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Wave decay in nonhomogeneous elastic rods

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THIS PAPER deals with the propagation and decay of shock and acceleration waves in nonhomogeneous linear hyperelastic constrained rods. The decay equations are integrated to find the growth-decay laws for all shock and acceleration waves which can propagate in the rod. Expressions for the form of the induced higher order waves are also obtained. Specific results are given for a simple example.

Praca poświęcona jest zagadnieniu propagacji i zanikania fal uderzeniowych i fal przyspieszenia w niejednorodnych hiposprężystych, ograniczonych prętach. Przeprowadzono całkowanie równań i znaleziono wyrażenia na wzrost i malenie amplitud fal uderzeniowych i fal przyspieszenia. Określono również postać fal wyższego rzędu. Szczegółowe rezultaty uzyskano dla prostego przykładu.

Работа посвящена задачи распространения и затухания ударных волн и волн ускорения в неоднородных гипоупругих, ограниченных стержнях. Проведено интегрирование уравнений и найдены выражения для роста и убывания амплитуд ударных волн и волн ускорения. Определен тоже вид волн высшего порядка. Подробные результаты получены для простого примера.

Introduction

THE PROBLEM of shock and acceleration waves, within the framework of the modern rod theory as proposed by COHEN [1] and GREEN and LAWS [2], has been dealt with recently in the paper by COHEN and WHITMAN [3]. Particular attention was paid in that study to the propagation and decay-induction equations for the case of uniform rods, a concept introduced by ERICKSEN [4], relating to the homogeneity of rod . For general hyperelastic constitutive relations, it was shown that weak shock waves in uniform rods propagate without growth or decay. Specific results were presented for the linear elastic rod constitutive equations in the form proposed by GREEN, KNOPS and LAWS [5]. In this case, wave speeds, wave modes and induced wave effects were presented for the various uniform rod geometries; emphasis was put on how rod geometry influences the results. In addition, the case of an arbitrary plane rod was treated in order to show the nature of growth-decay effects for non-uniform rods. The special case dealing with the constrained rod theory(1), as developed by WHITMAN and DESILVA [6], was discussed calling attention to the simplification that takes place due to the reduction in the number of degrees of freedom. Additionally, we pointed out that in this case, there exist a set of more easily interpretable linear and angular strain variables.

In the study to be presented here, our objective is to formulate and deal with the problem

(1) A constrained rod is one in which the cross-sections of the rod do not deform, and hence are restricted to execute only rigid motions.

of shock and acceleration waves, within the framework of the constrained linear theory of hyperelastic rods. Our goal is to carry out this study, while allowing the constitutive relations to be fully general and nonhomogeneous. By this we mean that none of the constitutive relations take on any simplified forms, due to special symmetries, material properties or rod geometries, but rather depend in a fully general way on these governing influences. Within this general setting, our ultimate aim is to solve the decay-induction equation, thereby obtaining explicit results for the growth-decay and induced wave character of all waves which can propagate in the rod.

We begin in Sect. 1 by giving the governing equations. These consist of the conservation of linear and angular momentum equations for both discontinuous and continuous motions, the strain-displacement and constitutive relations, as well as the jump compatibility relations. Following an idea introduced by ERICKSEN [7], these equations are then reformulated in terms of a six-dimensional Euclidean vector space V, which allows them to take on a concise form, convenient for algebraic manipulation. In terms of this abstract setting, we obtain the propagation condition and decay-induction equation for shock waves in a highly compressed form. Basic to these relations are two symmetric constitutive tensors $\boldsymbol{M}$ and $\boldsymbol{K}$, $\boldsymbol{K}$ positive-definite, which are linear operators on V, and which describe the elastic and inertial properties of the rod, respectively. The propagation condition is an eigenvalue problem in which the right eigenvectors of $\boldsymbol{K}^{-1}\boldsymbol{M}$, which is in general a non-symmetric tensor, define the possible modes of wave propagation. Moreover, the eigenvalues of this problem define the squares of the possible speeds of propagation. The decay-induction equation is a vector differential equation in the wave mode vector, dependent on $\boldsymbol{M}$, $\boldsymbol{K}$ and a tensor $\boldsymbol{X}$ associated with the direction of the rod axis. This equation governs the growth-decay characteristics of all shock waves, in addition to determining the acceleration waves induced by the various propagating shock waves.

In the next section, using the positive definite bilinear operator $\boldsymbol{K}$ as metric, we define a new inner product on V which symmetrizes $\boldsymbol{K}^{-1}\boldsymbol{M}$, thus yielding real eigenvalues for the squares of the shock wave speeds entering the aforementioned eigenvalue problem. Moreover, with respect to this new inner product the associated eigen or wave mode vectors can be made to constitute an orthonormal basis of V. By manipulating with this basis we are able, for the case of distinct wave speeds, to integrate explicitly the growth-decay equation so as to yield an expression for the amplitude of the wave mode vector. This leads to a result which we interpret physically in terms of the propagation of the kinetic energy jump.

In Sect. 3, utilizing the same abstract setting for our problem we obtain for the case of distinct wave speeds an expression for the deterministic component of the induced higher order wave vector. The component of this induced second-order wave, parallel to the wave mode vector being considered, turns out to be indeterminate. We ultimately express the induced wave vector in terms of its components along the orthonormal basis of wave mode vectors. These components are expressed in terms of the matrix of $\boldsymbol{X}$ with respect to the orthonormal basis of wave vectors and a set of scalars measuring the variation along the rod of this orthonormal system.

In Sect. 4 we examine briefly the related problem of acceleration waves in rods. We find that the propagation condition for acceleration waves is the same as that for shock waves, while the decay-induction equation for acceleration waves differs somewhat from the corresponding equation for shock waves. In a manner similar to that utilized in dealing with shock waves, we obtain for acceleration waves of distinct wave speeds both the growth-decay law and the form of the induced wave.

In Sect. 5 our aim is to illustrate the results by means of a simple example. We make assumptions on $\boldsymbol{M}$, $\boldsymbol{K}$ and the wave modes which are equivalent to choosing a special simple form for the constitutive equations, common in the classical literature on rods. These constitutive equations uncouple the mechanical response with respect to the various classical linear and angular strain effects, such as extension, transverse shear, bending and twisting. For this case we give explicit formulae for the wave speeds, wave modes and decay in terms of the constitutive coefficients. In addition, we give expressions for the induced waves. For these we restrict ourselves to the case based on an assumption which is equivalent to considering straight rods with uniform cross-sectional material properties.

Finally, in the appendix we record the form taken by the decay equations for multiple wave speeds. These decay equations comprise a coupled system of first-order differential equations in the components of the wave mode vector. For eigenvalues of multiplicity r, there are r equations governing the variation of the wave mode vector in the associated r-dimensional eigenspace. However, the magnitude of this vector is seen to obey precisely the same growth-decay laws found earlier for the case of distinct wave speeds. We conclude the paper by also recording an expression for the induced wave in the case of multiple wave speeds.

1. Formulation of the basic equations

Prior to stating the basic equations governing wave propagation in rods, we shall define some of the mathematical notation and terminology to be used in the sequel. We believe it to follow accepted common usage. We will denote three-dimensional Euclidean point space by E and the associated translation space of vectors by $\mathbf{V}$. We use $\mathbf{r}$ to denote the position vector to points in E relative to a chosen origin. Bold face roman lower case letters will denote vectors in $\mathbf{V}$, while bold face roman upper case letters will denote linear transformations or, equivalently, tensors on $\mathbf{V}$. The standard Euclidean inner product on $\mathbf{V}$ will be denoted by $\langle , \rangle$ so that for $\mathbf{u}, \mathbf{v} \in \mathbf{V}$, $\langle \mathbf{u}, \mathbf{v} \rangle \in R$ where R denotes the real numbers. The norm or length $||\mathbf{v}||$ of $\mathbf{v}$ is defined by $||\mathbf{v}|| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2}$. The transpose $\mathbf{M}^T$ of a tensor $\mathbf{M}$ is defined by $\langle \mathbf{M}^T\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{M}\mathbf{v} \rangle$, $\forall \mathbf{u}, \mathbf{v} \in \mathbf{V}$, and $\mathbf{M}$ is symmetric and skew-symmetric if $\mathbf{M} = \mathbf{M}^T$ and $\mathbf{M} = -\mathbf{M}^T$, respectively. We say that $\mathbf{M}$ is positive-definite, denoted $\mathbf{M} > 0$, if $\mathbf{M}$ is symmetric and $\langle \mathbf{v}, \mathbf{M}\mathbf{v} \rangle > 0$, $\forall \mathbf{v} \neq 0$. We assume $\mathbf{V}$ to be endowed with a fixed orientation so that to any vector $\mathbf{v}$ there corresponds a unique skew-symmetric tensor $\mathbf{V}$ satisfying $\mathbf{V}\mathbf{u} = \mathbf{v} \times \mathbf{u}$, $\forall \mathbf{u} \in \mathbf{V}$. If $\mathbf{e}_i$ $(i = 1, 2, 3)$ is a basis of V, then the matrix M_{ij} of $\mathbf{M}$ with respect to this basis is defined by $\mathbf{M}\mathbf{e}_i = \sum_{j=1}^{3} M_{ji}\mathbf{e}_j$.

The direct sum $V = \mathbf{V}_1 \oplus \mathbf{V}_2$ where $\mathbf{V}_1 = \mathbf{V}_2 = \mathbf{V}$ will consist of all pairs $(\mathbf{u}, \mathbf{v})$ made into a vector space in the standard way. Vectors and tensors in V will be denoted by bold-face italic lower and upper case letters, respectively, viz. $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{M}, \boldsymbol{T}$, etc. The natural injections $\mathbf{J}_\alpha$ ($\alpha = 1, 2$) and projections $\boldsymbol{P}_\alpha$ ($\alpha = 1, 2$) are linear maps from $\mathbf{V} \to V$ and $V \to \mathbf{V}$, respectively. These are defined by $\mathbf{J}_\alpha \mathbf{u} = (\mathbf{u}\delta_{1\alpha'}, \mathbf{u}\delta_{2\alpha})$ and $\boldsymbol{P}_\alpha \boldsymbol{u} = \sum_{\beta=1}^{2} \mathbf{u}_\beta \delta_{\alpha\beta}$ where $\boldsymbol{u} = (\mathbf{u}_1, \mathbf{u}_2)$ and $\delta_{\alpha\beta}$ is the Kronecker delta. The inner product $\langle\, , \rangle$ on $\mathbf{V}$ induces in a natural way an inner product $\{\, , \}$ on V, defined by $\{\boldsymbol{u}, \boldsymbol{v}\} = \langle \boldsymbol{P}_1 \boldsymbol{u}, \boldsymbol{P}_1 \boldsymbol{v}\rangle + \langle \boldsymbol{P}_2 \boldsymbol{u}, \boldsymbol{P}_2 \boldsymbol{v}\rangle$, making the two factors of V orthogonal complements of one another. The notions of norm, transpose, symmetry, skew-symmetry and positive-definiteness are all defined for V, with respect to the inner product $\{\, , \}$, in the same way as for $\mathbf{V}$. Similarly, with respect to any basis $\boldsymbol{e}_i$ ($i = 1, \ldots, 6$) of V one can define the matrix M_{ij} of $\boldsymbol{M}$ by $\boldsymbol{M}\boldsymbol{e}_i = \sum_{j=1}^{6} M_{ji} \boldsymbol{e}_j$. Other notations will be introduced as is needed in what follows.

We initiate our study by giving the basic equations governing the dynamical behaviour of constrained rods within the framework of a linear isothermal hyperelastic theory. These equations derive from either the work of WHITMAN and DESILVA [8], or of GREEN and LAWS [9](2). The modern definition of a rod in mathematical parlance is a "vector bundle" which here consists of a curve C called the rod axis, having a two-dimensional subspace $\overline{\mathbf{V}}$ of $\mathbf{V}$ attached at each point. $\overline{\mathbf{V}}$ is specified by choice of a basis $\mathbf{d}_\alpha$ ($\alpha = 1, 2$), called directors, which characterize the size and shape of the rod cross-section. We assume then that the undeformed configuration of the rod is specified by vector functions

$$\mathbf{r} = \mathbf{r}(s), \quad \mathbf{d}_\alpha = \mathbf{d}_\alpha(s), \tag{1.1}$$

where

$$\mathbf{d}_\alpha \cdot \mathbf{t} = 0, \quad \mathbf{t} = \mathbf{r}'(s). \tag{1.2}$$

In the relations (1.2) $\mathbf{t}$ denotes the unit tangent vector to C, while s denotes the arc length parameterization of C. Heretofore we shall use $'$ to denote $(\partial/\partial s)_t$ and $\cdot$ to denote $(\partial/\partial t)_s$, where t is the time.

The rod deformation is assumed to consist of general motions of C and linear transformations of $\overline{\mathbf{V}}$ in $\mathbf{V}$. If we restrict $\overline{\mathbf{V}}$ to be rigid, then the transformations of $\overline{\mathbf{V}}$ in $\mathbf{V}$ will be orthogonal. This restriction gives rise to the so-called constrained theory. The basic equations which govern the rod motion pertaining to the constrained theory are:

$$[\mathbf{n}] = -\varrho_0 U[\dot{\mathbf{u}}], \quad [\mathbf{m}] = -\varrho_0 U\mathbf{B}[\dot{\boldsymbol{\phi}}], \tag{1.3}$$

$$\mathbf{n}' + \varrho_0 \mathbf{f} = \varrho_0 \ddot{\mathbf{u}}, \quad \mathbf{m}' + \mathbf{t} \times \mathbf{n} + \varrho_0 \mathbf{l} = \varrho_0 \mathbf{B}\ddot{\boldsymbol{\phi}}, \quad \mathbf{B} = \mathbf{B}^T > 0, \tag{1.4}$$

$$\mathbf{z} = \mathbf{u}' + \mathbf{t} \times \boldsymbol{\phi}, \quad \boldsymbol{\varkappa} = \boldsymbol{\phi}', \tag{1.5}$$

$$\mathbf{n} = \mathbf{E}\mathbf{z} + \mathbf{H}\boldsymbol{\varkappa}, \quad \mathbf{m} = \mathbf{H}^T\mathbf{z} + \mathbf{G}\boldsymbol{\varkappa}, \quad \mathbf{E} = \mathbf{E}^T, \quad \mathbf{G} = \mathbf{G}^T. \tag{1.6}$$

Equations $(1.3)_1$ and $(1.3)_2$ represent the balance of linear and angular momentum, respectively, across a point D on C of discontinuity of the momenta. This point is assumed to move on C with a speed of propagation $U \neq 0$, thus representing a moving singular point or wave. The jump in any quantity Ψ across D is denoted by $[\Psi]$, and the order of

(2) See Sect. 7 of Ref. [3] for a brief discussion of constrained rod theories.

the wave corresponds to the lowest order discontinuous derivative of Ψ. Shock and acceleration waves correspond to first and second-order displacement waves, respectively. The quantities $\mathbf{n}$ and $\mathbf{m}$ are the force and moment resultants, respectively, while $\mathbf{u}$ and $\boldsymbol{\phi}$ are the infinitesimal displacement and rotation vectors of the rod axis and cross-section, respectively. The rod density is denoted by ϱ_0. The tensor $\mathbf{B}$ represents the rotational inertia of the rod and its specification is equivalent to a constitutive assumption(3). Equations $(1.4)_1$ and $(1.4)_2$ express the balance of linear and angular momentum for continuous motions, respectively, with $\mathbf{f}$ and $\mathbf{l}$ being the body force and moment vectors, respectively. Equations (1.5) define the strains $\mathbf{z}$ and $\boldsymbol{\varkappa}$, where $\mathbf{z}$ represents the rod extension and shear while $\boldsymbol{\varkappa}$ describes rod bending and twisting. Finally, the relations (1.6) represent the linear constitutive equations for nonhomogeneous rods. No assumption regarding the character of the constitutive tensors $\mathbf{E}$, $\mathbf{G}$, and $\mathbf{H}$ is made other than the assumed symmetry $(1.6)_{3,4}$. These tensors depend arbitrarily on s with the intention that the dependence represents the combined effect of rod material properties, the geometry of the axis as well as the geometry of the cross-section. The specification of this dependence for a theory developed via the direct approach is an open question, while for theories based on a development from the three-dimensional theory of elasticity, the question has been dealt with by WHITMAN and COHEN [10].

For shock wave propagation we require

$$[\mathbf{u}] = [\boldsymbol{\phi}] = 0 \tag{1.7}$$

with the assumed non-zero first and second-order jumps related by the following compatibility relations:

$$[\dot{\mathbf{u}}] = -U[\mathbf{u}'], \quad [\dot{\boldsymbol{\phi}}] = -U[\boldsymbol{\phi}']; \tag{1.8}$$

$$\begin{aligned} [\ddot{\mathbf{u}}] &= U^2[\mathbf{u}''] - 2U[\tilde{\mathbf{u}}'] - \tilde{U}[\mathbf{u}'], \\ [\ddot{\boldsymbol{\phi}}] &= U^2[\boldsymbol{\phi}''] - 2U[\tilde{\boldsymbol{\phi}}'] - \tilde{U}[\boldsymbol{\phi}'], \end{aligned} \tag{1.9}$$

where for any quantity Ψ

$$\tilde{\Psi} = \dot{\Psi} + U\Psi', \tag{1.10}$$

is just the directional derivative along the path or the wave in a space-time manifold.

If we introduce the skew-symmetric tensor $\mathbf{T}$ corresponding to the unit tangent vector $\mathbf{t}$, then we may rewrite Eqs. $(1.4)_2$ and $(1.5)_1$ as

$$\mathbf{m}' + \mathbf{T}\mathbf{n} + \varrho_0 \mathbf{l} = \varrho_0 \mathbf{B}\ddot{\boldsymbol{\phi}}, \quad \mathbf{z} = \mathbf{u}' + \mathbf{T}\boldsymbol{\phi}, \tag{1.11}$$

and from this point on, this form of these equations will be used.

We now reformulate our problem by recasting the basic equations in terms of an equivalent setting involving the vector space V. If we define

$$\boldsymbol{d} = (\mathbf{u}, \boldsymbol{\phi}), \quad \boldsymbol{s} = (\mathbf{z}, \boldsymbol{\varkappa}), \quad \boldsymbol{t} = (\mathbf{n}, \mathbf{m}), \quad \boldsymbol{h} = (\mathbf{f}, \mathbf{l}), \tag{1.12}$$

(3) Specific forms for $\mathbf{B}$ are proposed in [8] and [9].

then the basic field and compatibility equations (1.3)–(1.9) with Eqs. $(1.4)_2$ and $(1.5)_1$ replaced by the relations (1.11) take the form

$$
\begin{aligned}
&[\boldsymbol{t}] = -\varrho_0 U \boldsymbol{B}[\dot{\boldsymbol{d}}], \qquad \boldsymbol{t}' + \boldsymbol{T}\boldsymbol{t} + \varrho_0 \boldsymbol{h} = \varrho_0 \boldsymbol{B}\ddot{\boldsymbol{d}}, \\
&\boldsymbol{t} = \boldsymbol{M}\boldsymbol{s}, \qquad \boldsymbol{s} = \boldsymbol{d}' + \boldsymbol{X}\boldsymbol{d};
\end{aligned} \tag{1.13}
$$

$$
[\dot{\boldsymbol{d}}] = -U[\boldsymbol{d}'], \qquad [\ddot{\boldsymbol{d}}] = U^2[\boldsymbol{d}''] - 2U[\tilde{\dot{\boldsymbol{d}}}'] - \tilde{U}[\boldsymbol{d}'], \tag{1.14}
$$

where

$$
\begin{aligned}
&\boldsymbol{B} = \boldsymbol{B}^T = \mathbf{J}_1 \boldsymbol{P}_1 + \mathbf{J}_2 \mathbf{B} \boldsymbol{P}_2 > 0, \\
&\boldsymbol{T} = \mathbf{J}_2 \mathbf{T} \boldsymbol{P}_1, \qquad \boldsymbol{X} = \mathbf{J}_1 \mathbf{T} \boldsymbol{P}_2, \qquad \boldsymbol{X} = -\boldsymbol{T}^T, \\
&\boldsymbol{M} = \boldsymbol{M}^T = \mathbf{J}_1 \mathbf{E} \boldsymbol{P}_1 + \mathbf{J}_1 \mathbf{H} \boldsymbol{P}_2 + \mathbf{J}_2 \mathbf{H}^T \boldsymbol{P}_1 + \mathbf{J}_2 \mathbf{G} \boldsymbol{P}_2 .
\end{aligned} \tag{1.15}
$$

We note that the properties in Eqs. (1.15) of the above tensors are with respect to the $\{\,,\}$ inner product on V. We also point out that we may assume $\boldsymbol{B}$ in a more general form than that specified by Eq. $(1.15)_1$ so as to include coupling between linear and angular momentum effects. Such forms are suggested by the work of ERICKSEN [7] via a direct approach and by the previously cited paper [10] via the three-dimensional approach. Heretofore, we shall assume $\boldsymbol{B}$ to be completely arbitrary in its form and in its dependence on s, provided only $\boldsymbol{B} > 0$.

The propagation condition and decay-induction equation governing shock waves in rods result from Eqs. $(1.13)_1$ and $(1.13)_2$, respectively, when we write these equations in terms of displacements $\boldsymbol{d}$ by substituting from Eqs. $(1.13)_{3,4}$ and $(1.14)_{1,2}$ in these equations. Upon assuming $\boldsymbol{h}$ to be continuous and taking the jump of Eq. $(1.13)_2$, we obtain

$$
\begin{aligned}
&(\boldsymbol{M} - \lambda \boldsymbol{K})\boldsymbol{a} = \boldsymbol{0}, \\
&(\boldsymbol{M} - \lambda \boldsymbol{K})\boldsymbol{b} + (\boldsymbol{M}' + \boldsymbol{W} + \lambda^{-1/2}\tilde{\lambda}\boldsymbol{K}/2)\boldsymbol{a} + 2\lambda^{1/2}\boldsymbol{K}\tilde{\boldsymbol{a}} = \boldsymbol{0},
\end{aligned} \tag{1.16}
$$

where we have set

$$
\begin{aligned}
&\lambda = U^2, \qquad \boldsymbol{a} = [\boldsymbol{d}'], \qquad \boldsymbol{b} = [\boldsymbol{d}''], \\
&\boldsymbol{K} = \varrho_0 \boldsymbol{B}, \qquad \boldsymbol{W} = -\boldsymbol{W}^T = \boldsymbol{M}\boldsymbol{X} - \boldsymbol{X}^T\boldsymbol{M}.
\end{aligned} \tag{1.17}
$$

2. Shock wave decay in rods

The basic equations governing the propagation of weak shock waves in rods are Eqs. (1.16). Here, our aim is to use these equations to investigate the growth-decay behaviour of these waves. To begin with, we note that since $\boldsymbol{K} > 0$, the inverse $\boldsymbol{K}^{-1}$ exists and we may rewrite the basic equations (1.16) in the form

$$
\begin{aligned}
&\boldsymbol{L}_\lambda \boldsymbol{a} = \boldsymbol{0}, \\
&\boldsymbol{L}_\lambda \boldsymbol{b} + \boldsymbol{H}_\lambda \boldsymbol{a} + 2\lambda \boldsymbol{a}' = \boldsymbol{0},
\end{aligned} \tag{2.1}
$$

where

$$
\boldsymbol{L}_\lambda = \boldsymbol{K}^{-1}\boldsymbol{M} - \lambda \boldsymbol{I}, \qquad \boldsymbol{H}_\lambda = \boldsymbol{K}^{-1}\boldsymbol{M}' + \boldsymbol{K}^{-1}\boldsymbol{W} + \lambda'\boldsymbol{I}/2. \tag{2.2}
$$

In writing Eq. $(2.1)_2$ we have used the assumption that $\boldsymbol{M}$ and $\boldsymbol{K}$ and hence λ and $\boldsymbol{a}$ are not time dependent. We note that Eq. $(2.1)_1$ has the same eigenvalues and eigenvectors as Eq. $(1.16)_1$, from which it is derived. In Eqs. (2.2) $\boldsymbol{I}$ is the unit tensor.

We now find it convenient to introduce a new inner product $\{\ ,\ \}_K$ on V defined by

$$\{\boldsymbol{u}, \boldsymbol{v}\}_K = \{\boldsymbol{u}, \boldsymbol{K}\boldsymbol{v}\}. \tag{2.3}$$

It is easily verified that this definition satisfies the requirements for an inner product. Now, with respect to this new inner product

$$\{\boldsymbol{u}, \boldsymbol{K}^{-1}\boldsymbol{M}\boldsymbol{v}\}_K = \{\boldsymbol{K}^{-1}\boldsymbol{M}\boldsymbol{u}, \boldsymbol{v}\}_K, \tag{2.4}$$

i.e. we have that $\boldsymbol{K}^{-1}\boldsymbol{M}$ is symmetric. Thus, the characteristic equation

$$\det \boldsymbol{L}_\lambda = 0, \tag{2.5}$$

associated with the eigenvalue problem in Eq. $(2.1)_1$ has six real roots λ_i ($i = 1, \ldots, 6$) which determine the possible speeds of wave propagation. We assume that these root are all distinct, leaving for the appendix a discussion of the case of multiple roots. If $\boldsymbol{K}^{-1}\boldsymbol{M} > 0$ then all $\lambda_i > 0$ and there will exist six real non-zero speeds of propagation. We note that the eigenvectors $\boldsymbol{w}_i$ belonging to the eigenvalues λ_i define the possible wave modes, and furthermore that these constitute an orthornormal basis with respect to $\{\ ,\ \}_K$, i.e.

$$\{\boldsymbol{w}_i, \boldsymbol{w}_j\}_K = \delta_{ij}. \tag{2.6}$$

In order to deal with Eq. $(2.1)_2$ we state the following four easily proved results:

$$\mathscr{L}1: \quad \{\boldsymbol{w}_i, \boldsymbol{L}_{\lambda_i}\boldsymbol{v}\}_K = 0, \quad \boldsymbol{v} \in V; \tag{2.7}$$

$$\mathscr{L}2: \quad 2\{\boldsymbol{w}_i, \boldsymbol{w}_i'\}_K = -\{\boldsymbol{w}_i, \boldsymbol{K}'\boldsymbol{w}_i\}; \tag{2.8}$$

$$\mathscr{L}3: \quad \{\boldsymbol{w}_i, (\boldsymbol{M}' - \lambda_i \boldsymbol{K}')\boldsymbol{w}_i\} = \lambda_i'; \tag{2.9}$$

$$\mathscr{L}4: \quad \{\boldsymbol{w}_i, \boldsymbol{W}\boldsymbol{w}_i\} = 0. \tag{2.10}$$

R e m a r k s. $\mathscr{L}1$ is obvious since $\boldsymbol{L}_{\lambda_i}$ annihilates that part of $\boldsymbol{v}$ parallel to $\boldsymbol{w}_i$. $\mathscr{L}2$ follows on differentiating Eq. (2.6), while $\mathscr{L}3$ results on differentiating Eq. $(2.1)_1$ and using Eq. (2.6). Finally, $\mathscr{L}4$ is true since $\boldsymbol{W}$ is skew-symmetric.

We now set

$$\boldsymbol{a}_i = a_i \boldsymbol{w}_i, \tag{2.11}$$

where a_i is the amplitude of the ith wave mode vector and we substitute the relation (2.11) into the decay-induction equation $(2.1)_2$. We then take the $\{\ ,\ \}_K$ inner product of the resulting equation with $\boldsymbol{w}_i$, and utilize Eq. (2.2) and the relations (2.7)–(2.10) to find that the differential equation

$$a_i'/a_i = -3\lambda_i'/4\lambda_i \tag{2.12}$$

governs the amplitude of the ith wave mode vector. This equation yields the growth-decay relation

$$a_i/a_{i0} = (U_{i0}/U_i)^{3/2}, \tag{2.13}$$

where the subscript 0 indicates evaluation at a point $s = s_0$ on the rod. We note further, since $a_i = ||\boldsymbol{a}_i||_K$, that

$$||\boldsymbol{a}_i|| = a_i(K_{\bar{i}\bar{i}}^{-1})^{1/2}. \tag{2.14}$$

The above results may now be summarized in the form of the following theorem concerning the physics of waves in rods:

THEOREM. The discontinuity in the kinetic energy k associated with weak shock waves propagating into undeformed rods at rest is such that

$$[k_i] \propto U_i^{-1}, \tag{2.15}$$

where U_i is the speed of the ith wave mode w_i given by

$$U_i^2 = \{w_i, Mw_i\}. \tag{2.16}$$

Remark. The result in the relation (2.15) follows from Eqs. (1.8), (2.13) and the standard definition of k as

$$k = 1/2\varrho_0\langle\dot{\mathbf{u}}, \dot{\mathbf{u}}\rangle + 1/2\varrho_0\langle\dot{\boldsymbol{\phi}}, \mathbf{B}\dot{\boldsymbol{\phi}}\rangle = 1/2\{\dot{d}, \dot{d}\}_K, \tag{2.17}$$

and since for the region ahead of the wave at rest we have

$$[\|\dot{d}\|_K^2] = \|[\dot{d}]\|_K^2. \tag{2.18}$$

3. The induced wave

We now turn our attention to obtaining an expression for the induced second-order wave b_i associated with the shock wave a_i. We write $V = V_i \oplus V_i^{\perp}$, where V_i and $V_i^{\perp}$ denote respectively, the one-dimensional subspace of V spanned by w_i and the orthogonal complement of V_i in V with respect to $\{\ ,\ \}_K$. We define also the orthogonal projection operators $P_i\colon V \to V_i$ and $P_i^{\perp}\colon V \to V_i^{\perp}$, again with respect to $\{\ ,\ \}_K$. This decomposition of V leads us to write

$$b_i = b_i w_i + i_i, \qquad \{w_i, i_i\}_K = 0. \tag{3.1}$$

If we substitute Eq. $(3.1)_1$ into the basic decay-induction equation $(2.1)_2$, then it is apparent from Eq. (2.7) that b_i is indeterminate. Thus, the component of the acceleration wave parallel to the primary shock wave may be specified arbitrarily. In order to determine its behaviour, it is necessary to differentiate Eqs. $(1.13)_2$ and to use suitably the result to find an appropriate decay-induction equation.

On the other hand, the component i_i of b_i is fully determined by Eq. $(2.1)_2$ and is in fact the induced wave. If we define $G_i\colon V_i^{\perp} \to V_i^{\perp}$ by

$$G_i = L_{\lambda_i|V_i^{\perp}}, \tag{3.2}$$

then G_i is non-singular and G_i^{-1} exists, i.e.

$$G_i^{-1}G_i = G_iG_i^{-1} = I_{|V_i^{\perp}}. \tag{3.3}$$

We then find from Eq. $(2.1)_2$ that

$$i_i = -G_i^{-1}P_i^{\perp}(H_{\lambda_i}a_i + 2\lambda_i a_i'), \qquad i = 1, \ldots, 6. \tag{3.4}$$

In order to reduce Eq. (3.4) further we write

$$w_i' = \sum_{i=1}^{6}\Gamma_{ij}w_j, \qquad Xw_i = \sum_{j=1}^{6}X_{ji}w_j, \tag{3.5}$$

so that the Γ_{ij} are in differential geometric terms akin to connection coefficients, while the X_{ij} are the components of the matrix of X with respect to the w_i as basis. In addition, the following easily verified results will be of use to us:

$$\mathscr{L}5: \quad G_i^{-1} = \sum_{\substack{j=1 \\ j \neq i}}^{6} \frac{P_j}{(\lambda_j - \lambda_i)} = \sum_{\substack{j=1 \\ j \neq i}}^{6} \frac{w_j \{w_j, \}_K}{(\lambda_j - \lambda_i)}; \tag{3.6}$$

$$\mathscr{L}6: \quad \Gamma_{ij} = \{M' - \lambda_i K')w_i, w_j\}/(\lambda_i - \lambda_j), \quad i \neq j; \tag{3.7}$$

$$\mathscr{L}7: \quad \Gamma_{ii} = -\{w_i, K'w_i\}/2, \quad K' = K'^T; \tag{3.8}$$

$$\mathscr{L}8: \quad \{Ww_i, w_j\} = \lambda_j \{Xw_i, w_j\}_K - \lambda_i \{Xw_j, w_i\}_K. \tag{3.9}$$

Remarks. $\mathscr{L}5$ is easily verified by writing $G_i = \sum_{j=1}^{6} (\lambda_j - \lambda_i) P_j$ and substituting these into Eq. (3.3). $\mathscr{L}6$ results on differentiating Eq. $(2.1)_1$ and using Eqs. $(3.5)_1$ and (2.6). The result $\mathscr{L}7$ is not utilized here but given for the sake of interest only and results on differentiating Eq. (2.6) and using Eq. $(3.5)_1$. In fact, the symmetric part of Γ_{ij} as given by Eq. (3.7) may be obtained in this way. Finally, $\mathscr{L}8$ results on using Eqs. $(1.17)_5$ and $(2.1)_1$.

If we now substitute the relation (2.11) into Eq. (3.4) using Eqs. (2.6), (3.5), (3.6), (3.7) and (3.9), we find the induced waves to be given in the form

$$i_i = a_i \sum_{\substack{j=1 \\ j \neq i}}^{6} [\lambda_j \Gamma_{ij} - \lambda_i (2\Gamma_{ij} - \Gamma_{ji}) - (\lambda_j X_{ji} - \lambda_i X_{ij})] w_j / (\lambda_j - \lambda_i). \tag{3.10}$$

4. Acceleration waves

We now turn our attention briefly to the problem of acceleration waves within the framework of the theory of rods being considered. An acceleration wave is defined by the conditions

$$[d] = [\dot{d}] = 0, \quad [\ddot{d}] \neq 0. \tag{4.1}$$

For this situation Eq. $(1.13)_1$ is satisfied identically by virtue of Eqs. $(4.1)_{1,2}$, $(1.13)_{3,4}$ and $(1.14)_1$. If we now take the jump of the equations of motion $(1.13)_2$ and write the resulting equations in terms of displacements by using Eqs. $(1.13)_{3,4}$, $(1.14)_2$ and (4.1), then we obtain the propagation condition for acceleration waves which has precisely the same form as Eq. $(1.16)_1$, the corresponding propagation condition foı shock waves.

In order to obtain the decay-induction equation for acceleration waves, we differentiate the equations of motion $(1.13)_2$ with respect to time so as to obtain

$$[\dot{t}'] + T[\dot{t}] = \varrho_0 K[\dddot{d}], \tag{4.2}$$

on assuming the continuity of $\dot{h}$. The compatibility relations of second and third order which are necessary to the analysis are given by

$$\begin{aligned} [\dot{d}'] &= -U[d''], \quad [\dot{d}''] = -U[d'''] + \widetilde{[\dot{d}'']}, \\ [\dddot{d}] &= -U^3[d'''] + 3U^2 \widetilde{[d'']} + 3U\tilde{U}[d'']. \end{aligned} \tag{4.3}$$

If we now utilize Eqs. $(1.13)_{3,4}$ and (4.3) in Eq. (4.2), we obtain an equation in terms of displacements which is the decay-induction equation for acceleration waves, corresponding to Eq. $(1.16)_2$ for shock waves.

Rewriting the aforementioned propagation and decay-induction equations by multiplying by $\boldsymbol{K}^{-1}$, we obtain as analogues of Eqs. (2.1) for shock waves the basic equations for acceleration waves in the form

$$\begin{aligned} \boldsymbol{L}_\lambda \overline{\boldsymbol{a}} &= \boldsymbol{0}, \\ \boldsymbol{L}_\lambda \overline{\boldsymbol{b}} + (\boldsymbol{H}_\lambda + \lambda')\overline{\boldsymbol{a}} - (\boldsymbol{L}_\lambda - 2\lambda)\overline{\boldsymbol{a}}' &= \boldsymbol{0}, \end{aligned} \tag{4.4}$$

where

$$\overline{\boldsymbol{a}} = [\boldsymbol{d}]'', \quad \overline{\boldsymbol{b}} = [\boldsymbol{d}''']. \tag{4.5}$$

It is clear that the speeds and modes of propagation determined by the relation $(4.4)_1$ are precisely the same as for shock waves and that the relation (2.6) holds for the normalized eigenvectors. Following the procedure used in Sect. 2 to analyse the decay of shock waves, we write

$$\overline{\boldsymbol{a}}_i = \overline{a}_i \boldsymbol{w}_i, \tag{4.6}$$

in Eq. $(4.4)_2$. Then, we take the scalar product of Eq. $(4.4)_2$ with $\boldsymbol{w}_i$, with respect to $\{\ ,\ \}_K$, and use Eqs. (2.6)–(2.10) to obtain

$$\overline{a}_i'/\overline{a}_i = -5\lambda_i'/4\lambda_i, \tag{4.7}$$

resulting in

$$\overline{a}_i/\overline{a}_{i0} = (U_{i0}/U_i)^{5/2}, \tag{4.8}$$

for the determination of the growth-decay characteristics of acceleration waves. As usual, the subscript 0 indicates evaluation at some reference point on the rod.

Following the procedure in Sect. 3 we find the induced wave $\overline{\boldsymbol{i}}_i \in V_i^\perp$ corresponding to $\overline{\boldsymbol{a}}_i$ to be given by

$$\overline{\boldsymbol{i}}_i = -\boldsymbol{G}_i^{-1}\boldsymbol{P}_i^\perp[(\boldsymbol{H}_{\lambda_i} + \lambda_i')\overline{\boldsymbol{a}}_i + 2\lambda_i \overline{\boldsymbol{a}}_i'] + \boldsymbol{P}_i^\perp \overline{\boldsymbol{a}}_i'. \tag{4.9}$$

An equation analogous to Eq. (3.10) is readily found from Eq. (4.9), but we do not reproduce it here.

5. An example

In this section we consider an illustration of the foregoing analysis motivated by classical theories of rods which employ constitutive relations of a particularly simple form, as for example those given by LOVE [11] and utilized by WHITMAN and DESILVA [12]. We now proceed to make the following simplifying assumptions which pertain to restrictions on the constitutive tensors $\boldsymbol{M}$ and $\boldsymbol{K}$. These are:

(5.1) $\mathscr{A}1$: $\boldsymbol{MK} = \boldsymbol{KM}$;

(5.2) $\mathscr{A}2$: $\boldsymbol{w}_{\alpha i} \equiv \boldsymbol{P}_\alpha \boldsymbol{w}_i = \boldsymbol{0}$, if $\alpha = 1$, $i = 4, 5, 6$, and $\alpha = 2$, $i = 1, 2, 3$;

(5.3) $\mathscr{A}3$: $\boldsymbol{w}_{1i} \| \boldsymbol{w}_{2(i+3)}$, $\mathrm{i} = 1, 2, 3$, $\boldsymbol{w}_{23} \| \mathbf{t}$; $\Gamma_{ij} = 0$, $i \neq j$.

Remarks. Assumption $\mathscr{A}1$ states that $\boldsymbol{M}$ and $\boldsymbol{K}$ commute. Although the physical implications of this assumption are not totally clear, mathematically it implies that the matrices of $\boldsymbol{M}$ and $\boldsymbol{K}$, with respect to $\boldsymbol{w}_i$ as basis, are simultaneously diagonalized. In this case the squares of the speeds of propagation determined from the relation (2.16) are given by

$$U_i^2 = M_{ii}/K_{ii}. \tag{5.4}$$

Moreover, the $\boldsymbol{w}_i$ are now orthogonal with respect to the $\{\ ,\ \}$ inner product. Assumption $\mathscr{A}2$ implies that V splits into the direct sum $V = \boldsymbol{P}_1 V \oplus \boldsymbol{P}_2 V$ such that $\boldsymbol{w}_i$ $(i = 1, 2, 3)$ and $\boldsymbol{w}_i$ $(i = 4, 5, 6)$ are orthogonal bases of V_1 and V_2, respectively. The assumptions $\mathscr{A}1$ and $\mathscr{A}2$ taken together imply that linear strain (i.e. extension and shear) and angular strain (i.e. bending and twisting) are uncoupled with respect to the modes of propagation. In essence, this is so because the constitutive tensor $\mathbf{H}$ and any linear and angular momentum coupling are zero. In addition, the constitutive tensors $\mathbf{E}$ and $\mathbf{G}$, $\mathbf{B}$ have diagonal-matrices with respect to $\boldsymbol{w}_{1\mathrm{i}}$ and $\boldsymbol{w}_{2\mathrm{i}}$, respectively(4). The first part of $\mathscr{A}3$ along with $\mathscr{A}1$ and $\mathscr{A}2$ state simply that the rod axis and the cross-sectional plane perpendicular to it are associated with the directions of diagonalization of the constitutive matrices, and furthermore that these directions coincide for linear and angular strain effects. In fact, for this situation the modes of propagation are such as to completely uncouple all the usual strain effects such as axial extension, orthogonal transverse shearing and bending components, and axial twisting. The second part of $\mathscr{A}3$ pertains primarily to rod geometry and is taken so as to limit the nature of the induced wave effects we wish to present. This assumption restricts the analysis to rods with a straight axis provided the cross-sectional material properties of the rod are assumed to be homogeneous. However, physical situations may exist in which the rod axis is not straight, but cross-sectional material properties are such as to render the Γ_{ij} zero. On the other hand, there may exist rods with straight axes for which $\Gamma_{ij} \neq 0$. One of the interesting features of wave propagation in rods is how the Γ_{ij} effect the various primary and induced modes of propagation, a subject that has been dealt with by COHEN and WHITMAN in [3]. Here, we content ourselves with examining the restricted situation described by the above assumptions, leaving more general situations for future study.

Invoking the assumptions (5.1), (5.2) and (5.3) we find from Eqs. (5.4), (2.13), (2.14), (1.15) and $(1.17)_4$ that

$$U_{1\mathrm{i}}^2 = E_{\mathrm{ii}}/\varrho_0, \qquad \|\boldsymbol{a}_{1\mathrm{i}}\| \propto \varrho_0^{1/4} E_{\mathrm{ii}}^{-3/4}, \tag{5.5}$$

$$U_{2\mathrm{i}}^2 = G_{\mathrm{ii}}/\varrho_0 B_{\mathrm{ii}}, \qquad \|\boldsymbol{a}_{2\mathrm{i}}\| \propto (\varrho_0 B_{\mathrm{ii}})^{1/4} G_{\mathrm{ii}}^{-3/4}, \tag{5.6}$$

where $\mathrm{i} = 1, 2, 3$, which yield expressions for the wave speeds and amplitudes in terms of the constitutive tensors E, G and B. We note that there are two groups of waves pertaining to linear and angular strain effects, respectively. The linear waves consist of two transverse shear waves and an axial extension wave, while the angular waves are comprised of two bending waves and an axial twist wave.

If we set

$$\mathbf{d}_{\alpha\mathrm{i}} = \boldsymbol{w}_{\alpha\mathrm{i}}/\|\boldsymbol{w}_{\alpha\mathrm{i}}\|, \tag{5.7}$$

(4) From this point on we shall write $\boldsymbol{w}_{2\mathrm{i}}$ for $\boldsymbol{w}_{2(\mathrm{i}+3)}$.

then the induced waves $i_{\alpha j}$ are given by

$$
(5.8)\qquad
\begin{aligned}
&i_{11} = \|a_{11}\| E_{11} K_1 \mathbf{d}_{22}, && i_{12} = \|a_{12}\| E_{22} K_2 \mathbf{d}_{21}, && i_{13} = 0,\\
&i_{21} = \|a_{21}\| E_{22} B_{11} K_2 \mathbf{d}_{12}, && i_{22} = \|a_{22}\| E_{11} B_{22} K_1 \mathbf{d}_{11}, && i_{23} = 0,
\end{aligned}
$$

where

$$
(5.9)\qquad K_1 = (E_{11}B_{22}-G_{22})^{-1}, \quad K_2 = -(E_{22}B_{11}-G_{11})^{-1}.
$$

The results (5.8) show that a transverse shear wave in either possible direction induces a bending wave such as to produce a deformation in the same direction as the shear. Similarly, bending waves induce transverse shear waves which produce deformation in the same direction as the bending. On the other hand, axial extension and twisting waves induce no higher order waves. We note that the effect of non-zero Γ_{ij} will be to produce more coupling between the various types of wave.

6. Appendix. Multiple wave speeds

In this appendix we exhibit the form taken by the results of Sects. 2 and 3, dealing with decay-induction effects for shock waves, for the case of multiple wave speeds. Entirely analagous analyses and results may be obtained in the case of acceleration waves, which we do not reproduce here. The decay equations (2.12) are now replaced by coupled systems of equations. For a wave speed of multiplicity r there now exists a system of r coupled equations for the determination of the components of the associated wave mode vector. We remark that these systems display an interesting feature of the problem which we do not deal with here([5]). We conclude, however, that the amplitude of the wave mode vector follows the same decay law as found previously. Finally, we give an expression for the induced wave. Here the result is analagous to Eq. (3.10), the compiicating feature being purely notational, of the type usually associated with multiple eigenvalue problems.

Let λ_{i_α} and w_{i_α}, $(\alpha = 1, \dots, s;\ i_\alpha = \sum_{\beta=0}^{\alpha-1} r_\beta + 1, \sum_{\beta=0}^{\alpha-1} r_\beta + 2, \dots, \sum_{\beta=0}^{\alpha} r_\beta)$ denote the eigenvalues and their associated orthonormal eigenvectors, respectively, given by solving Eqs. $(2.1)_1$. We note that there are s distinct eigenvalues with λ_{i_α} being of multiplicity r_α. We define $r_0 = 0$, and note that $\sum_{\alpha=0}^{s} r_\alpha = 6$. Moreover, the w_{i_α} span the eigenspace V_α ($\dim V_\alpha = r_\alpha$) associated with λ_{i_α}. Corresponding to Eqs. (2.6) and (3.5) we have

$$
(6.1)\qquad \{w_{i_\alpha}, w_{j_\beta}\}_K = \delta_{i_\alpha j_\beta},
$$

sand

$$
(6.2)\qquad w'_{i_\alpha} = \sum_{j_\beta=1}^{6} \Gamma_{i_\alpha j_\beta} w_{j_\beta}, \quad X w_{i_\alpha} = \sum_{j_\beta=1}^{6} X_{j_\beta i_\alpha} w_{j_\beta},
$$

respectively.

([5]) The problem of multiple wave spaeds in connection with acceleration waves in three-dimensional simple elastic materials has been dealt with by WRIGHT [13].

We now consider a shock $\boldsymbol{a}_\alpha \in V_\alpha$ and write

$$(6.3) \qquad \boldsymbol{a}_\alpha = \sum_{i_\alpha=1}^{r_\alpha} a_{i_\alpha} \boldsymbol{w}_{i_\alpha},$$

assuming we have ordered the basis so that $\boldsymbol{w}_{i_\alpha}$ are the first r_α eigenvectors. On substituting Eqs. (6.3) and $(6.2)_1$ into Eq. $(2.1)_2$ we find, after taking the scalar product with $\boldsymbol{w}_{i_\alpha}$ and following the procedure of Sect. 2, that

$$(6.4) \qquad 2\lambda_{i_\alpha} a'_{i_\alpha} + \sum_{j_\alpha=1}^{r_\alpha} [\lambda_{i_\alpha}\{\boldsymbol{S}\boldsymbol{w}_{j_\alpha}, \boldsymbol{w}_{i_\alpha}\} + \lambda_{i_\alpha}(\Gamma_{j_\alpha i_\alpha} - \Gamma_{i_\alpha j_\alpha}) + \frac{3}{2}\lambda'_{i_\alpha}\delta_{i_\alpha j_\alpha}] a_{j_\alpha} = 0, \qquad i_\alpha = 1, \ldots, r_\alpha,$$

are a coupled system of r_α equations for the determination of a_{i_α}. In Eq. (6.4) we have defined $\boldsymbol{S}$ as

$$(6.5) \qquad \boldsymbol{S} = \boldsymbol{K}\boldsymbol{X} - \boldsymbol{X}^T\boldsymbol{K}.$$

We note that the orientation of the $\boldsymbol{w}_{i_\alpha}$ in V_α is arbitrary and that the skew part of $\Gamma_{i_\alpha j_\alpha}$ is no longer determined by the constitutive tensors through Eq. (3.7). Since the $\boldsymbol{w}_{i_\alpha}$ can be oriented arbitrarily in V_α, we may choose one of them to be always along $\boldsymbol{a}_\alpha$ .It then follows by an analysis paralleling that in Sect. 2 that the magnitude a_α of $\boldsymbol{a}_\alpha$ satisfies Eq. (2.13) even for the case of multiple wave speeds. Similarly, the amplitude of acceleration waves will still satisfy Eq. (4.8) even in the case of multiple speeds.

Finally, following the procedure utilized in Sect. 3 we find, for the induced wave $\boldsymbol{i}_\alpha \in V_\alpha^\perp$ corresponding to $\boldsymbol{a}_\alpha$, the expression

$$(6.6) \qquad \boldsymbol{i}_\alpha = \sum_{i_\alpha=1}^{r_\alpha} a_{i_\alpha} \sum_{\substack{j_\beta=1 \\ \beta\neq\alpha}}^{6} [\lambda_{j_\beta}\Gamma_{i_\alpha j_\beta} - \lambda_{i_\alpha}(2\Gamma_{i_\alpha j_\beta} - \Gamma_{j_\beta i_\alpha}) - (\lambda_{j_\beta}X_{j_\beta i_\alpha} - \lambda_{i_\alpha}X_{i_\alpha j_\beta})] \frac{\boldsymbol{w}_{j_\beta}}{(\lambda_{j_\beta} - \lambda_{i_\alpha})}.$$

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References

1. H. COHEN, *A non-linear theory of elastic directed curves*, Int. J. Engng. Sci., **4**, 511–524, 1966.
2. A. E. GREEN, N. LAWS, *A general theory of rods*, Proc. Roy. Soc. A, **293**, 145–155, 1966.
3. H. COHEN, A. B. WHITMAN, *Waves in elastic rods*, J. Sound and Vibr., **51**, 1977.
4. J. L. ERICKSON, *Simpler static problems in nonlinear theories of rods*, Int. J. Solids Structures, **6**, 371–377, 1970.
5. A. E. GREEN, R. J. KNOPS, N. LAWS, *Large deformation, superposed small deformation and stability of elastic rods*, Int. J. Solids Structures, **4**, 555–577, 1968.

6. A. B. WHITMAN, C. N. DESILVA, *Stability in a linear theory of elastic rods*, Acta Mech., **15**, 295–308, 1972.
7. J. L. ERICKSON, *Wave propagation in thin elastic shells*, Arch. Rat. Mech. Anal., **43**, 167–178, 1971.
8. A. B. WHITMAN, C. N. DESILVA, *A dynamical theory of elastic directed curves*, ZAMP, **20**, 200–212, 1969.
9. A. E. GREEN, N. LAWS, *Remarks on the theory of rods*, J. Elasticity, **3**, 179–184, 1973.
10. A. B. WHITMAN, H. COHEN, *Constitutive equations for curved and twisted, initially stressed elastic rods* [to appear in Acta Mech.].
11. A. E. H. LOVE, *A treatise on the mathematical theory of elasticity*, Dover 1944.
12. A. B. WHITMAN, C. N. DESILVA, *An exact solution in a nonlinear theory of rods*, J. Elasticity, **4**, 265–280, 1974.
13. T. W. WRIGHT, *Acceleration waves in simple elastic materials*, Arch. Rat. Mech. Anal., **50**, 237–277, 1973.

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