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On the solution of a certain class of spatial problems in the theory of plastic flow

II. Applications

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MAKING use of the general approach which was given in the first part of the paper (Arch. Mech., 1(78) Sects. 1–5), solutions are given for certain of some spatial problems of the theory of three-dimensional plastic flow. The known plane strain and axially-symmetric problems are also derived from the general analysis.

Korzystając z ogólnej metody przedstawionej w pierwszej części pracy (Arch. Mech., 1(78) punkty 1–5), podano kilka przykładów rozwiązań zagadnień przestrzennych teorii plastycznego płynięcia. Znane zagadnienia płaskie i osiowo-symetryczne otrzymano również jako przypadki szczególne.

Используя общий подход представленный в первой части работы (Архив прикладной механики, 1(78), главы 1–5), дается несколько примеров решений пространственных задач теории пластического течения. Известные плоские и осесимметричные задачи получены тоже из общего подхода как частные случаи.

6. Special cases: the plane strain and axially-symmetric problems

LET US ASSUME, that the plastic zone V is situated along a straight line — i.e., $\varkappa = 0$. The coordinate system z^1, z^2, z^3 is in this case rectilinear a (Carthesian system). Suppose all functions which describe the problem are independent of z^3 — i.e., that c, γ, ϱ, p^K, $\mathring{v}_K$ are functions of z^1, z^2 only. The functions σ, φ, obtained as the solutions of the static boundary value problem for the system (5.9) (in which $\omega = 0$), depend now exclusively on z^1, z^2. In viev of $\varkappa = 0$, we obtain from (5.10) the equality $\overline{f}^3 = 0$. Putting $n_3 = 0$ in (5.11), we also have $\overline{p}^3 = 0$. Analogously, if the kinematic boundary conditions (5.22) are independent of z^3, then the solution v_1, v_2 of the boundary value problem for the system (5.19) is independent of z^3. Thus from (5.21) it follows that $\overline{d}_{13} = 0$, $\overline{d}_{23} = 0$. This is the well known special case in which the three-dimensional problem of plastic flow reduces to a plane strain problem.

Now, let us assume that the plastic zone V is situated along a circular line — i.e., $\varkappa = \text{const.} \neq 0$, and let on the boundary of the region V satisfies the condition $n_3 = 0$. The coordinate system z^1, z^2, z^3 is in this case the cylindrical coordinate system. Moreover, let all known functions in formulation of the boundary value problem be independent of z^3. Then, the solution σ, φ of the boundary value problem for the Eqs. (5.9) is also independent of z^3. From the Eq. (5.10), we obtain $\overline{f}^3 = 0$ and from the Eq. (5.11) it follows that $\bar{p}^3 = 0$. Since the kinematic boundary conditions have also to be independent of z^3, then solution v_1, v_2 of the boundary value problem for the system (5.19) also depends on z^1, z^2 only. From the Eq. (5.21), we obtain $\bar{d}_{13} = 0$, $\bar{d}_{23} = 0$. Putting $\eta = 0$ or $\eta = 1$ (the case of complete plasticity is taken into account), we arrive at the second well known special case of plastic flow in which the three-dimensional problem of plastic flow reduces to an axially-symmetric problem.

For the plane strain problem as well as for the axially-symmetric problem, we have arrived at the conditions:

$$\bar{f}^3 = 0, \quad \bar{p}^3 = 0, \quad \bar{d}_{13} = 0, \quad \bar{d}_{23} = 0. \tag{6.1}$$

From Eqs. (6.1) and from the foregoing analysis (cf. the end of Sect. 5), it follows that in both special cases under consideration, the hypotheses (3.1) and (3.2) do not restrict the three-dimensional problem of plastic flow. This means that the solutions of this problem obtained by means of the Eqs. (3.1), (3.2) satisfy also the known equilibrium conditions $T^{\alpha\beta}|_\beta + f^\alpha = 0$ in V, the boundary conditions $T^{\alpha\beta} n_\beta = p^\alpha$ on S^α, and the kinematic equations $\xi_{\alpha\beta} = v_{(\alpha|\beta)}$ in V.

Let us also consider some special cases of plastic materials. Putting $\dot{\varrho} = \varrho$, we obtain the associated flow law. For the Coulomb medium, we have $\varrho > 0$, $h = c\,\mathrm{ctg}\varrho > 0$. For the cohesive medium, we have to put $c > 0$, $\varrho = 0$, and for the non-cohesive media — $c = 0$, $\varrho > 0$.

In the foregoing analysis, we have taken into account a rigid plastic material represented by the plastic potential (3.3) and by the yield condition (3.5). Such materials can also represent the model of the subsoil which was introduced in [8]. This model can be used here only if $\varkappa \neq 0$ and for $\eta = 0$ or $\eta = 1$, and it includes more specials models of subsoil which are used in soil mechanics.

7. Examples of solutions

7.1. Limiting states of subsoil along curvilinear retaining walls

Let us find the limit value of the load acting at the subsoil in the vicinity of the vertical curved retaining wall. The scheme of the problem is reprenseted in Fig. 4. We assume that the inner surface of the wall is the vertical cylindrical surface determined by the smooth curve L lying on the horizontal plane. We denote, as usually, by V the plastic zone of the subsoil, and we assume that the region V can be parametrized by the curvilinear coordinate system $\{z^\alpha\}$, $0 < -z^1 < a(z^3)$, $0 < z^2 < h(z^3)$, $0 < z^3 < l$. The inner surface of the wall coincides with the parametric surface $z^1 = 0$, and the plane $z^2 = 0$ coincides with the upper boundary of the subsoil, Fig. 4. The cross section of the region V by the parametric planes $z^3 =$ const. are triangles, that shape depending on z^3. The forces acting on V across the horizontal plane $z^2 = 0$, and across the cylindrical surface $z^1 = 0$ will be assumed as normal to the corresponding boundaries. Thus we shall carry on the limit analysis only if for each $z^3 =$ const. there appears either $\varphi = 0$ or $\varphi = \dfrac{\pi}{2}$ — i.e., if the forces of friction between the subsoil and the retaining wall can be disregarded. As a model of the subsoil, we shall take the homogeneous Coulomb medium, $\overset{*}{\varrho} = \varrho > c$, $c > 0$. To simplify calculations, we also disregard the influence of the weight of the subsoil

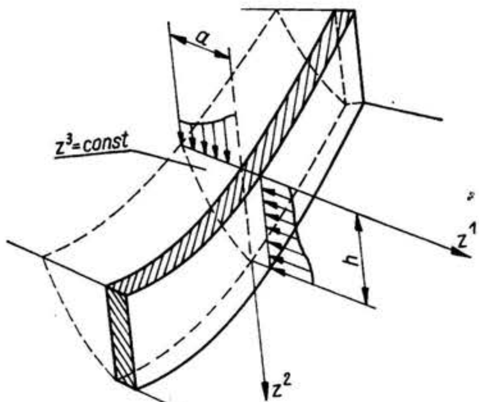

FIG. 4.

on the value of the limit load. We shall analyse separately two cases: the case in which either $\varphi = 0$, $\eta = 1$ or $\varphi = \frac{\pi}{2}$, $\eta = 0$ and the case in which either $\varphi = 0$, $\eta = 0$ or $\varphi = \frac{\pi}{2}$, $\eta = 1$.

1. *The special case*: $\varphi = 0$, $\eta = 1$ *or* $\varphi = \frac{\pi}{2}$, $\eta = 0$(¹).

Denoting by h the height of the plastic zone V, we obtain for the width of the plastic zone, measured on the plane $z^2 = 0$, the formula:

$$a = h\,\mathrm{tg}\left[\frac{\pi}{4} + (2\eta - 1)\frac{\varrho}{2}\right]. \tag{7.1}$$

The values of h and a can depend on z^3.

From the Eqs. (5.9), we obtain $\sigma_{,1} = 0$, $\sigma_{,2} = 0$. This means that $\sigma = \alpha(z^3)$, where α is an arbitrary differentiable function. The value of the state of stress can be calculated from the Eqs. (5.6), (5.8):

$$\begin{aligned} \sigma^{11} &= \alpha(z^3) + (2\eta - 1)\alpha(z^3)\sin\varrho - c\,\mathrm{ctg}\,\varrho, \\ \sigma^{22} &= \alpha(z^3) - (2\eta - 1)\alpha(z^3)\sin\varrho - c\,\mathrm{ctg}\,\varrho, \\ \sigma^{12} &= 0, \\ \sigma^{33} &= \alpha(z^3) - c\,\mathrm{ctg}\,\varrho + (2\eta - 1)\alpha(z^3)\sin\varrho. \end{aligned} \tag{7.2}$$

From the Eq. (3.1), we also have $\sigma^{13} = 0$ and $\sigma^{23} = 0$. We have assumed hitherto that the boundary horizontal plane $z^2 = 0$ is loaded by vertical forces — i.e., $p^1 = 0$,

(¹) We analyse here the passive as well as the active earth pressure, respectively, on the retaining wall, cf. [11] pp. 64–65.

$p^2 = p(z^3)$, $p^3 = 0$. On the vertical surface $z^1 = 0$ are given the kinematical boundary conditions. From the conditions on $z^2 = 0$ — i.e., from $-\sigma^{22} = p(z^3)$, $\sigma^{12} = 0$, $\sigma^{13} = 0$, we obtain:

$$\alpha(z^3) = \frac{c\,\mathrm{ctg}\varrho - p(z^3)}{1-(2\eta-1)\sin\varrho}. \tag{7.3}$$

Substituting the right-hand sides of the Eq. (7.3) into the Eqs. (7.2), we obtain the state of stress in the plastic zone V. On the retaining wall — i.e. for $z^1 = 0$ — by virtue of (5.5) we have: $\bar{p}^1 = \sigma^{11}$, $\bar{p}^2 = \sigma^{32} = 0$, $\bar{p}^3 = \sigma^{31} = 0$. It follows that

$$\bar{p}^1 = [c\,\mathrm{ctg}\varrho - p(z^3)]\frac{1+(2\eta-1)\sin\varrho}{1-(2\eta-1)\sin\varrho} - c\,\mathrm{ctg}\varrho. \tag{7.4}$$

From the Eq. (7.4), we conclude that the interaction between the retaining wall and the subsoil is uniformly distributed along the height of the wall. Substituting the right-hand side of (7.2) into (5.3), we arrive at the formula:

$$\bar{f}^3 = \left(\frac{c\,\mathrm{ctg}\varrho}{[1-z^1\varkappa(z^3)]^2} + \frac{p(z^3)-c\,\mathrm{ctg}\varrho}{[1-z^1\varkappa(z^3)]^2}\,\frac{1+(2\eta-1)\sin\varrho}{1-(2\eta-1)\sin\varrho}\right)_{,3}. \tag{7.5}$$

Provided that the external load $p(z^3)$ and the curvature $\varkappa(z^3)$ of the inner surface of the retaining wall are known, from the foregoing formula we obtain the value $\bar{f}^3$ of the internal force (cf. also the Eq. (4.6)).

Note that we disregard here the influence of the weight γ of the subsoil on the state of stress, assuming that it is sufficiently small with respect to the influence of external loads $p(z^3)$. This means that in the equations of equilibrium (5.2), we omit the terms (volume forces) of order γ. Thus, as the criterion of applicability of the solutions obtained (7.2)—(7.4) we can take the inequality $|\bar{f}^3| \leqslant \gamma$. This inequality enables us to take into account only such loading $p(z^3)$ of the plane $z^2 = 0$, and only such curvatures $\varkappa(z^3)$ of the inner surface of the retaining wall, for which the modulus of the expression on the right-hand sides of (7.5) does not exceed the weight by volume γ of the subsoil. For $\varkappa$ and p, which are independent of z^3, we obtain $\bar{f}^3 = 0$, and the Eqs. (7.2)—(7.4) reduce to the form given in [11] pp. 62–66 (in [11] the volumme density γ was also taken into account in the limit analysis). The equality $\bar{f}^3 = 0$ also occurs when the loading p is independent of z^3 and equal to

$$p = c\,\mathrm{ctg}\varrho\left\{1-\mathrm{tg}^2\left[\frac{\pi}{4}-(2\eta-1)\frac{\varrho}{2}\right]\right\} = c\,\mathrm{ctg}\varrho\left(1-\frac{a^2}{h^2}\right). \tag{7.6}$$

In this case, the plane curve L, which determines the vertical cylindrical surface $z^1 = 0$, is an arbitrary smooth curve. For the loading independent on z^3 from the condition $|\bar{f}^3| \leqslant \gamma$, and from (7.5) we obtain:

$$\left|\frac{\varkappa_{,3}}{(1-a\varkappa)^3}\right| \leqslant \frac{\gamma}{2a}\left|c\,\mathrm{ctg}\varrho + (p - c\,\mathrm{ctg}\varrho)\frac{1+(2\eta-1)\sin\varrho}{1-(2\eta-1)\sin\varrho}\right|^{-1}. \tag{7.7}$$

The Eq. (7.7) characterizes the admissible form of the curve L, provided that the loading p acting on the plane $z^2 = 0$ is independent of z^3.

The kinematics of the problem is trivial, since in the plastic zone V the relations $d_{11} = d_{12} = d_{22} = 0$ hold. The kinematical boundary conditions on the surface $z^1 = 0$ — i.e., the conditions between the subsoil and the retaining wall — have the form $v_1 = \mathring{v}_1(z^3)$, $v_2 = \mathring{v}_2(z^3)$. The rigid motion of any cross section $z^3 =$ const. of the plastic zone V can be determined in the manner detailed in [10] p. 173. By virtue of (5.17) we have:

$$\bar{d}_{13} = -\frac{\mathring{v}_{1,3}}{2[1-z^1\varkappa(z^3)]}, \quad \bar{d}_{23} = \frac{\mathring{v}_{2,3}}{2[1-z^1\varkappa(z^3)]}. \tag{7.8}$$

If the kinematical boundary conditions do not depend on z^3, then $\bar{d}_{13} = 0$, $\bar{d}_{23} = 0$; such a situation occurs in axially-symmetric problems.

2) *The special cases*: $\varphi = 0$, $\eta = 0$ *and* $\varphi = \frac{\pi}{2}$, $\eta = 1$.

As before denoting, by h the height of the plastic zone measured on the cylindrical surface $z^1 = 0$, we obtain:

$$a = h\,\mathrm{tg}\left[\frac{\pi}{4} - (2\eta-1)\frac{\varrho}{2}\right], \tag{7.9}$$

where a is a length of the plastic zone measured on the horizontal surface $z^2 = 0$. From Eqs. (5.9), we obtain now $\sigma_{,2} = 0$ and $[1-(2\eta-1)\sin\varrho]\sigma_{,1} + 2(2\eta-1)\omega\sin\varrho\,\sigma = 0$. It follows that $\sigma = \sigma(z^1, z^3)$ where σ is, for the time being, an arbitrary function satisfying the latter equation.

After taking into account Eq. (5.1), we arrive at the following differential equation:

$$\sigma_{,1} + \frac{2(2\eta-1)\sin\varrho}{1-(2\eta-1)\sin\varrho}\cdot\frac{\varkappa}{1-z^1\varkappa}\,\sigma = 0, \tag{7.10}$$

which can also be written in the form:

$$\sigma_{,1} + \frac{\chi\varkappa}{1-z^1\varkappa}\,\sigma = 0, \quad \chi \equiv \frac{2(2\eta-1)\sin\varrho}{1-(2\eta-1)\sin\varrho}. \tag{7.11}$$

The general solution of the Eq. (7.11) is given by:

$$\sigma = \beta(z^3)(1-\varkappa z^1)^{\chi}, \tag{7.12}$$

where $\beta(z^3)$ is an arbitrary differentiable function. Making use of (5.6) and (5.8), we have:

$$\begin{aligned} \sigma^{11} &= \beta(z^3)[1-z^1\varkappa(z^3)]^{\chi}\,[1-(2\eta-1)\sin\varrho] - c\,\mathrm{ctg}\varrho, \\ \sigma^{22} &= \beta(z^3)\,[1-z^1\varkappa(z^3)]^{\chi}\,[1+(2\eta-1)\sin\varrho] - c\,\mathrm{ctg}\varrho, \\ \sigma^{12} &= 0, \\ \sigma^{33} &= \beta(z^3)[1-z^1\varkappa(z^3)]^{\chi}\,[1+(2\eta-1)\sin\varrho] - c\,\mathrm{ctg}\varrho. \end{aligned} \tag{7.13}$$

By virtue of Eq. (3.1), in each of the problems under consideration we have $\sigma^{13} = \sigma^{23} = 0$(2).

The boundary conditions on the retaining wall are kinematic: $v_K = \mathring{v}_K$ for $z^1 = 0$. Suppose that the wall from the outer side is subjected to horizontal load, the density of which is the function of z^3 only. On the inner surface of the wall, by means of Eqs.

(2) Note that the latter relations hold in the special coordinate system $\{z^\alpha\}$ used throughout this paper.

(5.5) the static boundary conditions have the form $\sigma^{11} = \bar{p}^1$, $\sigma^{12} = \bar{p}^2$, where $\bar{p}^1$, $\bar{p}^2$ are unknown reactions of the wall. Applying the semi-inverse method, we assume that the external loading acting on the wall from the outer side is uniformely distributed as the reaction of the wall on the subsoil. This means that in Eq. (5.5) we may put $\bar{p}^1 = \mathring{p}(z^3)$ and $\bar{p}^2 = 0$, where $\mathring{p}(z^3)$ is the known function. The forces of friction between the wall and the subsoil have been disregarded here. For $z^1 = 0$ we have $\sigma^{11} = \mathring{p}(z^3)$, and by virtue of the Eqs. (7.13) we obtain:

$$\beta(z^3) = \frac{\mathring{p}(z^3) + c\,\mathrm{ctg}\,\varrho}{1-(2\eta-1)\sin\varrho}. \tag{7.14}$$

Substituting the right-hand side of (7.14) into (7.13), we obtain the formulas determining the state of stress in the subsoil. Taking into account Eqs. (7.13), we can calculate the loading q of the subsoil acting on the horizontal plane $z^2 = 0$, which is necessary to maintain the limit state of the part V of the subsoil:

$$q = -\sigma^{22} = c\,\mathrm{ctg}\,\varrho - (\mathring{p} + c\,\mathrm{ctg}\,\varrho)\,(1 - z^1\varkappa(z^3))]^{\varkappa}\,\frac{1+(2\eta-1)\sin\varrho}{1-(2\eta-1)\sin\varrho}. \tag{7.15}$$

Substituting (7.13) into (5.3), we also obtain:

$$\bar{f}^3 = \left(\frac{c\,\mathrm{ctg}\,\varrho}{[1 - z^1\varkappa(z^3)]^2} - \frac{\mathring{p}(z^3) + c\,\mathrm{ctg}\,\varrho}{1 - z^1\varkappa(z^3)]^2}\cdot\frac{1+(2\eta-1)\sin\varrho}{1-(2\eta-1)\sin\varrho}\right)_{,3}. \tag{7.16}$$

Thus for the reaction forces $\mathring{p}(z^3)$ postulated a priori, of the wall on the subsoil, and for the known curvature $\varkappa$ of the inner cylindrical surface of the wall, we can calculate from (7.16) the value of the density of the internal force $\bar{f}^3$. Using the same procedure as before, we shall take the condition $|\bar{f}^3| \leqslant \gamma$ as the necessary condition of applicability of the solutions obtained. For $\varkappa$ and $\mathring{p}$ independent of z^3, we obtain from Eq. (7.16) the equality $\bar{f}^3 = 0$. In this case, the problem under consideration reduces to an axially-symmetric problem, and Eqs. (7.13)–(7.15) take the form which is given in [13] pp. 66–69.

The kinematic analysis of the problem is analogous to that which was studied before. If for both the cases detailed above we have $|a\varkappa| \ll 1$ (i.e., if the length a of the plastic zone, Fig. 4, is small as compared with the radius of curvature $|\varkappa|^{-1}$ of the cylindrical surface of the retaining wall), then, disregarding the value $|a\varkappa|$ as sufficiently small with respect to 1, we obtain $\bar{f}^3 = 0$, provided that the loading p of the horizontal plane $z^2 = 0$ is independent of z^3. In this case, the curvature $\varkappa$ in Eqs. (7.13) will be absent. It follows that the solutions here obtained for an arbitrary $\varkappa$, are sufficiently good approximations, provided that $|a\varkappa|$ is sufficiently small with respect to 1.

7.2. Analysis of shapes of spatial slopes

Now let us consider the problem of the shape of a spatial slope. Let the slope be loaded by the vertical load $p = \alpha z^3 + \beta$, $\alpha = \text{const.}$, $\beta = \text{const}$, $0 < z^3 < l$, cf. Fig. 5. Let us assume also, that the volume density of the subsoil is constant, and that the material of the slope is homogeneous and perfectly cohesive. Under these conditions, it can be proved that the equation of the surface of the slope is given by

$$z^1 = -\frac{2c}{\gamma} \ln \frac{\sin\left(\frac{\alpha}{2\beta} z^3 + \frac{\gamma}{2\beta} z^2 + \frac{\pi}{2}\right)}{\sin\left(\frac{\alpha}{2\beta} z^3 + \frac{\pi}{2}\right)}, \tag{7.17}$$

where $0 < z^3 < l$, $0 < z^2 < h(z^3)$, and

$$z^2 = h(z^3) \equiv -\frac{\alpha}{\gamma} z^3 + \frac{\pi\beta}{\gamma}, \quad 0 < z^3 < l, \tag{7.18}$$

where $z^2 = h(z^3)$ for each $z^3 \in (0, l)$ represents the equation of the asymptote for the curve given by Eq. (7.17). The foregoing relations constitute a generalization of the classical

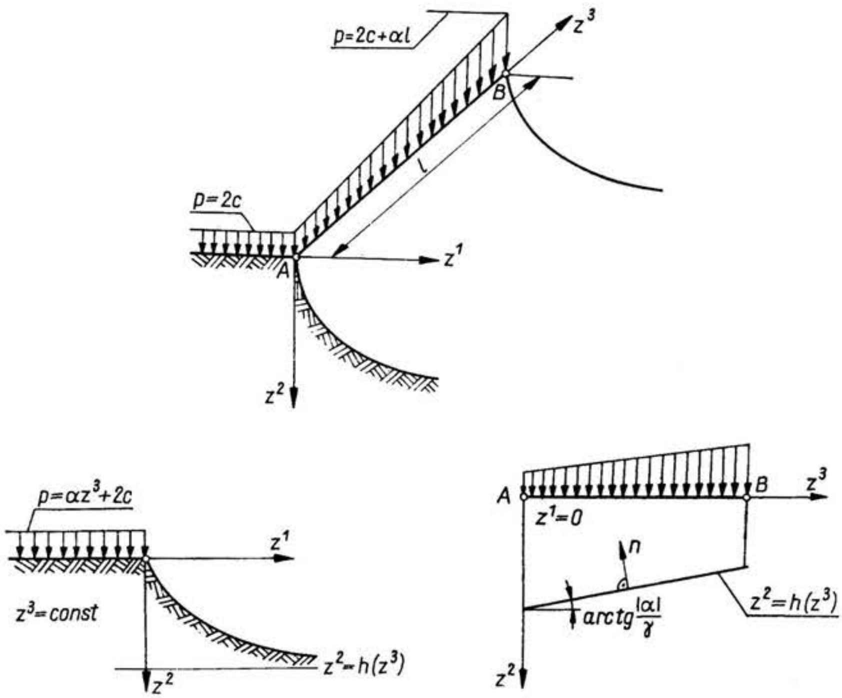

FIG. 5.

solutions given by W. Sokolovsky (cf. [10] pp. 128–129), which have been obtained under assumptions of plane strain. In the region of the subsoil situated in the vicinity of the plane $z^2 = 0$, the state of stress is given by:

$$\begin{aligned} \sigma^{11} &= \alpha z^3 + \gamma z^2, \\ \sigma^{12} &= 0, \\ \sigma^{22} &= \alpha z^3 + \gamma z^2 + 2\beta. \end{aligned} \tag{7.19}$$

Taking into account Eq. (3.6), and putting $\eta = 0.5$, we can write:

$$\sigma^{33} = 0.5\ (\sigma^{11}+\sigma^{22}) = \alpha z^3+\gamma z^2+\beta. \tag{7.20}$$

It follows that (see Eq. (5.3))

$$\overline{f}^3 = -\sigma^{33}{}_{,3} = -\alpha. \tag{7.21}$$

The state of stress in the vicinity of the slope is given by:

$$\sigma_1 = 0, \quad \sigma_2 = 2\beta, \quad \sigma_3 = \sigma^{33} = 0.5\ (\sigma_1+\sigma_2) = \beta. \tag{7.22}$$

Thus we obtain:

$$\overline{f}^3 = -\sigma^{33}{}_{,3} = 0. \tag{7.23}$$

In the spatial problem under consideration, we also deal with the component $\bar{p}^3$ of the boundary force on the surface of the slope. In accordance with (5.11), the latter is equal to

$$\bar{p}^3 = \sigma^{33} n_3 = \beta n_3, \tag{7.24}$$

where (n_1, n_2, n_3) is the unit vector normal to the surface of the slope. Let us calculate the value:

$$\sup \bar{p}^3 = \beta \sup n_3 = \beta |n_3|_{z^2=h}. \tag{7.25}$$

Because of (cf. Fig. 6c)

$$n_3|_{z^2-h} = \frac{dh(z_3)}{dz^3} = -\frac{\alpha}{\gamma}, \tag{7.26}$$

we can write:

$$\sup \bar{p}^3 = \frac{\beta(\alpha)}{\gamma}. \tag{7.27}$$

To simplify our calculations, we shall not determine here the value $\overline{f}^3$ of internal force in the region where $\sigma^{12} \neq 0$.

Let us assume now that the weight by volume γ of the subsoil can be determined with approximation which is not greater then $\varepsilon\gamma$, where ε is the positive number given a priori, small with respect to unity: $\varepsilon \ll 1$. Then from (7.19) we calculate that the stresses σ^{11}, σ^{22} are determined with approximation which does not exceed $\varepsilon\gamma \sup z^2 = \varepsilon\gamma h_{\max} = \varepsilon\pi\beta$, provided that $\alpha \geqslant 0$. It follows that the volume forces can be determined here with approximation of order $\varepsilon\gamma$. Thus we conclude that the internal volume forces $(0, 0, \overline{f}^3)$ can be treated as sufficiently small if their values are of order $\varepsilon\gamma$:

$$|\overline{f}^3| \leqslant \varepsilon\gamma. \tag{7.28}$$

Taking into account Eq. (7.21), we see that $|\alpha| \leqslant \varepsilon\gamma$. Analogously, the internal surface forces $(0, 0, \bar{p}^3)$ may be disregarded when they are of order $\varepsilon\pi\beta$, the latter term being the approximation of the boundary kinetic conditions on the surface on the slope. Thus we have:

$$\sup \bar{p}^3 \leqslant \varepsilon\pi\beta. \tag{7.29}$$

Taking into account Eq. (7.27), we obtain $|\alpha| \leqslant \varepsilon\gamma\pi$. From the two foregoing inequalities, it follows that

$$|\alpha| \leqslant \varepsilon\gamma. \tag{7.30}$$

Thus the solution (7.17) of the spatial problem can be applied only if the condition (7.30) holds. For a uniformely distributed load $\alpha = 0$, the condition (7.30) becomes an identity, and we pass to the well known plane problem (cf. [10] p. 127).

The foregoing analysis is slightly changed if the weight of subsoil can be disregarded. Such approximation is valid only if the value γz^2 in Eq. (7.20) is sufficiently small. From Eq. (7.17), for $\gamma \to 0$ we arrive at:

$$z^1 = \operatorname{tg}\left(\frac{\alpha}{2\beta} z^3\right) z^2, \quad z^2 \geqslant 0. \tag{7.31}$$

For $0 \leqslant z^3 \leqslant l = \dfrac{\pi\beta}{\alpha}$, the shape of the slope is given in Fig. 6. Similarly as before, we have now $|\bar{f}^3| \leqslant \gamma$. The Eq. (7.31) is now valid only if

$$|\alpha| \leqslant \gamma. \tag{7.32}$$

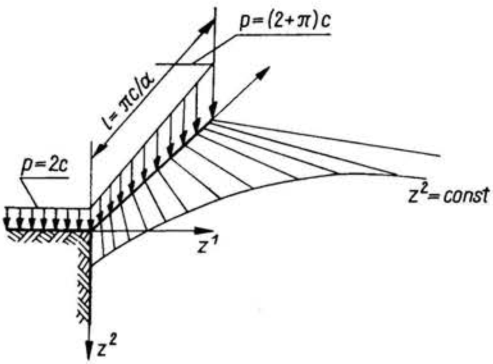

FIG. 6.

In Eq. (7.20), we can now disregard the term γz^2 — i.e., $\sigma^{33} = \alpha z^3 + \beta$; the same term has to be disregarded in the formulas determining the boundary forces. Thus we can write:

$$|\bar{p}^3| \leqslant \gamma z^2, \quad z^2 \geqslant 0. \tag{7.33}$$

Making use cf. (7.24), we obtain:

$$\beta |n_3| \leqslant \gamma z^2. \tag{7.34}$$

After some calculations, we arrive at the following formulas for the component n_3 of the unit vector normal to the slope:

$$n_3 = -\frac{\alpha z^2}{2\beta} \frac{1}{\sqrt{\cos^2\left(\frac{\alpha}{2\beta} z^3\right) + \left(\frac{\alpha}{2\beta} z^2\right)^2}}, \quad z^2 \geqslant 0, \quad 0 < z^3 < l. \tag{7.35}$$

The condition (7.34) is, by virtue of (7.35), fulfilled for $z^2 = 0$. For $z^2 > 0$, we obtain:

$$\frac{|\alpha|}{\sqrt{\cos^2\left(\frac{\alpha}{2\beta} z^3\right) + \left(\frac{\alpha}{2\beta} z^2\right)^2}} \leqslant 2\gamma, \quad 0 < z^3 < l. \tag{7.36}$$

The Eqs. (7.32), (7.36) represent the necessary conditions under which the solution (7.31) of the spatial problem can be applied.

7.3. Compression of a thin layer

Let us analyse the problem of compresion of a thin homogenous isotropic layer, situated between two cylindrical surfaces $z^1 = \pm h(z^3)$, cf. Fig. 7. The coordinate system $\{z^\alpha\}$ will be assumed here as an orthogonal Carthesian coordinate system, in which the coordinate plane $z^1 = 0$ is the plane of symmetry of the layer, $-h(z^3) \leqslant z^1 \leqslant +h(z^3)$, $2a$ is the height of the layer, $0 \leqslant z^2 \leqslant 2a$, and l is the length of the layer, $0 \leqslant z^3 \leqslant l$. We assume that $h \ll a$, $h \ll l$—i.e., we assume that the layer is thin. In the problem under

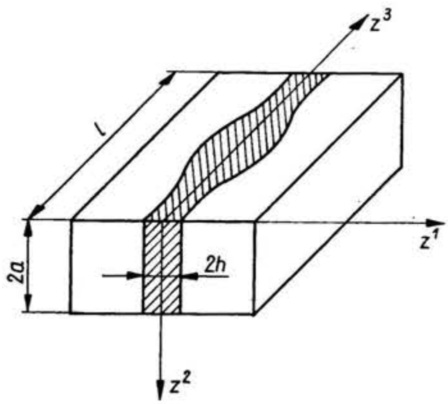

FIG. 7.

consideration the curvature $\varkappa$ of $z^1 = 0$ is equal to zero, $\varkappa = 0$, and we have to assume that the Levy-Mises flow law associated with the Huber-Mises yield condition holds, cf. Sect. 5. The influence of the volume forces on the limiting state of the layer in the following analysis is disregarded.

The solution of the equilibrium equations (5.2) for $\gamma = 0$, $\omega = 0$ has the form:

$$\sigma^{11} = -c\frac{\pi}{2} - c\frac{z^2}{h(z^3)},$$

(7.37)
$$\sigma^{12} = c\frac{z^1}{h(z^3)},$$

$$\sigma^{22} = -c\frac{\pi}{2} - c\left[\frac{z^2}{h(z^3)} - 2\sqrt{1-(\zeta)^2}\right], \quad \xi \equiv \frac{z^1}{h(z^3)}.$$

The right-hand sides of Eqs. (7.37) satisfy the Huber-Mises yield condition, provided, that $\eta = 0.5$. The components of the flow velocity vector can be obtained as solutions of Eqs. (5.19) for $\mathring{\varrho} = 0$, $\omega = 0$:

(7.38)
$$v_1 = -u\frac{z^1}{h(z^3)}, \quad v_2 = u\left(\frac{\pi}{2} - \frac{a}{h(z^3)}\cdot\frac{z^2}{h(z^3)} - 2\sqrt{1-(\zeta)^2}\right),$$

where $u = u(z^3)$ is an arbitrary differentiable function. From the boundary conditions, we calculate that $v_1 = \pm u$ for $z^1 = \pm h$, respectively. The physical components of the strain rate tensor can be calculated from Eqs. (5.18):

$$
\begin{aligned}
&d_{11} = -\frac{u}{h}, \quad d_{33} = 0,\\
(7.39)\qquad &d_{22} = \frac{u}{h}, \quad d_{12} = \frac{u}{h}\frac{\zeta}{\sqrt{1-(\zeta)^2}}; \quad |z^1| < h, \quad \zeta < 1.
\end{aligned}
$$

The solution given by Eqs. (7.1)–(7.3) represents a certain generalization of the well known Prandtl solution of the compresion of a thin layer of constant thickness, $h =$ const. (cf. [6] pp. 214–216). Denoting by $q = q(z^3)$ the intensity of the limiting value of the compressive force (measured on the unit length along z^3-axis), we can write (cf. [6] p. 216):

$$2q(z^3) = -ca\left(\frac{a}{h} + \pi\right),$$

the limiting value of the compresion force being equal to

$$p = -c\int_0^l a\left(\frac{a}{h(z^3)} + \pi\right)dz^3.$$

To determine the scope of applicability of the solution (7.37)–(7.39) to the spatial problems, we shall determine the maximal values of $|\overline{f}^3|$ $|\overline{d}^{13}|$, $|\overline{d}_{23}|$. In view of $\eta = 0.5$, $\varkappa = 0$ from (5.3) we obtain:

$$(7.40)\qquad \overline{f}^3 = -\frac{c}{h}\left(\frac{z^2}{h} + \frac{(\zeta)^2}{\sqrt{1-(\zeta)^2}}\right)h_{,3}; \quad |z^1| < h, \quad \zeta < 1.$$

Let us analyse separately the case in which the surfaces $z^1 = \pm h(z^3)$ are absolutly rigid, and the case in which these surfaces are rigid only in the planes $z^1 =$ const. (in the latter they can be subjected to bending in the planes $z^2 =$ const.).

If the surfaces $z^1 = \pm h(z^3)$ are absolutly rigid, then in view of $u = v_1$ for $z^1 = -h(z^3)$, we can assume that $u =$ const. Making use of (5.21) and (7.38), we obtain:

$$
\begin{aligned}
&\overline{d}_{13} = \frac{uz^1}{2h^2}h_{,3},\\
(7.41)\qquad &\overline{d}_{23} = \frac{u}{h^2}\left(\frac{\zeta^2}{\sqrt{1-\zeta^2}}h - \frac{a}{2}\right)h_{,3}.
\end{aligned}
$$

It can be observed that the absolute value $|\overline{f}^3| \equiv |-\sigma^{33}{}_{,3}|$ is small as compared with $|\sigma^{22}{}_{,1}|$, where by means of Eq. (7.1) we have:

$$\sigma^{22}{}_{,1} = -\frac{2c}{h}\frac{\zeta}{\sqrt{1-\zeta^2}}; \quad |z^1| < h, \quad |\zeta| < 1,$$

and where $|h_{,3}|$ is small with respect to 1. Analogously, $|\overline{d}_{13}|$ is small with respect to $|d_{11}|$, and $|\overline{d}_{23}|$ is small with respect to $|d_{12}|$, provided that $|h_{,3}| \ll 1$. Denoting by $\varepsilon\%$ an admissible error which arises from the postulates (3.1), (3.2), we have to restrict the form of the function $h = h(z^3)$ by the condition:

$$|h_{,3}| < \frac{\varepsilon}{100}. \tag{7.42}$$

The foregoing condition also assures that $\bar{p}^3$ can be disregarded as sufficiently small with respect to σ^{33}. Thus Eq. (7.42) represents the condition of applicability of the solution of spatial problems under consideration.

Now suppose that the surfaces $z^1 = \pm h(z^3)$ are rigid only in the planes $z^1 = \text{const.}$, and that they can be bent in the planes $z^2 = \text{const}$. This assumption can be applied if $a \ll l$. The scalar u is in this case dependent of z^3. Instead of the Eqs. (7.41), we obtain now:

$$\begin{aligned} \bar{d}_{13} &= -\frac{u}{h}\left(\frac{hu_{,1}}{u} - \frac{1}{2}h_{,3}\right)\zeta, \\ \bar{d}_{23} &= \frac{u}{h}\left(\frac{\zeta^2}{\sqrt{1-\zeta^2}}h - \frac{1}{2}a\right)h_{,3} - \frac{1}{2}u_{,3}\left(\frac{\pi}{2} - \frac{a}{h} + \frac{z^2}{h} - 2\sqrt{1-\zeta^2}\right). \end{aligned} \tag{7.43}$$

Since $|h_{,3}|$ has to be sufficiently small with respect to 1, then

$$\frac{hu_{,3}}{u} \ll 1. \tag{7.44}$$

The Eq. (7.44) is a condition imposed on the values of the velocity field $v_1 = \pm u$ on the surfaces $z^1 = \pm h(z^3)$, respectively, under which the foregoing solutions of the spatial problem are valid.

7.4. Cylindrical tube under radial pressure

Now let us analyse the problem of limit state of a thick walled shell, of variable thickness, a fragment of which is given in Fig. 8. The problem we have to investigate is a certain generalization of the well known problem of the yield state of a thick walled cylindrical shell (cf. [6] pp. 165–166 and 125–127) of constant thickness. Let z^1, z^2, z^3 be Carthesian orthogonal coordinates, and let as define the cylindrical coordinates r, φ, z, putting $r = \sqrt{(z^1)^2+(z^2)^2}$, $\varphi = \operatorname{arc\,tg}\frac{z^2}{z^1}$, $z = z^3$, (cf. Fig. 8). Assuming that the yield condition has the form $\sigma_2 - \sigma_1 = 2c$, and taking into account only axially-symmetric external loads $p = p(z)$, we obtain:

$$\sigma_1 = \sigma_r = 2c\ln\frac{r}{b(z)}, \quad \sigma_2 = \sigma_\varphi = 2c\left(1+\ln\frac{r}{b(z)}\right), \tag{7.45}$$

and

$$\sigma_3 = \sigma_z = \frac{\sigma_1+\sigma_2}{2} = c\left(1+2\ln\frac{r}{b(z)}\right), \quad a(z) \leqslant r \leqslant b(z). \tag{7.46}$$

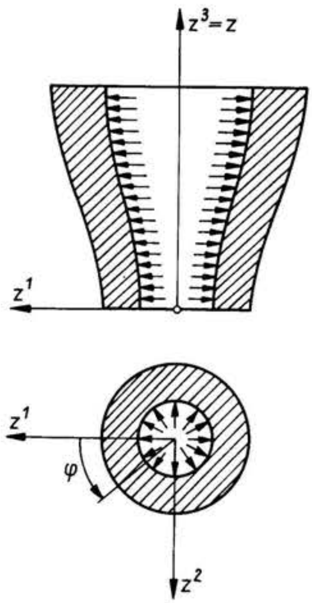

FIG. 8.

The limit value of external load is equal to

$$p(z) = 2c \ln \frac{b(z)}{a(z)}, \tag{7.47}$$

where $r = a(z)$, $r = b(z)$ are equations of the inner and outer surfaces of the shell, respectively (Fig. 8). From the Eq. (7.47) it follows that $p(z) = \text{const.}$ when $b(z)/a(z) = \text{const.}$ From Eqs. (5.5) and (5.11) we obtain:

$$\begin{aligned} \bar{f}^3 &= -\sigma_{3,3} = 2c\frac{d(\ln b)}{dz}, \\ \bar{p}^3|_{r=a} &= \sigma_3 n_3|_{r=a} = c\left(1+2\ln\frac{a(z)}{b(z)}\right)\frac{da}{dz}, \\ \bar{p}^3|_{r=b} &= \sigma_3 n_3|_{r=b} = c\frac{db}{dz}, \end{aligned} \tag{7.49}$$

where the weight of the material has been disregarded. Thus we conclude that we may also disregard the internal volume forces, the modulus of which does not exceed γ:

$$|\bar{f}^3| \leqslant \gamma. \tag{7.50}$$

Moreover, suppose that the coefficient c in the yield condition is determined with approximation $\pm\varepsilon c$, where ε is a positive number, $\varepsilon \ll 1$ given a priori. The stress σ_3 for $r = a(z)$ is, by virtue of (7.46), then determined with approximation $\pm\varepsilon c\left(1+\ln\frac{a}{b}\right)$, and for $r =$

$= b(z)$ it is determined with approximation $\pm\varepsilon c$. It follows that the values of internal surface forces $\bar{p}^3|_{r=a}$, $\bar{p}^3|_{r=b}$ may be disregarded if their moduli satisfy the condition:

$$(7.51) \qquad |\bar{p}^3|_{r=a} \leqslant \varepsilon c \left|1+2\ln\frac{a}{b}\right|, \quad |\bar{p}^3|_{r=b} \leqslant \varepsilon c.$$

From the inequalities (7.50), (7.51), and from Eqs. (7.49), we obtain the following criteria of applicability of Eqs. (7.47)

$$(7.52) \qquad \left|\frac{d\ln b(z)}{dz}\right| \leqslant \frac{\gamma}{2c}, \quad \left|\frac{da(z)}{dz}\right| \leqslant \varepsilon, \quad \left|\frac{db(z)}{dz}\right| \leqslant \varepsilon.$$

When the outer boundary surface of the shell is cylindrical — i.e. $b(z)$ = const. — then the first and the last relations from Eqs. (7.42) become identities.

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