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Nonaxisymmetric Stokes flow past a torus in the presence of a wall

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THIS PAPER presents the exact solutions for the creeping motion of an open torus of arbitrary size in the presence of a wall for the following conditions: asymmetrical translation of a torus in the parallel direction to a stationary wall, rotation about the axis perpendicular to the axis symmetry of the torus, shear flow past a rigidly held torus. The boundary collocation technique was applied to solve these problems. The associated resisting force, torque and wall correction factor are computed for a torus and compared with the exact solutions for the same problems for the sphere and with the approximation theory for a particle of an arbitrary shape [3].

W pracy przedstawiono rozwiązania ścisłe dotyczące opływu otwartego torusa w przepływie Stokesa w obecności ścianki dla następujących warunków zadania: a) asymetryczne przemieszczenie torusa w kierunku równoległym do stacjonarnej ścianki, b) obrót wokół osi prostopadłej do osi symetrii torusa, c) przepływ ścinający wokół torusa sztywno zamocowanego. Do rozwiązania zastosowano metodę kollokacji brzegowej. Obliczono siłę oporu, moment obrotowy i współczynnik oddziaływania ścianki oraz porównano rezultaty z wynikami ścisłymi dotyczącymi podobnego zagadnienia dla sfery jak również z przybliżonymi rozwiązaniami dla cząsteczki o dowolnym kształcie [3].

В работе представлены точные решения, касающиеся обтекания открытого тора для течении Стокса в присутствии стенки для следующих условий задачи: а) асимметри перемещения тора в направлении параллельном к стационарной стенке, б) вращение вокруг оси перпендикулярной к оси симметрии тора, в) течение со сдвигом вокруг жестко закрепленного тора. Для решения применен метод граничной колокации. Вычитлены сила сопротивления, крутящий момент и поправочный коэфициент стенки, а также сравнены результаты с точными результатами, касающимися аналогичной задачи для сферы, как тоже с приближенными решениями для частицы произвольной формы [3].

1. Introduction

THE LOW REYNOLDS number flow past a body in the presence of a wall has been investigated for many years: LORENTZ (1907) [12], BRENNER (1964) [3], HO i LEAL (1977) [9], but the sphere has been the main object of research.

The method of reflection was generally used to solve these problems. It requires such a natural coordinate system in which one could simultaneously satisfy the no-slip boundary conditions on the surface of the body and along the wall. The point — force approximation has been used for treating slow flow problems, too.

The combined analytical-numerical solution procedure used in this study is the collocation technique first developed by GLUCKMAN, PFEFFER and WEINBAUM [6] for the unbounded multispherical flows and then by GANATOS, PFEFFER and WEINBAUM [5] for the motion of a sphere between plane parallel boundaries.

This method is based on the concept that for the Stokes flow, disturbances from all boundaries may be treated simultaneously. A cardinal rule for the successful application of the collocation technique is that the velocity disturbances produced by each coordinate boundary may be represented by an ordered sequence of fundamental solutions appropriate to the constant orthogonal coordinate surfaces.

This paper presents the exact solutions for the three-dimensional creeping motions of a torus in the presence of a wall with the application of the collocation technique. The quasi-steady flow with planar symmetry is considered.

The associated resisting force, torque and wall correction factor are computed and compared with the exact solutions for the same problems for the sphere [3].

This study is an extension of [10], in which the axisymmetrical motion of a torus in the presence of a wall is considered.

2. Mathematical formulation

A rigid, open torus of the geometrical ratio $k = R/a$ ($R-a$ being the smallest radius of the open hole and $R+a$ being the radius to the outermost rim of the torus) (Fig. 1) moves in the presence of the wall in a homogeneous incompressible fluid of density ϱ and viscosity μ which, being far removed from the torus, has a shear velocity profile.

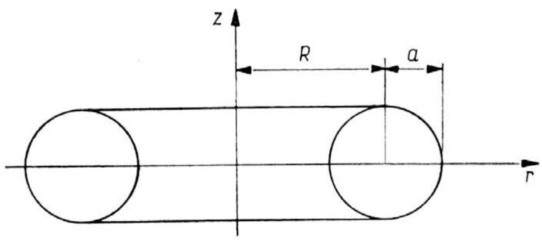

FIG. 1. Torus.

The torus moves with a constant velocity **W** in a direction parallel to the infinity plane; moreover, it rotates about an axis perpendicular to its axis of symmetry with a constant velocity **Ω** (Fig. 2).

In a system of Cartesian coordinates (x, y, z) in which z is the axis of symmetry and the plane $z = 0$ is the plane of symmetry of the body, not decreasing the generality of the

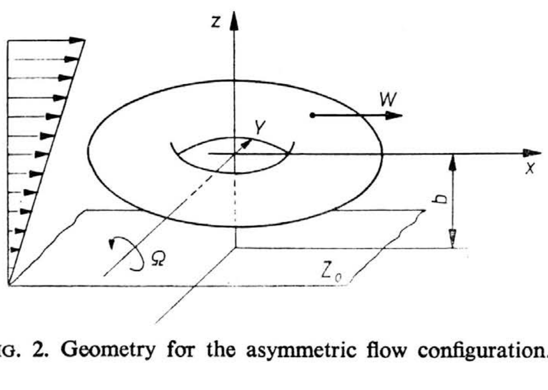

FIG. 2. Geometry for the asymmetric flow configuration.

problem, it is supposed that the torus translates with the velocity $\mathbf{W}$ along the x-axis and rotates with the velocity $\mathbf{\Omega}$ about the y-axis.

The shear flow velocity profile may be written $\mathbf{v} = S(z+b)i_x$.

We assume that the Reynolds numbers

$$\frac{TW\varrho}{\mu} \quad \text{and} \quad \frac{\Omega T^2\varrho}{\mu}, \quad \frac{ST^2\varrho}{\mu} \quad \text{where} \quad T = (R+a)\cdot 2$$

are sufficiently small to allow us to neglect the nonlinear inertia terms in the Navier–Stokes equations of the fluid motion

Accordingly, the equations governing the flow are the quasi-stationary Stokes creeping-flow equations

$$\mu\nabla^2\mathbf{v} = \nabla p, \tag{2.1}$$

and the equation of continuity

$$\nabla\mathbf{v} = 0, \tag{2.2}$$

where $\mathbf{v}$ and p are, respectively, the fluid velocity and the pressure fields.

The velocity satisfies the following conditions:

Boundary condition on the surface of the torus

$$\mathbf{v} = \mathbf{W}+(\mathbf{\Omega}\times\mathbf{r}). \tag{2.3}$$

Condition at infinity

$$\mathbf{v} = S(z+b)i_x. \tag{2.4}$$

Boundary condition on the wall

$$\mathbf{v} = 0.$$

For the geometry of the problem at hand, owing to the linearity of the Stokes equation, the velocity field is linearly composed of three parts:

$$\mathbf{v} = \mathbf{w}_w+\mathbf{u}_t+\mathbf{V}_\infty, \tag{2.5}$$

where $\mathbf{w}_w$ denotes the fundamental solution (2.1) in halfspace, $\mathbf{u}_t$ represents the fundamental solution (2.1) outside the torus, in space, $\mathbf{V}_\infty$ is the velocity far removed from the torus, $\mathbf{V}_\infty = S(z+b)i_x$.

In this paper we solve the problem at the moment t_0, when the axis of symmetry of the torus is perpendicular to the wall. In this case, the boundary conditions and solution have the simplest form following from the planar symmetry of the motion.

The boundary conditions on the surface of the torus (2.3) and at infinity are conveniently expressed in terms of the cylindrical polar coordinates (r, θ, z) with corresponding velocities (v_r, v_t, v_z) which are related to the Cartesian values in the ordinary way.

For this case, the no-slip boundary condition (2.3) has the form

$$\begin{aligned} v_r &= (W+\Omega z)\cos\theta, \\ v_t &= -(W+\Omega z)\sin\theta, \\ v_z &= -\Omega r\cos\theta \end{aligned} \tag{2.6}$$

and condition at infinity (2.4)

$$
\begin{aligned}
v_r &= S(z+b)\cos\theta,\\
v_t &= -S(z+b)\sin\theta,\\
v_z &= 0.
\end{aligned}
\tag{2.7}
$$

The part $\mathbf{w}_w$ represents a double integral of all separable solutions of Eq. (2.1) [5] in rectangular coordinates which produced finite velocities everywhere in the flow field and is given by the double Fourier integral

$$\mathbf{w}_w = (w_x, w_y, w_z)$$

$$
\begin{aligned}
w_x(x, y, z) &= \int_0^\infty\int_0^\infty \hat{w}_x(\alpha, \beta, z)\cos\alpha x\cos\beta y \; d\alpha d\beta,\\
w_y(x, y, z) &= \int_0^\infty\int_0^\infty \hat{w}_y(\alpha, \beta, z)\sin\alpha x\sin\beta y \; d\alpha d\beta,\\
w_z(x, y, z) &= \int_0^\infty\int_0^\infty \hat{w}_z(\alpha, \beta, z)\sin\alpha x\cos\beta y \; d\alpha d\beta,\\
p(x, y, z) &= \int_0^\infty\int_0^\infty \hat{p}(\alpha, \beta, z)\sin\alpha x\cos\beta y \; d\alpha d\beta,
\end{aligned}
\tag{2.8}
$$

where

$$
\begin{aligned}
\hat{w}_x(\alpha, \beta, z) &= \left(A\left(1-\frac{\alpha^2}{k}z\right)+\frac{\alpha\beta B}{k}z-\alpha Cz\right)e^{-kz},\\
\hat{w}_y(\alpha, \beta, z) &= \left(\frac{\alpha\beta}{k}\cdot Az+B\left(1-\frac{\beta^2}{k}z\right)+Cz\beta\right)e^{-kz},\\
\hat{w}_z(\alpha, \beta, z) &= (\alpha Az-\beta Bz+C(1+kz))e^{-kz},\\
\hat{p}(\alpha, \beta, z) &= (-\alpha A+\beta B-kC).
\end{aligned}
\tag{2.9}
$$

Here the A, C, B coefficients are unknown functions of the separation variable α and β. By proper choice of these functions, $\mathbf{w}_w$ is capable of exactly cancelling disturbances produced by the torus along the plane boundary.

The $\mathbf{u}_t$ is conveniently expressed in terms of toroidal coordinates (η, θ, ξ) (Fig. 3) which are related to the cylindrical coordinates (r, θ, z) by the formula

$$
r = c\,\frac{\operatorname{sh}\eta}{\operatorname{ch}\eta-\cos\xi}, \quad z = c\,\frac{\sin\xi}{\operatorname{ch}\eta-\cos\xi}.
\tag{2.10}
$$

The factor c is a constant with the dimension of length whose significance will be apparent.

Guided by the dependence of the velocity components on θ by the boundary conditions (2.6), by the work of MAJUMDAR, O'NEILL [13] for the axisymmetric translation of the torus and by the analysis of GOREN, O'NEILL [7] for the asymmetric translation and rotation of the torus, we obtain the solution $\mathbf{u}_t$ in the following form: in cylindrical coordinates.

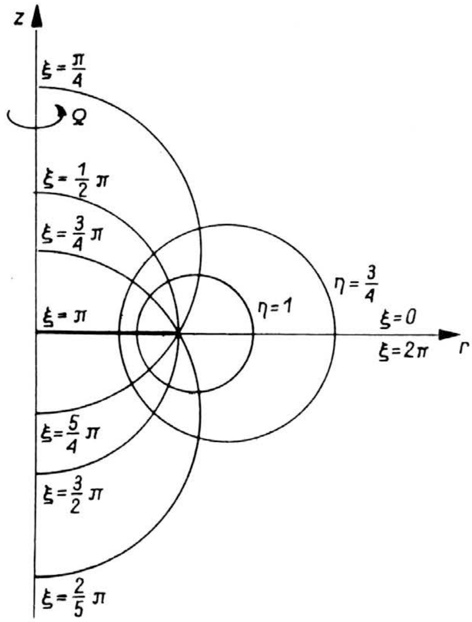

FIG. 3. Toroidal coordinates.

(2.11)
$$\begin{aligned}
\mathbf{u}_t &= (u_r, u_t, u_z),\\
u_r &= \frac{1}{2}(U+V+rQ/c)\cos\theta,\\
u_t &= -\frac{1}{2}(U-V)\sin\theta,\\
u_z &= \frac{1}{2}(2W+zQ/c)\cos\theta,
\end{aligned}$$

where

(2.12)
$$\begin{aligned}
U &= (\mathrm{ch}\,\eta-\cos\xi)^{1/2}\sum_{n=0}^{\infty}(A_n'\cos n\xi+A_n\sin n\xi)P^2_{n-1/2}(\mathrm{ch}\,\eta),\\
Q &= (\mathrm{ch}\,\eta-\cos\xi)^{1/2}\sum_{n=0}^{\infty}(B_n'\cos n\xi+B_n\sin n\xi)P^1_{n-1/2}(\mathrm{ch}\,\eta),\\
V &= (\mathrm{ch}\,\eta-\cos\xi)^{1/2}\sum_{n=0}^{\infty}(D_n'\cos n\xi+D_n\sin n\xi)P_{n-1/2}(\mathrm{ch}\,\eta),\\
W &= (\mathrm{ch}\,\eta-\cos\xi)^{1/2}\sum_{n=0}^{\infty}(C_n'\cos n\xi+C_n\sin n\xi)P^1_{n-1/2}(\mathrm{ch}\,\eta),
\end{aligned}$$

$P^m_{n-1/2}$ denotes the associated Legendre function of the first kind of order $n-1/2$ and degree m.

The coefficients A_n, A'_n, B_n, B''_n, ..., D'_n are independent of η, ξ and the continuity equation provides a relationship between them.

$$-\left(n-\frac{3}{2}\right)B'_{n-1}-5B'_n+(n+3/2)B'_{n+1}-(n-4)(n-3/2)A'_{n-1}$$
$$+2(n-9/4)A'_n-(n+3/2)(n+5/2)A'_{n+1}-(2n-1)C_{n-1}+4nC_n$$
$$+(2n-1)C_{n+1}-\frac{1}{2}D'_{n+1}+D'_n-\frac{1}{2}D'_{n+1} = 0,$$

(2.13)
$$\left(n-\frac{3}{2}\right)B_{n-1}+5B_n-\left(n+\frac{3}{2}\right)B_{n+1}+(n-4)(n-3/2)A_{n-1}$$
$$-2(n-9/4)A_n+(n+3/2)(n+5/2)A_{n+1}-(2n-1)C'_{n-1}+4nC'_n$$
$$+(2n-1)C'_{n+1}+\frac{1}{2}D_{n-1}-D_n+\frac{1}{2}D_{\substack{n+1\\ n\geq 1}} = 0.$$

Each of the fundamental solution $\mathbf{w}_w$ (2.8), $\mathbf{u}_t$ (2.11) already satisfies the governing equation (2.1) at each point in the field, the proper boundless conditions at infinity and the requirements of the planar symmetry.

Owing to the linearity of Stokes equations, we separately seek the solution for translation without rotation of the torus in the direction parallel to the wall,

rotation without any translation of the torus,

shear flow past a rigidly held torus.

Now we solve the problem for the translational motion.

An application of the boundary conditions $\mathbf{v} = 0$ along the wall $z = -b$ permits to express the disturbance produced at the wall by a disturbance produced by the torus

$$\mathbf{u}_t = (u_x, u_y, u_z), \qquad \mathbf{w}_w = (w_x, w_y, w_z),$$

(2.14)
$$\int_0^\infty\int_0^\infty \hat{w}_x(\alpha, \beta, -b)\cos\alpha x\cos\beta y\, d\alpha\, d\beta = -u_x(x, y, -b),$$
$$\int_0^\infty\int_0^\infty \hat{w}_y(\alpha, \beta, -b)\sin\alpha x\sin\beta y\, d\alpha\, d\beta = -u_y(x, y, -b),$$
$$\int_0^\infty\int_0^\infty \hat{w}_z(\alpha, \beta, -b)\sin\alpha x\cos\beta y\, d\alpha\, d\beta = -u_z(x, y, -b).$$

These equations may be inverted, and integration may be performed analytically using the results for the Hankel transforms found in [1] and based on expressing the associated toroidal Legendre function by its polynomial representation. The results are given in Appendix A and we now may write

$$w_x(\alpha, \beta, -b) = \sum_{n=0}^{\infty} B_nB_n^* + B'_nB_n^{*\prime} + D_nD_n^* + D'_nD_n^{*\prime} + C_nC_n^* + C'_nC_n^{*\prime},$$

(2.15)
$$w_y(\alpha, \beta, -b) = \sum_{n=0}^{\infty} B_nB_n^{**} + B'_nB_n^{**\prime} + D_nD_n^{**} + D'_nD_n^{**\prime},$$

$$w_z(\alpha, \beta, -b) = \sum_{n=0}^{\infty} B_n B_n^{***} + B_n' B_n^{***\prime} + A_n A_n^* + A_n' A_n^{*\prime}.$$

The coefficients $A_n, A_n' \dots D_n, D_n'$ satisfy Eq. (2.13).

Equations (2.15) give the w_x, w_y, w_z functions evaluated at $z = -b$ in terms of the still unknown coefficients $A_n, A_n', C_n, \dots, D_n$.

To obtain the w_x, w_y, w_z at any value of z, one must determine the unknown functions A, B, C, in Eqs. (2.9).

Using Eqs. (2.8), (2.9) and (2.15) as the final result, we have in the cylindrical coordinates $\mathbf{w}_w = (w_r, w_t, w_z)$

$$\begin{aligned} w_r(r, \theta, z) &= \sum_{n=0}^{\infty} (B_n BR_n + B_n' BR_n' + A_n AR_n + A_n' AR_n' + C_n CR_n + C_n' CR_n' \\ &\qquad + D_n DR_n + D_n' DR_n')\cos\theta, \\ w_t(r, \theta, z) &= \sum_{n=0}^{\infty} (B_n BT_n + B_n' BT_n' + A_n AT_n + A_n' AT_n' + C_n ST_n \\ &\qquad + C_n' CT_n' + D_n DT_n + D_n' DT_n')\sin\theta, \\ w_z(r, \theta, z) &= \sum_{n=0}^{\infty} (B_n BZ_n + B_n' BZ_n' + A_n AZ_n + A_n' AZ_n' + C_n CZ_n + C_n' CZ_n')\cos\theta. \end{aligned} \tag{2.16}$$

The functions $BR_n, \dots, CZ_n$ are listed in Appendix C.

We have now expressions for the solution $\mathbf{v} = \mathbf{u}_t + \mathbf{w}_w$, which are still in terms of the unknown toroidal coefficients $A_n, A_n' \dots D_n, D_n'$. In this manner the infinite domain boundary value problem has been reduced to a much simpler finite domain problem in which the infinite array of the unknown coefficients describing the toroidal disturbance need to be determined so as to satisfy the appropriate boundary conditions on the surface of the torus.

To satisfy the boundary conditions

$$\mathbf{v} = \mathbf{W} \tag{2.17}$$

exactly on the surface of the torus for the translational motion, one would require the solution of the entire infinite array of some unknown coefficients.

The collocation technique satisfies the boundary conditions at a finite number of discrete points of the torus generating arc and reduces the infinite series to a finite one. If the no-slip conditions are to be satisfied at M points, the infinite series are truncated after M terms. Together with the continuity equation, there is a set of $4M$ simultaneous linear algebraic equations for the $4M$: A_n, B_n, $C_n' \dots D_n'$ unknown coefficients of the truncated solution, which may be solved by any standard matrix reduction technique. Once these constants are determined, the solution for the velocity field is completely known.

In the same way the solutions for rotational motion of the torus and for the shear flow were obtained. In these cases the boundary conditions (2.17) were replaced by

$$\begin{aligned} \mathbf{v} &= \mathbf{\Omega} \times \mathbf{r} \quad \text{for rotational motion,} \\ \mathbf{v} &= 0 \quad \text{for shear flow,} \end{aligned} \tag{2.18}$$

respectively.

3. Forces and torque acting on the torus

The force **F** acting on the torus is given by

$$\mathbf{F} = \int_S \mathbf{R}_n ds, \tag{3.1}$$

where $\mathbf{R}_n$ is the stress vector associated with the direction n of the outward normal at any point on the surface of the torus.

Substituting the derived expressions for the pressure and velocity gradients evaluated on the surface η_0 of the torus and carrying out the indicated integration, we get the following results: $\mathbf{F} = (F_x, 0, 0)$

$$F_x = \frac{1}{2}\sqrt{2}\,\pi\mu H_f[c]\sum_{n=0}^{\infty}\big(4D'_n + (4n^2-1)B'_n + nQ_{n-1/2}P^1_{n-1/2} + (B_n + C_n) + Q'_{n-1/2}A_n\big)\sinh\eta_0. \tag{3.2}$$

The torque M acting on the torus is given by

$$\mathbf{M} = \int_S (\mathbf{r}\times\mathbf{R}_n)ds \tag{3.3}$$

when the moments of the surface stress R_n are taken about the origin $r = z = 0$. The resisting torque has Cartesian components $(0, -M_y 0)$ with

$$M_y = \pi\sqrt{2}\,H_g\mu[c]^3\sum_{n=0}^{\infty}\big((4n^2-1)C'_n - 4\mathrm{n}D_n + 2nQ'_{n-1/2}P^1_{n-1/2} + (B'_n + (n-1)C_n) + nQ'_{n-1/2}P^2_{n-1/2}A'_n\big)\sinh\eta_0. \tag{3.4}$$

For translational motion of the torus we put $H_f = W$, $H_g = W/T$, for rotational motion $H_f = \Omega T$, $H_g = \Omega$, for shear flow past a torus we have $H_f = S\cdot T$, $H_g = S$.

In the case when the wall tends to infinity $(b\to\infty)$ the formulas (3.2) and (3.3), are similar to those given by GOREN, O'NEILL [7]. In computing and presenting the results, it is convenient to use the nondimensional physical quantities.

A dimensionless wall drag correction factor is defined as

$$\lambda_f = \frac{F_x}{F_\infty}, \tag{3.5}$$

the ratio of F_x (3.2) to the force F_∞ acting on a torus in the same flow in unbounded flow.

Analogically we have defined the wall correction factor for the torque

$$\lambda_g = \frac{M_y}{M_\infty}. \tag{3.6}$$

The calculation was made for various geometrical ratios of the torus: $k \in \{1, 2, 3, 5, 10, 80\}$ and for various dimensionless distances $\bar{b}$ defined by the formula

$$\bar{b} = \frac{b}{T}, \tag{3.7}$$

$$T = 2(a+R), \tag{3.8}$$

$$\bar{b} \in \{1.13,\ 1.54,\ 2.35,\ 3.76,\ 6.13,\ 10.1\}.$$

The calculated values of λ_f and λ_g are given in Figs. 4 and 5 for the translational motion of a torus, in Figs. 6 and 7 for the rotational motion of a torus and in Figs. 8 and 9 for rigidly immersed torus in shear flow.

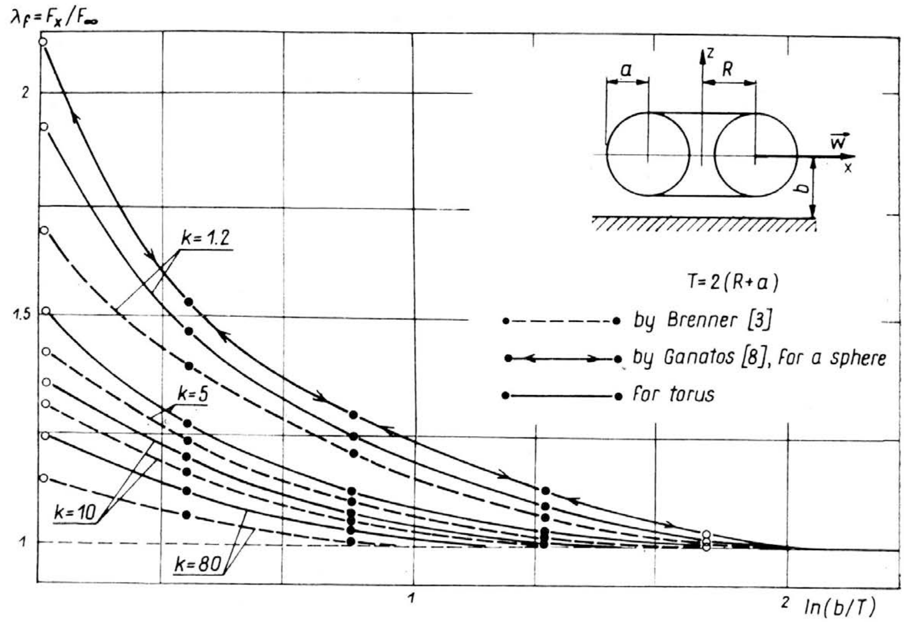

FIG. 4. Force acting on a torus in translational, nonaxisymmetrical motion.

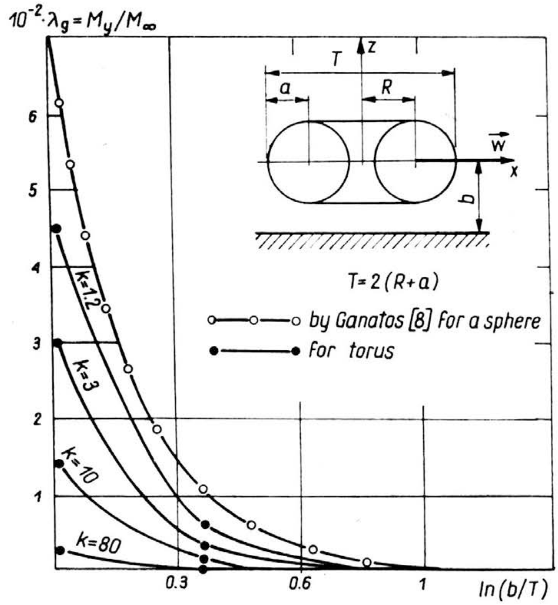

FIG. 5. Torque acting on the torus in translational nonaxisymmetrical motion.

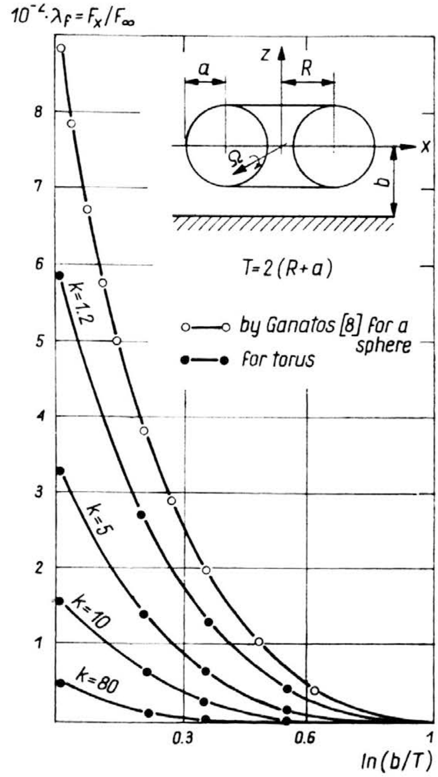

FIG. 6. Force acting on the torus in rotational, nonaxisymmetrical motion, at this time t_0, when the axis of symmetry of the torus is perpendicular to the plane wall.

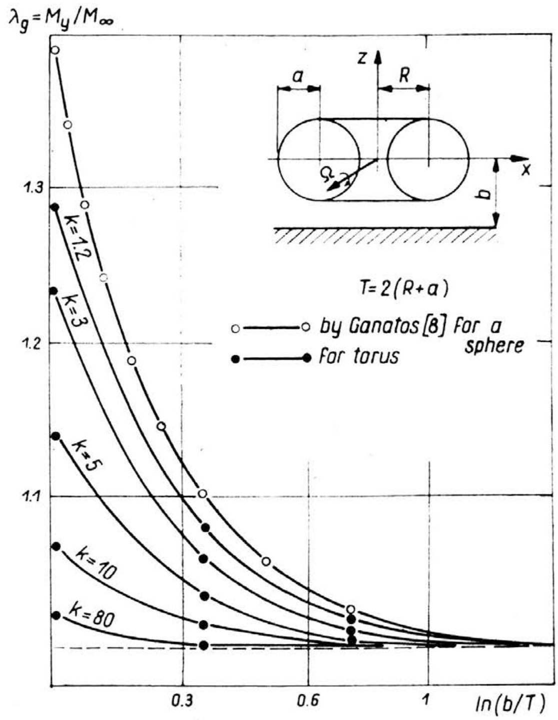

FIG. 7. Torque acting on the torus in rotational, nonaxisymmetrical motion, at this time t_0, when the axis of symmetry of the torus is perpendicular to the plane wall.

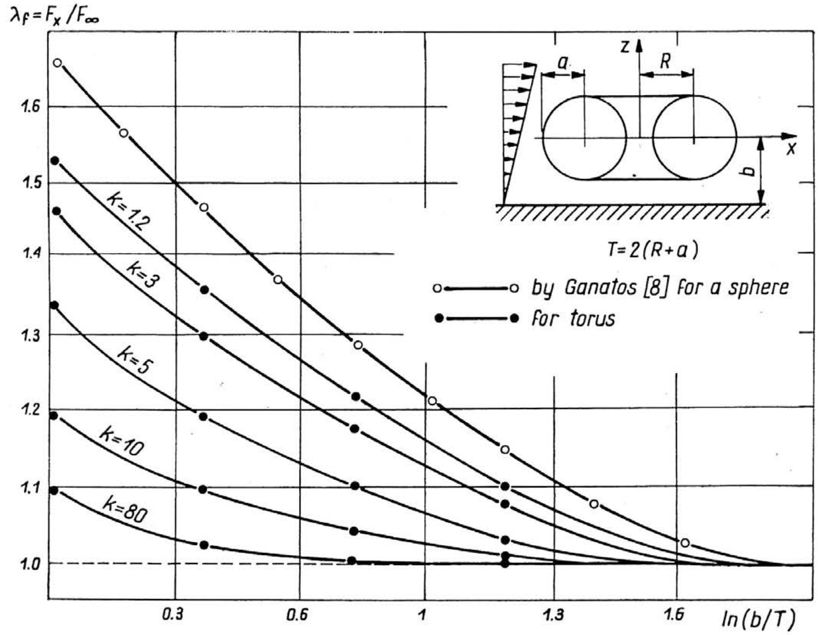

FIG. 8. Force acting on the torus immersed in shear flow, at the time t_0, when the axis of symmetry of the torus is perpendicular to the plane wall.

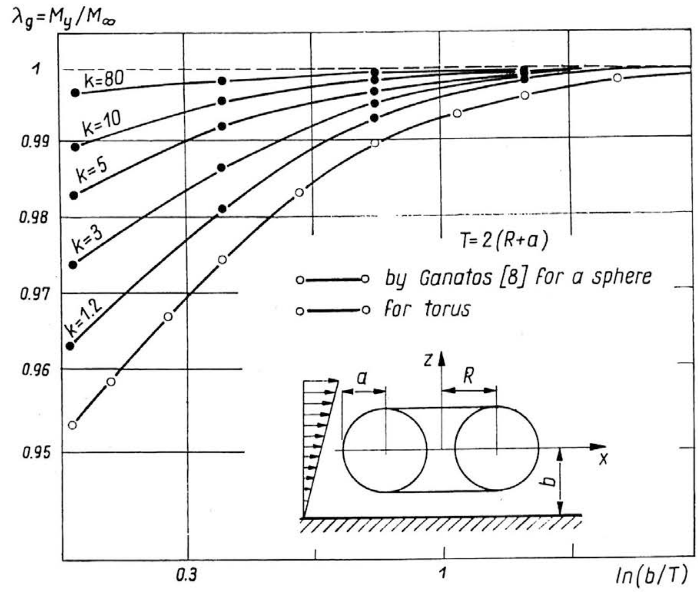

FIG. 9. Torque acting on the torus immersed in shear flow, at the time t_0, when the axis of symmetry of the torus is perpendicular to the plane wall.

The above results were compared with the ones for the same problems for the sphere given by GANATOS [5] and with the approximate theory.

The accuracy and convergence of this method of solution was tested for all the motions considered in this paper and compared with the published exact solutions for a torus in the unbounded flow due to the lack of other exact solutions for the torus in the bounded flow. The scheme for spacing the points on the surface of the torus is based on the papers [4, 5] in which the corresponding problem for the motion of a sphere in the presence of a wall is considered.

In the results the calculations of the wall correction factor in this study were made using the points $\{0°, 22.5°, 45°, 67.5°, 90°, 113°, 135°, 157.5°, 178°\}$. It is noteworthy that each boundary point represents a ring, owing to the nature of the problem.

Using this collocation scheme, solutions were obtained for various M-numbers of the points, different values of the torus shape factor k and torus-to-wall spacings b. For $b \to \infty$ the solutions were compared with the exact solutions of GOREN and O'NEILL [7]. These results show that the collocation solutions converge monotonically to the exact solutions. Convergence is very rapid as k increases and becomes slow for $k \to 1$ (see Table 1, 2).

Table 1. Comparison of exact and approximate dimensionless force coefficients for translation of a torus along a transverse axis in unbounded flow.

M	$k = 10$	$k = 2.5$	$k = 1.2$
2	0.4182	0.5327	0.7243
4	0.4537	0.5982	0.8945
6	0.4621	0.6011	0.9524
8	0.4621	0.6274	0.9741
10	0.4621	0.6284	0.9841
12	0.4621	0.6284	0.9841
exact, by [7]	0.4621	0.6284	0.9841

Table 2. Comparison of exact and approximate dimensionless torque coefficients for rotational motion of a torus in unbounded flow.

M	$k = 10$	$k = 2.5$	$k = 1.2$
2	0.3994	0.5982	0.8163
4	0.4295	0.6163	0.8274
6	0.4318	0.6111	0.8352
8	0.4318	0.6120	0.8624
10	0.4318	0.6292	0.8763
12	0.4318	0.6243	0.8765
exact, by [7]	0.4318	0.6243	0.8765

Examination of the accuracy and convergence of this method for various b shows that when the torus is located near the wall (b is small), then the convergence is slow and we must use a large number of points in order to obtain the solution accurate enough (Fig. 10).

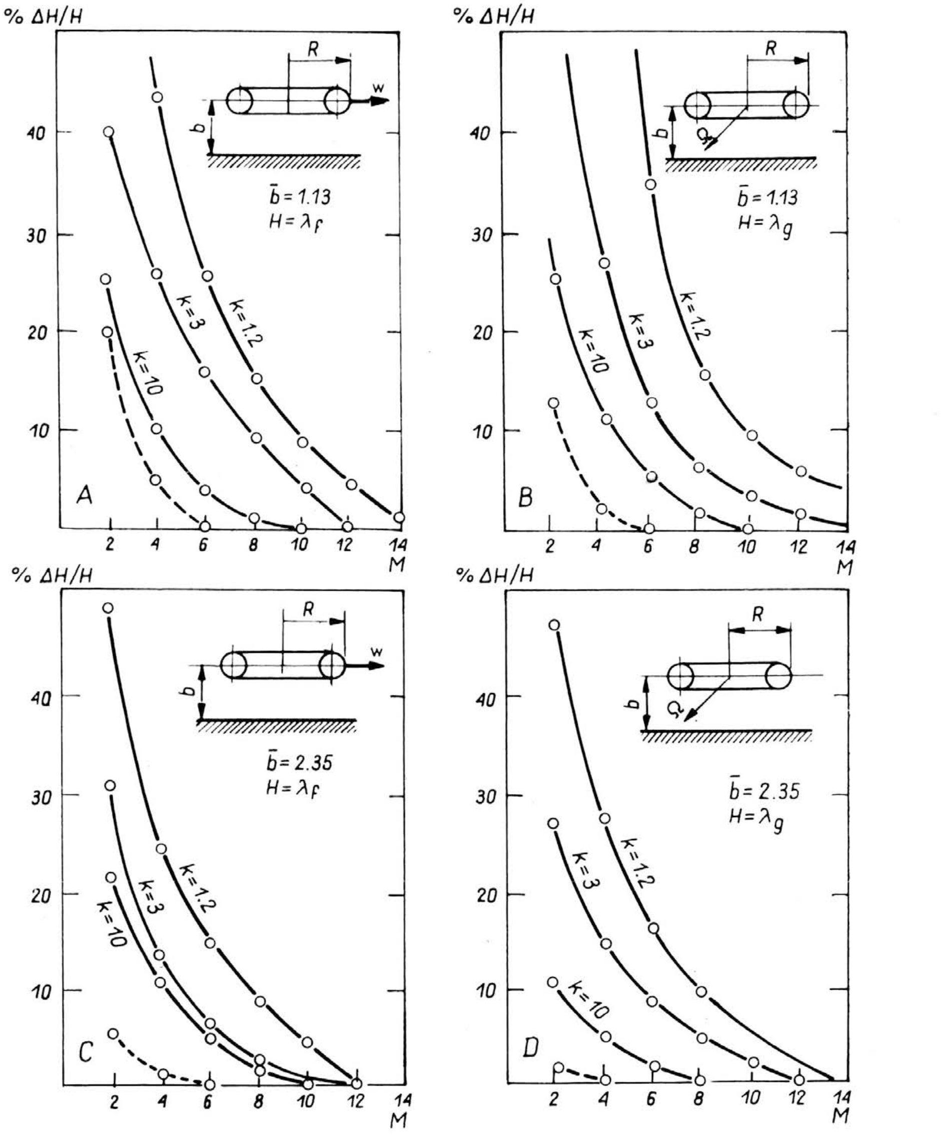

FIG. 10. Convergence of a wall correction factor for translational *A*, *C* and rotational *B*, *D* motion of a torus —— in the presence of a wall; - - - - wall correction factor for a sphere, by [5] ($R = T/2$).

4. Main conclusions

1. The behaviour of the wall correction factor λ for the flows considered in this paper is similar to that for a sphere and depends on various geometrical ratios of the torus k as well. When k tends to 1, then the values λ_f, λ_g obtained for the torus tend to values λ_f, λ_g for the sphere.

2. For all motions considered in this paper when the value b, distance from the wall, is fixed, with the increasing k, $\lambda_{\bar{b}}$ decreases and when the geometrical shape factor of a torus k is constant, with decreasing b $\lambda_{\bar{b}}$ increases ($\bar{b} = f, g$).

3. Solutions for any combinations of the motions described in this paper may be obtained by a simple superposition of solutions given above.

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Appendix A

This appendix contains a list of functions contained in Eqs. (2.15)

$$B_n^* = -\frac{1}{\pi}\left(\frac{\alpha^2}{(\alpha^2+\beta^2)^{3/2}} f_{n,1,2,0,0} + \frac{\beta^2-\alpha^2}{(\alpha^2+\beta^2)^2} f_{n,1,1,1,0}\right),$$

$$B_n^{*\prime} = -\frac{1}{\pi}\left(\frac{\alpha^2}{(\alpha^2+\beta^2)^{3/2}} f_{n,1,2,0,1} + \frac{\beta^2-\alpha^2}{(\alpha^2+\beta^2)^2} f_{n,1,1,1,1}\right),$$

$$D_n^* = -\frac{2}{\pi}\left(\frac{2\alpha^2-1}{2(\alpha^2+\beta^2)^{3/2}} f_{n,2,1,0,0} + \frac{\beta^2-\alpha^2}{(\alpha^2+\beta^2)^2} f_{n,2,0,1,0}\right),$$

$$D_n^{*\prime} = -\frac{2}{\pi}\left(\frac{2\alpha^2-1}{2(\alpha^2+\beta^2)^{3/2}} f_{n,2,1,0,1} + \frac{\beta^2-\alpha^2}{(\alpha^2+\beta^2)^2} f_{n,2,0,1,1}\right),$$

$$C_n^* = -\frac{1}{\pi} f_{n,0,1,0,0} \frac{1}{(\alpha^2+\beta^2)^{1/2}},$$

$$C_n^{*\prime} = -\frac{1}{\pi} f_{n,0,1,0,1} \frac{1}{(\alpha^2+\beta^2)^{1/2}},$$

$$B_n^{**} = \frac{1}{\pi} \frac{\alpha\beta}{(\alpha^2+\beta^2)^{3/2}}\left(f_{n,1,2,0,0} - \frac{2}{(\alpha^2+\beta^2)^{1/2}} f_{n,1,1,0,0}\right),$$

$$B_n^{**\prime} = \frac{1}{\pi} \frac{\alpha\beta}{(\alpha^2+\beta^2)^{3/2}}\left(f_{n,1,2,0,1} - \frac{2}{(\alpha^2+\beta^2)^{1/2}} f_{n,1,1,0,1}\right),$$

$$D_n^{**} = \frac{2}{\pi} \frac{\alpha\beta}{(\alpha^2+\beta^2)^{3/2}}\left(f_{n,2,1,0,0} - \frac{2}{(\alpha^2+\beta^2)^{1/2}} f_{n,2,0,1,0}\right),$$

$$D_n^{**\prime} = \frac{2}{\pi} \frac{\alpha\beta}{(\alpha^2+\beta^2)^{3/2}}\left(f_{n,2,1,0,1} - \frac{2}{(\alpha^2+\beta^2)^{1/2}} f_{n,2,0,1,1}\right),$$

$$B_n^{***} = -\frac{1}{\pi} \frac{\alpha}{(\alpha^2+\beta^2)} (-b) f_{n,1,1,1,0},$$

$$B_n^{***\prime} = -\frac{1}{\pi} \frac{\alpha}{\alpha^2+\beta^2} (-b) f_{n,1,1,1,1},$$

$$A_n^{*\prime} = -\frac{2}{\pi}\frac{\alpha}{\alpha^2+\beta^2} f_{n,1,1,1,1},$$

$$A_n^{*} = -\frac{2}{\pi}\frac{\alpha}{\alpha^2+\beta^2} f_{n,1,1,1,0},$$

where

$$f_{n,m,s,\nu,1} = \int\limits_0^\infty t^s P_{n-1/2}^m(\cosh\bar{\eta})\big(\cos(n\bar{\xi})(\cosh\bar{\eta}-\cos\bar{\xi})\big)^{1/2} J_\nu(\alpha t)\,dt,$$

$$f_{n,m,s,\nu,0} = \int\limits_0^\infty t^s P_{n-1/2}^m(\cosh\bar{\eta})\sin(n\bar{\xi})(\cosh\bar{\eta}-\cos\bar{\xi})^{1/2} J_\nu(\alpha t)\,dt,$$

$$\bar{\eta} = \ln\frac{R_2}{R_1},$$

$$\cos\bar{\xi} = \frac{R_1^2+R_2^2-2c^2}{2R_1R_2},$$

$$R_1^2 = b^2+(t-c)^2,$$

$$R_2^2 = b^2+(t+c)^2.$$

Appendix B

This appendix contains a summary of the formulas used in evaluating the integrals required by the relations (2.14). These formulas were obtained using the results and general formulas for Fourier transforms found in ERDELYI [2]:

$$\int\limits_0^{\pi/2} \cos(g\cos\gamma)\cos(d\sin\gamma)\,d\gamma = \frac{\pi}{2}J_0(u),$$

$$\int\limits_0^{\pi/2} \cos^2\gamma\cos(g\cos\gamma)\cos(d\sin\gamma)\,d\gamma = \frac{\pi}{2u^2}\left(g^2J_0(u)+\frac{d^2-g^2}{u}J_1(u)\right),$$

$$\int\limits_0^{\pi/2} \cos\gamma\sin\gamma\sin(g\cos\gamma)\sin(d\sin\gamma)\,d\gamma = -\frac{\pi}{2}\frac{gd}{u^2}\left(J_0(u)-\frac{2}{u}J_1(u)\right),$$

$$\int\limits_0^{\pi/2} \cos\gamma\sin(g\cos\gamma)\cos(d\sin\gamma)\,d\gamma = \frac{\pi}{2}\frac{g}{u}J_1(u),$$

where $u = \sqrt{g^2+d^2}$.

Appendix C

This appendix contains a list of the functions contained in Eqs. (2.16) where:

$$AR_n = 2\int\limits_0^\infty e^{-x(z+b)}\big(Ef_{n,1,1,1,0}\,x(z+b)\,x^{1/2}\big)dx,$$

$$AT_n = 2\int_0^\infty e^{-x(z+b)}F\cdot\big(f_{n,1,1,1,0}\,x(z+b)\big)x^{1/2}dx,$$

$$AZ_n = -2\int_0^\infty \big(1+x(z+b)x^{1/2}\big)e^{-x(z+b)}f_{n,1,1,1,0}\,D\,dx,$$

$$BR_n = -\int_0^\infty e^{-x(z+b)}\Bigg(f_{n,1,1,0,0}\left(E(1-x)(z+b)+\frac{f_{n,1,1,1,0}}{x}E\right)\big(2\cdot x(z+b)-1\big)$$

$$+bx^2(z+b)+A\Bigg)dx,$$

$$BT_n = -\int_0^\infty e^{-x(z+b)}\left(f_{n,1,2,0,0}F(1-x)(z+b)+\frac{f_{n,1,1,1,0}}{x}F\right)\big(2\big(x(z+b)-1\big)$$

$$+bx^2(z+b)-A\big)dx,$$

$$BZ_n = -\int_0^\infty e^{-x(z+b)}\Big(x^{1/2}Bf_{n,1,2,0,0}\,x(z+b)+f_{n,1,1,1,0}\big(1+x(z+b)-(z+b)\big)\Big)dx,$$

$$CR_n = \int_0^\infty e^{-x(z+b)}\Big(f_{n,0,1,0,0}\big(x(z+b)E-A\big)x^{1/2}\Big)dx,$$

$$CT_n = \int_0^\infty e^{-x(z+b)}\Big(f_{n,0,1,0,0}\big(x(z+b)F+A\big)x^{1/2}\Big)dx,$$

$$CZ_n = -\int_0^\infty e^{-x(z+b)}x^{1/2}Df_{n,0,1,0,0}\,dx,$$

$$DR_n = -\int_0^\infty e^{-x(z+b)}\Bigg(x^{-1/2}\left(E\left(f_{n,2,1,0,0}-\frac{2f_{n,2,0,1,0}}{x}\right)(2-x)(z+b)\right)$$

$$+\left(f_{n,2,1,0,0}-\frac{2f_{n,2,0,1,0}}{x}\right)A\Bigg)dx,$$

$$DT_n = -\int_0^\infty e^{-x(z+b)}x^{-1/2}\Bigg(F\left(f_{n,2,1,0,0}-\frac{2f_{n,2,0,1,0}}{x}\right)\big(2-x(z+b)\big)$$

$$+\left(f_{n,2,1,0,0}-\frac{2f_{n,2,0,1,0}}{x}\right)A\Bigg)dx,$$

$$DZ_n = \int_0^\infty e^{-x(z+b)}x^{3/2}\left(\frac{f_{n,2,0,1,0}}{x}-f_{n,2,1,0,0}\right)(z+b)D\,dx.$$

The symbols A, E, F, D denote respectively

$$A = \frac{\pi}{2} J_0(xr),$$

$$D = \frac{\pi}{2} J_1(xr),$$

$$E = \frac{\pi}{2} J_0(xr) - \frac{1}{xr} J_1(xr),$$

$$F = \frac{\pi}{2r_0 r} J_1(xr).$$

References

1. M. ABRAMOVITZ, L. STEGUN, *Handbook of mathematical functions*, Dover 1965.
2. A. ERDELYI *Higher transcendental functions*, vol. I, II, New York 1953.
3. H. BRENNER, *Effect of finite boundaries on the Stokes resistance of an arbitrary particle*, J. Fluid Mech., **18**, 144, 1964.
4. P. GANATOS, R. PFEFFER, S. WEINBAUM, *A strong interaction theory for the creeping motion of a sphere between plane parallel boundaries; Part 1. Perpendicular motion*, J. Fluid Mech., **99**, 739–753, 1980.
5. P. GANATOS, R. PFEFFER, S. WEINBAUM, *A strong interaction theory of a sphere between plane parallel boundaries; Part* **2**. *Parallel motion*, J. Fluid Mech., **99**, 755–783, 1980.
6. M. GLUCKMAN, R. PFEFFER, S. WEINBAUM, *A new technique for treating multiparticle slow viscous flow: axisymmetric flow past spheres and spheroids*, J. Fluid Mech., **50**, 705–740, 1971.
7. L. GOREN, M. O'NEILL, *Asymmetric creeping motion of an open torus*, J. Fluid Mech., **101**, 97–110, 1980.
8. J. HAPPEL, H. BRENNER, *Low Reynolds number hydrodynamics*, Prentice-Hall 1965.
9. B. P. HO, L. G. LEAL, *Internal migration of rigid spheres in two-dimensional unidirectional flows*, J. Fluid Mech., **65**, 356, 1974.
10. A. KUCABA-PIĘTAL, *Axi-symmetric quasi-stationary motion of a torus in the presence of a wall; The Stokes approximation*, Engng. Trans., **31**, 151–161, 1983 [in Polish].
11. S. LEICHTBERG, R. PFEFFER, S. WEINBAUM, *Stokes flow past finite coaxial dusters of spheres in a circular cylinder*, Int. J. Multiphase Flow, **3**, 147–169, 1976.
12. H. LORENTZ, Abhand. Theor. Phys., 1, 23.
13. S. MAJUMDAR, M. O'NEILL, *On axisymmetric Stokes flow past a torus*, ZAMP, **28**, 541–550, 1977.

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