Archives of Mechanics • Archiwum Mechaniki Stosowanej • 30, 2, pp. 217-224, Warszawa 1978

Some recent mathematical results concerning the Navier-Stokes equations(*)

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THE WHOLE set of limit-states for the Navier-Stokes equations in a bounded domain of E_2 for an arbitrary Reynolds number is described. For the three-dimensional case the solvability of the Cauchy problem for the statistical Hopf's equation is established. The results concerning the unique solvability of an initial boundary value problem for viscous incompressible inhomogeneous liquids and of some problems with free (unknown) boundaries are enumerated. Some new equations describing the dynamics of viscous incompressible liquids with large gradients of velocities are discussed.

Przedstawiono pełny układ stanów granicznych dla równań Naviera-Stokesa w obszarze ograniczonym przestrzeni E_2 dla dowolnej liczby Reynoldsa. Wykazano rozwiązalność problemu Cauchy'ego w przypadku trójwymiarowym dla statystycznego równania Hopfa. Podano wyniki dotyczące jednoznaczności rozwiązania problemu początkowo-brzegowego dla lepkich, nieściśliwych, niejednorodnych cieczy oraz niektórych zagadnień ze swobodnymi (nieznanymi) brzegami. Przedyskutowano niektóre nowe równania dynamiki lepkich nieściśliwych płynów z dużymi gradientami prędkości.

Рассматривается полная система предельных состояний для уравнений Навье-Стокса в ограниченной области пространства E_2 для произвольного числа Рейнольдса. Для трехмерного случая доказана разрешенность уравнения Хопфа. Приводятся результаты в области однозначности решения начально-краевой задачи для вязких несжимаемых неоднородных жидкостей и для некоторых задач со свободными (неизвестными) границами. Обсуждаются некоторые новые уравнения динамики вязких несжимаемых жидкостей при больших градиентах скорости.

1. Introduction

WE SHALL describe some results concerning the Navier-Stokes equations

$$\mathbf{v}_t - \nu \Delta \mathbf{v} + \sum_{k=1}^{n} v_k \mathbf{v}_{x_k} = -\operatorname{grad} \varrho + \mathbf{f}, \tag{1.1}$$

$$\operatorname{div} \mathbf{v} = 0, \tag{1.2}$$

for the cases $n = 2$ or 3. Here $x = (x_1, \ldots, x_n)$ is a point of the Euclidean space E_n with the Cartesian coordinates x_k, $\mathbf{v} = (v_1, \ldots, v_n)$ — the velocity field, $\mathbf{v}_t$ and $\mathbf{v}_x$ — the derivatives of $\mathbf{v}$, $\Delta \mathbf{v} = \sum_{k=1}^{n} \mathbf{v}_{x_k x_k}$ — the Laplacian of $\mathbf{v}$, ϱ the pressure, $\nu = \text{const} > 0$ — the coefficient of the viscosity, and $\mathbf{f} = (f_1, \ldots, f_n)$ — the external forces. Let x belong to a bounded domain $\Omega \subset E_n$ and $\mathbf{v}$ satisfies the boundary condition

$$\mathbf{v}|_{\partial\Omega} = 0 \tag{1.3}$$

(*) This paper was accepted as a Sectional Paper for the 1976 IUTAM Congress.

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(the cases of the inhomogeneous and periodic boundary conditions are treated in a similar way). We shall use three Hilbert spaces: $\mathbf{L}_2(\Omega)$, $\dot{I}(\Omega) \equiv Y$ and $H(\Omega)$. The space $\mathbf{L}_2(\Omega)$ consists of all vector-functions $\mathbf{u}(x)$, square-summable over the domain Ω. The scalar product and the norm in it is defined as follows:

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \sum_{k=1}^n u_k(x) v_k(x)\, dx, \qquad ||u|| = (\mathbf{u}, \mathbf{u})^{1/2}.$$

Y is the subspace of $\mathbf{L}_2(\Omega)$ and it is defined as the closure in the norm of $\mathbf{L}_2(\Omega)$ of the set $\dot{I}(\Omega)$ of all smooth solenoidal vector-functions $\mathbf{u}(x)$, which is zero near the boundary $\partial\Omega$. Last $H(\Omega)$ is the closure of the set $\dot{I}(\Omega)$ in the norm of the Dirichlet integral

$$||\mathbf{u}||_1 \equiv \Big(\int_\Omega \sum_{i,k=1} u^2_{ix_k}(x)\, dx\Big)^{1/2}.$$

The scalar ptoduct in it is

$$(\mathbf{u}, \mathbf{v})_1 = \int_\Omega \sum_{i,k=1}^n u_{ix_k}(x) v_{ix_k}(x)\, dx.$$

We shall make use of some results concerning the spectral problem

$$\text{(1.4)} \qquad \begin{aligned} -\Delta\mathbf{u} + \operatorname{grad} q &= \lambda\mathbf{u},\\ \operatorname{div}\mathbf{u} = 0, \quad \mathbf{u}|_{\partial\Omega} &= 0. \end{aligned}$$

It is known (see LADYŽENSKAYA [3]) that the spectrum of this problem consists of an accountable set of positive numbers $\{\lambda_k\}_{k=1}^\infty$, which may be ordered in such a way: $0 < \lambda_1 \leqslant \leqslant \lambda_2 \leqslant \ldots$. Each λ_k has a finite multiplicity and λ_k goes to infinity when $k \to \infty$. The corresponding eigenfunctions $\{\varphi^{(k)}(x)\}_{k=1}^\infty$ form an orthogonal basis in Y and in $H(\Omega)$ and may be normalized in such a way that

$$(\boldsymbol{\varphi}^k, \boldsymbol{\varphi}^l) = \delta_k^l, \qquad (\boldsymbol{\varphi}^k, \boldsymbol{\varphi}^l)_1 = \lambda_k \delta_k^l.$$

The letter are infinitely differentiable in Ω and their smothness near the boundary $\partial\Omega$ depend on the smoothness of $\partial\Omega$.

2. On the limit-states of the problem (1.1)–(1.3) for the case $n = 2$

We want to know what kind of regimes could be observed in the problem (1.1)–(1.3) "after a long time" if the $\mathbf{f} = \mathbf{f}(x)$ and Ω (with smooth $\partial\Omega$) are fixed and the initial data

$$\text{(2.1)} \qquad \mathbf{v}|_{t=0} = \mathbf{a}(x)$$

is an arbitrary element of Y. Let us take the ball $Y_R = \{\mathbf{a}: ||\mathbf{a}|| \leqslant R\}$ in Y. If its radius $R \geqslant R_0 = (\lambda_1 \nu)^{-1}||\mathbf{f}||$, then from each point $\mathbf{a}$ of K_R will come out a unique trajectory $V_t(\mathbf{a})$, $t \geqslant 0$, (i.e. the solution $v(x, t)$ ot the problem (1.1)–(1.3), (2.1)) and this trajectory never leaves the ball K_R (see [3]). Let us follow the immage $K_R(t)$ of K_R under the nonlinear transformator V_t. It is obvious that $K_R(t_2) \subset K_R(t_1)$ for $t_2 > t_1$. The sets $K_R(t)$, $t > 0$ are compacts in Y and their elements are smooth vector-functions if f and $\partial\Omega$ are sufficiently

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smooth. Consider the intersection $\mathfrak{M} = \cap_{t\geq 0} K_R(t)$. The elements of $\mathfrak{M}_R$ are the velocity fields observed in the flow "after infinite interval of time".

The following properties of $\mathfrak{M}_R$ have been proved (Ladyženskaya [4]): 1) $\mathfrak{M}_R = \mathfrak{M}_{R_0}$ for all $R \geq R_0$; 2) the set $\mathfrak{M}_R$ is compact in Y; 3) $\mathfrak{M}_R$ consists of those and only those elements $\mathbf{a}(x)$ of Y_R for which Eqs. (1.1)–(1.3), Eq. (2.1) are uniquely solvable both for $t \in [0, \infty)$ and for $t \in (-\infty, 0]$; 4) the set $\mathfrak{M}_R$ is an invariant of Eqs. (1.1)–(1.3) that is, if $\mathbf{a} \in \mathfrak{M}_R$, then the full trajectory $V_t(\mathbf{a})$, $t \in (-\infty, \infty)$ belongs to $\mathfrak{M}_R$; 5) the problem (1.1)–(1.3) defines a dynamical system over $\mathfrak{M}_R$. In particular, the trajectories $V_t(\mathbf{a})$, $V_t(\mathbf{a}')$ starting at different points $\mathbf{a}$ and $\mathbf{a}'$ never cross (i.e. $V_t(\mathbf{a}) \neq V_t(\mathbf{a}')$ for all t) and $V_t(\mathbf{a})$ depends on $\mathbf{a}$ continuously over any finite interval of time. 6) Moreover, the dynamical system (1.1)–(1.3) "behaves on $\mathfrak{M}$ as a finite-dimensional one", this means that numbers ν, $||\mathbf{f}||$ and some characteristics of Ω define some number m such that if one considers the m-dimensional linear subspace $Y^{(m)}$ of the space Y spanned on the first m eigenfunctions $\{\boldsymbol{\varphi}^k\}$, $k = 1, \ldots, m$, of the spectral problem (1.4), and if one denotes by P_m the orthogonal operator projecting Y^m onto $Y^{(m)}$, then projection $P_m V_t(\mathbf{a})$ of any complete trajectory $V_t(\mathbf{a})$, $t \in (-\infty, \infty)$, belonging to $\mathfrak{M}_R$ defines the trajectory $V_t(\mathbf{a})$ itself. Besides, if $P_m V_t(\mathbf{a})$ is a time-independent, ω — periodic or almost periodic function of t, so is $V_t(\mathbf{a})$.

The set $\mathfrak{M}_R$ definitely contains all stationary, periodic and almost periodic solutions of the problem (1.1)–(1.3). According to the Bogolyubov-Krylov theory there exist invariant measures which may be determined with the help of the procedure described by these authors. The structure ot the set $\mathfrak{M}_R$ essentially depends upon the Reynolds number. In particular, foı a small Reynolds number $\mathfrak{M}_R$ consists of one point — the unique stationary solution of Eqs. (1.1)–(1.3).

3. A statistical approach to the study of the Navier-Stokes equations

All attempts to prove the unique solvability "in the large" (i.e. for all $t \geq 0$ and for an arbitrary Reynolds number) of the problem (1.1)–(1.3) (2.1) for the case $n = 3$ failed. Therefore, it was natural to try to investigate this problem statistically, studying the evolution μ_t of the probability measure μ determined on the set of the initial data (2.1). This approach was suggested by E. Hopf [2]. The stistical Hopf's equation can be written in the form

$$\frac{\partial \mathscr{F}}{\partial t} = -\nu \sum_{m=1}^{\infty} \lambda_m \theta_m \frac{\partial \mathscr{F}}{\partial \theta} - i \sum_{j,k,m=1}^{\infty} a^{j,k,m} \theta_m \frac{\partial^2 \mathscr{F}}{\partial \theta_j \partial \theta_k} + i \sum_{m}^{\infty} f_m \theta_m \mathscr{F} \tag{3.1}$$

for the characteristic function $\mathscr{F}(\boldsymbol{\theta}, t)$ of the measure μ_t. Here $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots)$ and t are the arguments of the $\mathscr{F}(\boldsymbol{\theta}, t)$, $\lambda_1, \lambda_2, \ldots$ — the eigenvalues of the problem (1.4), $\hat{f}_m = (f, \boldsymbol{\varphi}^m)$ and $a^{jk,m} = ((\boldsymbol{\varphi}^j, \nabla)\boldsymbol{\varphi}^m, \boldsymbol{\varphi}^k)$ — the known constants. For the equation (3.1) it is necessary to solve the Cauchy problem with the initial data:

$$\mathscr{F}(\boldsymbol{\theta}, t)|_{t=0} = \int_Y \exp[i(\boldsymbol{\theta}, \mathbf{a})] d\mu(\mathbf{a}), \tag{3.2}$$

where $(\boldsymbol{\theta}, \mathbf{a}) = \sum_{m=1}^{\infty} \theta_m a_m$, $a_m = (\mathbf{a}, \boldsymbol{\varphi}^m)$.

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Formally, Eq. (3.1) is simply derived from the law of the evolution of measures

$$\mu_t(\mathbf{a}) = \mu\big(V_t^{-1}(\mathbf{a})\big) \tag{3.3}$$

and from the system of the ordinary differential equations

$$\frac{dv_m(t,\mathbf{a})}{dt} = -\nu\lambda_m v_m(t,\mathbf{a}) + \sum_{j,k=1}^{\infty} a^{jk,m} v_j(t,\mathbf{a}) v_k(t,\mathbf{a}) + \hat{f}_m, \qquad m = 1, 2, \ldots \tag{3.4}$$

for the Fourier coefficients $v_m(t, \mathbf{a}) = \big(V_t(\mathbf{a}), \boldsymbol{\varphi}^m\big)$ of the solution $V_t(\mathbf{a})$ of the problem (1.1)–(1.3), (2.1). If we knew that the "sufficiently good" evolution operator V_t of the problem (1.1)–(1.3) does exist, then the unique solution $\mathscr{F}(\boldsymbol{\theta}, t)$ of the Cauchy problem (3.1), (3.2) would be given by the formula

$$\mathscr{F}(\boldsymbol{\theta}, t) = \int_Y \exp[i(\boldsymbol{\theta}, V_t(\mathbf{a}))]\, d\mu(\mathbf{a}) = \int_Y \exp[i(\boldsymbol{\theta}, \mathbf{a})]\, d\mu_t(\mathbf{a}). \tag{3.4}$$

But we do not dispose of such a "good" operator V_t. Instead of it we have proved (LADYŽENSKAYA [5]) that there exist the operators W_t^s, $s = 1, 2, \ldots$, which give, for $\forall t \in [0, T]$, the measurable transformations of a σ-algebra $\Sigma(Y_R)$ defined on the ball $Y_R (R \geqslant R_0$, — see Point 2), considered as a metric space (in details, see [5]). For each W_t^s the function

$$\mathscr{F}^{(s)}(\boldsymbol{\theta}, t) = \int_{Y_R} \exp[i(\boldsymbol{\theta}, W_t^{(s)}(\mathbf{a}))]\, d\mu(\mathbf{a}) \tag{3.5}$$

has all derivatives incoming in Eq. (3.1) and satisfies Eq. (3.1) for all $\boldsymbol{\theta} \in H$ and $t \in \mathscr{T}$, where $\mathscr{T}$ is a set of $[0, T]$, having the Lebesque's measure T. All sums of Eq. (3.1) are convergent. $\Big($Here H is the Hilbert space of all sequences $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots)$ with $\|\boldsymbol{\theta}\|_H = \big(\sum_{m=1}^{\infty} \lambda_m \theta_m^2\big)^{1/2} < \infty\Big)$. The functions $\mathscr{F}^{(s)}(\boldsymbol{\theta}, t)$ are continuous on $H \times [0, T]$ and satisfy the condition (3.2). We have thus found, generally speaking, many solutions $\mathscr{F}^{(s)}(\boldsymbol{\theta}, t)$ of the problem (3.1), (3.2). The sum $\mathscr{F}^d(\boldsymbol{\theta}, t) = \sum_{s=1}^{\infty} \alpha_s \mathscr{F}^{(s)}(\boldsymbol{\theta}, t)$ with arbitrary $\alpha_s \geqslant 0$, satisfying the condition $\sum_{s=1}^{\infty} \alpha_s = 1$, is also the solution of the problem (3.1), (3.2). Each solution $\mathscr{F}^{\alpha}(\boldsymbol{\theta}, t)$ determines the evolution μ_t^{α} of the initial measure μ. If the problem (1.1)–(1.3), (2.1) really has more than one weak (Hopf's) solution, then the problem (3.1), (3.2) has not a unique solution either and it is necessary to find some additional principle which would choose among all solutions $\mathscr{F}$ of Eqs. (3.1), (3.2) the only one.

From the physical point of view it is reasonable to seek for a principle which would select not the unique solution of Eqs. (3.1) and (3.2) but the unique averaging velocity field $\langle \mathbf{v}(\cdot, t)\rangle = \int_{Y_R} \mathbf{a}\, d\mu_t(\mathbf{a})$. (Let us note that each averaging velocity field $\langle \mathbf{v}^{\alpha}(\cdot, t)\rangle = \int_{Y_R} \mathbf{a}\, d\mu_t^{\alpha}(\mathbf{a})$ satisfies its own Reynolds equations). One of such principles has been proposed by C. FOIAŞ and G. PRODI [1]. For the case when Eq. (3.1) has the "stationary" solutions, i.e. the solutions for which $\mu_t^{\alpha} = \mu^{\alpha}$ does not depend on t, the principle of Foiaş

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and Prodi make it imperative to take among the $\langle \mathbf{v}^\alpha \rangle$ such a one which minimizes the Dirichlet integral $\int_\Omega \sum_{j,k=1}^{3} v_{jx_k}^2(x)\,dx$. Since the set $\{\mu^\alpha\}$ of all stationary (invariant) measures is convex, this variational problem is uniquelly solvable. For the general case, when the measures $\{\mu_t^\alpha\}$ depend on t, it is also desirable to find the principle which would select among the averaged velocity fields $\langle \mathbf{v}^\alpha(\cdot, t)\rangle$ the only one $\langle \mathbf{v}(\cdot, t)\rangle$. Besides, choice must not depend on the length T of the time-interval $[0, T]$ of the observation, i.e. if $\langle \mathbf{v}(\cdot, t)\rangle_{T_1}$ and $\langle \mathbf{v}(\cdot, t)\rangle_{T_2}$ are the selecting averaged velocity fields for the time-intervals $[0, T_1]$ and $[0, T_2]$ and $T_2 > T_1$, then $\langle \mathbf{v}(\cdot, t)\rangle_{T_1} = \langle \mathbf{v}(\cdot, t)\rangle_{T_2}$ for $t \in [0, T_1]$. The principles suggested in [1] do not satisfy the last property.

4. The investigation of the unique solvability of the boundary value problems for viscous incompressible inhomogeneous fluids

In the paper [6] submitted by V. A. SOLONNIKOV and the author, the problem

$$\varrho\Big[\mathbf{v}_t + \sum_{k=1}^{n} v_k v_{x_k}\Big] - \nu\Delta\mathbf{v} = -\nabla p + \varrho\mathbf{f},$$

$$\text{(4.1)} \qquad \operatorname{div}\mathbf{v} = 0, \quad \varrho_t + \sum_{k=1}^{n} v_k \varrho_{x_k} = 0,$$

$$\mathbf{v}|_{\partial\Omega} = 0, \quad \mathbf{v}|_{t=0} = \mathbf{a}(x), \quad \varrho|_{t=0} = \underset{0}{\varrho}(x) > 0,$$

in a bounded $\Omega \subset E_n$, $n = 2, 3$ was considered. Here $\mathbf{f}$, $\mathbf{a}$ and $\underset{0}{\varrho}$ are known functions and ϱ, $\mathbf{v}$, p have to be found. In the main, our results concerning the unique solvability of this problem are the same as for the problem (1.1)–(1.3), (2.1): a) for $n = 2$ the problem (4.1) is uniquely solvable "in the large"; b) for $n = 3$ the problem (4.1) has a unique solution for all $t \geq 0$ if $\partial\Omega$ and the known functions are "sufficiently smooth" and if $\mathbf{a}$ and $\mathbf{f}$ are "sufficiently small". If $\mathbf{a}$ and $\mathbf{f}$ are not "small", then the unique solution exists for some positive time-interval $[0, T]$. I shall not give here the exact formulations of our theorems and shall mention only that we have considered the problem (4.1) in the functional spaces: $\mathbf{v} \in W_q^{2,1}(Q_T)$, $\nabla p \in L_q(Q_T)$, $Q_T = \Omega \times (0, T)$, $q > n$, $\varrho \in C^1(\bar{Q}_T)$. The same may be done in other functional spaces and for unbounded domains Ω and under the inhomogeneous or periodic boundary conditions.

5. On the solvability of some problems with the free (unknown) boundaries

In the last years some problems for the stationary and non-stationary Navier-Stokes equations in which the boundary $S \equiv \delta\Omega$ of the domain Ω, occupied by the fluid, or one part S_1 of S is unknown have been investigated. On S_1 we have to satisfy $n+1$ ($n = 2$ or 3) boundary conditions and on $S_2 = S \setminus S_1$ n boundary conditions (for example: $\mathbf{v}|_{S_2} = \mathbf{a}$).

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Some years ago I proposed the following program for the investigation of stationary problems: at the beginning one should solve the "auxiliary" problem

$$-\nu\Delta\mathbf{v}+\sum_{k=1}^{n}v_k\mathbf{v}_{x_k} = -\nabla p+\mathbf{f}(x),$$

$$\text{(5.1)}\qquad \operatorname{div}\mathbf{v} = 0, \quad \mathbf{v}|_{S_2} = a,$$

$$\mathbf{t}-\mathbf{n}(\mathbf{t},\mathbf{n})|_{S_1} = \mathbf{b}, \quad (\mathbf{v},\mathbf{n})|_{S_1} = d,$$

in a fixed domain Ω. Here $\mathbf{t} = (t_1, \ldots, t_n)$, $t_i = \sum_{k=1}^{n} t_{ik}n_k$, $t_{ik} = -p\delta_i^k+\nu(v_{kx_i}+v_{ix_k})$ and $n = (n_1, \ldots, n)$ — the unit outer normal to S. After that one uses the $(n+1)$th boundary condition on the S_1 for the determination of S_1. The realization of this program made it necessary to prove that the problem (5.1) has the solution $\mathbf{v}\in C^{(2+\alpha)}(\bar{\Omega})$, $\nabla p \in C^{(\alpha)}(\bar{\Omega})$ if $S \in C^{(3+\alpha)}$, $\mathbf{a} \in C^{(2+\alpha)}(S_2)$, $\mathbf{b} \in C^{(1+\alpha)}(S_1)$, $\alpha \in C^{(2+\alpha)}(S_1)$, $\mathbf{f} \in C^{(\alpha)}(\bar{\Omega})$. This fact had been proved by V. A. SOLONNIKOV [13] for the case when $S_1\cap S_2 = \phi$ (in [13] only the linearized problem (5.1) had been considered; the nonlinear problem (5.1) is investigated following the same method which I used in [3] for the system (5.1) with the first boundary condition (2.1)) and was used by V. V. PUKHNACHOV [10, 11, 12], myself and V. OSMOLOVSKII [7] for some problems with an unknown part S_1 of the boudary S. In all these problems we searched Ω, $\mathbf{v}$ and p slightly distinguishing from the known Ω_0, $\mathbf{v}^\circ$ and p°. But in many real situations the condition $S_1\cap S_2 = \phi$ of the Solonnikov's theorem is not satitisfied and it is very interesting to understand the exact dependence of the smoothness of the solutions of the problem (5.1) in the vicinity of $S_1\cap S_2 = \phi$ on S_1, S_2 and on known functions.

V. A. SOLONNIKOV [14, 15] also studied the non-stationary problem in which besided $\mathbf{v}(x, t)$ and $p(x, t)$, the domain Ω_t occupied by the fluid at the moment $t > 0$ has to be found. He proved the unique solvability of this initial-boundary value problem for a small interval of time $t \in [0, T]$ if $\partial\Omega_0$, $\mathbf{v}(x, 0)$ and external forces $\mathbf{f}(x, t)$ are smooth enough.

6. Some generalizations of Navier-Stokes equations

The system (1.1), (1.2) has been proposed for the description of the motion of fluids when the derivatives $|\mathbf{v}_{x_k}|$ are "comparatively small". It is known that in some cases an equation or a system, derived from some physical principle under a hypothesis that some characteristics of the medium are small enough, prove to be applicable for somewhat larger values of these characteristics and have sufficiently good properties from the mathematical point of view. For a long time it was believed that the Navier-Stokes equations belong just to that sort of equations and that the presence in Eq. (1.1) of the term $-\nu\Delta\mathbf{v}$ with $\nu > 0$ eliminates the possibility of the infinite increase of the $|\mathbf{v}_{x_k}(x, t)|$ on a finite interval of time (if $\partial\Omega$, $\mathbf{v}(x, 0)$, and $\mathbf{f}(x, t)$ are smooth) and thereby guarantees the unique solvability of the problem (1.1)–(1.3), (2.1) "in the large". It is really so for the case $n = 2$ (see [3]). But for the case $n = 3$ when the Reynolds number is not small, the solvability of the problem (1.1)–(1.3), (2.1) for all $t \geqslant 0$ has been proved only in a class of discontin-

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uous functions. For its weak (Hopf's) solutions the integral $||\mathbf{v}(\cdot, t)||_1$ may be equal to ∞ for some times (in spite of the smoothness of $\partial\Omega$, $\mathbf{a}$ and $\mathbf{f}$) and the theorem of uniqueness for them is not true (see LADYŽENSKAYA [8, 3]). I think that the system (1.1), (1.2) really allows infinite values of $||\mathbf{v}(\cdot, t)||_1$ for some times and its solutions may be branched out. In Point 3 I have explained one of the possible ways to search a principle which could give together with the system (1.1)–(1.3) a deterministic description of the dynamics in fluids. But for the real fluids, as it seems to me, it is necessary to change Eqs. (1.1) when $|\mathbf{v}_{x_k}|$ or $||\mathbf{v}||_1$ are large.

In the papers [9] (see also [3]) I have suggested the systems (1.1) which satisfy the Stokes postulates and for which the same as for Eq. (1.1), initial-boundary value pıoblems are uniquely solvable "in the large". One class here for the incompressible fluids in E_3 has the form

$$\mathbf{v}_t + \sum_{k=1}^{3} v_k \mathbf{v}_{x_k} - \sum_{i,k=1}^{3} \frac{\partial}{\partial x_k} [\beta(\hat{v}^2) v_{ik}] = -\nabla p + \mathbf{f}, \tag{6.1}$$

where $\hat{v}^2 = \sum_{i,k=1}^{3} (v_{ix_k} + v_{kx_i})^2$ and $\beta(\tau)$ is a non-decreasing function of $\tau \geqslant 0$ satisfying the inequalities $\nu_0 + c_0\tau^\mu \leqslant \beta(\tau) \leqslant \nu_1 + c_1\tau^\mu$, $\tau \geqslant 0$ with positive numbers ν_i, c_i and $\mu \geqslant \frac{1}{4}$. In particular, $\beta(\tau)$ may be constant on an interval $\tau \in [0, \tau_0]$ and thereby the equations (6.1) coincide with Eq. (1.1) for $\hat{v}^2 \leqslant \tau_0$.

The other class here has the same form as Eq. (6.1), but the $\beta(\hat{v}^2)$ in it is replaced by $\gamma(||\hat{v}||^2)$, where $||\hat{v}||^2 = \int_\Omega \hat{v}^2(x, t)dx$ and $\gamma(\tau)$ is a non-decreasing function of $\tau \geqslant 0$ satisfying the inequalities $\nu_0 + c_0\tau \leqslant \gamma(\tau) \leqslant \nu_1 + c_1\tau$, $\tau \geqslant 0$, with positive numbers ν_i, c_i. The function $\gamma(\tau)$ may also be constant for $\tau \in [0, \tau_0)$ and for this our equations coincide with Eq. (1.1) when $||\hat{v}(x, t)||^2 \leqslant \tau_0$.

From the system for $\mathbf{v}$, p and the temperature T derived from the Boltzman equation and some physical hypotheses on the connection between the coefficients $\mu(T)$ and $\varkappa(T)$ which enter in this system, two-sided estimates for T may be done (see [9]). These estimates show that it is reasonable to take the function $\beta(\tau)$ in the form $\nu_0(1+\varepsilon\tau)^\mu$, where the positive numbers ν_0, ε, μ depend on the character of molecule interaction.

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Received September 13, 1977.

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