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Strong discontinuity wave in initially strained elastic medium

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AT THE SURFACE of discontinuity the strain gradient and the velocity are discontinuous. The equations of conservation of momentum and moment of momentum, the compatibility conditions and the stress-strain law constitute a system of equations governing the wave propagation problem. Assuming the adiabatic wave to be propagated into an initially strained medium, the jump of entropy and the propagation condition are determined. The propagation speed and the direction of amplitude are represented by power series of the amplitude. The resulting infinite set of algebraic equations is used to derive consecutive approximations of the propagation velocity and the direction of amplitude. The propagation speed is shown to increase with increasing absolute values of the amplitude, and for infinitesimal amplitudes it becomes equal to the velocity of sound. The strong discontinuity wave is supersonic for the region in front of the wave and subsonic for the region behind it. The condition of existence of the strong discontinuity wave is the mutual approach of two acceleration wave fronts (or two sound waves) propagating in initially strained material.

Na powierzchni nieciągłości gradient odkształcenia i prędkości są nieciągłe. Równania zachowania pędu i energii, warunki zgodności i związek naprężenie-odkształcenie tworzą układ równań rządzący propagacją fali. Rozważa się propagację fali adiabatycznej. Po rozłożeniu prędkości propagacji i amplitudy na szeregi potęgowe otrzymano rozwiązanie. Pokazano, że prędkość propagacji rośnie o ile rośnie amplituda fali. Pokazano też, że fala jest falą naddźwiękową w ośrodku znajdującym się przed frontem i falą poddźwiękową w ośrodku znajdującym się za frontem.

На поверхности разрыва градиент деформации и скорости имеют разрыв. Уравнения сохранения импульса и энергии, условия совместности и соотношение напряжение-деформация образуют систему уравнений описывающую распространение волны. Рассматривается распространение адиабатической волны. После разложения скорости распространения и амплитуды в степенные ряды получено решение. Показано, что скорость распространения растет, если растет амплитуда волны. Тоже показано, что волна является сверхзвуковой волной в среде находящейся перед фронтом и дозвуковой волной в среде находящейся за фронтом.

THE PROBLEM of propagation of weak discontinuity waves in non-linear elastic materials was recently considered in numerous papers. Many results in that area are given e.g. in the monograph [1]. The problem of strong discontinuity waves was considered in a limited scope, mainly as regards the one-dimensional phenomena [2]. It should be stressed that the problem of propagation of strong discontinuity waves in gaseous media has been discussed in detail; the same applies to the general theory of solution of differential equations, cf. e.g. [3].

In this paper we shall discuss the propagation of the strong discontinuity wave in an initially strained elastic material. The results derived are then compared with those concerning the acceleration wave. Similar solution for incompressible material is given in [4].

1. Discontinuity surface

Let us denote by $\mathscr{S}$ the surface dividing the reference configuration B_R into two parts. The equation of surface $\mathscr{S}$ being

$$t = \Psi(X^\alpha), \tag{1.1}$$

the unit normal N_α and velocity U (in the direction of N_α) are given by the formulae

$$N_\alpha = \frac{\Psi_{,\alpha}}{\sqrt{\Psi_{,\varrho}\Psi_{,}^{\varrho}}}, \qquad U = \frac{1}{\sqrt{\Psi_{,\varrho}\Psi_{,}^{\varrho}}}. \tag{1.2}$$

The physical fields may suffer jumps at $\mathscr{S}$. The value of a field H at the surface $\mathscr{S}$ measured at the front side of N_α is denoted by H^F, and at the opposite side — by H^B. The jump of the field value H at $\mathscr{S}$ is denoted by

$$[\![H]\!] = H^B - H^F. \tag{1.3}$$

If a certain magnitude H is continuous at $\mathscr{S}$ and its first and higher derivatives are discontinuous, then the compatibility conditions hold true:

$$[\![H_{,\alpha}]\!] = AN_\alpha, \qquad [\![H_{,t}]\!] = -AU, \tag{1.4}$$

where A is a parameter characterizing the jump magnitude. A detailed derivation of the relations may be found, for example, in [4]. Index t following a comma denotes differentiation with respect to time t.

Motion of the medium is described by the function

$$x^i = \xi^i(X^\alpha, t). \tag{1.5}$$

The derivatives (1.5) with respect to X^α and t are the strain gradient x^i_α and the velocity $\dot{x}^i$, respectively.

Let us consider the surface $\mathscr{S}_v$ at which the function (1.5) is continuous, and discontinuous are its derivatives:

$$x^i_\alpha = \frac{\partial \xi^i}{\partial X^\alpha}, \quad v^i = \dot{x}^i = \frac{\partial \xi^i}{\partial t}, \tag{1.6}$$

that is, the strain gradient and velocity of the medium. All the phenomena occurring at such a surface constitute the wave of strong discontinuity. The velocity of propagation of weak discontinuity waves is traditionally denoted by U. To stress the distinction between the two magnitudes, the propagation speed of the strong discontinuity wave will be denoted by U_v, index v symbolizing the discontinuity of the velocity v^i. The strong discontinuity wave itself will be called the velocity wave.

In compliance with the compatibility conditions (1.4), the following expressions for the strain gradient and velocity jumps are obtained in the case of strong discontinuity waves:

$$\begin{aligned} [\![x^i_{\alpha}]\!] &= H^i N_{\alpha}, \\ [\![\dot{x}^i]\!] &= -H^i U_v. \end{aligned} \tag{1.7}$$

The value of H^i characterizes the magnitudes of jumps of the strain gradients x^i_{α} and velocity $\dot{x}^i$; it is called the amplitude of a strong discontinuity wave.

Let us pass to the momentum and the moment of momentum conservation laws. Consider two positions of the surface $\mathscr{S}_v$ at the instants t_1 and t_2 and construct a curvi-

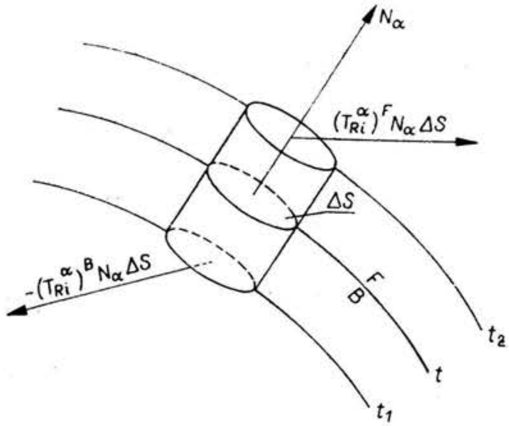

FIG. 1.

linear cylinder based on $\mathscr{S}_v$ (Fig. 1). In the time interval $t_1 < t < t_2$ the front surfaces of the cylinder are acted on by the force

$$(T_{Ri}{}^{\alpha})^F N_{\alpha} \Delta S + (T_{Ri}{}^{\alpha})^B (-N_{\alpha}) \Delta S,$$

$T_{Ri}{}^{\alpha}$ being the Piola-Kirchhoff stress tensor. The forces acting on the lateral surface are, for small $t_2 - t_1$, small values of a higher order of magnitude. The impulse of that force is equal to the momentum increment. Since the mass of the cylinder equals $\varrho_R U_v (t_2 - t_1) \Delta S$ the following relation holds true:

$$[(T_{Ri}{}^{\alpha})^F - (T_{Ri}{}^{\alpha})^B] N_{\alpha} \Delta S (t_2 - t_1) = \varrho_R U_v (t_2 - t_1) \Delta S [(\dot{x}_i)^B - (\dot{x}_i{}^F)]. \tag{1.8}$$

Passing to the limit with $\Delta S \to 0$, $t_2 \to t_1$, we obtain the momentum conservation law in the form

$$[\![T_{Ri}{}^{\alpha}]\!] N_{\alpha} = -\varrho_R U_v [\![\dot{x}_i]\!]. \tag{1.9}$$

According to the energy conservation requirements, the following equation must be satisfied:

$$\delta K + \delta \Sigma = \delta L + \delta Q. \tag{1.10}$$

Here δK denotes the increment of kinetic energy, $\delta\Sigma$ — the increment of potential energy L — work, Q — heat. All the magnitudes are referred to the cylinder under consideration and to the time interval $t_1 < t < t_2$.

In considering the cylinder shown in Fig. 1 we obtain the following expressions for δK, $\delta\Sigma$, δL and δQ;

$$\delta K = \frac{1}{2}\varrho_R U_v(t_2-t_1)\Delta S[(\dot{x}_i)^B(\dot{x}^i)^B-(\dot{x}_i)^F(\dot{x}^i)^F] = \frac{1}{2}\varrho_R U_v[\![\dot{x}_i\dot{x}^i]\!]\Delta S(t_2-t_1),$$

$$\delta\Sigma = \varrho_R U_v(t_2-t_1)\Delta S(\sigma^B-\sigma^F) = \varrho_R U_v[\![\sigma]\!]\Delta S(t_2-t_1),$$

$$\delta L = (T_{Ri}{}^{\alpha})^F N_\alpha \Delta S(\dot{x}^i)^F(t_2-t_1)+(T_{Ri}{}^{\alpha})^B(-N_\alpha)\Delta S(\dot{x}^i)^B(t_2-t_1) = -[\![T_{Ri}{}^{\alpha}\dot{x}^i]\!]N_\alpha\Delta S(t_2-t_1), \tag{1.11}$$

$$\delta Q = -Q_\alpha{}^F N^\alpha \Delta S(t_2-t_1)-Q_\alpha{}^B(-N^\alpha_\alpha)\Delta S(t_2-t_1) = [\![Q^\alpha]\!]N_\alpha\Delta S(t_2-t_1).$$

Here σ denotes the potential energy of a unit mass. In elastic materials $\sigma = \sigma(x^i{}_\alpha, \eta)$, η being the entropy. Q_α is the heat flux referred to the configuration B_R.

Substitution of Eqs. (1.11) into Eq. (1.10) and passing to the limit with $\Delta S \to 0$, $t_2 \to t_1$ yields the equation of energy conservation

$$\frac{1}{2}\varrho_R U_v[\![\dot{x}_i\dot{x}^i]\!]+\varrho_R U_v[\![\sigma]\!] = -[\![T_{Ri}{}^{\alpha}\dot{x}^i]\!]N_\alpha+[\![Q^\alpha]\!]N_\alpha. \tag{1.12}$$

It will be shown later that if $\dot{x}^i$ suffers a jump, then the entropy η must also suffer a jump. According to the second law of thermodynamics, the following relation holds true (T denotes the absolute temperature):

$$\varrho_R U_v\Delta S(t_2-t_1)\eta^B-\varrho_R U_v\Delta S(t_2-t_1)\eta^F \geqslant \left[\frac{(Q^\alpha)^B}{T^B}-\frac{(Q^\alpha)^F}{T^F}\right]N_\alpha\Delta S(t_2-t_1).$$

Passing to the limit $\Delta S \to 0$, $t_2 \to t_1$, we obtain

$$\varrho_R U_v[\![\eta]\!] \geqslant \left[\!\!\left[\frac{Q^\alpha}{T}\right]\!\!\right]N_\alpha. \tag{1.13}$$

If the process is adiabatic, then $Q^\alpha = 0$ and the inequality (1.13) is reduced to an inequality concerning the entropy jump

$$[\![\eta]\!] \geqslant 0. \tag{1.14}$$

The equations derived should be complemented by the appriopriate constitutive relations. In the case of a non-linear elastic material such relations have the form (cf. e.g. [1])

$$\sigma = \sigma(x^i{}_\alpha, \eta), \tag{1.15}$$

$$T_{Ri}{}^{\alpha} = \varrho_R\frac{\partial\sigma}{\partial x^i{}_\alpha}, \qquad T = \frac{\partial\sigma}{\partial\eta}. \tag{1.16}$$

The above enables us to express $[\![T_{Ri}{}^{\alpha}]\!]$ is terms of $[\![x^i{}_\alpha]\!]$ and $[\![\eta]\!]$.

2. Adiabatic wave

Let us confine our considerations to adiabatic waves. Since in such cases there is no heat flow, Eqs. (1.7), (1.9), (1.12), (1.15) and (1.16) are reduced to the following equations:

$$(2.1)\qquad [\![T_{Ri}{}^{\alpha}]\!]N_{\alpha} = -\varrho_R U_v [\![\dot{x}_i]\!],$$

$$(2.2)\qquad \varrho_R U_v [\![\sigma]\!] + \frac{1}{2}\varrho_R U_v [\![\dot{x}_i \dot{x}^i]\!] = -[\![T_{Ri}{}^{\alpha}\dot{x}^i]\!]N_{\alpha},$$

$$(2.3)\qquad [\![x^i{}_{\alpha}]\!] = H^i N_{\alpha},$$

$$(2.4)\qquad [\![\dot{x}^i]\!] = -H^i U_v,$$

$$(2.5)\qquad T_{Ri}{}^{\alpha} = \varrho_R \frac{\partial \sigma}{\partial x^i{}_{\alpha}},$$

$$(2.6)\qquad \sigma = \sigma(x^i{}_{\alpha}, \eta),$$

$$(2.7)\qquad [\![\eta]\!] \geqslant 0.$$

Equations (2.1)–(2.6) form a system of 26 equations with 27 unknowns $[\![T_{Ri}{}^{\alpha}]\!]$, $[\![x^i{}_{\alpha}]\!]$, $[\![\dot{x}^i]\!]$, $[\![\sigma]\!]$, $[\![\eta]\!]$, H^i and U_v. Thus the solution will depend on a single parameter; the parameter, which will be introduced later, will be shown to have a simple physical interpretation. From all the solutions we should select those which satisfy the inequality (2.7).

Let us consider a wave which is propagated into an initially strained material. The following relations hold

$$(2.8)\qquad \begin{aligned} [\![\dot{x}_i \dot{x}^i]\!] &= [\![\dot{x}_i]\!][\![\dot{x}^i]\!] + 2(\dot{x}^i)^F [\![\dot{x}_i]\!], \\ [\![T_{Ri}{}^{\alpha}\dot{x}^i]\!] &= [\![T_{Ri}{}^{\alpha}]\!][\![\dot{x}^i]\!] + (\dot{x}^i)^F [\![T_{Ri}{}^{\alpha}]\!] + (T_{Ri}{}^{\alpha})^F [\![\dot{x}^i]\!]. \end{aligned}$$

Equation (2.2) is then reduced to

$$(2.9)\qquad \varrho_R U_v [\![\sigma]\!] + \frac{1}{2}\varrho_R [\![\dot{x}_i]\!][\![\dot{x}^i]\!] = -[\![T_{Ri}{}^{\alpha}]\!][\![\dot{x}^i]\!]N_{\alpha} - (T_{Ri}{}^{\alpha})^F [\![\dot{x}^i]\!]N_{\alpha}.$$

Multiplication of Eq. (2.1) by $[\![\dot{x}^i]\!]$ yields

$$(2.10)\qquad [\![T_{Ri}{}^{\alpha}]\!][\![\dot{x}^i]\!]N_{\alpha} = -\varrho_R U_v [\![\dot{x}_i]\!][\![\dot{x}^i]\!].$$

On substituting the result into Eq. (2.9) we obtain two equivalent equations

$$(2.11)\qquad \varrho_R U_v [\![\sigma]\!] = \frac{1}{2}\varrho_R U_v [\![\dot{x}_i]\!][\![\dot{x}^i]\!] - (T_{Ri}{}^{\alpha})^F [\![\dot{x}^i]\!]N_{\alpha},$$

$$(2.12)\qquad 2\varrho_R U_v [\![\sigma]\!] + [\![T_{Ri}{}^{\alpha}]\!][\![\dot{x}^i]\!]N_{\alpha} = -2(T_{Ri}{}^{\alpha})^F [\![\dot{x}^i]\!]N_{\alpha}.$$

Let us assume $\sigma(x^i{}_{\alpha}, \eta)$ to be an analytical function of its arguments. In compliance with Eq. (2.5) the stress $T_{Ri}{}^{\alpha}$ is also an analytical function of the same arguments, and hence we have

$$\sigma^B = \sigma^F + \left(\frac{\partial \sigma}{\partial x^i{}_{\alpha}}\right)^F [\![x^i{}_{\alpha}]\!] + \left(\frac{\partial \sigma}{\partial \eta}\right)^F [\![\eta]\!] + \frac{1}{2}\left(\frac{\partial^2 \sigma}{\partial x^i{}_{\alpha}\partial x^k{}_{\beta}}\right)^F [\![x^i{}_{\alpha}]\!][\![x^k{}_{\alpha}]\!] + \frac{1}{2}\left(\frac{\partial^2 \sigma}{\partial x^i{}_{\alpha}\partial \eta}\right)^F \times$$

$$\times [\![x^i{}_{\alpha}]\!][\![\eta]\!] + \frac{1}{2}\left(\frac{\partial^2 \sigma}{\partial \eta^2}\right)^F [\![\eta]\!]^2 + \frac{1}{6}\left(\frac{\partial^3 \sigma}{\partial x^i{}_{\alpha}\partial x^k{}_{\beta}\partial x^m{}_{\gamma}}\right)^F [\![x^i{}_{\alpha}]\!][\![x^k{}_{\beta}]\!][\![x^m{}_{\gamma}]\!] + \ldots,$$

$$(T_{Ri}{}^{\alpha})^{B} = (T_{Ri}{}^{\alpha})^{F} + \varrho_R\left(\frac{\partial^2\sigma}{\partial x^i{}_\alpha \partial x^k{}_\alpha}\right)^F [\![x^k{}_\alpha]\!] + \varrho_R\left(\frac{\partial^2\sigma}{\partial x^i{}_\alpha \partial\eta}\right)^F [\![\eta]\!]$$
$$+\frac{1}{2}\varrho_R\left(\frac{\partial^3\sigma}{\partial x^i{}_\alpha \partial x^k{}_\beta \partial x^m{}_\gamma}\right)^F [\![x^k{}_\beta]\!][\![x^m{}_\gamma]\!] + \frac{1}{2}\varrho_R\left(\frac{\partial^3\sigma}{\partial x^i{}_\alpha \partial x^k{}_\beta \partial\eta}\right)^F [\![x^k{}_\beta]\!][\![\eta]\!]$$
$$+\frac{1}{2}\varrho_R\left(\frac{\partial^3\sigma}{\partial x^i{}_\alpha \partial\eta^2}\right)^F [\![\eta]\!]^2 + \frac{1}{6}\varrho_R\left(\frac{\partial^4\sigma}{\partial x^i{}_\alpha \partial x^k{}_\beta \partial x^m{}_\gamma \partial x^p{}_\delta}\right)^F [\![x^k{}_\beta]\!][\![x^m{}_\gamma]\!][\![x^p{}_\delta]\!] + \ldots .$$

These expressions are now substituted into Eqs. (2.1) and (2.14), taking into account the relations (2.3) and (2.4). For the sake of brevity the following notations are introduced:

$$\underbrace{\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma{}_{\ldots}}_{M} = \left(\frac{\partial^M\sigma}{\partial x^i{}_\alpha \partial x^k{}_\beta \partial x^m{}_{\gamma\ldots}}\right)^F,$$

(2.13)
$$\underbrace{\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma{}_{\ldots}}_{M}\underbrace{{}_{\eta\ldots\eta}}_{N} = \left(\frac{\partial^{M+N}\sigma}{\partial x^i{}_\alpha \partial x^k{}_\beta \partial x^m{}_{\gamma\ldots} \partial\eta^N}\right)^F,$$

$$[\![\eta]\!] = S.$$

In such a manner we obtain two equations:

(2.14)
$$\Big\{\sigma_i{}^\alpha{}_k{}^\beta H^k N_\beta + \sigma_i{}^\alpha{}_\eta S + \frac{1}{2}[\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma H^k H^m N_\beta N_\gamma + \sigma_i{}^\alpha{}_k{}^\beta{}_\eta H^k N_\beta S + \sigma_i{}^\alpha{}_{\eta\eta} S^2]$$
$$+\frac{1}{6}[\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma{}_n{}^\delta H^k H^m H^n N_\alpha N_\beta N_\delta + \sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma{}_\eta H^k H^m N_\beta N_\gamma S$$
$$+\sigma_i{}^\alpha{}_k{}^\beta{}_{\eta\eta} H^k N_\beta S^2 + \sigma_i{}^\alpha{}_{\eta\eta\eta} S^3] + \ldots\Big\} N_\alpha = U_v^2 H_i,$$

(2.15)
$$2\varrho_R U_v\Big\{\sigma_i{}^\alpha H^i N_\alpha + \sigma_\eta S + \frac{1}{2}[\sigma_i{}^\alpha{}_k{}^\beta H^i H^k N_\alpha N_\beta + \sigma_i{}^\alpha{}_\eta H^i N_\alpha S + \sigma_{\eta\eta} S^2]$$
$$+\frac{1}{6}[\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma H^i H^k H^m N_\alpha N_\beta N_\gamma + \sigma_i{}^\alpha{}_k{}^\beta{}_\eta H^i H^k N_\alpha N_\beta S + \sigma_i{}^\alpha{}_{\eta\eta} H^i N_\alpha S^2 + \sigma_{\eta\eta\eta} S^3]$$
$$+\frac{1}{24}[\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma{}_n{}^\delta H^i H^k H^m H^n N_\alpha N_\beta N_\gamma N_\delta + \ldots]\Big\} - \varrho_R U_v H^i N_\alpha\Big\{\sigma_i{}^\alpha{}_k{}^\beta H^k N_\beta + \sigma_i{}^\alpha{}_\eta S$$
$$+\frac{1}{2}[\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma H^k H^m N_\beta N_\gamma + \sigma_i{}^\alpha{}_k{}^\beta{}_\eta H^k N_\beta S + \sigma_i{}^\alpha{}_{\eta\eta} S^2]$$
$$+\frac{1}{6}[\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma{}_n{}^\delta H^k H^m H^n N_\beta N_\gamma N_\delta + \ldots]\Big\} = 2(T_{Ri}{}^\alpha)^F U_v H^i N_\alpha .$$

According to the formula (2.5)

(2.16)
$$(T_{Ri}{}^\alpha)^F = \varrho_R\left(\frac{\partial\sigma}{\partial x^i{}_\alpha}\right)^F = \varrho_R \sigma_i{}^\alpha .$$

It follows that the first and last terms of Eq. (2.15) are reduced and the expression $(T_{Ri}{}^{\alpha})^F$ does not appear in Eqs. (2.14) and (2.15). Dividing Eq. (2.15) by $\varrho_R U_v$ and ordering the result according to the powers we obtain

$$(2.17)\quad -\frac{1}{6}\sigma_i{}^{\alpha}{}_k{}^{\beta}{}_m{}^{\gamma}H^iH^kH^mN_\alpha N_\beta N_\gamma-\frac{1}{12}\sigma_i{}^{\alpha}{}_k{}^{\beta}{}_m{}^{\gamma}{}_n{}^{\delta}H^iH^kH^mH^nN_\alpha N_\beta N_\gamma N_\delta$$
$$+S\left(2\sigma_\eta-\frac{1}{6}\sigma_i{}^{\alpha}{}_k{}^{\beta}{}_\eta H^iH^kN_\alpha N_\beta-\frac{1}{12}\sigma_i{}^{\alpha}{}_k{}^{\beta}{}_m{}^{\gamma}{}_\eta H^iH^kH^mN_\alpha N_\beta N_\gamma\right)$$
$$+S^2\left(\sigma_{\eta\eta}-\frac{1}{6}\sigma_i{}^{\alpha}{}_{\eta\eta}H^iN_\alpha-\frac{1}{12}\sigma_i{}^{\alpha}{}_k{}^{\beta}{}_{\eta\eta}H^iH^kN_\alpha N_\beta\right)$$
$$+S^3\left(\frac{1}{3}\sigma_{\eta\eta\eta}-\frac{1}{12}\sigma_i{}^{\alpha}{}_{\eta\eta\eta}H^iN_\alpha\right)+S^4\left(\frac{1}{12}\sigma_{\eta\eta\eta\eta}\right)+\ldots=0.$$

Equations (2.14) and (2.17) constitute a system of four equations for the five unknowns H^i, U_v and S.

3. The propagation condition

Let us pass to the derivation of the propagation condition of the strong discontinuity wave. Equations (2.17) allows, in principle, for the determination of the function $S=S(H^i)$; however, since the order of that equation is infinite, the closed-form solution can not be determined. The solution will be sought for in the form of the following power series:

$$(3.1)\qquad S=C+C_iH^i+C_{ij}H^iH^j+C_{ijk}H^iH^jH^k+\ldots.$$

Let us observe that if $H^i=0$, then also $[\![x^i]\!]=0$, $[\![x^i_\alpha]\!]=0$ and, according to Eq. (2.13), $[\![\sigma]\!]$ must be equal to zero and $[\![\eta]\!]=S=0$. It follows that $C=0$. Inserting Eq. (3.1) into Eq. (2.17) and ordering the result according to the products $H^iH^j\ldots$ we obtain the equation

$$(3.2)\quad H^i(2C_i\sigma_\eta)+H^iH^k(2C_{ik}\sigma_\eta+C_iC_k\sigma_{\eta\eta})+H^iH^kH^m\left(-\frac{1}{6}\sigma_i{}^{\alpha}{}_k{}^{\beta}{}_m{}^{\gamma}N_\alpha N_\beta N_\gamma+2C_{ikm}\sigma_\eta\right.$$
$$\left.+\frac{1}{3}C_iC_kC_m\sigma_{\eta\eta}-\frac{1}{6}C_i\sigma_k{}^{\alpha}{}_m{}^{\beta}{}_\eta N_\alpha N_\beta-\frac{1}{6}C_iC_k\sigma_m{}^{\alpha}{}_{\eta\eta}N_\alpha+2C_iC_{km}\sigma_{\eta\eta}\right)$$
$$+H^iH^kH^mH^n\left(-\frac{1}{12}\sigma_i{}^{\alpha}{}_k{}^{\beta}{}_m{}^{\gamma}{}_n{}^{\delta}N_\alpha N_\beta N_\gamma N_\delta-\frac{1}{6}C_{ik}\sigma_m{}^{\alpha}{}_n{}^{\beta}{}_\gamma N_\alpha N_\beta\right.$$
$$-\frac{1}{12}C_iC_k\sigma_m{}^{\alpha}{}_n{}^{\beta}{}_{\eta\eta}N_\alpha N_\beta-\frac{1}{3}C_iC_{km}\sigma_n{}^{\alpha}{}_{\eta\eta}N_\alpha+2C_iC_{kmn}\sigma_{\eta\eta}+C_{ik}C_{mn}\sigma_{\eta\eta}$$
$$\left.-\frac{1}{12}A_iA_kA_m\sigma_n{}^{\alpha}{}_{\eta\eta\eta}N_\alpha+\frac{1}{12}C_iC_kC_mC_n\sigma_{\eta\eta\eta\eta}+2C_{ikmn}\sigma_\eta\right)+\ldots=0.$$

This equation must be satisfied for each H^i; this means that each of the coefficients of the consecutive products $H^iH^kH^m$... equals zero. It follows that

$$C_i = C_{ik} = 0,$$

$$C_{ikm} = \frac{1}{3}\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma N_\alpha N_\beta N_\gamma/\sigma_\eta, \tag{3.3}$$

$$C_{ikmn} = \frac{1}{24}\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma{}_n{}^\delta N_\alpha N_\beta N_\gamma N_\delta/\sigma_\eta;$$

and consequently, according to Eqs. (3.1) and (2.7), with the accuracy up to the terms $(H^i)^4$ we have

$$S = \frac{1}{12}\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma H^iH^kH^m N_\alpha N_\beta N_\gamma/\sigma_\eta + \frac{1}{24}\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma{}_n{}^\delta H^iH^kH^mH^n N_\alpha N_\beta N_\gamma N_\delta/\sigma_\eta + \ldots \geqslant 0. \tag{3.4}$$

The jump of entropy η is then of the order of m^3, $m = \sqrt{H^iH_i}$. Obviously S^2 is of the order of m^6, S^3 of the order of m^9 etc. Observe, moreover, that σ_η is equal to the absolute temperature and hence $\sigma_\eta > 0$.

Let us now pass to the determination of the propagation condition, the terms of orders which exceed m^4 being disregarded (all the derivatives are assumed to be of the same order). According to the above remark the expressions $S^2, S^3, \ldots$ are disregarded. Inserting the expression (3.4) into Eq. (2.14) we obtain the equation in which the only unknowns are the amplitude H^i and the propagation speed U_v of the strong discontinuity wave

$$\begin{aligned}(3.5)\quad & \sigma_i{}^\alpha{}_k{}^\beta H^k N_\alpha N_\beta + \frac{1}{2}\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma H^kH^m N_\alpha N_\beta N_\gamma + \frac{1}{6}\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma{}_n{}^\delta H^kH^mH^n N_\alpha N_\beta N_\gamma N_\delta \\ & + \frac{1}{24}\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma{}_n{}^\delta{}_p{}^\lambda H^kH^m \ldots N_\lambda + (\sigma_i{}^\alpha{}_\eta N_\alpha + \sigma_i{}^\alpha{}_k{}^\beta{}_\eta H^k N_\alpha N_\beta)\Big(\frac{1}{3}\sigma_m{}^\gamma{}_n{}^\delta{}_p{}^\lambda H^mH^nH^p N_\gamma N_\delta N_\lambda \\ & + \frac{1}{4}\sigma_m{}^\gamma{}_n{}^\delta{}_p{}^\lambda{}_q{}^\mu H^mH^nH^pH^q N_\gamma N_\delta N_\lambda N_\mu\Big)\Big/\sigma_\eta = H_i U_v^2.\end{aligned}$$

This is the propagation condition of the strong discontinuity wave. If the absolute value of the jump is prescribed,

$$m = (H_iH^i)^{1/2} = ([\![x^i_{\alpha}]\!][\![x_i{}^\alpha]\!])^{1/2}, \tag{3.6}$$

then Eq. (3.5) constitutes a system of three equations with three unknowns: the propagation speed U_v and two directional coefficients of the amplitude H_i. The system may be solved by means of, say, the method of consecutive approximations or by numerical methods. Once the direction of H_i is determined, we should verify whether S given by Eq. (3.4) satisfies the inequality (2.7). If $S \geqslant 0$, the wave can be propagated; if, in contrast, $S < 0$, then the solution is of a purely formal character and the strong discontinuity wave does not exist. The absolute value m of the jump of H^i is now found to be the parameter which was mentioned before in the discussion on the number of equations and unknowns. The solution depends on the wave intensity, its measure being m. Another problem arises in determining the equations of transport which could express m as a function of the position of $\mathcal{S}_v$. This problem will not be dealt with here; it may be mentioned solely that the magnitude of m is influenced not only by the initial conditions but also by the boundary values.

Observe that if H^i, U_v satisfy the condition (3.5) and the inequality (3.4), then also H^i, $-U_v$ must satisfy Eqs. (3.5) and (3.4). The strong discontinuity wave propagating in a certain direction may hence also be propagated in the opposite direction. It should be stressed that in spite of H^i and $[\![x^i_{\alpha}]\!]$ being the same for both waves, the velocity jumps $[\![\dot{x}^i]\!]$ of those waves differ by the sign; this follows from Eq. (2.4).

4. Propagation speed and the amplitude

The condition of propagation (3.5) is a non-linear system of three algebraic equations for four unknowns: the amplitude H^i and speed U_v. Its solution must then constitute a one-parameter family of magnitudes (H^i, U_v). Equation (3.5) is an algebraic equation of the order of infinity, thus it can not be determined in a closed form. In order to find an approximate solution let us observe that H^i/m is a unit vector the direction of which depends on m. The vector and the propagation speed are now assumed to be expandable into power series of the parameter m,

$$\frac{H^i(m)}{m} = \overset{0}{H^i}+m\overset{1}{H^i}+m^2\overset{2}{H^i}+ \ldots, \tag{4.1}$$

$$U_v(m) = \overset{0}{U}+m\overset{1}{U}+m^2\overset{2}{U}+ \ldots. \tag{4.2}$$

Let us now substitute the expressions (4.1) and (4.2) into the propagation condition (3.5), the parameter m being treated as small. The procedure yields the following system of equations:

$$(\sigma_i{}^{\alpha}{}_k{}^{\beta}N_{\alpha}N_{\beta}-\overset{0}{U}{}^2g_{ik})\overset{0}{H^k} = 0, \tag{4.3}$$

$$(\sigma_i{}^{\alpha}{}_k{}^{\beta}N_{\alpha}N_{\alpha}-\overset{0}{U}{}^2g_{ik})\overset{1}{H^k}-2\overset{0}{H_i}\overset{0}{U}\overset{1}{U}+\frac{1}{2}\sigma_i{}^{\alpha}{}_k{}^{\beta}{}_m{}^{\gamma}\overset{0}{H^k}\overset{0}{H^m}N_{\alpha}N_{\beta}N_{\gamma} = 0, \tag{4.4}$$

$$\begin{aligned}(\sigma_i{}^{\alpha}{}_k{}^{\beta}N_{\alpha}N_{\beta}-\overset{0}{U}{}^2g_{ik})\overset{2}{H^k}-2\overset{0}{H_i}\overset{0}{U}\overset{2}{U}-(2\overset{1}{H_i}\overset{0}{U}\overset{1}{U}+\overset{0}{H_i}\overset{1}{U}\overset{1}{U})+\sigma_i{}^{\alpha}{}_k{}^{\beta}{}_m{}^{\gamma}\overset{0}{H^k}\overset{1}{H^m}N_{\alpha}N_{\beta}N_{\gamma}\\ +\frac{1}{6}\sigma_i{}^{\alpha}{}_k{}^{\beta}{}_m{}^{\gamma}{}_n{}^{\delta}\overset{0}{H^k}\overset{0}{H^m}\overset{0}{H^n}N_{\alpha}N_{\beta}N_{\gamma}N_{\delta}+\frac{1}{12}\sigma_i{}^{\alpha}{}_{\eta}N_{\alpha}\sigma_k{}^{\beta}{}_m{}^{\gamma}{}_n{}^{\delta}\overset{0}{H^k}\overset{0}{H^m}\overset{0}{H^n}N_{\beta}N_{\gamma}N_{\delta}/\sigma_{\eta} = 0.\end{aligned} \tag{4.5}$$

These equations should be complemented by the condition $H^i/H_i = m^2$. According to Eq. (4.1) we have

$$\overset{0}{H^i}\overset{0}{H_i} = 1, \tag{4.6}$$

$$\overset{1}{H^i}\overset{0}{H_i} = 0, \tag{4.7}$$

$$2\overset{2}{H^i}\overset{0}{H_i}+\overset{1}{H^i}\overset{1}{H_i} = 0. \tag{4.8}$$

Equations (4.3) and (4.6) constitute a system of four equations for four unknowns $\overset{0}{H}{}^i$ and $\overset{0}{U}_v$. Once the unknowns are found, Eqs. (4.4) and (4.7) are treated as a set of four equations for the unknowns $\overset{1}{H}{}^i$ and $\overset{1}{U}_v$. A similar procedure leads to the determination of all subsequent coefficients $\overset{K}{H}{}^i$, $\overset{K}{U}_v$ of the series (4.1) and (4.2). From Eq. (4.3) it follows that $\overset{0}{H}{}^i$ is the eigenvector, and $\overset{0}{U}{}^2$ — the eigenvalue of the tensor $\sigma_i{}^\alpha{}_k{}^\beta N_\alpha N_\beta$.

Let us multiply Eqs. (4.4) by H^i. The expression in parenthesis is symmetric in i and k and thus, in view of Eq. (4.3), the first term vanishes. Using Eq. (4.6) we obtain

$$2\overset{0}{U}\overset{1}{U} = \frac{1}{2}\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma \overset{0}{H}{}^i\overset{0}{H}{}^k\overset{0}{H}{}^m N_\alpha N_\beta N_\gamma. \tag{4.9}$$

For small amplitudes the second term in Eq. (4.3) is neglegibly small when compared with the first one, and from Eq. $(1.14)_2$ it follows that $\sigma_\eta > 0$; hence we have

$$\sigma_i{}^\mu{}_k{}^\beta{}_m{}^\gamma \overset{0}{H}{}^i\overset{0}{H}{}^k\overset{0}{H}{}^m N^\alpha N_\beta N_\gamma \geqslant 0. \tag{4.10}$$

In view of Eqs. (4.9) and (4.10) the inequality

$$\overset{1}{U} = \frac{1}{4\overset{0}{U}}\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma \overset{0}{H}{}^i\overset{0}{H}{}^k\overset{0}{H}{}^m N_\alpha N_\beta N_\gamma \geqslant 0 \tag{4.11}$$

is always satisfied. The propagation speed of the strong discontinuity wave is then found to increase with increasing values of the velocity jump.

Substitution of the expression (4.9) into Eq. (4.4) yields the non-homogeneous system of algebraic equations

$$(\sigma_i{}^\alpha{}_k{}^\beta N_\alpha N_\beta - \overset{0}{U}{}^2 g_{ik})H^k = \frac{1}{2}\sigma_r{}^\alpha{}_k{}^\beta{}_m{}^\beta \overset{0}{H}{}^k\overset{0}{H}{}^m(\overset{0}{H}{}^r\overset{0}{H}_i - \delta_i^r)N_\alpha N_\beta N_\gamma, \tag{4.12}$$

which uniquely determines this part of $\overset{1}{H}{}^k$ which is orthogonal to $\overset{0}{H}{}^k$; the part of $\overset{1}{H}{}^k$ parallel to $\overset{0}{H}{}^k$ must be assumed, in view of Eq. (4.7), to be equal to zero. In turn, on multiplying the expression (4.5) by $\overset{0}{H}{}^i$ the value of $\overset{2}{H}{}^k$ is eliminated, what yields an equation for $\overset{2}{U}$, and then we may determine $\overset{2}{H}$ satisfying Eqs. (4.5) and (4.8). Such procedure makes it possible to determine all the coefficients of the expansions (4.1) and (4.2).

The function (1.5) maps the surface $\mathscr{S}_v$ in B_R onto a surface $\bar{\mathscr{S}}_v$ in B. This surface is propagated at a velocity u_v different from U_v. The normal n_i to the surface $\bar{\mathscr{S}}_v$ is determined by the formula

$$n_i = \frac{u_v}{U_v}N_\alpha(x^{-1})^\alpha{}_i. \tag{4.13}$$

If H^i is parallel to n_i, the wave is longitudinal, and if it is perpendicular to n_i, the wave is transverse. From the above considerations it is evident that the direction of H^i varies with variable intensity of the strong discontinuity wave. For instance, if for a certain m_0 the wave is longitudinal, then, in general; for $m < m_0$ and $m > m_0$ the wave ceases to be longitudinal.

In the particular case in which all the derivatives of σ are of the form

$$\sigma_{i}{}^{\alpha}{}_{k}{}^{\beta}{}_{m}{}^{\gamma}{}_{\dots} = \sigma_{\hat{i}}{}^{\alpha}{}_{\hat{k}}{}^{\beta}{}_{\hat{m}}{}^{\gamma} h_i^{\hat{i}} h_k^{\hat{k}} h_m^{\hat{m}} \dots, \tag{4.14}$$

in which $h_i^{\hat{i}}$ is an arbitrary tensor and $\sigma_{\hat{i}}{}^{\alpha}{}_{\hat{k}}{}^{\beta}{}_{\hat{m}}{}^{\gamma}$ is zero provided $\alpha \neq \hat{i}$ or $\beta \neq \hat{k}$ or $\gamma \neq m$, then $\overset{1}{H}{}^k = \overset{2}{H}{}^k = \dots = 0$ and the direction of the amplitude is independent of the wave intensity.

Let us now demonstrate certain connections between the waves of strong and weak discontinuity. The weak discontinuity wave embraces the totality of phenomena connected with the surface at which ξ^i, x^i_α and $\dot{x}^i$ are continuous, and all second derivatives of the function ξ^i are discontinuous. Such a wave is usually called an acceleration wave. The corresponding jumps of the second derivatives are determined by the formulae (cf. e.g. [1])

$$\begin{aligned} [\![x^i{}_{,\alpha,\beta}]\!] &= A^i N_\alpha N_\beta, \\ [\![\dot{x}^i{}_{,\alpha}]\!] &= -A^i N_\alpha U, \\ [\![\ddot{x}^i]\!] &= A^i U^2, \end{aligned} \tag{4.15}$$

and the propagation condition has the form

$$\left(\frac{\partial^2 \sigma}{\partial x^i{}_\alpha \partial x^k{}_\beta} N_\alpha N_\beta - U^2 g_{ik}\right) A^k = 0. \tag{4.16}$$

The higher order waves for which the n-th derivative of $\xi^i(X^\alpha, t)$ is discontinuous have the same direction of propagation, Eq. (4.16), and the same propagation speed U. U is called the sound velocity since the sound wave constitutes a superposition of such waves for $n \geqslant 2$.

The propagation speed U and the amplitude A^k of an acceleration wave are independent of the absolute value of that amplitude $(A^k A_k)^{1/2}$. Thus the situation is entirely different from that corresponding to the velocity wave.

Let us denote by U^F and $(A^k)^F$ the speed and amplitude of an acceleration wave propagating in front of the velocity wave $\mathscr{S}$. In view of Eq. (4.16) the condition of propagation of that wave is

$$\left[\left(\frac{\partial^2 \sigma}{\partial x^i{}_\alpha \partial x^k{}_\beta}\right)^F N_\alpha N_\beta - (U^F)^2 g_{ik}\right](A^k)^F = 0,$$

or, using the notations of Eq. (2.13),

$$[\sigma_i{}^\alpha{}_k{}^\beta N_\alpha N_\beta - (U^F)^2 g_{ik}](A^k)^F = 0. \tag{4.17}$$

Let us observe that Eq. (4.3) is identical with Eq. (4.17). At $m \to 0$ we have $U_v = U$ and hence Eq. (4.3) describes the strong discontinuity wave with an infinitesimal value of m. This observation yields an important conclusion, namely, that a velocity wave with infinitesimal intensity propagates in the same manner as the acceleration wave, and the following conditions are satisfied:

$$\begin{aligned} \overset{0}{U} &= U^F, \\ \overset{0}{H}{}^k &= (A^k)^F / |(A^k)^F|. \end{aligned} \tag{4.18}$$

In view of Eqs. (4.2), (4.3) and (4.18) we have the approximate equality

$$U_v = \overset{0}{U} + m\overset{1}{U} = U^F + \frac{m}{4U^F}\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma \overset{0}{H^i}\overset{0}{H^k}\overset{0}{H^m} N_\alpha N_\beta N_\gamma, \tag{4.19}$$

the last term of which is positive.

Let us then consider the acceleration wave propagating immediately behind the strong discontinuity wave front. Its propagation speed is denoted by U^B. The strain gradient right behind the strong discontinuity wave is approximately equal to

$$(x^i_\alpha)^B = (x^i_\alpha)^F + m\overset{0}{H^i} N_\alpha,$$

and so the following approximate equation holds true:

$$\left(\frac{\partial^2\sigma}{\partial x^i_\alpha \partial x^k_\beta}\right)^B = \left(\frac{\partial^2\sigma}{\partial x^i_\alpha \partial x^k_\beta}\right)^F + \left(\frac{\partial^3\sigma}{\partial x^i_\alpha \partial x^k_\beta \partial x^m_\gamma}\right)^F m\overset{0}{H^l} N_\alpha = \sigma_i{}^\alpha{}_k{}^\beta + \sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma m\overset{0}{H^m} N_\gamma.$$

The propagation condition (4.16) of the acceleration wave under consideration takes now the form

$$[(\sigma_i{}^\alpha{}_k{}^\beta + m\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma \overset{0}{H^m} N_\gamma) N_\alpha N_\beta - (U^B)^2 g_{ik}](A^k)^B = 0. \tag{4.20}$$

Due to the approximate relation $(A^k)^B = (A^k)^F = \overset{0}{H^k}$ we may write another approximate equation

$$\begin{aligned} (U^B)^2 &= (\sigma_i{}^\alpha{}_k{}^\beta + m\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma \overset{0}{H^m} N_\gamma)\overset{0}{H^i}\overset{0}{H^k} N_\alpha N_\beta = (U^F)^2 + m\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma \overset{0}{H^i}\overset{0}{H^k}\overset{0}{H^m} N_\alpha N_\beta N_\gamma, \\ U^B &= U^F + \frac{m}{2U^F}\sigma_i{}^\alpha{}_k{}^\beta{}_m{}^\gamma \overset{0}{H^i}\overset{0}{H^k}\overset{0}{H^m} N_\alpha N_\beta N_\gamma. \end{aligned} \tag{4.21}$$

The inequality (3.4) proves that the last term in Eq. (4.21) is positive. The relations (4.19) and (4.21) lead to the conclusion that

$$U^F \leqslant U_v \leqslant U^B. \tag{4.22}$$

The speeds U^F and U^B are the propagation speeds of the acceleration waves moving immediately in front of and behind the strong discontinuity wave which itself is propagated at the speed U_v. The sound waves are propagated at the speed equal to that of the acceleration wave. Thus the inequalities (4.22) yield an important qualitative result: the strong discontinuity wave represents a supersonic wave in the medium in front of the wave, and a subsonic wave in the medium behind it. Thus the velocity wave catches up with the acceleration waves running in front of it, thereby increasing (or decreasing) its intensity. On the other hand, it is being caught up by the acceleration waves running behind it; those increase or decrease its intensity.

Let us finally consider two acceleration waves propagating in the region $B_R{}^F$ at a small distance c from each other. Let us denote by $\overset{1}{X^\beta}$ a point lying at the instant t at the front of the first wave. The point $\overset{2}{X}{}^{\beta}$ lying at the front of the second wave is determined by the equation

$$\overset{2}{X}{}^{\beta} = \overset{1}{X}{}^{\beta} - cN^{\beta}.$$

Denote by $(x^i_{\alpha})_1$ the strain gradient at the front of the first wave, and by $(A^i)_1$ — its amplitude. The strain gradient and velocities of the second wave are approximately equal to

$$(x^i_{\alpha})_2 = (x^i_{\alpha})_1 + (x^i_{\alpha,\beta})_1(-cN^{B}),$$
$$(\dot{x}^i)_2 = (\dot{x}^i)_1 + (\dot{x}^i_{,\alpha})(-cN^{\beta}),$$

and hence, in view of Eqs. (4.15), we have

$$(4.23)\qquad \begin{aligned}(x^i_{\alpha})_2 &= (x^i_{\alpha})_1 - c(A^i)_1 N_{\alpha},\\ (\dot{x}^i)_2 &= (\dot{x}^i)_1 + c(A^i)_1 U_1.\end{aligned}$$

Let us also consider two waves for which $(\dot{x}^i)_2 - (\dot{x}^i)_1$ has the same sign as the jump $[\![\dot{x}^i]\!]$ of the velocity wave. If, for instance, the velocity wave is dilatational, then we consider the dilatational acceleration waves. According to Eq. (2.4) we have $[\![x^i]\!] = -H^i U_v$ and hence we should assume

$$(4.24)\qquad A^i = -\overset{0}{H}{}^i.$$

Let us denote by $(\sigma_i{}^{\alpha}{}_k{}^{\beta})_1$ the value of $\partial^2\sigma/\partial x^i_{\alpha}\partial x^k_{\beta}$ at the point $\overset{1}{X}{}^{\beta}$. At the point $\overset{2}{X}{}^{\beta}$ the function $(\sigma_i{}^{\alpha}{}_k{}^{\beta})$ is then approximately equal to

$$(\sigma_i{}^{\alpha}{}_k{}^{\beta})_2 = (\sigma_i{}^{\alpha}{}_k{}^{\beta})_1 + (\sigma_i{}^{\alpha}{}_k{}^{\beta}{}_m{}^{\gamma})_1(-c)(A^m)_1 N_{\gamma}.$$

Denoting by U_2 the speed of the second wave and taking into consideration the fact that the amplitudes are approximately equal, we obtain the following propagation conditions of the first and second waves:

$$(4.25)\qquad \begin{aligned}&[(\sigma_i{}^{\alpha}{}_k{}^{\beta})_1 N_{\alpha} N_{\beta} - U_1^2 g_{ik}](-\overset{0}{H}{}^k) = 0,\\ &[(\sigma_i{}^{\alpha}{}_k{}^{\beta})_1 N_{\alpha} N_{\beta} + (\sigma_i{}^{\alpha}{}_k{}^{\beta}{}_m{}^{\gamma})_1 \overset{0}{H}{}^m N_{\gamma} c - U_2^2 g_{ik}](-\overset{0}{H}{}^k) = 0.\end{aligned}$$

It follows immediately that

$$(4.26)\qquad \begin{aligned}U_1^2 &= (\sigma_i{}^{\alpha}{}_k{}^{\beta})_1 \overset{0}{H}{}^i \overset{0}{H}{}^k N_{\alpha} N_{\beta},\\ U_2^2 &= (\sigma_i{}^{\alpha}{}_k{}^{\beta})_1 \overset{0}{H}{}^i \overset{0}{H}{}^k N_{\alpha} N_{\beta} + c(\sigma_i{}^{\alpha}{}_k{}^{\beta}{}_m{}^{\gamma})_1 \overset{0}{H}{}^i \overset{0}{H}{}^k \overset{0}{H}{}^m N_{\alpha} N_{\beta} N_{\gamma}.\end{aligned}$$

In view of the inequality (3.4) we obtain

$$U_2 \geqslant U_1.$$

The condition of existence of the velocity wave is the mutual approach of two acceleration waves for which the sign of the difference $(\dot{x}^i)_2 - (\dot{x}^i)_1$ is the same as the sign of the jump $[\![x^i]\!]$ of the velocity wave.

References

1. Z. Wesołowski, *Dynamic problems of nonlinear elasticity* [in Polish], Warszawa 1974.
2. D. R. Bland, *Nonlinear dynamic elasticity*, Waltham 1969.
3. A. Hanyga, *On the solution to the Riemann problem for arbitrary hyperbolic system of laws*, Dissertation, Warsaw 1975.
4. C. Truesdell, R. Toupin, *The classical field theories*, Handbuch der Physik III/1, Berlin 1960.
5. Z. Wesołowski, W. Bürger, *Shock waves in incompressible elastic solids*, Rheol. Acta, **16**, 1977.

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