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A contribution to contact problems for a class of solids and structures

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IN THIS PAPER two dual problems are derived and discussed. Firstly, for a linear elastic body in the presence of Signorini's boundary conditions with friction the dual problem in the sense of Mosco is formulated. The friction condition may be nonconvex and anisotropic, while the subdifferential friction law is not associated with the friction condition. Secondly, a novel dual problem expressed in the terms of static fields is proposed for the obstacle problem of a von Kármán plate.

W pracy wyprowadzono i przedyskutowano dwa zagadnienia dualne. Po pierwsze, dla ośrodka liniowo-sprężystego w przypadku warunków Signoriniego z tarciem sformułowano problem dualny w sensie Mosco. Warunek tarcia może być niewypukły i anizotropowy, zaś subróżniczkowe prawo tarcia jest niestowarzyszone z tym warunkiem. Po drugie, dla płyty von Kármána, w przypadku zagadnienia z przeszkodą, sformułowano nowe zadanie dualne w terminach pól statycznych.

В работе выведены и обсуждены две дуальные задачи. Во-первых, для линейно-упругой среды, в случае граничных условий Синьорини с трением, сформулирована дуальная задача в смысле Моско. Условие трения возможно невыпукло и анизотропно, тогда как субдифференциальный закон трения не ассоциирован с этим условием. Во-вторых, для пластины фон Кармана, в случае проблемы с препятствием, сформулирована новая дуальная задача только при помощи статических полей.

1. Introduction

IN MOST cases contact problems belong to the so-called free surface problems due to the inherent behaviour of contacting bodies. Methods of convex analysis and variational inequalities proved to be very useful in studying such problems, cf. Refs. [9, 11, 40]. An up-to-date and rather exhaustive survey of applications of those methods to various contact problems for solids and structures is presented in the paper by the second author [40].

The purpose of this work is twofold and, accordingly, the paper is divided into two parts. In the first part we shall primarily derive the implicit variational inequality, being the weak (variational) formulation of the boundary value problem for a linear elastic body in a friction contact with a rigid support on a part of the boundary. This implicit variational inequality, expressed in terms of displacement, is very general since the friction condition can be neither convex nor isotropic and the subdifferential friction law is not associated with this condition. In the existing literature only Signorini's contact problems with Coulomb's friction have so far been investigated [5, 6, 7, 9, 15–19, 22, 28, 34, 35, 37]. LICHT [26] considers viscous friction but in the case of a bilateral contact only. For all these specific cases no dual formulations seem to have been proposed. Since at the present state of knowledge it is not possible to derive an implicit variational inequality from an extremum principle, the methods of the theory of duality [1, 11, 42] are not applicable. This assertion does not exclude an a priori formulation of a contact problem in terms of stresses in the form of a quasi-variational inequality. However, such problems will not be investigated here. MOSCO [30] has proved that for an arbitrary, well-posed variational inequality a dual formulation is always available. A generalization of Mosco's theory to a large class of implicit variational problems has been proposed by CAPUZZO DOLCETTA and MATZEU [3], cf. Appendix B. Applying, in Sect. 3 of Part 1, this general duality theory to Signorini's problem with friction, we shall obtain the quasi-variational inequality defined on the surface of a possible contact only and expressed in terms of stresses.

The second part of the present paper concerns the dual formulation of the obstacle problem for a von Kármán plate. Various unilateral problems, including the obstacle problem, for von Kármán plates have already been studied from both the theoretical and numerical point of view, cf. Refs [8, 12, 20, 23, 32, 33, 36, 38, 40]. Yet the only contribution dealing with the dual problem is our paper [2] where the dual obstacle problem has been formulated in terms of static *and* kinematic fields. Moreover, a linear operator Λ playing an important role in Rockafellar's theory of duality (see Ref. [11], and Appendix A) depends parametrically on this kinematic field, entering into the equilibrium equation. Hence the need of further investigation of the dual formulation. In Sect. 6 we shall propose a novel approach to the same dual obstacle problem. Now the operator Λ is different and exhibits no parametric dependence on a kinematic field. As a result, in the dual obstacle problem a kinematic field is not explicitly present.

To facilitate the reading of this paper two appendices are attached to it. In Appendix A we present the results of convex analysis indispensable for our considerations. Appendix B deals with a concise presentation of the duality theory for implicit variational problems. Due to the limitation of the number of acceptable pages in the whole paper, we shall not enter into mathematical details.

Part 1. Dual formulation, in the sense of Mosco, of a Signorini's contact problem with friction for a linear elastic solid

2. Formulation of the unilateral boundary value problem with friction

Let $\Omega \subset R^3$ be a bounded, sufficiently regular domain. Its boundary $\partial\Omega$, denoted by Γ, consists of three nonoverlapping parts: $\Gamma_0, \Gamma_1, \Gamma_2$, and $\partial\Omega = \overline{\Gamma} = \overline{\Gamma}_0 \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_2$. Here $\overline{\Gamma}_0$ denotes the closure of Γ_0, etc. By $\mathbf{n} = (n_i)$ we denote the outer unit normal Γ. Throughout Part 1 Latin indices run from 1 to 3.

A vector $\mathbf{v} = (v_i)$, defined on Γ, decomposes as follows:

$$\mathbf{v} = v_N \mathbf{n} + \mathbf{v}_T, \tag{2.1}$$

where $v_N = \mathbf{v}_i n_i$ denotes the normal component of $\mathbf{v}$, while $v_{Ti} = v_i - v_N n_i$ are tangential components of $\mathbf{v}$.

If $\boldsymbol{\sigma} = (\sigma_{ij})$ is a stress tensor, then a similar decomposition holds for the stress vector $(\sigma_{ij}n_j)$, defined on Γ, that is

$$\sigma_{ij}n_j = \sigma_N n_i + \sigma_{Ti}, \tag{2.2}$$

where

$$\sigma_N = \sigma_{ij}n_i n_j, \qquad \sigma_{Ti} = \sigma_{ij}n_j - \sigma_N n_i.$$

We assume that the friction condition, defined on Γ_2, is given by

$$f(\sigma_N, \boldsymbol{\sigma}_T) \leqslant 0. \tag{2.3}$$

Specifically, Coulomb's friction condition has the form

$$f(\sigma_N, \boldsymbol{\sigma}_T) = |\boldsymbol{\sigma}_T| - \nu|\sigma_N| \leqslant 0, \tag{2.4}$$

where $\nu = \nu(\mathbf{x})$, $\mathbf{x} \in \Gamma_2$, is the coefficient of friction, and $|\sigma_T| = \sqrt{\sigma_{Ti}\sigma_{Ti}} = \sqrt{\boldsymbol{\sigma}_T \cdot \boldsymbol{\sigma}_T}$. We observe that the function (2.4) is convex and isotropic, whereas in the general case (2.3) it is neither convex nor isotropic.

We set

$$K(\sigma_N) = \{\boldsymbol{\tau}_T | f(\sigma_N, \boldsymbol{\tau}_T) \leqslant 0\} \tag{2.5}$$

and assume that for each σ_N the set $K(\sigma_N)$ is convex and closed. This assumption implies that the set

$$K = \{(\sigma_N, \boldsymbol{\sigma}_T) | f(\sigma_N, \boldsymbol{\sigma}_T) \leqslant 0\} \tag{2.6}$$

is not necessarily convex. Several nonconvex friction conditions have been obtained in [29]. To find the conditions relative to anisotropic friction, the reader should refer to the papers [29, 41, 43].

The friction law is assumed in the subdifferential form

$$-\mathbf{u}_T \in \partial\chi_{K(\sigma_N)}(\boldsymbol{\sigma}_T) \tag{2.7}$$

or

$$(\boldsymbol{\tau}_T - \boldsymbol{\sigma}_T) \cdot \mathbf{u}_T \geqslant 0 \qquad \forall\, \boldsymbol{\tau}_T \in K(\sigma_N). \tag{2.8}$$

The indicator function $\chi_{K(\sigma_N)}$ of the set $K(\sigma_N)$ is defined as follows [39]:

$$\chi_{K(\sigma_N)}(\boldsymbol{\tau}_T) = \begin{cases} 0, & \text{if} \quad \boldsymbol{\tau}_T \in K(\sigma_N), \\ \infty, & \text{if} \quad \boldsymbol{\tau}_T \notin K(\sigma_N). \end{cases} \tag{2.9}$$

The support function $s(\sigma_N, -\mathbf{u}_T)$ of $K(\sigma_N)$ is given by

$$s(\sigma_N, -\mathbf{u}_T) = \sup\,\{\boldsymbol{\tau}_T \cdot (-\mathbf{u}_T) | \boldsymbol{\tau}_T \in K(\sigma_N)\}. \tag{2.10}$$

It is easy to prove that

$$\chi_{K(\sigma_N)}(\boldsymbol{\sigma}_T) + s(\sigma_N, -\mathbf{u}_T) = \boldsymbol{\sigma}_T \cdot (-\mathbf{u}_T). \tag{2.11}$$

The function $s(\sigma_N, -\mathbf{u}_T)$ represents the work of the friction stress σ_T on the displacement $\mathbf{u}_T$. We define

$$j(\sigma_N, \mathbf{u}_T) = s(\sigma_N, -\mathbf{u}_T). \tag{2.12}$$

It can readily be verified that in the case of Coulomb's friction we have [41]

$$j(\sigma_N, \mathbf{u}_T) = \nu|\sigma_N||\mathbf{u}_T|. \tag{2.13}$$

The properties of the indicator function $\chi_{K(\sigma_N)}$ imply that the function j is convex and subdifferentiable with respect to the second argument. The subdifferentiability means that

$$j(\sigma_N, \mathbf{v}_T) - j(\sigma_N, \mathbf{u}_T) \geqslant (-\boldsymbol{\sigma}_T)\cdot(\mathbf{v}_T - \mathbf{u}_T), \quad \forall \mathbf{v}_T. \tag{2.14}$$

More elaborate analysis of local friction laws, presented above, is given in the paper [41]. Here we confine ourselves to local laws only.

The unilateral (Signorini) boundary value problem with friction is now formulated as

PROBLEM 1

Find a displacement field $\mathbf{u} = \mathbf{u}(\mathbf{x})$, $\mathbf{x} \in \Omega$, such that

$$\sigma_{ij,j} + b_i = 0, \quad \text{in } \Omega, \tag{2.15}$$

$$\sigma_{ij}(\mathbf{u}) = a_{ijkl} e_{kl}(\mathbf{u}), \tag{2.16}$$

$$\mathbf{u} = \mathbf{0}, \quad \text{on } \Gamma_0, \tag{2.17}$$

$$\sigma_{ij} n_j = F_i, \quad \text{on } \Gamma_1, \tag{2.18}$$

$$u_N \leqslant 0, \quad \sigma_N \leqslant 0, \quad \sigma_N u_N = 0, \quad \text{on } \Gamma_2, \tag{2.19}$$

$$-\mathbf{u}_T \in \partial \chi_{K(\sigma_N)}(\boldsymbol{\sigma}_T), \quad \text{on } \Gamma_2. \tag{2.20}$$

Here $e_{ij}(\mathbf{u}) = \dfrac{1}{2}(u_{i,j} + u_{j,i}) = u_{(i,j)}$ and the usual symmetry and ellipticity conditions are imposed on the elasticity tensor (a_{ijkl}), cf. Ref. [9]. Moreover, we assume that

$$a_{ijkl} \in L^\infty(\Omega), \quad (\mathbf{b}, \mathbf{F}) \in V^*. \tag{2.21}$$

The definitions of function spaces used in the present paper can be found in [27].

Suppose that V is a real reflexive Banach space such that $\mathbf{u} \in V$ and $j(\sigma_N(\mathbf{u}), \mathbf{u}_T)$ make sense. A dependence of the function j on σ_N is a delicate matter even in the simpler case of Coulomb's friction law, cf. Refs. [6, 7, 18, 19, 34, 35, 37].

A plausible choice of the space V is the following:

$$V = \{\mathbf{v} = (v_i) | v_i \in H^1(\Omega),\ \sigma_{ij}(\mathbf{v}) n_j \in L^2(\Gamma_2)\}. \tag{2.22}$$

Let us set

$$\mathcal{K} = \{\mathbf{v} | \mathbf{v} \in V,\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0;\ v_N \leqslant 0 \text{ on } \Gamma_2\}, \tag{2.23}$$

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega a_{ijkl} e_{ij}(\mathbf{u}) e_{kl}(\mathbf{v}) dx, \tag{2.24}$$

$$J(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_2} j(\sigma_N(\mathbf{u}), \mathbf{v}_T) d\Gamma, \tag{2.25}$$

$$L(\mathbf{v}) = \int_\Omega b_i v_i dx + \int_{\Gamma_1} f_i v_i d\Gamma. \tag{2.26}$$

Problem 1 can now be formulated in the variational, or weak, form as

PROBLEM ($\mathscr{P}$)

Find $\mathbf{u} \in \mathcal{K}$ such that

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + J(\mathbf{u}, \mathbf{v}) - J(\mathbf{u}, \mathbf{u}) \geqslant L(\mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{K}. \tag{2.27}$$

The proof is straightforward. Multiplying Eq. (2.15) by $\mathbf{v}-\mathbf{u} \in V$ and integrating, we obtain

$$a(\mathbf{u}, \mathbf{v}-\mathbf{u})+\int_{\Gamma_2} (-\boldsymbol{\sigma}_T) \cdot (\mathbf{v}_T-\mathbf{u}_T)dx = L(\mathbf{v}-\mathbf{u})+\int_{\Gamma_2} \sigma_N (v_N-u_N)d\Gamma. \tag{2.28}$$

Now, taking account of the Signorini's conditions (2.19) and the friction law (2.20), or inequality (2.14), we arrive at the *implicit variational inequality* = I.V.I., given by the relation (2.27).

Existence of a solution $\mathbf{u} \in \mathscr{K}$ solving the problem $(\mathscr{P})$ results from the general theory of implicit variational problems [31] and will not be discussed here. Such a solution is not unique in general, even in the case of Coulomb's friction law. For this latter case uniqueness is assured if the coefficient of friction is sufficiently small.

3. Dual formulation of the I.V.I. (2.27)

To adjust our I. V. I. to the theory of duality outlined in Appendix B, we set

$$g(\mathbf{u}, \mathbf{w}) = a(\mathbf{u}, \mathbf{w})-L(\mathbf{w}) \tag{3.1}$$

and

$$\varphi(A\mathbf{v}, \mathbf{w}) = J(\mathbf{v}, \mathbf{w})+\chi_{\mathscr{K}}(\mathbf{w}), \tag{3.2}$$

where $A\colon V \ni \mathbf{v} \to \sigma_N(\mathbf{v})$ on Γ_2. Accordingly, the assumptions $(H_1)-(H_3)$, formulated in Appendix B, are satisfied.

Making use of Lemma A.1 we have

$$\begin{aligned}\varphi^*(A\mathbf{v}, \mathbf{w}^*) &= \sup_{\mathbf{w}\in V}\{\langle \mathbf{w}^*, \mathbf{w}\rangle - J(\mathbf{v}, \mathbf{w})-\chi_{\mathscr{K}}(\mathbf{w})\} \\ &= \sup_{\mathbf{w}=(w_N, \mathbf{w}_T)}\{\langle w_N^*, w_N\rangle+\langle \mathbf{w}_T^*, \mathbf{w}_T\rangle - J(\mathbf{v}, \mathbf{w})-\chi_{\mathscr{K}}(w_N)\} \\ &= \chi_{C(\sigma_N(\mathbf{v}))}(\mathbf{w}_T^*)+\chi_{\mathscr{K}^*}(w_N^*)\end{aligned} \tag{3.3}$$

and

$$g^*(\mathbf{v}, \mathbf{w}^*) = \sup_{\mathbf{w}\in V}\{\langle \mathbf{w}^*, \mathbf{w}\rangle - a(\mathbf{v}, \mathbf{w})+L(\mathbf{w})\} = \begin{cases}0, & \text{if} \quad \mathbf{w}^* = B\mathbf{v}-L,\\ \infty, & \text{otherwise}.\end{cases} \tag{3.4}$$

Here $C(\sigma_N)(\mathbf{x}) = K(\sigma_N(\mathbf{x}))$ for almost every $\mathbf{x} \in \Gamma_2$ and L can be identified with $(\mathbf{b}, \mathbf{F})$, whereas the continuous linear and invertible operator B is defined as follows:

$$\langle B\mathbf{v}, \mathbf{w}\rangle = \int_{\Omega} \sigma_{ij}(\mathbf{v})e_{ij}(\mathbf{w})dx. \tag{3.5}$$

$\mathscr{K}^*$ is the polar cone of $\mathscr{K}$, that is

$$\mathscr{K}^* = \{\mathbf{v}^* \in V^* | \langle \mathbf{v}^*, \mathbf{v}\rangle \leqslant 0, \quad \forall \mathbf{v} \in \mathscr{K}\}. \tag{3.6}$$

The definition (3.1) of the functional g and the assumption (H_3) imply

$$Dg(\mathbf{u}, \mathbf{u})+L = B\mathbf{u}. \tag{3.7}$$

Setting $G = B^{-1}$, we arrive at

$$\mathbf{u} = G(\boldsymbol{\Sigma})+\mathbf{u}, \tag{3.8}$$

where

$$\hat{\mathbf{u}} = GL = G(\mathbf{b}, \mathbf{F}), \tag{3.9}$$

$$\mathbf{\Sigma} = (\sigma_{ij} n_j) = Dg(\mathbf{u}, \mathbf{u}), \quad \text{on } \Gamma_2, \tag{3.10}$$

Obviously, G is the Green operator for the mixed boundary value problem of linear elasticity.

From Eqs. $(2.19)_2$, (3.3) and (3.6) we infer that now $\mathbf{u}^* = -\mathbf{\Sigma}$, on Γ_2. Hence we have $(Dg)^{-1}(-\mathbf{u}^*) = \mathbf{u} = G(\mathbf{\Sigma})+\mathbf{u}$. The dual problem of $(\mathscr{P})$ is eventually expressed as

PROBLEM $(\mathscr{P}^*)$

Find $\mathbf{\Sigma} = (\sigma_N, \boldsymbol{\sigma}_T) \in [-\mathscr{K}^*] \times \mathrm{C}(\sigma_N)$ such that

$$\langle \boldsymbol{\tau} - \mathbf{\Sigma}, G(\mathbf{\Sigma}) + \hat{\mathbf{u}} \rangle \geqslant 0, \quad \forall \boldsymbol{\tau} \in [-\mathscr{K}^*] \times C(\sigma_N) \tag{3.11}$$

or

$$\langle \tau_N - \sigma_N, [G(\mathbf{\Sigma})] \cdot \mathbf{n} + \hat{u}_N \rangle + \langle \boldsymbol{\tau}_T - \boldsymbol{\sigma}_T, [G(\mathbf{\Sigma})]_T + \hat{\mathbf{u}}_T \rangle \geqslant 0; \quad \forall \tau_N \in -\mathscr{K}^*, \quad \forall \boldsymbol{\tau}_T \in C(\sigma_N). \tag{3.11'}$$

Moreover, the extremality condition (B.4) yields

$$J(\mathbf{u}, \mathbf{u}) = -\langle \boldsymbol{\sigma}_T, \mathbf{u}_T \rangle = -\int_{\Gamma_2} \boldsymbol{\sigma}_T \cdot \mathbf{u}_T d\Gamma, \quad \langle \sigma_N, u_N \rangle = \int_{\Gamma_2} \sigma_N u_N d\Gamma = 0. \tag{3.12}$$

We note that the relation (3.11) or the relation (3.11′) is the *quasi-variational inequality* = = Q. V. I. defined on Γ_2 only. For instance, such Q. V. I. can be useful for Hertz-like problems with friction when it is desirable to determine the normal stresses σ_N and tangential (frictional) stresses $\boldsymbol{\sigma}_T$ on Γ_2.

REMARK 3.1. For Signorini's problem without friction $\boldsymbol{\sigma}_T = \boldsymbol{\tau}_T = \mathbf{0}$ and the relation (3.11′) yields

$$\sigma_N : \langle \tau_N - \sigma_N, [G(\sigma_N)]_N + \hat{u}_N \rangle \geqslant 0, \ \forall \tau_N \in -\mathscr{K}^*. \tag{3.13}$$

KIKUCHI [21] calls the inequality (3.13) the "reciprocal variational inequality". He derived this inequality by a different approach, not referring to the Mosco's theory of duality.

REMARK 3.2. The primal problem $(\mathscr{P})$ is formulated in terms of displacements. Another pair of the dual problems in the sense of Mosco is available for the same contact problem if the primal problem is formulated in terms of stresses. Such a primal problem results in a quasi-variational inequality defined on Ω. We shall study this Q. V. I. and its dual separately.

Part. 2. A novel approach to the duality of the obstacle problem for von Kármán plates

THE SECOND part of the present paper is concerned with the dual formulation of the obstacle problem for a clamped von Kármán plate. A similar problem was already studied in our note [2]. However, the approach we use here is different and the results obtained are more complete. The dual problem formulated in [2] depends explicitly on a kinematic field — the transverse displacement of the plate. The approach we employ here overcomes this difficulty and the results obtained are more satisfactory, at least from the viewpoint of the theory of duality.

4. Basic relations

A lucid account of the theory of von Kármán plates can be found in [4, 13].

Let ω be a bounded sufficiently regular domain of R^2. By $\mathbf{u} = (u_1, u_2)$, w and ϱ we denote the in-plane displacement vector, the transverse displacement of the plate and the density of an external loading, respectively. We assume that $\varrho \in L^2(\omega)$, and the rigid obstacle is given by a function $\psi \in H^2(\omega)$. The set

$$K_1 = \{v \in H_0^2(\omega) | v \geqslant \psi\} \tag{4.1}$$

is closed and convex. The sense of the inequality in the definition (4.1) of K_1 is clear due to the implication $v \in H^2 \Rightarrow v$ is continuous.

The curvature tensor is denoted by $\boldsymbol{\varkappa} = (\varkappa_{\alpha\beta})$. Greek indices take the values 1 and 2 The curvature-displacement relation is given by

$$\varkappa_{\alpha\beta}(w) = -\frac{\partial^2 w}{\partial x_\alpha \partial x_\beta} = -w_{,\alpha\beta}. \tag{4.2}$$

If $\boldsymbol{\epsilon} = (\varepsilon_{\alpha\beta})$ is the strain tensor, then the strain-displacement relation reads

$$\varepsilon_{\alpha\beta}(\mathbf{u}, w) = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha} + w_{,\alpha} w_{,\beta}). \tag{4.3}$$

According to the von Kármán theory of isotropic elastic plates, the constitutive equations are

$$M_{\alpha\beta} = S_{\alpha\beta\lambda\mu} \varkappa_{\lambda\mu}, \quad \text{or} \quad \mathbf{M} = \mathbf{S}\boldsymbol{\varkappa} \tag{4.4}$$

and

$$N_{\alpha\beta} = Z_{\alpha\beta\lambda\mu} \varepsilon_{\lambda\mu}, \quad \text{or} \quad \mathbf{N} = \mathbf{Z}\boldsymbol{\epsilon}, \tag{4.5}$$

where

$$S_{\alpha\beta\lambda\mu} = \frac{Eh^3}{12(1-\nu^2)}[(1-\nu)\delta_{\alpha\lambda}\delta_{\beta\mu} + \nu\delta_{\alpha\beta}\delta_{\lambda\mu}], \tag{4.6}$$

$$Z_{\alpha\beta\lambda\mu} = \frac{Eh}{1-\nu^2}[(1-\nu)\delta_{\alpha\lambda}\delta_{\beta u} + \nu\delta_{\alpha\beta}\delta_{\lambda\mu}]. \tag{4.7}$$

Here $\mathbf{M}$ and $\mathbf{N}$ denote the bending moment tensor and the membrane force tensor, respectively; E is Young's modulus, ν is Poisson's ratio, while h stands for the thickness of the plate.

We observe that the constitutive equations (4.4) and (4.5) are linear, whereas the strain-displacement relation (4.3) is nonlinear.

We assume that $S_{\alpha\beta\lambda\mu} \in L^\infty(\omega)$, hence also $Z_{\alpha\beta\lambda\mu} \in L^\infty(\omega)$. Straightforward calculations show the existence of constants $\lambda_0 > 0$ and $\lambda_1 > 0$ such that for all $(t_{\alpha\beta})$, $t_{\alpha\beta} = t_{\beta\alpha}$, we have

$$\begin{aligned} S_{\alpha\beta\lambda\mu}(\mathbf{x}) t_{\alpha\beta} t_{\lambda\mu} &\geqslant \lambda_0 t_{\alpha\beta} t_{\alpha\beta}, \\ Z_{\alpha\beta\lambda\mu}(\mathbf{x}) t_{\alpha\beta} t_{\lambda\mu} &\geqslant \lambda_1 t_{\alpha\beta} t_{\alpha\beta}. \end{aligned} \quad \text{for almost every} \quad \mathbf{x} \in \omega, \tag{4.8}$$

5. Primal obstacle problem

For the sake of simplicity, we shall study the obstacle problem for a clamped plate only. Thus the space

$$X = [H_0^1(\omega)]^2 \times H_0^2(\omega), \tag{5.1}$$

is suitable for the primal or ($\mathscr{P}$) problem.

The total potential energy is expressed as follows:

$$I(\mathbf{u}, w) = \frac{1}{2} \int_\omega Z_{\alpha\beta\lambda\mu} \varepsilon_{\alpha\beta}(\mathbf{u}, w) \varepsilon_{\lambda\mu}(\mathbf{u}, w) d\mathbf{x} + \frac{1}{2} \int_\omega S_{\alpha\beta\lambda\mu} \varkappa_{\alpha\beta}(w) \varkappa_{\lambda\mu}(w) d\mathbf{x} - \int_\omega \varrho w d\mathbf{x}. \tag{5.2}$$

In virtue of the inequalities (4.8), the quadratic functional

$$I_1(\boldsymbol{\epsilon}, \boldsymbol{\varkappa}) = \frac{1}{2} \int_\omega Z_{\alpha\beta\lambda\mu} \varepsilon_{\alpha\beta} \varepsilon_{\lambda\mu} d\mathbf{x} + \frac{1}{2} \int_\omega S_{\alpha\beta\lambda\mu} \varkappa_{\alpha\beta} \varkappa_{\lambda\mu} d\mathbf{x} \tag{5.3}$$

is strictly convex over the space $[L^2(\omega)]^4 \times [L^2(\omega)]^4$. Unfortunately, due to the nonlinearity of the relation (4.3), the functional $I_1(\boldsymbol{\epsilon}(\mathbf{u}, w), \boldsymbol{\varkappa}(w))$ is no longer convex over the space X.

The primal or ($\mathscr{P}$) — obstacle problem for the clamped plate means evaluating

$$\inf \{I(\mathbf{u}, w) | \, (\mathbf{u}, w) \in X_1\}, \tag{5.4}$$

where $X_1 = [H_0^1(\omega)]^2 \times K_1$.

The existence of a solution of the problem ($\mathscr{P}$) follows from the results obtained in [36]. We note that in the fundamental paper by DUVAUT and LIONS [8] on unilateral problems for von Kármán plates the possibility of formulating such problems as minimization problems is not discussed.

The solution $(\bar{\mathbf{u}}, \bar{w})$ of the problem ($\mathscr{P}$) fulfills the following variational inequality:

$$\langle DI(\bar{\mathbf{u}}, \bar{w}), \, (\mathbf{v} - \bar{\mathbf{u}}, w - \bar{w}) \rangle \geqslant 0 \quad \forall (\mathbf{v}, w) \in X_1, \tag{5.5}$$

where DI $(\mathbf{u}, w)$ denotes the Gâteaux derivative of the functional I at the point $(\mathbf{u}, w)$. It can readily be verified that the inequality (5.5) results in the variational inequality and the variational equation studied already in [8].

6. Dual problem and its properties

6.1. The formulation of the dual problem

The primal problem ($\mathscr{P}$) given by the relation (5.4) is formulated in terms of kinematic fields. For the purpose of deriving the dual problem in terms of static fields, we shall apply Rockafellar's theory of duality outlined in Appendix A.

The linear operator Λ playing an important role in this theory is now defined as follows:

$$\Lambda(\mathbf{u}, w) = (\operatorname{sym} \nabla \mathbf{u}, \nabla w, -\nabla^2 w) = (e_{\alpha\beta}(\mathbf{u}), w_{,\alpha}, -w_{,\alpha\beta}), \tag{6.1}$$

where $e_{\alpha\beta} = \frac{1}{2}(\mathbf{u})(u_{\alpha,\beta}+u_{\beta,\alpha})$. For later convenience we set

(6.2) $$\Lambda(\mathbf{u}, w) = (\Lambda_1 \mathbf{u}, \Lambda_2 w), \quad \theta_\alpha = w_{,\alpha},$$

where

(6.3) $$\Lambda_1 \mathbf{u} = \operatorname{sym}\nabla\mathbf{u}, \quad \Lambda_2 w = (\nabla w, -\nabla^2 w).$$

We have

(6.4) $$\Lambda: X \to Y = Y_1 \times [H_0^1(\omega)]^2 \times Y_1,$$

where $Y_1 = Y_1^*$ is the space of symmetric matrices $\mathbf{a} = (a_{\alpha\beta})$, $a_{\alpha\beta} = a_{\beta\alpha} \in L^2(\omega)$. The operator Λ^*, adjoint of Λ, maps

$$Y^* = Y_1 \times [H^{-1}(\omega)]^2 \times Y_1 \quad \text{into} \quad X^* = [H^{-1}(\omega)]^2 \times H^{-2}(\omega).$$

Let $\mathbf{P}^* = (\mathbf{p}^*, \mathbf{q}^*) = (p^*_{\alpha\beta}, t^*_\alpha, r^*_{\alpha\beta})$ be an element of Y^*. Then we have

(6.5) $$\langle \Lambda(\mathbf{u}, w), \quad \mathbf{P}^*\rangle = \int_\omega u_{\alpha,\beta} p^*_{\alpha\beta} d\mathbf{x} + \int_\omega w_{,\alpha} t^*_\alpha d\mathbf{x} - \int_\omega w_{,\alpha\beta} r^*_{\alpha\beta} d\mathbf{x}$$

where the second integral is to be taken in the sense of duality between $H_0^1(\omega)$ and $H^{-1}(\omega)$. Integrating in the right-hand side of Eq. (6.5) by parts, we arrive at the following form of the operator $\Lambda^* = (\Lambda_1^*, \Lambda_2^*)$:

(6.6) $$(\Lambda_1^* \mathbf{p}^*) = -\operatorname{div}\mathbf{p}^* = (-p^*_{\alpha\beta,\beta}), \ \Lambda_2^*(\mathbf{t}^*, \mathbf{r}^*) = -t^*_{\alpha,\alpha} - r^*_{\alpha\beta,\beta\alpha},$$

where the derivatives are to be taken in the sense of distributions.

The functional I, given by Eq. (5.2), can be decomposed as follows:

(6.7) $$I(\mathbf{u}, w) = J\big((\mathbf{u}, w), \Lambda(\mathbf{u}, w)\big) = G(\Lambda_1 \mathbf{u}, \Lambda_2 w) + F(w),$$

where

(6.8) $$G(\Lambda_1 \mathbf{u}, \Lambda_2 w) = \frac{1}{2}\int_\omega Z_{\alpha\beta\lambda\mu}\left[e_{\alpha\beta}(\mathbf{u}) + \frac{1}{2}\theta_\alpha(w)\theta_\beta(w)\right]\left[e_{\lambda\mu}(\mathbf{u}) + \frac{1}{2}\theta_\lambda(w)\theta_\mu(w)\right]d\mathbf{x} + \frac{1}{2}\int_\omega S_{\alpha\beta\lambda\mu}\varkappa_{\alpha\beta}(w)\varkappa_{\lambda\mu}(w)d\mathbf{x},$$

and

(6.9) $$F(w) = -\langle \varrho, w\rangle + \chi_{K_1}(w) = -\int_\omega \varrho w d\mathbf{x} + \chi_{K_1}(w).$$

Obviously, χ_{K_1} is the indicator function of K_1.

To formulate the dual problem for the problem (5.4), we take the functional Φ, now defined by

(6.10) $$\Phi\big((\mathbf{u}, w), (\mathbf{p}, \mathbf{q})\big) = G(\Lambda_1 \mathbf{u} - \mathbf{p}, \Lambda_2 w - \mathbf{q}) + F(w),$$

where

$$\mathbf{p} = (p_{\alpha\beta}), \quad \mathbf{q} = (t_\alpha, r_{\alpha\beta}) \quad \text{and} \quad (\mathbf{p}, \mathbf{q}) \in Y.$$

The dual problem means evaluating

(6.11) $$\sup\{-\Phi\big((0, 0), (\mathbf{p}^*, \mathbf{q}^*)\big) | (\mathbf{p}^*, \mathbf{q}^*) \in Y^*\}$$

The definition of the polar functional gives

(6.12) $$\Phi^*((\mathbf{0},0),(\mathbf{p}^*,\mathbf{q}^*)) = \sup_{\substack{(\mathbf{u},w)\in X\\(\mathbf{p},q)\in Y}} \{\langle \mathbf{p}^*,\mathbf{p}\rangle + \langle \mathbf{q}^*,\mathbf{q}\rangle - G(\Lambda_1\mathbf{u}-\mathbf{p},\Lambda_2 w-\mathbf{q}) - F(w)\}$$
$$= G^*(-\mathbf{p}^*,-\mathbf{q}^*) + F^*(\Lambda_2^*\mathbf{q}^*) + \sup\{\langle \mathbf{u},\Lambda_1^*\mathbf{p}^*\rangle \mid \mathbf{u}\in [H_2^1(\omega)]^2\}.$$

The dual problem can be rewritten as follows:

(6.13) $$(\mathscr{P}^*)\qquad \sup\{-G^*(-\mathbf{p}^*,-\mathbf{q}^*) - F^*(\Lambda_2^*\mathbf{q}^*) \mid (\mathbf{p}^*,\mathbf{q}^*)\in Y^*\},$$

subject to

(6.14) $$\Lambda_1^*\mathbf{p}^* = 0.$$

To complete the formulation of the dual problem, we must find the functionals G^* and F^*. Since $\mathbf{q}^* = (t_\alpha^*, r_{\alpha\beta}^*)$, hence from the definition of the polar functional we obtain

(6.15) $$G^*(\mathbf{p}^*,\mathbf{t}^*,\mathbf{r}^*) = \sup_{(\mathbf{p},\mathbf{t},\mathbf{r})\in Y}\Big\{\langle \mathbf{p}^*,\mathbf{p}\rangle + \langle \mathbf{t}^*,\mathbf{t}\rangle + \langle \mathbf{r}^*,\mathbf{r}\rangle$$
$$-\frac{1}{2}\int_\omega Z_{\alpha\beta\lambda\mu}\Big(p_{\alpha\beta}+\frac{1}{2}t_\alpha t_\beta\Big)\Big(p_{\lambda\mu}+\frac{1}{2}t_\lambda t_\mu\Big)d\mathbf{x} - \frac{1}{2}\int_\omega S_{\alpha\beta\lambda\mu}r_{\alpha\beta}r_{\lambda\mu}d\mathbf{x}\Big\}$$
$$= \sup_{\mathbf{p},\mathbf{t}}\Big\{\langle \mathbf{p}^*,\mathbf{p}\rangle + \langle \mathbf{t}^*,\mathbf{t}\rangle - \frac{1}{2}\int_\omega Z_{\alpha\beta\lambda\mu}\Big(p_{\alpha\beta}+\frac{1}{2}t_\alpha t_\beta\Big)\Big(p_{\lambda\mu}+\frac{1}{2}t_\lambda t_\mu\Big)d\mathbf{x}\Big\}$$
$$+\sup_{\mathbf{r}}\Big\{\langle \mathbf{r}^*,\mathbf{r}\rangle - \frac{1}{2}\int_\omega S_{\alpha\beta\lambda\mu}r_{\lambda\mu}r_{\alpha\beta}d\mathbf{x}\Big\}.$$

We set

(6.16) $$G_1^*(\mathbf{p}^*,\mathbf{t}^*) = \sup_{(\mathbf{p},\mathbf{t})\in Y_1\times Y_2}\Big\{\langle \mathbf{p}^*,\mathbf{p}\rangle + \langle \mathbf{t}^*,\mathbf{t}\rangle - \frac{1}{2}\int_\omega Z_{\alpha\beta\lambda\mu}\Big(p_{\alpha\beta}+\frac{1}{2}t_\alpha t_\beta\Big)\Big(p_{\lambda\mu}$$
$$+\frac{1}{2}t_\lambda t_\mu\Big)d\mathbf{x}\Big\},$$

(6.17) $$G_2^*(\mathbf{r}^*) = \sup_{\mathbf{r}\in Y_1}\Big\{\langle \mathbf{r}^*\,\mathbf{r}\rangle - \frac{1}{2}\int_\omega S_{\alpha\beta\lambda\mu}r_{\alpha\beta}r_{\lambda\mu}d\mathbf{x}\Big\},$$

where $Y_2 = [H_0^2(\omega)]^2$. Simple calculation leads to

(6.18) $$G_2^*(\mathbf{r}^*) = \frac{1}{2}\int_\omega C_{\alpha\beta\lambda\mu}r_{\alpha\beta}^*r_{\lambda\mu}^*d\mathbf{x},\qquad \mathbf{C} = \mathbf{S}^{-1}.$$

The derivation of the explicit form of the functional G_1^* is rather lenghty and therefore is omitted in this paper. The ultimate form of G_1^* is given by

(6.19) $$G_1^*(\mathbf{p}^*,t^*) = \begin{cases} \frac{1}{2}\int_\omega H_{\alpha\beta\lambda\mu}p_{\alpha\beta}^*p_{\lambda\mu}^*dx + \frac{1}{2}\int_\omega R_{\alpha\beta}t_\alpha^*t_\beta^*dx, \quad \text{where}\quad \mathbf{R} = (p_{\alpha\beta}^*)^{-1} \\ \qquad\qquad \text{if}\quad \mathbf{p}^* \text{ is positive definite;} \\ \frac{1}{2}\int_\omega H_{\alpha\beta\lambda\mu}p_{\alpha\beta}^*p_{\lambda\mu}^*dx + \frac{1}{2}\int_\omega [p_{\sigma\sigma}^*]^{-1}(t_\sigma^*)^2dx, \quad (\sigma \text{ not summed!}), \\ \qquad \text{if}\quad \operatorname{rank}\mathbf{p}^* = 1,\ p_{\sigma\sigma} > 0 \quad\text{and}\quad \sqrt{p_{11}^*}\,t_2^* = \sqrt{p_{22}^*}\,t_1^*; \\ 0,\quad \text{if}\quad \mathbf{p}^* = 0 \quad\text{and}\quad \mathbf{t}^* = 0; \\ \infty,\quad \text{otherwise} \end{cases}$$

where $\mathbf{H} = \mathbf{Z}^{-1}$.

Now we pass to the derivation of the functional F^*. We have

$$(6.20)\qquad F^*(\Lambda_2^*\mathbf{q}^*) = \sup_{w\in H_0^2(\omega)} \{\langle \Lambda_2^*\mathbf{q}^*, w\rangle + \langle \varrho, w\rangle - \chi_{K_1}(w)\} = \sup_{w\in K_1}\{\langle \Lambda_2^*\mathbf{q}^* + \varrho, w\rangle\}.$$

We set $w = \psi + w_1$, where ψ determines the obstacle and $w_1 \in \mathscr{C} = \{z \in H_0^2(\omega) | z \geqslant 0\}$. Substituting into Eq. (6.20) and taking account of Lemma A. 1, we obtain

$$(6.21)\qquad F^*(\Lambda_2^*\mathbf{q}^*) = \begin{cases} \langle \Lambda_2^*\mathbf{q}^* + \varrho, \psi\rangle, & \text{if} \quad \Lambda_2^*\mathbf{q}^* + \varrho \leqslant 0, \\ \infty, & \text{if} \quad \Lambda_2^*\mathbf{q}^* + \varrho > 0. \end{cases}$$

Consequently, the final form of the dual problem can be written as follows:

$$(6.22)\qquad \sup_{(\mathbf{p}^*, \mathbf{t}^*, \mathbf{r}^*)\in Y^*} \left\{ -G_1^*(-\mathbf{p}^*, -\mathbf{t}^*) - G_2^*(\mathbf{r}^*) - \int_\omega t_\alpha^* \psi_{,\alpha} dx + \int_\omega r_{\alpha\beta}^* \psi_{,\alpha\beta}\, dx - \int_\omega \varrho\psi dx \right\}$$

subject to

$$(6.23)\qquad \Lambda_1^*\mathbf{p}^* = 0$$

and

$$(6.24)\qquad \Lambda_2^*\mathbf{q}^* + \varrho \leqslant 0.$$

6.2. Relationship between the primal and dual problems. Extremality conditions

We set $\mathbf{U} = (\mathbf{u}, w)$, $\mathbf{P}^* = (\mathbf{p}^*, \mathbf{q}^*) = (\mathbf{p}^*, \mathbf{t}^*, \mathbf{r}^*)$. The primal problem can be written as

$$(6.25)\qquad (\mathscr{P})\quad \inf_{\mathbf{U}\in X} J(\mathbf{U}, \Lambda\mathbf{U}) = \inf_{\mathbf{U}\in X} \Phi(\mathbf{U}, \mathbf{0}) = \inf_{\mathbf{U}\in X} \{G(\Lambda\mathbf{U}) + F(w)\}$$

whereas the dual problem has the form

$$(6.26)\qquad (\mathscr{P}^*)\quad \sup_{\mathbf{P}^*\in Y^*}\{-J^*(\mathbf{P}^*)\} = \sup_{\mathbf{P}^*\in Y^*}\{-\Phi^*(\mathbf{0}, \mathbf{P}^*)\} = \sup_{\mathbf{P}^*\in Y^*}\{-G^*(-\mathbf{P}^*) - F^*(\Lambda_2^*\mathbf{q}^*)\}.$$

We shall also make use of the relaxed problem, here given by the bidual problem $(\mathscr{P}^{**})$, see Appendix A:

$$(6.27)\qquad \inf_{\mathbf{U}\in X} J^{**}(\mathbf{U}) = \inf_{\mathbf{U}\in X} \Phi^{**}(\mathbf{U}, \mathbf{0}) = \inf_{\mathbf{U}\in X} \{G^{**}(\Lambda\mathbf{U}) + F(w)\}.$$

We observe that though the functional J is not convex, both J^* and J^{**} are convex functionals.

Suppose that the problem $(\mathscr{P})$ has a solution (minimizer) $\overline{\mathbf{U}} = (\overline{\mathbf{u}}, \overline{w})$. Making use of Theorem A.2 we infer that

$$(6.28)\qquad \Phi^{**}(\overline{\mathbf{U}}, \mathbf{0}) = \Phi(\overline{\mathbf{U}}, \mathbf{0}) = \inf \Phi^{**}(\mathbf{U}, \mathbf{0}).$$

Next, applying Theorem A.1 we deduce that if Φ^{**} attains a finite infimum at a point $(\overline{\mathbf{U}}, \mathbf{0})$, say, then

$$(6.29)\qquad \inf_{\mathbf{U}\in X} \Phi^*(\mathbf{U}, \mathbf{0}) = \sup_{\mathbf{P}^*\in Y^*} \{-\Phi^{***}(\mathbf{0}, \mathbf{P}^*)\}.$$

and there exists at least one element $\tilde{\mathbf{P}}^* \in Y^*$ such that

$$\sup_{\mathbf{P}^* \in Y^*} \{-\Phi^{***}(\mathbf{0}, \mathbf{P}^*)\} = -\Phi^{***}(\mathbf{0}, \tilde{\mathbf{P}}^*). \tag{6.30}$$

Since $\Phi^{***} = \Phi^*$ hence, in virtue of Theorems A.1 and A.2 we have, cf. also Theorem A.3,

$$\inf_{\mathbf{U} \in X} \Phi(\mathbf{U}, \mathbf{0}) = \inf_{\mathbf{U} \in X} \Phi^{**}(\mathbf{U}, \mathbf{0}) = \sup_{\mathbf{P}^* \in Y^*} \{-\Phi^{***}(\mathbf{0}, \mathbf{P}^*)\} = \sup_{\mathbf{P}^* \in Y^*} \{-\Phi^*(\mathbf{0}, \mathbf{P}^*\}. \tag{6.31}$$

Thus we have proved that

$$\inf(\mathscr{P}) = \sup(\mathscr{P}^*). \tag{6.32}$$

We pass now to the discussion of extremality conditions, see Appendix A. Let $\bar{\mathbf{U}} = (\bar{\mathbf{u}}, \bar{w})$ and $\bar{\mathbf{P}}^* = (\bar{\mathbf{p}}^*, \bar{\mathbf{q}}^*)$ be solutions of the problems $(\mathscr{P})$ and $(\mathscr{P}^*)$, respectively. Then we have

$$\Phi(\bar{\mathbf{U}}, \mathbf{0}) + \Phi^*(\mathbf{0}, \bar{\mathbf{P}}^*) = 0 \tag{6.33}$$

or, since the functional Φ is given by Eq. (6.10),

$$G(\Lambda(\bar{\mathbf{u}}, \bar{w})) + G^*(-\bar{\mathbf{p}}^*, -\bar{\mathbf{q}}^*) = \langle -(\bar{\mathbf{p}}^*, \bar{\mathbf{q}}^*), \Lambda(\bar{\mathbf{u}}, \bar{w})\rangle \tag{6.34}$$

and

$$F(\bar{\mathbf{u}}, \bar{w}) + F^*(\Lambda^*(\bar{\mathbf{p}}^*, \bar{\mathbf{q}}^*)) = \langle \Lambda^*(\bar{\mathbf{p}}^*, \bar{\mathbf{q}}^*), (\bar{\mathbf{u}}, \bar{w})\rangle. \tag{6.35}$$

The extremality condition (6.34) implies the following global constitutive relation:

$$(-\bar{\mathbf{p}}^*, -\bar{\mathbf{q}}^*) \in \partial G(\Lambda(\bar{\mathbf{u}}, \bar{w})) \tag{6.36}$$

which eventually yields

$$-\bar{p}^*_{\alpha\beta} = Z_{\alpha\beta\lambda\mu}\, \varepsilon_{\lambda\mu}(\bar{\mathbf{u}}, \bar{w}), \tag{6.37}$$

$$-\bar{t}^*_{\alpha} = Z_{\alpha\beta\lambda\mu}\, \varepsilon_{\lambda\mu}(\bar{\mathbf{u}}, \bar{w})\, \theta_\beta(\bar{w}), \tag{6.38}$$

$$-\bar{r}^*_{\alpha\beta} = S_{\alpha\beta\lambda\mu}\, \varkappa(\bar{w}). \tag{6.39}$$

The second extremality condition, given by Eq. (6.35), implies

$$\Lambda^*(\bar{\mathbf{p}}^*, \bar{\mathbf{q}}^*) \in \partial F(\bar{\mathbf{u}}, \bar{w}) = \partial F(\bar{w}). \tag{6.40}$$

Setting $\bar{w} = \psi + z$, $z \in \mathscr{C}$, we have

$$\partial \chi_{K_1}(\bar{w}) = \partial \chi_{K_1}(\psi + z) = \partial \chi_{\mathscr{C}}(z) \tag{6.41}$$

where the cone $\mathscr{C}$ has been defined in the subsection 6.1. Employing Eqs. (6.40) and (6.41) and Lemma A.2 we arrive at

$$\Lambda_1^* \bar{\mathbf{p}}^* = 0, \quad \text{and} \quad \Lambda_2^* \bar{\mathbf{q}}^* + \varrho \leqslant 0. \tag{6.42}$$

6.3. Mechanical interpretation

The extremality conditions (6.37)–(6.39) suggest that it is quite natural to assume

$$N_{\alpha\beta} = -p^*_{\alpha\beta}, \qquad M_{\alpha\beta} = -r^*_{\alpha\beta}. \tag{6.43}$$

Moreover, we set $Q_\alpha = -t^*_\alpha$. Then the dual problem $(\mathscr{P}^*)$ takes on the form

$$\sup_{(\mathbf{N}, \mathbf{Q}, \mathbf{M}) \in Y^*} \left\{ -G_1^*(\mathbf{N}, \mathbf{Q}) - G_2^*(\mathbf{M}) - \int_\omega Q_\alpha \psi_{,\alpha} d\mathbf{x} - \int_\omega M_{\alpha\beta} \psi_{,\alpha\beta} d\mathbf{x} - \int_\omega \varrho \psi d\mathbf{x} \right\}, \tag{6.44}$$

subject to

(6.45) $$N_{\alpha\beta,\beta} = 0$$

and

(6.46) $$M_{\alpha\beta,\beta\alpha} + Q_{\alpha,\alpha} + \varrho \leqslant 0.$$

From the relation (6.46) we conclude that finally $Q_\alpha = N_{\alpha\beta} w_{,\beta}$. Thus the kinematic quantity, or the transverse displacement field, enters *implicitly* into the dual problem ($\mathscr{P}^*$). However, it should be noted that the supremum in the problem (6.44) is taken over **M**, **N** *and* **Q**. In our contribution [2] the operator Λ depends parametrically on w and therefore is denoted by Λ_w. It has the following form:

$$\Lambda_w(\mathbf{u}, y, z) = (\Lambda_1 y, \Lambda_2 \mathbf{u} + \Lambda_{3w} z), \quad \text{where} \quad \Lambda_1 y = (-y_{,\alpha\beta}), \quad y \in H_0^2(\omega),$$
$$\Lambda_2(\mathbf{u}) = \big(e_{\alpha\beta}(\mathbf{u})\big), \quad \Lambda_{3w} z = (w_{,\alpha} z_{,\beta}),$$

and w is treated as a parameter. Such a choice is admissible and implies a parametric dependence of the dual problem on w and taking of the supremum over **M** and **N** only.

REMARK 6.1. Above we have assumed that $\psi \in H_0^2(\omega)$. Less smooth obstacles, such as $\psi \in H^1(\omega)$, or $\psi \in C(\overline{\omega})$, say, can likewise be considered. In the latter case we arrive at $\Lambda^*\mathbf{q}^* + \varrho \in M^1(\omega)$, where $M^1(\omega) = [C(\overline{\omega})]^*$ is the space of bounded measures, see [10].

REMARK 6.2. The dual problem for von Kármán plates has also been studied by LABISCH [25], but only in the case of classical boundary conditions, without taking into account unilateral conditions. Yet some of his assumptions are stronger than ours, and the complementary energy is not convex. Our approach is rigorous and the results in the convex complementary energy are defined on the whole space.

Appendix A. Elements of convex analysis

This appendix is provided for a brief review of ideas of convex analysis used throughout the paper. For additional details the reader should refer to [11, 39].

Let V be a real reflexive Banach space and V^* its topological dual. Let $\langle \cdot, \cdot \rangle : V^* \times \times V \to R$ be a duality pairing and $f: V \to \overline{R} = R \cup \{-\infty, \infty\}$ a functional, not necessarily convex.

The Fenchel transformation

(A.1) $$f^*(u^*) = \sup_{u \in V} \{\langle u^*, u\rangle - f(u)\}, \quad u^* \in V^*$$

defines the polar (conjugate) functional f^*. The polar functional f^* is convex and lower semi-continuous (l.s.c.) i.e. $f^* \in \Gamma_0$ (V^*) in the terminology used in [11]. The formula (A.1) implies

(A.2) $$f^*(u^*) + f(u) \geqslant \langle u^*, u\rangle, \quad \forall u \in V, \quad \forall u^* \in V^*.$$

An element $u^* \in V^*$ such that

(A.3) $$f(v) \geqslant f(u) + \langle u^*, v-u\rangle, \quad \forall v \in V$$

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is called a subgradient of the functional f at u. The set of all elements u^* satisfying the relation (A.3) is denoted by $\partial f(u)$, and called subdifferential. We write $u^* \in \partial f(u)$. Particularly it may happen that $\partial f(u) = \phi$, for instance if $f(u) = \infty$; here ϕ denotes the empty set.

The following property is important:

$$f^*(u^*)+f(u) = \langle u^*, u\rangle \Leftrightarrow u^* \in \partial f(u), \quad \text{or} \quad u \in \partial f^*(u^*). \tag{A.4}$$

Applying the Fenchel transform to f^* we obtain the bipolar (bidual) functional f^{**} of f^*, that is

$$f^{**} = (f^*)^*. \tag{A.5}$$

Next we define the polar of f^{**}

$$f^{***} = (f^{**})^*. \tag{A.6}$$

The functional f^* maps V^* into $\overline{R}$, and due to the reflexivity of V we have $f^{**}: V \to \overline{R}$. The bipolar functional f^{**} is the convex envelope of f, that is $f^{**}(u) \leqslant f(u)$, $\forall u \in V$. Since f^{***} is convex, hence we obtain

$$f^{***} = f^*. \tag{A.7}$$

Suppose that $\partial f(u) \neq \phi$, then

$$f(u) = f^{**}(u). \tag{A.8}$$

The following minimization problem, which means evaluating

$$\inf \{f(u) | u \in V\}, \tag{A.9}$$

will be called the primal problem and is denoted by $(\mathscr{P})$.

The dual problem of $(\mathscr{P})$ denoted by $(\mathscr{P}^*)$ can be derived using Rockafellar's approach which is briefly presented below.

Let $\Phi = \Phi(u, p)$ be a so-called perturbed functional defined on $V \times Y$, such that $\Phi(u, 0) = f(u)$; hence also $\inf f(u) \equiv \inf \Phi(u, 0)$. Here Y is a Hausdorff topological space, for instance a normed space. Then the dual problem $(\mathscr{P}^*)$ is formulated as follows:

$$\sup \{-\Phi^*(0, p^*) | p^* \in Y^*\}. \tag{A.10}$$

In applications, for instance in the calculus of variations, the following functional arises:

$$f(u) = J(u, \Lambda u) \tag{A.11}$$

where Λ is a continuous linear operator $\Lambda: V \to Y$. Very often

$$J(u, \Lambda u) = G(\Lambda u) + F(u), \tag{A.12}$$

where G and F are given functionals. Then the following perturbed functional Φ may be considered:

$$\Phi(u, p) = G(\Lambda u - p) + F(u). \tag{A.13}$$

The problem $(\mathscr{P}^*)$ takes on the form

$$\sup_{p^* \in Y^*} \{-G^*(-p^*) - F^*(\Lambda^* p^*)\}, \tag{A.14}$$

where $\Lambda^*: Y^* \to V^*$ is the adjoint operator of Λ. In virtue of the relation (A.2) we have

$$\Phi(u, 0) + \Phi^*(0, p^*) \geqslant \langle (u, 0), (0, p^*)\rangle = 0, \quad \forall u \in V, \quad \forall p^* \in Y^* \tag{A.15}$$

Hence we obtain

(A.16) $$\inf \Phi(u, 0) \geqslant \sup\{-\Phi^*(0, p^*)\},$$

or in the usually assumed concise notation

(A.17) $$\inf(\mathscr{P}) \geqslant \sup(\mathscr{P}^*).$$

We also make use of the following theorems.

THEOREM A.1. *Assume that the functional J is convex and let* $\inf(\mathscr{P})$ *be finite. Suppose that $u_0 \in V$ exists such that $J(u_0, \Lambda u_0) < +\infty$, the functional $p \to J(u_0, p)$ being continuous at Λu_0. Then*

(A.18) $$\inf(\mathscr{P}) = \sup(\mathscr{P}^*)$$

and the problem $(\mathscr{P}^*)$ has at least one solution $\bar{p}^* \in Y^*$.

THEOREM A.2. (see also [14]). *Let the functional J be given by Eq.* (A.12). *Suppose that the $(\mathscr{P})$-problem has a solution $\bar{u} \in V$. Let the element $u_0 \in V$ exist such that the functional G is finite in the neighbourhood of Λu_0. Then*

(A.19) $$J^{**}(\bar{u}, \Lambda\bar{u}) = J(\bar{u}, \Lambda\bar{u}) = \inf J^{**}(u, \Lambda u).$$

A direct consequence of Theorems A.1 and A.2 is

THEOREM A.3. *Let the functional J be given by Eq.* (A. 12) *and assume that $\bar{u} \in V$ and $u_0 \in V$ exist such that*

(i) $$J(\bar{u}, \Lambda\bar{u}) = \inf\{J(u, \Lambda u) | u \in V\},$$

(ii) *G is finite in the neighbourhood of Λu_0.*

Then the element $\bar{p}^* \in Y^*$ exists such that

(A.20) $$-J^*(\Lambda^*\bar{p}^*, \bar{p}^*) = \sup\{-J(\Lambda^* p^*, p^*) | p^* \in Y^*\} = J(\bar{u}, \Lambda\bar{u}).$$

At some points $u \in V$, $p^* \in Y^*$ the relation (A. 15) can turn into an equality which is then called the extremality condition. If the functional J is given by Eq. (A. 12), then under the assumptions of Theorem A.3, we obtain two extremality conditions, namely

(A.21) $$G(\Lambda\bar{u}) + G^*(-\bar{p}^*) = \langle -\bar{p}^*, \Lambda\bar{u}\rangle,$$

and

(A.22) $$F(\bar{u}) + F^*(\Lambda^*\bar{p}^*) = \langle \Lambda^*\bar{p}^*, \bar{u}\rangle,$$

which are equivalent to

(A.23) $$-\bar{p}^* \in \partial G(\Lambda\bar{u}),$$

(A.24) $$\Lambda^*\bar{p}^* \in \partial F(\bar{u}),$$

respectively.

Finally, we recall two useful lemmas.

LEMMA A.1. If V is a reflexive Banach space and $\mathscr{C} \subset V$ a closed convex cone such that $0 \in \mathscr{C}$, then

(A.25) $$(\chi_{\mathscr{C}})^*(v^*) = \chi_{\mathscr{C}^*}(v^*),$$

where $\mathscr{C}^*$ is the polar cone of $\mathscr{C}$, that is

(A.26) $$\mathscr{C}^* = \{v^* | v^* \in V^*,\ v^* \leqslant 0\}.$$

Here $v^* \leqslant 0$ means that $\langle v^*, v\rangle \leqslant 0$, $\forall v \in \mathscr{C}$.

LEMMA A.2. Let $\mathscr{C}$ be a nonempty closed convex cone. Then $u^* \in \partial\chi_{\mathscr{C}}(u)$ if and only if $u \in \partial\chi_{\mathscr{C}^*}(u^*)$. These conditions are equivalent to

$$u \in \mathscr{C}, \quad u^* \in \mathscr{C}^*, \quad \langle u^*, u\rangle = 0. \tag{A.27}$$

An alternative approach to the theory of duality for nonconvex problems was proposed in [1] and [42]. We hope to use it in our future investigations.

Appendix B. Dual formulation of implicit variational problems

A general theory of implicit variational problems = I.V.Ps is presented in the paper by MOSCO [31]. We observe that variational and quasi-variational inequalities are specific cases of I.V.Ps. A duality theory for I.V.Ps has been developed by CAPUZZO DOLCETTA and MATZEU [3]. These authors ingeniously extended to I.V.Ps the duality theory proposed by MOSCO [30] for variational inequalities. Therefore, in the case of I.V.Ps the term "duality in Mosco's sense" will also be interchangeably used. It is interesting to note that for, say, variational inequalities derivable from a minimum principle two quite different dual problems are available. The first dual problem can be formulated using the theory outlined in the previous appendix. The second dual problem is a problem in the sense of MOSCO [30].

Below we shall present essential aspects of the theory of duality for I.V.Ps, yet in a slightly more general setting than in [3], indispensable for our purposes.

Let us consider the following I.V.P., denoted by

$$(\mathscr{P}) \quad \left| \begin{array}{l} \text{find } u \in V \text{ such that} \\ \varphi(Au, u)+g(u,u) \leqslant \varphi(Au, w)+g(u, w), \quad \forall w \in V. \end{array} \right. \tag{B.1}$$

Here V is a real reflexive Banach space, and

(H_1) | A is a continuous linear operator from V into another Banach space Y.

(H_2) | $w \to \varphi(Av, w)$ is, for every $v \in V$, a convex lower semi-continuous function on V, $\varphi \neq \infty$.

(H_3) | $w \to g(v, w)$ is, for every $v \in V$, a real valued convex function on V which is continuous when $w = v$.
$w \to g(v, w)$ has, for every $v \in V$, a Gâteaux derivative with respect to the second variable $Dg(v, w)$ at $w = v$ such that for every $w^* \in V^*$ the set
$\{v \in V | Dg(v, v) = w^*\}$
contains at the most one element $(Dg)^{-1}(w^*)$.

If $Y = V$ and A is the identity operator, then we recover the results obtained in [3].

The Fenchel conjugate of φ taken with respect to the second variable is defined as follows:

$$\varphi^*(Av, w) = \sup_{w \in V}\{\langle w^*, w\rangle - \varphi(Av, w)\} \tag{B.2}$$

The functional $g^*(v, w^*)$ is defined similarly.

Then the dual problem of ($\mathscr{P}$) given by the inequality (B.1) is formulated as

PROBLEM ($\mathscr{P}^*$)

Find $u^* \in V^*$ such that

(B.3) $$\varphi^*(A[(Dg)^{-1}(-u^*)], u) - \langle u^*, (Dg)^{-1}(-u^*)\rangle \leqslant \varphi^*(A[(Dg)^{-1}(-u^*)], w) - \langle w^*, (Dg)^{-1}(-u^*)\rangle, \quad \forall w^* \in V^*.$$

Problems ($\mathscr{P}$) and ($\mathscr{P}^*$) are interrelated by

THEOREM B.1. *Let V be a real reflexive Banach space and assume that the hypotheses (H_1), (H_2) and (H_3) hold. Then $u \in V$ is a solution of ($\mathscr{P}$) if and only if $u^* = -Dg(u, u)$, $u^* \in V^*$ is a solution of ($\mathscr{P}^*$). Moreover, the following extremality condition is satisfied:*

(B.4) $$\varphi(Au, u) + \varphi^*(Au, u^*) = \langle u^*, u\rangle = -g(u, u) - g^*(u, -u^*).$$

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(B.3) $\varphi^*\left(A[(Dg)^{-1}(-u^*)], u\right)-\langle u^*, (Dg)^{-1}(-u^*)\rangle \leqslant$

$$\varphi^*\left(A[(Dg)^{-1}(-u^*)], w\right)-\langle w^*, (Dg)^{-1}(-u^*)\rangle, \quad \forall w^* \in V^*.$$

Problems ($\mathscr{P}$) and ($\mathscr{P}^*$) are interrelated by

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