^*(\sigma_m, \varepsilon_m)\mathring{\sigma}_m},$$

where σ_1 is the intermediate principal stress and δ_{58}, ω_0 and ω_1 are the material constants.

Equations (5.27) to (5.30) involve the conditions of the same type as those of the subsequent yield condition specifically when any change of loading path never occurs throughout the loading history, i.e. for a proportional (or simple) loading. However, we should recognize that the stress-rate vector $\mathring{\boldsymbol{\sigma}}$ may generally affect the fracture conditions by the terms of its direction in the stress space, which has again never been pointed out by any worker on fracture criterion.

6. A brief discussion on the proposed fracture condition

To clarify the meaning of a new aspect of the fracture condition proposed above we consider the special case when the loading path is kept proportional and then subjected to an abrupt change at the nearly critical state of fracture. Then the following three types of situation are expected to occur as a result of the influence of the terms of stress vector involved in the fracture conditions. Figure 2 shows them schematically, for convenience, in the two-dimensional stress space. In a) curve *1* shows the fracture locus correspond-

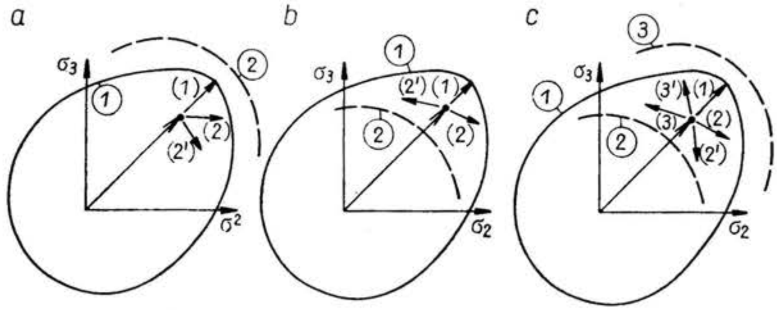

FIG. 2. Various influences of an abrupt change of the loading path on the subsequent fracture locus (Schematic illustrations in a two-dimensional stress space).

ing to the continued proportional loading (1) and the curve *2* shows the one corresponding to the other loading increment (2) which lies on the outer side of *1*. Thus the effect of the change of the loading path could prolong the material life.

In b) the curves *1* and *2* have the same meaning as those in a), but the resulting phenomenon cannot be the same at least. On the contrary, we should understand that a catastrophic fracture would occur at the instant of any abrupt change of the loading path, because the curve *2* lies on the inner side of *1*. This case seems most dangerous in our engineering sense.

Figure c) shows the intermediate situation. Namely, some kinds of change of the direction of the loading path would induce a catastrophic fracture and others would prolong the material life.

Of course, the occurrence of a) to c) depends on the form of the material function and constants involved in the fracture condition which should be determined by some appropriately prescribed experiments.

Evidently, similar conclusions would be drawn with respect to an abrupt change of the straining path by referring to the strain space.

7. Concluding remarks

The conditions of constitutive instability of elastoplastic materials are examined in detail with the aid of the representation theorems for isotropic functions. A rate-type elastoplastic constitutive equation which is an extension of the idea of hypoelasticity and was introduced by the author earlier is adopted as the basic expression of the material constitution. These conditions can be interpreted as the initial and subsequent yield and/or fracture conditions, according to the reference situation. The general form of the subsequent yield condition established here may give us a powerful clue to find in a rational manner any concrete form of such a condition. When $\boldsymbol{\sigma}$ (stress) and $\boldsymbol{\epsilon}$ (plastic strain) are co-axial, the subsequent yield condition is decomposed into two types, i.e. normal and shear ones, and involves Yoshimura's function as a normal type and an extended Tresca function as a shear type.

The fracture condition is expressed in two manners, i.e. 1) by the terms of $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$ and $\mathbf{D}$ (strain rate) and 2) by the terms of $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$ and $\mathring{\boldsymbol{\sigma}}$ (an objective stress rate), where $\mathbf{D}$ or $\mathring{\boldsymbol{\sigma}}$ plays its role through the direction of its vectorial form ($\hat{\mathbf{D}}$ or $\hat{\mathring{\boldsymbol{\sigma}}}$) in the corresponding strain or stress space. When co-axiality between the variables holds, the condition is again decomposed into two types, i.e. normal and shear ones, and involves some simple forms. Formally, the same condition as that of subsequent yielding might be a fracture condition specifically when the direction of $\hat{\mathbf{D}}$ or $\hat{\mathring{\boldsymbol{\sigma}}}$ is kept unchanged throughout the deformation history. However, we should note that the fracture condition newly proposed here implies some important roles of any change of the external straining or loading conditions. For example, an abrupt change of the loading (or straining) path could induce a catastrophic fracture of the material which would otherwise continue to deform stably or could prolong the life of the material which would otherwise cease to deform stably. The author hopes that this new proposition on the fracture criterion will enable the present theory of fracture to make a breakthrough in future development.

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