Archives of Mechanics • Archiwum Mechaniki Stosowanej • 29, 2, pp. 229–249, Warszawa, 1977

A rational approach to creep mechanics

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SOME attractive ideas contained in the literature on time dependent inelasticity are identified and combined to form a more rational basis for the solution of boundary value problems in creep. Tnis is achieved by deriving two new fundamental equations which describe the inelastic response of a body. These equations provide a definitive means of studying the properties of solution of complex continuum creep problems, as well as a natural means of approximation. The solution of a simple structural problem is developed by means of illustration, and an alternative, variational approach is derived.

Przeprowadzono identyfikację i syntezę pewnych interesujących idei i pomysłów, które znaleźć można w literaturze naukowej dotyczącej zależnego od czasu zachowania się ciał niesprężystych, tworząc w ten sposób bardziej racjonalną podstawę dla rozwiązywania problemów brzegowych w teorii pełzania. Osiągnięto to drogą wyprowadzenia dwóch nowych, podstawowych równań opisujących niesprężyste zachowanie się ciała. Równania te pozwalają w sposób definitywny analizować własności rozwiązań złożonych problemów pełzania ośrodków ciągłych, a również stanowią naturalną podstawę do opracowania metod aproksymacyjnych. Dla ilustracji podano rozwiązanie prostego problemu konstrukcyjnego, przy czym podano również alternatywne podejście wariacyjne.

Проведены идентификация и синтез некоторых интересных идей и концепций, которые можно найти в научной литературе касающейся, зависящего от времени, поведения неупругих тел, образуя таким образом более рациональную основу для решения краевых задач в теории ползучести. Это достигнуто путем вывода двух новых, основных уравнений, описывающих неупругое поведение тела. Эти уравнения позволяют решающим образом анализировать свойства решений сложных задач ползучести сплошных сред и тоже составляют естественную основу для разработки аппроксимационных методов. Для иллюстрации приведено решение простой конструктивной задачи, причем дается тоже альтернативный вариационный подход.

1. Introduction

THE PHENOMENON of creep in structural materials, particularly in metals, is of continually increasing importance. The prediction of its long term effect on the behaviour of structures operating at elevated temperatures has become the theme of numerous studies over the past half century. Diverse techniques have been employed in the resolution of these inherently time dependent problems and have proven successful in many applications by comparison with those elementary analytical solutions which are available on the one hand, and with experiment on the other. However, there appears to be no mathematical basis for an investigation into the validity and relevance of these methods, which arose out of the engineering demands of the time. Nevertheless, the fundamental components of such a "rational" approach which would systematically review and incorporate achievements as well as clarify ultimate goals have already been postulated in the literature. In this paper a preliminary attempt is made to resolve these components into a rigorous foundation for a rational creep mechanics by formulating compact equations for the time dependent response of a continuum to creep — these equations can be identified as orthodox forms in functional, rather than classical, analysis. The premise is that the important aspects of the creep behaviour are isolated in a manner useful for analysis and discussion.

The needs of high temperature engineering forced the genesis of practical methods for the numerical resolution of complex structural problems. Here, the most popular is the "method of elastic solutions" (or "initial strains" or "successive approximations") formed from concepts out of thermoelasticity applied to elasto-plasticity by A. A. ILYUSHIN [1] and extended to creep by H. PORITSKY and F. A. FEND [2], A. MENDELSON, M. H. HIRSCHBERG and S. S. MANSON [3] and P. S. KURATOV and V. I. ROZENBLIUM [4]. These methods are essentially numerical "algorithms" based on simple approximate integration of the temporal response. Attempts to construct more accurate temporal integrations convinced several authors that some sort of mathematical formulation of the process of stress redistribution should be possible. The publications of W. C. CARPENTER [5] Z. P. BARANT [6] and O. C. ZIENKIEWICZ and I. C. CORMEAU [7], confirmed this speculation for complex problems modelled by finite elements — the step to a general "equation of stress redistribution" was then taken by the writer [8] and the results were applied to some problems of engineering importance [9, 10]. It was also pointed out in [8] that it was possible to derive an equation for the evolution of inelastic growth in a continuum, and that this could be identified as the principle of an existence proof for viscoelasticity derived some years earlier by I. BABUSKA and I. HLAVACEK [11]. It is the main purpose of this paper to bring these theories together as a unified whole and thus develop a more rational approach to creep mechanics, the publication of Babuska and Hlavacek providing a major influence in the presentation of this work.

2. Motivation

Consider rationally the process of time dependent creep. A body in equilibrium with itself and its environment is suddenly subjected to a series of external influences such that an initial stress and strain pattern is generated. As time progresses the body creeps and the stress and strain patterns evolve from this initial state. Thus, quantitatively, what has to be resolved is a problem of evolution — is it not then reasonable to expect that this process should be represented by an equation of evolution, or initial value problem? Indeed by discretizing in space (via finite elements), W. C. CARPENTER [5], Z. P. BAZANT [6] and O. C. ZIENKIEWICZ and I. C. CORMEAU [7] confirmed this. Moreover, it is fairly easy to establish an equation of evolution for simple problems.

Consider the simplest problem of a beam in bending, after W. J. GOODEY [12]. Suppose that the cross section of area A, has symmetry about one principal axis of bending and that the longitudinal stress depends solely on the height above the centroidal plane, x, and time, t. Let $\varepsilon = \varepsilon(x, t)$ be the longitudinal strain, and $\varkappa = \varkappa(t)$ the curvature change. The fundamental field equations are:

a) Equilibrium with external bending moment M

$$\int_A \sigma(x, t) x dA = M.$$

b) Strain-displacement

$$\varepsilon(x,t) = \varkappa(t)x.$$

c) Constitutive relations

$$\varepsilon(x,t) = \varepsilon_e(x,t)+\varepsilon_v(x,t),$$

$$\varepsilon_e(x,t) = \sigma(x,t)/E,$$

$$\frac{d}{dt}\varepsilon_v(x,t) = B(t)\sigma(x,t)^n,$$

$$\varepsilon_v(x,0) = 0,$$

where E is Young's modulus, and assuming for simplicity that the material creep response can be described by a time hardening law of the Norton type [13] where $B(t)$ and n are material parameters.

For convenience it is assumed that the applied bending moment M is constant in time. Firstly differentiate all equations with respect to time

$$\int_A \dot{\sigma} x dA = 0, \quad \dot{\varepsilon} = \dot{\varkappa} x, \quad \dot{\varepsilon} = \frac{\dot{\sigma}}{E}+\dot{\varepsilon}_v,$$

thus

$$\dot{\sigma} = E(\dot{\varepsilon}-\dot{\varepsilon}_v)$$

and on substituting into equilibrium and using strain displacement there results

$$\dot{\sigma} = E\left\{\frac{x}{I}\int_A \dot{\varepsilon}_v x dA-\dot{\varepsilon}_v\right\}, \tag{2.1}$$

where $I = \int_A x^2 dA$.

Hence, on using the creep law

$$\frac{d\sigma}{dt} = E\left\{\frac{x}{I}\int_A B(t)\sigma^n x dA-B(t)\sigma^n\right\}, \tag{2.2}$$

which is an equation of evolution for the stress field taken with the initial condition

$$\sigma(x,0) = \sigma_0(x), \tag{2.3}$$

where σ_0 is the initial elastic stress

$$\sigma_0(x) = \frac{Mx}{I}.$$

Further, if Eq. (2.1) is integrated with respect to time, then

$$\sigma = E\left\{\frac{x}{I}\int_A \varepsilon_v x dA-\varepsilon_v\right\}+\sigma_0$$

using the initial conditions.

But from the constitutive relations

$$\frac{d}{dt}\varepsilon_v = B(t)\sigma^n$$

hence, on substitution,

$$\frac{d}{dt}\varepsilon_v = B(t)\left[\sigma_0 + E\left\{\frac{x}{I}\int_A \varepsilon_v x dA - \varepsilon_v\right\}\right]^n, \tag{2.4}$$

which is an equation of evolution for the growth of inelastic strain taken with the initial condition

$$\varepsilon_v(x, 0) = 0. \tag{2.5}$$

Hence, for this simple problem the existence of equations of evolution for the creep response, (2.2) and (2.4), have been established. Such an equation similarly prevails for other creep laws (for example, consult the work of B. EINARSSON [14] on thick spheres and cylinders using the "strain hardening" theory [13]).

The question is: does the procedure described above always operate? At first sight it seems impossible, relying on an ability to express $\dot{\sigma}$ in terms of $\dot{\varepsilon}_v$ (2.1). However, consider the procedure in a more abstract fashion. What in fact has been shown is that

$$\dot{\sigma} = R(\dot{\varepsilon}_v), \tag{2.6}$$

where

$$R(\varepsilon^*) = E\left\{\frac{x}{I}\int \varepsilon^* x dA - \varepsilon^*\right\}$$

for some function, or "operator" R (which maps a function of x into another function of x) and which is obviously linear. Then, using the creep constitutive law

$$\frac{d}{dt}\sigma = B(t)R[\sigma^n], \tag{2.7}$$

which is equation (2.2).

Similarly, the equation for inelastic growth (2.4) can be written

$$\frac{d}{dt}\varepsilon_v = B(t)[\sigma_0 + R(\varepsilon_v)]^n. \tag{2.8}$$

Thus for any problem, even though an explicit expression for R may not be available, this operator can still be formally defined. By examining the course by which Eqs. (2.7) and (2.8) were derived it is obvious that R may be defined in the following manner:

Let ε^* be some strain, then $R(\varepsilon^*)$ is the stress resulting from the solution of an elastic problem for an identical beam, but with an initial strain pattern ε^* imposed upon it, giving zero resultant moment. These results may be readily extended to the variable loading situation. Then the equation of stress redistribution becomes

$$\frac{d}{dt}\sigma = B(t)R[\sigma^n] + \dot{\sigma}_0 \tag{2.9}$$

whilst that of inelastic growth remains unaltered, except that now $\sigma_0(x, t)$ is the stress resulting from the solution of an equivalent elastic problem to the creep problem with inelasticity ignored, i.e.

$$\sigma_0(x, t) = \frac{M(t)x}{I}$$

with the initial condition

$$\sigma(x, 0) = \sigma_0(x, 0) = M(0)x/I. \tag{2.10}$$

The above simple observations provide the motivation for the following discussion.

3. The continuum problem

At this point, before proceeding any further, there should be an appraisal of the mechanical laws which characterise creep deformation. Of the representations which are available few are concerned with the atomic structure of the material, involving a stochastic and kinetic description; thus most constitutive approximations are of a purely deterministic, phenomenological nature.

3.1. A constitutive model

The majority of the more sophisticated constitutive models are based on the so-called Axioms of Constitutive theory [15] being a set of fundamental rules to which the models should adhere. One of these axioms, otherwise known as the Principle of Determinism, is open to criticism of a practical, rather than a physical, nature. The principle asserts that the stress in a body at some instant shall be determined by the history of deformation of the body up to that instant. Although intended to eliminate the (unrealistic) influence of future events, it is often used to imply a constitutive model with stress expressed in terms of strain history. Simple observation of the methods and possibilities of large scale material testing and documentation for creep highlights the excessive practical limitations of such an approach — essentially restricting any experimental program to short term tensile tests at variable load producing data on strain change in terms of stress level. Unless a sound physical theory of material creep behaviour on the atomistic level is forthcoming, and this does not appear to be remotely accessible, any mathematical constitutive model shall be largely speculative, and the problems involved thus seem insurmountable. Nevertheless, an alternative view of creep constitutive theory has recently been expressed by F. A. LECKIE [16]. The underlying philosophy is the treatment of the material as a "black box" with an input of stress and an output of strain rate — thus the effect of the black box may be assessed by performing suitable tests, even though the actual mechanism of the box is unknown.(1) A suitable mathematical model of the black box is a form of the so-called

(1) Indeed the concept leads to more fruitful ideas with the direct application of information theory to creep mechanics in which the writer is now involved to an extent eliminating accurate deterministic material models.

internal, or "hidden", state variable theory [17, 18, 19] which is of the uniaxial form

$$\dot{\varepsilon}_v = F(\sigma, \alpha_j),$$
(3.1)
$$\dot{\alpha}_i = G_i(\sigma, \alpha_j), \quad i = 1, 2, \dots, N,$$

where σ is the stress, ε_v the viscous (creep) strain and α_i, $i = 1, 2, \dots, N$ a set of internal state variables; the functions F and G_i are determined by experiment — indeed LECKIE has shown [16, 19, 20] how the pertinent parameters which result may be obtained by "operating" the black box.

Thus in this paper a rational mechanics for creep is developed on this theme of black boxes using the internal state variable theory as model.

3.2. The creep problem

Consider a body $\mathcal{B}$ occupying a volume $\mathcal{V}$ in a three-dimensional space, bounded by a surface $\mathcal{S}$; points of $\bar{\mathcal{V}} = \mathcal{V} \cup \mathcal{S}$ are described by the triplet $(X_1, X_2, X_3) = \mathbf{X}$. It is supposed that at time $t = 0^-$ the body is in an unstressed and unstrained state in equilibrium with itself and its environment. At time $t = 0$ the state of the body is suddenly changed by the application of external influences in the form of internal body forces within $\mathcal{B}$, of prescribed surface tractions on a part of its boundary $\mathcal{S}_s \subset \mathcal{S}$ and prescribed displacements on the remainder $\mathcal{S}_U = \mathcal{S} \setminus \mathcal{S}_s$. The environment is such that the body creeps inelastically in time.

It shall be assumed that isothermal conditions are maintained so that thermal strains can *be* discounted; furthermore, the deformation is infinitesimal and quasi-static so that inertial and accelerative effects can also be ignored. The initial deformation is supposed to be purely elastic so that instantaneous plasticity, for example, is not taken into account. Finally it is assumed that the portions $\mathcal{S}_s$ and $\mathcal{S}_U$ of the bounding surface do not alter in time. None of these suppositions are restrictive to the theory which can be readily extended to cope with their inclusion — they serve only to add notational complications. Firstly, some definitions are required to identify properly the field variables of stress, strain and displacement in the body [21]:

A continuum field process $[\mathbf{u}, \boldsymbol{\epsilon}, \boldsymbol{\sigma}]$ for the body $\mathcal{B}$ is a function defined on the real line R whose values at time $t \in$ R are ordered triplets $(\mathbf{u}(t), \boldsymbol{\epsilon}(t), \boldsymbol{\sigma}(t))$ such that $\mathbf{u}(t)$ is a vector, displacement field with components $u_i(\mathbf{X}, t)$, $i = 1, 2, 3$, $\mathbf{X} \in \bar{\mathcal{V}}$ and $\boldsymbol{\epsilon}(t)$ and $\boldsymbol{\sigma}(t)$ are second order strain and stress fields tensors with components $\varepsilon_{ij}(\mathbf{X}, t)$, $\sigma_{ij}(\mathbf{X}, t)$, $i, j = 1, 2, 3$, $\mathbf{X} \in \bar{\mathcal{V}}$, respectively. This definition is intended to clarify the difference between the whole history of displacement, strain and stress $[\mathbf{u}, \boldsymbol{\epsilon}, \boldsymbol{\sigma}]$ and the state of displacement, strain and stress at a particular instant, t, say $(\mathbf{u}(t), \boldsymbol{\epsilon}(t), \boldsymbol{\sigma}(t))$. However, frequently explicit dependence on t shall be dropped where there is no confusion. Nevertheless not all of the possible states of deformation and stress for the body are satisfactory on more general physical or geometrical grounds.

The triplet $(\mathbf{u}^A, \boldsymbol{\epsilon}^A, \boldsymbol{\sigma}^A)$ is an *admissible* state for $\mathcal{B}$ if $u^A(\mathbf{X})$ is smooth on $\mathcal{V}$ with $\mathbf{u}^A$, $\hat{\nabla}\mathbf{u}^A$ continuous on $\bar{\mathcal{V}}$, where $\hat{\nabla}$ is the tensor

$$\hat{\nabla}\mathbf{u}^A = \frac{1}{2}(u^A_{i,j} + u^A_{j,i}), \quad i, j = 1, 2, 3,$$

$\boldsymbol{\epsilon}^A(\mathbf{X})$ is a symmetric tensor continuous on $\mathscr{V}$, and $\boldsymbol{\sigma}^A(\mathbf{X})$ is a symmetric tensor, smooth on $\mathscr{V}$ and such that $\boldsymbol{\sigma}^A$ and $\operatorname{div}\boldsymbol{\sigma}^A$ are continuous on $\bar{\mathscr{V}}$ where div is the vector

$$\operatorname{div}\boldsymbol{\sigma}^A = \sigma^A_{ij,j}, \quad i = 1, 2, 3$$

with $(\ )_{,i} = \frac{\partial}{\partial X_i}(\ )$, $i = 1, 2, 3$ and using the Einstein summation convention.

Then $[\mathbf{u}, \boldsymbol{\epsilon}, \boldsymbol{\sigma}]$ is an *admissible* process for $\mathscr{B}$ if its values $(\mathbf{u}(t), \boldsymbol{\epsilon}(t), \boldsymbol{\sigma}(t))$ are admissible states for $\mathscr{B}$.

The inelastic constitutive equation (3.1) for the internal state variable theory generalises to

$$\text{(3.2)} \qquad \begin{aligned} \dot{\boldsymbol{\epsilon}}_v &= \mathbf{F}(\boldsymbol{\sigma}(t), \boldsymbol{\alpha}_j(t)), \\ \dot{\boldsymbol{\alpha}}_i &= \mathbf{G}_i(\boldsymbol{\sigma}(t), \boldsymbol{\alpha}_j(t)), \quad i, j = 1, 2, \ldots, N \end{aligned}$$

with $\mathbf{F}$ and $\mathbf{G}_i$ non-linear tensor valued operators, $\boldsymbol{\alpha}_i$ being a process of internal change whose value at time t is the tensor field $\boldsymbol{\alpha}_i(t)$ with components $\alpha_{ijk}(\mathbf{X}, t)$ $i = 1, 2, 3, \ldots, N$, $j, k = 1, 2, 3$, $\mathbf{X} \in \bar{\mathscr{V}}$ representing the N hidden state variables.

The inelastic strain is related to the actual (total) strain through the decoupling identity

$$\text{(3.3)} \qquad \boldsymbol{\epsilon}(t) = \boldsymbol{\epsilon}_e(t) + \boldsymbol{\epsilon}_v(t),$$

where $\boldsymbol{\epsilon}_e(t)$ is an elastic strain related to stress through the generalised Hooke's law

$$\text{(3.4)} \qquad \boldsymbol{\epsilon}_e(t) = \mathbf{C}(\boldsymbol{\sigma}(t)),$$

where $\mathbf{C}$ is a linear operator whose components C_{ijkl}, $i, j, k, l = 1, 2, 3$ form the fourth order "compliance" tensor (it is possible to require $\mathbf{C}$ to depend on t but for convenience this shall not be done). Since the initial response is supposed to be purely elastic the constitutive relations are completed by the identity

$$\text{(3.5)} \qquad \boldsymbol{\epsilon}_v(0) = \mathbf{0}.$$

Now suppose that the body is subjected to an internal force process $\bar{\mathbf{f}}$ with values $\bar{\mathbf{f}}(t)$ and components $\bar{f}_i(\mathbf{X}, t)$, $i = 1, 2, 3$, $\mathbf{X} \in \mathscr{V}$. Then by a *quasi-static creep process for the body* $\mathscr{B}$ is meant an admissible process $[\mathbf{u}, \boldsymbol{\epsilon}, \boldsymbol{\sigma}]$ for $\mathscr{B}$ whose values are such that the stress field $\boldsymbol{\sigma}(t)$ is in equilibrium with the body force $\bar{\mathbf{f}}(t)$

$$\text{(3.6)} \qquad \operatorname{div}\boldsymbol{\sigma}(t) + \bar{\mathbf{f}}(t) = \mathbf{0} \quad \text{in } \mathscr{V}$$

while being related to the strain field $\boldsymbol{\epsilon}(t)$ through the constitutive relations (3.2)–(3.5) which is compatible with the displacement field through the strain-displacement identity

$$\text{(3.7)} \qquad \boldsymbol{\epsilon}(t) = \hat{\nabla}\mathbf{u}(t) \quad \text{in } \bar{\mathscr{V}}.$$

Let a surface displacement vector field process $\bar{\mathbf{u}}$ be given on $\mathscr{S}_U$, and a surface traction vector field process $\bar{\mathbf{s}}$ be given on $\mathscr{S}_s$, then by a solution of the *mixed quasi-static creep problem corresponding to the boundary data* $[\bar{\mathbf{u}}, \bar{\mathbf{s}}]$ is meant a quasi-static creep process for $\mathscr{B}$ such that

$$\text{(3.8)} \qquad \mathbf{u}(t) = \bar{\mathbf{u}}(t) \quad \text{on} \quad \mathscr{S}_U$$

$$\text{)3.9)} \qquad \boldsymbol{\sigma}(t)\mathbf{n} = \bar{\mathbf{s}}(t) \quad \text{on} \quad \mathscr{S}_s$$

where $\mathbf{n}$ is the unit normal vector to the surface $\mathscr{S}_s$.

3.3. The equivalent elastic problem

For each creep problem it is possible to define an "equivalent" elastic problem by ignoring inelasticity in the formulation. This equivalent elastic problem [11] is useful in the presentation of the rational approach.

The process $[\mathbf{u}_0, \boldsymbol{\epsilon}_0, \boldsymbol{\sigma}_0]$ satisfies the equivalent elastic problem corresponding to the boundary data $[\bar{\mathbf{u}}, \bar{\mathbf{s}}]$ and the body process $\bar{\mathbf{f}}$ if its values satisfy

$$\begin{aligned} &\operatorname{div} \boldsymbol{\sigma}_0(t) + \bar{\mathbf{f}}(t) = 0 \qquad \text{in} \quad \mathscr{V};\\ &\boldsymbol{\epsilon}_0(t) = \hat{\nabla} \mathbf{u}_0(t),\\ &\boldsymbol{\epsilon}_0(t) = \mathbf{C}[\boldsymbol{\sigma}_0(t)] \quad \text{in} \quad \bar{\mathscr{V}};\\ &\mathbf{u}_0(t) = \bar{\ }(t) \quad \text{on} \quad \mathscr{S}_U;\\ &\boldsymbol{\sigma}(t)\mathbf{n} = \bar{\mathbf{s}}(t) \quad \text{on} \quad \mathscr{S}_s. \end{aligned}$$

It is important to notice that the solution of the initial elastic problem is the value of the process $[\mathbf{u}_0, \boldsymbol{\epsilon}_0, \boldsymbol{\sigma}_0]$ at time $t = 0$, i.e. $(\mathbf{u}_0(0), \boldsymbol{\epsilon}_0(0), \boldsymbol{\sigma}_0(0))$.

4. The equations of inelastic growth and stress redistribution

In the examination of the simple beam in bending equations of evolution for the growth of inelasticity and redistribution of stress were derived. These shall be generalised to the continuum problem.

4.1. Statement of the basic equations

It is fairly obvious from Eqs. (2.8) and (2.9), bearing in mind modifications to allow for an internal state variable constitutive model, that the generalised "equation of inelastic growth" should take the form

$$\begin{aligned} &\frac{d}{dt}\boldsymbol{\epsilon}_v(t) = \mathbf{F}\big[\boldsymbol{\sigma}_0(t) + \mathbf{R}[\boldsymbol{\epsilon}_v(t)], \boldsymbol{\alpha}_j(t)\big],\\ &\frac{d}{dt}\boldsymbol{\alpha}_i(t) = \mathbf{G}_i\big[\boldsymbol{\sigma}_0(t) + \mathbf{R}[\boldsymbol{\epsilon}_v(t)], \boldsymbol{\alpha}_j(t)\big] \quad i, j = 1, 2, \ldots, N \end{aligned} \tag{4.1}$$

subject to the initial conditions

$$\begin{aligned} &\boldsymbol{\epsilon}_v(0) = \mathbf{0},\\ &\boldsymbol{\alpha}_i(0) = \boldsymbol{\alpha}_{i0}, \end{aligned} \tag{4.2}$$

where $\boldsymbol{\alpha}_{i0}$ is the initial state of the internal variables. Similarly the generalised "equation of stress redistribution" should be

$$\begin{aligned} &\frac{d}{dt}\boldsymbol{\sigma}(t) = \mathbf{R}\big[\mathbf{F}\big(\boldsymbol{\sigma}(t), \boldsymbol{\alpha}_j(t)\big)\big] + \dot{\boldsymbol{\sigma}}_0(t),\\ &\frac{d}{dt}\boldsymbol{\alpha}_i(t) = \mathbf{G}_i\big(\boldsymbol{\sigma}(t), \boldsymbol{\alpha}_j(t)\big) \quad i, j = 1, 2, \ldots, N \end{aligned} \tag{4.3}$$

subject to

$$\sigma(0) = \sigma_0(0),\qquad \alpha_i(0) = \alpha_{i0} \tag{4.4}$$

noting that σ_0 is the stress associated with the equivalent elastic problem. The operator **R**, called here the "residual operator", maps second order tensor fields into second order tensor fields and can be defined through the solution of a further elastic problem, which shall be called the "residual elastic problem":

4.2. The residual elastic problem and the residual operator

Let ϵ^* be a strain process whose values are admissible strains (i.e. symmetric and continuous on $\mathscr{V}$) then $[\mathbf{u}_R, \epsilon_R, \sigma_R]$ is a solution of the residual elastic problem for ϵ^* if its values satisfy

$$\operatorname{div} \sigma_R(t) = \mathbf{0} \quad \text{in} \quad \mathscr{V};$$

$$\epsilon_R(t) = \tilde{\nabla} \mathbf{u}_R(t),$$

$$\epsilon_R(t) = \mathbf{C}[\sigma_R(t)] + \epsilon^*(t) \quad \text{in} \quad \bar{\mathscr{V}};$$

$$\mathbf{u}_R(t) = \mathbf{0} \quad \text{on} \quad \mathscr{S}_U;$$

$$\sigma_R(t)\mathbf{n} = \mathbf{0} \quad \text{on} \quad \mathscr{S}_s.$$

Therefore the residual elastic problem is an elastic initial strain problem such that $\mathscr{S}_s$ is free and $\mathscr{S}_U$ is fixed with no body forces. Thus the residual operator is defined as

$$\sigma_R(t) = \mathbf{R}[\epsilon^*(t)].$$

It turns out that this operator has several important properties(2) [8] (some are discussed in Appendix 1).

4.3. Representation theorem for inelastic growth

Although it is tolerably straightforward to define these equations it is not at all clear that they, together with the auxiliary elastic problems, yield the solution of the quasi-static creep problem. This needs to be proven.

Firstly it should be established that the process of inelastic strain indeed complies with the equations of inelastic growth.

THEOREM 1 (a). *Let* $[\mathbf{u}, \epsilon, \sigma]$ *be a solution of the quasi-static creep problem with* $[\mathbf{u}_0, \epsilon_0, \sigma_0]$ *a solution of the equivalent elastic problem, then* ϵ_v *must satisfy the equations of inelastic growth.*

P r o o f. It is fairly obvious that $[\mathbf{u}-\mathbf{u}_0, \epsilon-\epsilon_0, \sigma-\sigma_0]$ satisfies a residual elastic problem for ϵ_v. By definition

$$\dot{\epsilon} = \mathbf{C}(\dot{\sigma}) + \mathbf{F}(\sigma, \alpha_j), \qquad \dot{\alpha}_i = \mathbf{G}_i(\sigma, \alpha_j)$$

hence, since $\epsilon_0 = \mathbf{C}(\sigma_0)$

$$\dot{\epsilon} - \dot{\epsilon}_0 = \mathbf{C}(\dot{\sigma} - \dot{\sigma}_0) + \mathbf{F}(\sigma, \alpha_j), \qquad \dot{\alpha}_i = \mathbf{G}_i(\sigma, \alpha_j). \tag{i}$$

(2) In particular it should be noted that it is linear.

But $\sigma - \sigma_0$ satisfies the residual elastic problem for ϵ_v so

(ii) $$\dot{\sigma} - \dot{\sigma}_0 = \mathbf{R}(\dot{\epsilon}_v),$$

(iii) $$\epsilon - \dot{\epsilon}_0 = \mathbf{C}(^{\cdot} - \dot{\sigma}_0) + \dot{\epsilon}_v.$$

Comparing (i) and (iii)

$$\dot{\epsilon}_v = \mathbf{F}(\sigma, \alpha_j), \quad \alpha_i = \mathbf{G}_i(\sigma, \alpha_j)$$

and from (ii)

$$\dot{\epsilon}_v = \mathbf{F}[\sigma_0 + \mathbf{R}(\epsilon_v), \alpha_j], \quad \dot{\alpha}_i = \mathbf{G}_i[\sigma_0 + \mathbf{R}(\epsilon_v), \alpha_j].$$

Since it is readily verified that

$$\epsilon_v(0) = \mathbf{0}, \quad \alpha_i(0) = \alpha_{i0}$$

the result is proven, QED.

Secondly it needs to be established that if a solution, ϵ^* say, of the equations of inela stic growth is attainable then a solution of the creep problem can be constructed.

THEOREM 1 (b). *Let* $[\mathbf{u}_0, \epsilon_0, \sigma_0]$ *satisfy the equivalent elastic problem and* $[\mathbf{u}_R, \epsilon_R, \sigma_R]$ *satisfy the residual elastic problem for* ϵ^* *where* ϵ^* *is a solution of the equations of inelastic growth. Then* $[\mathbf{u}_0 + \mathbf{u}_R, \epsilon_0 + \epsilon_R, \sigma_0 + \sigma_R]$ *is a solution of the creep problem.*

P r o f. By definition

$$\epsilon_0 = \mathbf{C}(\sigma_0), \quad \epsilon_R = \mathbf{C}(\sigma_R) + \epsilon^*, \quad \sigma_R = \mathbf{R}(\epsilon^*)$$

and

$$\epsilon^* = \mathbf{F}[\sigma_0 + \mathbf{R}(\epsilon^*), \alpha_j], \quad \dot{\alpha}_i = \mathbf{G}_i[\sigma_0 + \mathbf{R}(\epsilon^*), \alpha_j].$$

Combining these

$$\epsilon_0 + \dot{\epsilon}_R = \mathbf{C}(\dot{\sigma}_0 + \dot{\sigma}_R) + \mathbf{F}(\sigma_0 + \sigma_R, \alpha_j), \quad \alpha_i = \mathbf{G}_i(\sigma_0 + \sigma_R, \alpha_j)$$

and similarly

$$\operatorname{div}(\sigma_0 + \sigma_R) + \bar{\mathbf{f}} = \mathbf{0} \quad \text{in} \quad \mathscr{V},$$

$$\epsilon_0 + \epsilon_R = \hat{\nabla}(\mathbf{u}_0 + \mathbf{u}_R) \quad \text{in} \quad \bar{\mathscr{V}},$$

$$\mathbf{u}_0 + \mathbf{u}_R = \bar{\mathbf{u}} \quad \text{on} \quad \mathscr{S}_U,$$

$$(\sigma_0 + \sigma_R)\mathbf{n} = {}^{-} \quad \text{on} \quad \mathscr{S}_s.$$

Hence since

$$\mathbf{u}_R(0) = \mathbf{0}, \quad \epsilon_R(0) = \mathbf{0}, \quad \sigma_R(0) = \mathbf{0},$$

$[\mathbf{u}_0 + \mathbf{u}_R, \epsilon_0 + \epsilon_R, \sigma_0 + \sigma_R]$ is a solution of the creep problem, QED.

Thus, Theorems 1(a) and 1(b) establish the representation of the quasi-static creep process by the equations of inelastic growth. It is interesting to note that these "representation theorem" qualify the classical decoupling of a creep process into an equivalent elastic component and a residual component [22] — and that, furthermore, this residual part is obtainable through the solution of another elastic problem together with an appropriate equation of evolution [11].

4.4. Representation theorem for stress redistribution

It has been shown that viscous strain obeys the equation of inelastic growth and that once a solution of this is available it is possible to construct a solution of the creep problem. It can also be demonstrated that the stress associated with the creep problem satisfies the redistribution Eqs. (4.3) whether the exact solution or the constructed solution is available. Unless this stress is unique it is not possible to construct a solution of the creep problem by resolving the redistribution equations alone unless additional conditions on the constitutive equations are specified. However,

THEOREM 2 (a). *Let* $[\mathbf{u}, \boldsymbol{\epsilon}, \boldsymbol{\sigma}]$ *be a solution of the creep problem such that* $[\mathbf{u}_0, \boldsymbol{\epsilon}_0, \boldsymbol{\sigma}_0]$ *is a solution of the equivalent elastic problem, then* $\boldsymbol{\sigma}$ *satisfies the redistribution equations.*

Proof. By definition $\boldsymbol{\sigma}-\boldsymbol{\sigma}_0 = \mathbf{R}(\boldsymbol{\epsilon}_v)$ and from Theorem 1 (a), $\boldsymbol{\epsilon}_v$ is a solution of the equation of inelastic growth

$$\dot{\boldsymbol{\epsilon}}_v = \mathbf{F}[\boldsymbol{\sigma}_0+\mathbf{R}(\boldsymbol{\epsilon}_v), \boldsymbol{\alpha}_j],$$
$$\dot{\boldsymbol{\alpha}}_i = \mathbf{G}_i[\boldsymbol{\sigma}_0+\mathbf{R}(\boldsymbol{\epsilon}_v), \boldsymbol{\alpha}_j]$$

consequently, $\dot{\boldsymbol{\epsilon}}_v = \mathbf{F}(\boldsymbol{\sigma}, \boldsymbol{\alpha}_j)$ and

$$\dot{\boldsymbol{\sigma}} = \mathbf{R}\big(\mathbf{F}(\boldsymbol{\sigma}, \boldsymbol{\alpha}_j)\big)+\dot{\boldsymbol{\sigma}}_0, \qquad \dot{\boldsymbol{\alpha}}_i = \mathbf{G}_i(\boldsymbol{\sigma}, \boldsymbol{\alpha}_j),$$
$$\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0(0), \qquad \boldsymbol{\alpha}_i(0) = \boldsymbol{\alpha}_{i0}$$

and the result is proven, QED.

THEOREM 2 (b). *Let* $[\mathbf{u}_0, \boldsymbol{\epsilon}_0, \boldsymbol{\sigma}_0]$ *be a solution of the equivalent elastic problem and* $[\mathbf{u}_R, \boldsymbol{\epsilon}_R, \boldsymbol{\sigma}_R]$ *satisfy the residual elastic problem for* $\boldsymbol{\epsilon}^*$ *where* $\boldsymbol{\epsilon}^*$ *satisfies the equations of inelastic growth. Then, the constructed solution* $\boldsymbol{\sigma}_0+\boldsymbol{\sigma}_R$ *satisfies the redistribution equations.*

Proof. By definition $\boldsymbol{\sigma}_R = \mathbf{R}(\boldsymbol{\epsilon}^*)$ and

$$\dot{\boldsymbol{\epsilon}}^* = \mathbf{F}(\boldsymbol{\sigma}_0+\mathbf{R}(\boldsymbol{\epsilon}^*), \boldsymbol{\alpha}_j), \qquad \dot{\boldsymbol{\alpha}}_i = \mathbf{G}_i(\boldsymbol{\sigma}_0+\mathbf{R}(\boldsymbol{\epsilon}^*), \boldsymbol{\alpha}_j)$$

hence

$$\dot{\boldsymbol{\sigma}}_R = \mathbf{R}[\mathbf{F}(\boldsymbol{\sigma}_0+\boldsymbol{\sigma}_R, \boldsymbol{\alpha}_j)].$$

Consequently

$$\dot{\boldsymbol{\sigma}}_0+\dot{\boldsymbol{\sigma}}_R = \mathbf{R}[\mathbf{F}(\boldsymbol{\sigma}_0+\boldsymbol{\sigma}_R, \boldsymbol{\alpha}_j)]+\dot{\boldsymbol{\sigma}}_0, \qquad \dot{\boldsymbol{\alpha}}_i = \mathbf{G}_i(\boldsymbol{\sigma}_0+\boldsymbol{\sigma}_R, \boldsymbol{\alpha}_j)$$

but from Theorem 1(b) $\boldsymbol{\sigma}_0+\boldsymbol{\sigma}_R$ satisfies the creep problem, so

$$\boldsymbol{\sigma}_0(0)+\boldsymbol{\sigma}_R(0) = \boldsymbol{\sigma}_0(0)$$

and the result is proven, QED.

4.5. The equations of evolution for deformation

The preceding analysis has successfully established the existence of initial value problems for inelastic strain and stress. Similar formulations can be inferred for the deformation process, coupled with either of these "primitive" equations.

From Theorem 1 (b)

$$\mathbf{u}(t) = \mathbf{u}_R(t)+\mathbf{u}_0(t), \qquad \boldsymbol{\epsilon}(t) = \boldsymbol{\epsilon}_R(t)+\boldsymbol{\epsilon}_0(t)$$

and on defining

$$\mathbf{u}_R = \mathbf{r}^{\mathrm{I}}(\boldsymbol{\epsilon}^*),$$
$$\boldsymbol{\epsilon}_R = \mathbf{R}^{\mathrm{II}}(\boldsymbol{\epsilon}^*) = \mathbf{C}\big(\mathbf{R}(\boldsymbol{\epsilon}^*)\big)+\boldsymbol{\epsilon}^*$$

there results on rearranging and employing Theorem 1

$$\frac{d}{dt}\mathbf{u}(t) = \mathbf{r}^{\mathrm{I}}\big(\mathbf{F}(\boldsymbol{\sigma}(t), \boldsymbol{\alpha}_j(t))\big)+\dot{\mathbf{u}}_0(t),$$

(4.5) $$\frac{d}{dt}\boldsymbol{\epsilon}(t) = \mathbf{R}^{\mathrm{II}}\big(\mathbf{F}(\boldsymbol{\sigma}(t), \boldsymbol{\alpha}_j(t))\big)+\dot{\boldsymbol{\epsilon}}_0(t),$$

(4.6) $$\mathbf{u}(0) = \mathbf{u}_0(0), \quad \boldsymbol{\epsilon}(0) = \boldsymbol{\epsilon}_0(0),$$

which should be integrated along with Eqs. (4.3) and (4.4) to obtain a solution. Theorems with similar implications to Theorem 2 may be deduced.

5. Implication

There are two main implications of this treatment of creep mechanics. Firstly the inelasticity has been isolated and shown to obey a certain equation of evolution; the creep process may be constructed from this growth law. Secondly, if it can be established that a unique solution exists, the creep process may be alternatively described by the redistribution equations and the equations of evolution of deformation without resorting to the growth law.

This "rational" approach is, in a sense, an exact statement of the well-known "initial strain" algorithm [2–4]. Discrete forms of the redistribution equations have been identified by W. C. CARPENTER [5] and Z. P. BAZANT [6] and of the evolution of deformation by O. C. ZIENKIEWICZ and I. C. CORMEAU [7] on approximating the field space variable by finite elements. The inelastic growth law can in fact be recognised as the basis of the existence proof for viscoelasticity, described by a hereditary integral constitutive equation, as given by I. BABUSKA and I. HLAVÁČEK [11].

It is the writer's contention that the equations of inelastic growth are fundamental to the creep process — and that familiar results concerning the mathematical formulation of creep and its numerical resolution should be directly related to the properties of these equations, or the associated redistribution equations. For example, Ref. [8] considers the existence and uniqueness of solution to the creep problem using the equations of inelastic growth, whilst Ref. [23] employs the redistribution equations in a discussion of the behaviour of creeping bodies at large times. Moreover this rational approach can be used not only in a systematisation of the study of properties of solution of creep problems, but also in the clarification of approximate solution methods, and practical examples. For example, the stress redistribution equations can be used to derive stress bounds for creep [24], or to resolve complex problems in structural design for high temperature [9, 10]. In order to demonstrate the possible applications of the approach two examples shall be presented the first of which illustrates the scheme of the method in problem solving while the second illustrates some mathematical manipulations of the equations.

6. Examples

6.1. Forward creep of a straight pipe

Consider a long, thin, constant thickness circular cross section cylindrical shell of mean radius r, thickness 2h and length L, as shown in Fig. 1. It is supposed that $h/r << 1$ so that the radial stress components are negligibly small, with the remaining longitudinal stress $\sigma(\phi, t)$ being constant through the thickness.

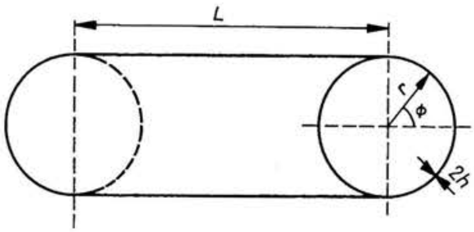

FIG. 1. Geometry of a straight pipe.

Let the pipe be loaded by a bending moment M which is kept fixed as the pipe creeps; the problem is to determine the resultant curvature change $\varkappa(t)$.

The *field equations* are those of a simple beam

$$\varepsilon = \varkappa r \sin\phi,$$

$$M = \int_0^{2\pi} (2hr^2)\sigma \sin\phi\, d\phi,$$

where ε is the longitudinal strain composed of an elastic and a viscous part

$$\varepsilon = \varepsilon_e + \varepsilon_v$$

related to stress through the constitutive equations

$$\varepsilon_e = \sigma/E, \quad \dot{\varepsilon}_v = B(t)\sigma^n,$$

E being Young's modulus, and $B(t)$, n material parameters associated with Norton's power law of creep [13].

With reference to Eq. (4.3) the *redistribution equations* can be written down (remembering that R is linear)

$$(6.1) \qquad \frac{d}{dt}\sigma = B(t)R[\sigma^n] + \dot{\sigma}_0, \quad \sigma(0) = \sigma_0(0)$$

as well as the equation of evolution for curvature

$$(6.2) \qquad \frac{d}{dt}\varkappa = B(t)r^{\mathrm{I}}[\sigma^n] + \dot{\varkappa}_0, \quad \varkappa(0) = \varkappa_0(0),$$

where σ_0 and $\varkappa_0$ are the stress and curvature from the *equivalent elastic problem*

$$\varepsilon_0 = \varkappa_0 r \sin\phi,$$

$$M = \int_0^{2\pi} (2hr^2)\sigma_0 \sin\phi \, d\phi,$$

$$\varepsilon_0 = \sigma_0/E,$$

with the solution

$$\sigma_0 = \frac{M}{2hr^2}\frac{\sin\phi}{\pi}, \qquad \varkappa_0 = \frac{M}{2hr^2}\frac{1}{Er\pi}.$$

For an arbitrary initial strain ε^*, $R(\varepsilon^*)$ and $r^{\mathrm{I}}(\varepsilon^*)$ are the stress and curvature from the *residual elastic problem for* ε^*

$$\varepsilon_R = \varkappa_R r \sin\phi,$$

$$0 = \int_0^{2\pi} (2hr^2)\sigma_R \sin\phi \, d\phi,$$

$$\varepsilon_R = \sigma_R/E + \varepsilon^*,$$

with solution

$$R(\varepsilon^*) = \sigma_R = E\left\{\frac{\sin\phi}{\pi}\int_0^{2\pi} \varepsilon^* \sin\phi \, d\phi - \varepsilon^*\right\},$$

$$r^{\mathrm{I}}(\varepsilon^*) = \varkappa_R = \frac{1}{\pi r}\int_0^{2\pi} \varepsilon^* \sin\phi \, d\phi.$$

On adopting the normalised time scale

$$\tau = E\sigma_d^{n-1}\int B(t)\,dt,$$

where $\sigma_d = M/2h\pi r^2$, Eqs. (6.1) and (6.2) become

$$\frac{dS}{d\tau} = \frac{\sin\phi}{\pi}\int_0^{2\pi} S^n \sin\phi \, d\phi - S^n,$$

(6.3)
$$\frac{dK}{d\tau} = \frac{1}{\pi}\int_0^{2\pi} S^n \sin\phi \, d\phi,$$

$$S(\phi, 0) = \sin\phi, \qquad K(0) = 1,$$

where

$$S(\phi, \tau) = \sigma(\phi, t)/\sigma_a, \qquad K(\tau) = \varkappa(t)/\varkappa_0$$

and noting that M, and therefore σ_0 and $\varkappa_0$ are constant in time. By replacing the integrals in Eqs. (6.3) by a finite sum using, for example, Simpson's quadrature rule, the continuous initial value problem (6.3) may be replaced by a finite system of first order ordinary differential equations in time which may be resolved numerically by means of, say, a Runge-Kutta method. For interest, results for K with creep index $n = 3.0$, 5.0 and 7.0 are given in Fig. 2; K is a measure of the increasing "flexibility" of the pipe with creep compared to its initial, elastic flexibility in bending.

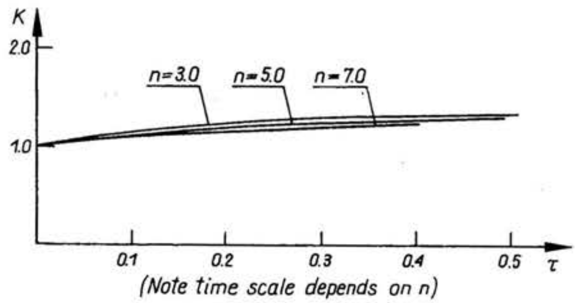

FIG. 2. Flexibility of a straight pipe in creep.

6.2. Nguyen's variational inequality of evolution

Variational formulations of problems in structural mechanics, encompassing in particular the Theorems of minimum total potential energy and complementary energy, have played an important role in describing global properties of the whole system rather than its component parts leading often to information on bounds to solution and providing a natural means of approximation [25]. However, there has been growing recognition that only a very restricted number of problems possess a variational formulation in the classical sense [26, 27] leading to a reexamination of the principles of the calculus of variations [28, 29]. In particular the quasi-static creep problem has no such formulation [8]. Nevertheless it has been argued that the essence of the variational technique lies in the theory of convex analysis and "variational inequalities" [30, 31]. A rough sketch of this theory is given in Appendix 2, but more information, together with many examples from mechanics, can be found in the book by G. DUVAUT and J. L. LIONS [31]. This theory has been applied to the so-called "steady" phase of material creep behaviour by B. NAYROLLES [32]. However, Q. S. NGUYEN [33, 34] has derived a variational inequality of evolution for a class of materials known as "standard, generalised" with a time dependent viscoplastic response described by an internal variable theory [34]. In this sense the quasi-static creep problem has a variational form, and it shall be demonstrated how it may be deduced from the rational approach.

The inner product between second order tensors is defined by

$$\langle \mathbf{\Phi}, \mathbf{\Psi} \rangle = \int_{\mathscr{V}} \mathbf{\Phi} \cdot \mathbf{\Psi} \, d\mathbf{X}.$$

The set of all possible tensor fields for which this can be defined forms an inner product space, denoted by $\mathscr{I}$: it is possible to refine this space to the Hilbert space $L_2(\mathscr{V})$ being that subset of $\mathscr{I}$ such that

$$||\mathbf{\Phi}|| < \infty, \quad \forall \mathbf{\Phi} \in L_2(\mathscr{V})$$

if the integral is in terms of the Lebesque integration.

It shall be assumed that the linear elastic operator $\mathbf{C}$ is symmetric and positive in $L_2(\mathscr{V})$ that is,

$$\langle \mathbf{C}(\mathbf{\Phi}), \mathbf{\Psi}\rangle = \langle \mathbf{\Phi}, \mathbf{C}(\mathbf{\Psi})\rangle,$$

$$\langle \mathbf{C}(\mathbf{\Phi}), \mathbf{\Phi}\rangle \geqslant 0, = 0 \quad \text{iff} \quad \mathbf{\Phi} = \mathbf{0}, \quad \mathbf{\Phi}, \mathbf{\Psi} \in L_2(\mathscr{V})$$

and furthermore that $\mathbf{C}^{-1}$ exists. With these assumptions it is possible to define another inner product on $\mathscr{I}$ given by

$$[\mathbf{\Phi}, \mathbf{\Psi}] = \langle \mathbf{C}(\mathbf{\Phi}), \mathbf{\Psi}\rangle$$

which can be called the "elastic energy inner product" since the associated norm

$$||\mathbf{\Phi}|| = \sqrt{[\mathbf{\Phi}, \mathbf{\Phi}]}$$

is twice the elastic energy of the body with stress pattern $\mathbf{\Phi}$.

Define the convex set $S(t)$ of statically admissible stresses

$$S(t) = \{\boldsymbol{\sigma}^s(t) \in L_2(\mathscr{V})/\operatorname{div}\boldsymbol{\sigma}^s(t) + \bar{\mathbf{f}}(t) = \mathbf{0} \quad \text{in} \quad \mathscr{V}, \ \boldsymbol{\sigma}^s(t)\mathbf{n} = \bar{\mathbf{s}}(t) \quad \text{on} \quad \mathscr{S}_s\}$$

and let $\psi_{s(t)}$ be the indicatrix of $S(t)$; on $S(t)$ define the elastic energy inner product as structure. Then, the subdifferential of $\psi_{s(t)}$ is the set

$$\partial\psi_{s(t)}(\boldsymbol{\sigma}) = \{\mathbf{\Phi} \in L_2(\mathscr{V})/[\mathbf{\Phi}, \boldsymbol{\sigma}^s - \boldsymbol{\sigma}] \leqslant 0, \quad \forall \boldsymbol{\sigma}^s \in S(t)\}.$$

Suppose that $\hat{\boldsymbol{\sigma}}$ is self equilibrating, i.e.

$$\operatorname{div}\hat{\boldsymbol{\sigma}} = \mathbf{0} \quad \text{in} \quad \mathscr{V},$$

$$\hat{\boldsymbol{\sigma}}\mathbf{n} = \mathbf{0} \quad \text{on} \quad \mathscr{S}_s$$

and that $\boldsymbol{\epsilon}^* \in L_2(\mathscr{V})$. Then, if $[\mathbf{u}_R, \boldsymbol{\epsilon}_R, \boldsymbol{\sigma}_R]$ is the solution of the residual elastic problem correponding to $\boldsymbol{\epsilon}^*$

$$[\mathbf{R}(\boldsymbol{\epsilon}^*), \hat{\boldsymbol{\sigma}}] = \langle \mathbf{C}[\mathbf{R}(\boldsymbol{\epsilon}^*)], \hat{\boldsymbol{\sigma}}\rangle = \langle \boldsymbol{\epsilon}_R - \boldsymbol{\epsilon}^*, \hat{\boldsymbol{\sigma}}\rangle = \langle \boldsymbol{\epsilon}_R, \hat{\boldsymbol{\sigma}}\rangle - \langle \boldsymbol{\epsilon}^*, \hat{\boldsymbol{\sigma}}\rangle.$$

But using Green's identity

$$\langle \boldsymbol{\epsilon}_R, \hat{\boldsymbol{\sigma}}\rangle = \int_{\mathscr{V}} \boldsymbol{\epsilon}_R \cdot \hat{\boldsymbol{\sigma}} d\mathbf{X} = \int_{\mathscr{V}} (\hat{\nabla}\mathbf{u}_R) \cdot \hat{\boldsymbol{\sigma}} d\mathbf{X} = -\int_{\mathscr{V}} \operatorname{div}\hat{\boldsymbol{\sigma}} \cdot \mathbf{u}_R d\mathbf{X} + \int_{\mathscr{S}} (\boldsymbol{\sigma}\mathbf{n}) \cdot \mathbf{u}_R d\mathbf{X} = 0$$

from the properties of the solution of the residual elastic problem.

Hence,

$$[\mathbf{R}(\boldsymbol{\epsilon}^*), \hat{\boldsymbol{\sigma}}] = -\langle \boldsymbol{\epsilon}^*, \hat{\boldsymbol{\sigma}}\rangle$$

and thus

$$[\mathbf{R}(\boldsymbol{\epsilon}^*) + \mathbf{C}^{-1}(\boldsymbol{\epsilon}^*), \hat{\boldsymbol{\sigma}}] = 0$$

and if $\boldsymbol{\sigma}, \boldsymbol{\sigma}^s \in S(t), \hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^s - \boldsymbol{\sigma}$

$$[\mathbf{R}(\boldsymbol{\epsilon}^*) + \mathbf{C}^{-1}(\boldsymbol{\epsilon}^*), \boldsymbol{\sigma}^s - \boldsymbol{\sigma}] = 0$$

so that, by definition,

$$\mathbf{R}(\boldsymbol{\epsilon}^*)+\mathbf{C}^{-1}(\boldsymbol{\epsilon}^*) \in \partial\psi_{s(t)}(\boldsymbol{\sigma}).$$

Indeed if $\boldsymbol{\sigma}$ is identified with the stress from the creep problem and $\boldsymbol{\epsilon}^*$ with $\dot{\boldsymbol{\epsilon}}_v$, then, from the redistribution Eq. (4.3)

$$\mathbf{R}(\dot{\boldsymbol{\epsilon}}_v)+\mathbf{C}^{-1}(\dot{\boldsymbol{\epsilon}}_v) = \dot{\boldsymbol{\sigma}}-\dot{\boldsymbol{\sigma}}_0+\mathbf{C}^{-1}[\mathbf{F}(\boldsymbol{\sigma}, \boldsymbol{\alpha}_j)]$$

and there results the *multivalued equation of evolution for creep*

$$\frac{d}{dt}\boldsymbol{\sigma}-\dot{\boldsymbol{\sigma}}_0+\mathbf{C}^{-1}[\mathbf{F}(\boldsymbol{\sigma}, \boldsymbol{\alpha}_j)] \in \partial\psi_{s(t)}(\boldsymbol{\sigma}),$$

$$\frac{d}{dt}\boldsymbol{\alpha}_i = \mathbf{G}_i(\boldsymbol{\sigma}, \boldsymbol{\alpha}_j),$$

which is equivalent to the *variational inequality of evolution for creep*

$$[\mathbf{C}^{-1}\mathbf{F}(\boldsymbol{\sigma}, \boldsymbol{\alpha}_j)+\dot{\boldsymbol{\sigma}}_0-\frac{d}{dt}\boldsymbol{\sigma}, \boldsymbol{\sigma}^*-\boldsymbol{\sigma}]+\psi(\boldsymbol{\sigma}^*)-\psi(\boldsymbol{\sigma}) \geqslant 0, \quad \forall\boldsymbol{\sigma}^* \in S(t),$$

$$\frac{d}{dt}\boldsymbol{\alpha}_i = \mathbf{G}_i(\boldsymbol{\sigma}, \boldsymbol{\alpha}_j).$$

Thus the approach of Q. S. NGUYEN [33, 34] in viscoplasticity is regained for creep.

7. Conclusions and comments

In this essay a rational approach to creep mechanics has been adopted, through the derivation of the equations of inelastic growth and stress redistribution, with the temporal behaviour admitted to a primary role rather than emphasising the inherent nonlinearity of the response. There has been a tendency in the past to believe that this aspect of creep is routine by applying any of the initial strain algorithms shunned to an extent due to excessive computational costs and the feeling that expensive sledge hammers were being employed in the destruction of rather ill-defined nuts. While the phase is certainly only transient, this view is possessed of an amount of truth in the solution of basic problems of engineering importance. But looking ahead, the future of creep research should lie, not in a progression from its current state, but rather in a search for a new basis. Perhaps the hypothesis of the black box, and the systematisation of this rational approach, shall indicate a means by which this basis could be achieved? Meanwhile, the main result of this rational approach has been to identify and combine several attractive ideas which have remained virtually unnoticed in the literature and illustrate the diversity of applica-ation of the end product.

The body of this essay has been concerned with the small deformation problem — although it should be a fairly straightforward matter to extend the results to large deformations. However, considerations of gross material deterioration such as rupture, fatigue and ratchetting unfortunately cannot be included because of lack of space. It is hoped that this position shall be rectified in the near future.

Appendix 1

Some properties of the residual operator

It is of interest to record here some attractive properties of the residual operator. As in Sect. 6.2, the inner product space $\mathscr{S}$ (and $L_2(\mathscr{V})$) are defined, and it is supposed that $\mathbf{C}$ is symmetric and positive definite.

It can be shown that $\mathbf{R}$ is *symmetric and negative* on $\mathscr{S}$ i.e.

$$\langle \mathbf{R}(\mathbf{\Phi}), \mathbf{\Psi} \rangle = \langle \mathbf{\Phi}, \mathbf{R}(\mathbf{\Psi}) \rangle,$$

$$\langle \mathbf{R}(\mathbf{\Phi}), \mathbf{\Phi} \rangle \leqslant 0$$

for all $\mathbf{\Phi}, \mathbf{\Psi} \in \mathscr{S}$ however, it should be noted that it is not necessarily true that

$$\langle \mathbf{R}(\mathbf{\Phi}), \mathbf{\Phi} \rangle = 0 \quad \text{if} \quad \mathbf{\Phi} = \mathbf{0},$$

that is, $\mathbf{R}$ is not negative definite.

Let $\mathbf{\Phi}, \mathbf{\Psi} \in \mathscr{S}$ and $(\mathbf{u}_\phi, \boldsymbol{\epsilon}_\phi, \boldsymbol{\sigma}_\phi)$ $(\mathbf{u}_\psi, \boldsymbol{\epsilon}_\psi, \boldsymbol{\sigma}_\psi)$ be solutions of corresponding residual elastic problems for $\mathbf{\Phi}$ and $\mathbf{\Psi}$, respectively.

Then,

$$\langle \mathbf{R}(\mathbf{\Phi}), \mathbf{\Psi} \rangle = \langle \boldsymbol{\sigma}_\phi, \mathbf{\Psi} \rangle = \langle \boldsymbol{\sigma}_\phi, \boldsymbol{\epsilon}_\psi - \mathbf{C}(\boldsymbol{\sigma}_\psi) \rangle = \langle \boldsymbol{\sigma}_\phi, \boldsymbol{\epsilon}_\psi \rangle - \langle \boldsymbol{\sigma}_\phi, \mathbf{C}(\boldsymbol{\sigma}_\psi) \rangle.$$

But $\langle \boldsymbol{\sigma}_\phi, \boldsymbol{\epsilon}_\psi \rangle = 0$ on application of Green's identity (virtual work). Hence,

$$\langle \mathbf{R}(\mathbf{\Phi}), \mathbf{\Psi} \rangle = -\langle \boldsymbol{\sigma}_\phi, \mathbf{C}(\boldsymbol{\sigma}_\psi) \rangle.$$

Similarly,

$$\langle \mathbf{R}(\mathbf{\Psi}), \mathbf{\Phi} \rangle = -\langle \boldsymbol{\sigma}_\psi, C(\boldsymbol{\sigma}_\phi) \rangle.$$

Hence, since $\mathbf{C}$ is symmetric it follows that $\mathbf{R}$ is symmetric; and since $\mathbf{C}$ is positive, $\mathbf{R}$ is negative.

Appendix 2

Variational inequalities

Consider the operator equation

$$\text{(i)} \qquad A(U) = F,$$

where U is a set of unknowns such that the set of all possible U forms an *inner product space* $\mathscr{S}$ i.e. there exists a real valued, bilinear function $\langle .,. \rangle$ with the properties

$$\langle U_1, U_2 \rangle = \langle U_2, U_1 \rangle,$$

$$\langle U, U_, \rangle \geqslant 0, = 0 \quad \text{iff} \quad U = 0,$$

$$\langle U_1, \mu U_2 + \lambda U_3 \rangle = \mu \langle U_1, U_2 \rangle + \lambda \langle U_1, U_3 \rangle,$$

where $U_1, U_2, U_3 \in \mathscr{S}$. The operator A, which can be differential, integral, algebraic and so on, transforms members of $\mathscr{S}$ into another member of $\mathscr{S}$; it is therefore supposed that $F \in \mathscr{S}$. The classical aim of the calculus of variations [27] is to determine if (i) has a variational formulation; this is possible if there exists a real valued function $W(U)$, (mapping $\mathscr{I}$ onto the real line — usually called a functional) whose *gradient*, grad $W(U)$, given by the Gateaux derivative

$$\lim_{s\to 0}\frac{W(U+sH)-W(U)}{s} = \langle \operatorname{grad} W(U), H\rangle, \quad \forall H\in\mathscr{I}$$

exists and is such that

(ii) $$\operatorname{grad} W(U) = A(U)-F.$$

It can be shown that if W is also *convex*, i.e.

$$W(U)-W(V)-\langle \operatorname{grad} W(U), U-V\rangle \geqslant 0, \quad \forall U, V\in\mathscr{I}$$

then it is minimised by the solution of (i) and may in consequence be bounded, i.e.

(iii) $$W(U)\leqslant W(U^*),$$

where U minimizes W (and thus solves (i)) and U^* is any other member of $\mathscr{I}$.

Often this bounding can only be realised if U, $U^*\in K$ is a *convex subset* of $\mathscr{I}$ i.e.

$$\forall U, V\in K\subseteq\mathscr{I},\ \lambda\in[0,1],\quad \lambda U+(1-\lambda)V\in K$$

so (iii) is equivalent to the condition

$$\langle \operatorname{grad} W(U), U^*-U\rangle \geqslant 0, \quad \forall U^*\in K$$

or, using (ii)

(iv) $$\langle A(U)-F, U^*-U\rangle \geqslant 0, \quad \forall U^*\in K$$

which is called a *variational inequality* [31].

Thus it has been suggested [28, 29] that instead of searching for a variational formulation, which may not even exist, it would be better to study the inequality (iv) which can be postulated for a much wider range of problems.

If the *indicatrix* of a convex set K

$$\psi_K(U) = \begin{cases} +\infty & U\notin K,\\ 0 & U\in K\end{cases}$$

is introduced [30], then (iv) is identical to

(v) $$\langle A(U)-F, U^*-U\rangle+\psi_K(U^*)-\psi_K(U)\geqslant 0, \quad \forall U^*\in K$$

which may be further generalised on letting ψ be a *proper convex function* [30] (i.e. convex, semi-continuous from below and not identically equal to $+\infty$) so that (v) becomes

(vi) $$\langle A(U)-F, U^*-U\rangle+\psi(U^*)-\psi(U)\geqslant 0, \quad \forall U^*\in K.$$

The problem is to find a U such that (vi) is satisfied for all $U^*\in K$. It is possible to define the subdifferential $\partial\psi$ of ψ as the set [30]

$$\partial\psi(U) = \{\Psi/\psi(U^*)-\psi(U)-\langle\Psi, U^*-U\rangle\geqslant 0\}$$

and then (vi) is equivalent to

(vii) $$-(A(U)-F)\in\partial\psi(U),$$

which is an equation involving *many-valued* functions.

Acknowledgment

The author would like to acknowledge support from the Science Research Council during the course of these investigations.

References

1. A. A. ILYUSHIN, *Some problems in the theory of plastic deformations*, PMM, **7**, 245–272, 1943.
2. H. PORITSKY, F. A. FEND, *Relief of thermal stress through creep*, Trans. ASME **80**, J. Appl. Mech., 589–597, 1958.
3. A. MENDELSON, M. H. HIRSCHBERG, S. S. MANSON, *A general approach to the practical solution of creep problems*, Trans. ASME, **81**, J. Basic Engin., 589–589, 1959.
4. P. S. KURATOV, V. I. ROZENBLIUM, *On the integration of the equations of unsteady creep of solid bodies*, PMM, **24**, 146–148, 1960.
5. W. C. CARPENTER, *Viscoelastic stress analysis*, Internat. J. Numerical Methods in Engin., **4**, 357–366, 1972.
6. Z. P. BAZANT, *Matrix differential equations and higher order numerical methods for problems of nonlinear creep viscoelasticity and elasto-plasticity*, Int. J. Numerical Methods in Engin., **4**, 11–15, 1972.
7. O. C. ZIENKIEWICZ, I. C. CORMEAU, *Viscoplasticity and plasticity. An alternative for finite element solution of material nonlinearities*, Proc. Int. Symposium "Computer Methods in Applied Science and Engineering", Versailles, 1973. Lecture notes in Computer Science No. 10 Springer, 1974.
8. J. T. BOYLE, *Rational creep mechanics*, Doctoral dissertation, University of Strathclyde, 1975.
9. J. T. BOYLE, J. SPENCE, *A comparison of approximate methods for the creep relaxation of a curved pipe*, Int. J. Pressure Vessels and Piping (in press).
10. J. T. BOYLE, J. SPENCE, *The flexibility of curved pipes in creep* (to be published).
11. I. BABUSKA, I. HLAVACEK, *On the existence and uniqueness of solution in the theory of viscoelasticity*, Arch. of Mech., **18**, 47–84, 1966.
12. W. J. GOODEY, *Creep deflection and stress distribution in a beam*, Aircraft Engin., **30**, 170–171, 1958.
13. R. K. PENNY, D. J. MARRIOTT, *Design for creep*, McGraw-Hill, 1971.
14. B. EINARSSON, *Numerical treatment of integro-differential equations with a certain maximum property*, Numerishe Mathematik, **18**, 267–288, 1971.
15. A. C. ERINGEN, *Nonlinear theory of continuous media*, McGraw-Hill, 1962.
16. F. A. LECKIE, *Black box, black magic*, Lecture presented at the University of Glasgow, May 1975.
17. E. T. ONAT, *Description of the mechanical behaviour of inelastic solids*, Proc. 5th U.S. National Congress on "Applied Mechanics" Minneapolis, 1966.
18. E. T. ONAT, F. F. FARDSHISHEH, *Representation of creep of metals*, ORNL REP No. 4783, 1972.
19. F. A. LECKIE, A. R. S. PONTER, *On the state variable description of creeping materials*, Ingenieur Archiv, **43**, 158–167, 1974.
20. J. J. WILLIAMS, F. A. LECKIE, *A method for quantifying creep strain due to cyclic stress*, Trans ASME, J. Applied Mech., **41**, 953–958, 1975.
21. M. J. LEITMAN, G. M. C. FISHER, *The linear theory of viscoelasticity*, Handbuch der Physic, Vol. VIa/3 Mechanics of Solids III, Ed. S. Flugge, Springer 1975.
22. A. R. S. PONTER, *On the stress analysis of creeping structures subjected to variable loading*, Trans. ASME, J. Applied Mech., **40**, 589–594, 1973.
23. J. T. BOYLE, *The behaviour of creeping structures at large times* (Submitted for publication).
24. J. T. BOYLE, *Stress bounds for creeping structures* (To be published).
25. J. T. ODEN, *Finite elements of nonlinear continua*, McGraw-Hill, 1972.
26. M. M. VAINBERG, *Variational methods and the method of monotone operators in the theory of nonlinear equations*, Wiley, 1974.
27. E. TONTI, *The inverse problem of the calculus of variations*, Academic Press, 1975.

28. J. L. LIONS, *Quelques methods de resolution des problemes aux limites non lineaires*, Dunod 1969.
29. F. E. BROWDER, *Problemes nonlineaires*, University of Montreal Press, 1966.
30. R. T. ROCKEFELLER, *Convex analysis*, Princeton University Press, 1970.
31. G. DUVAUT, J. L. LIONS, *Les inequations en mecanique et en physique*, Dunod 1972.
32. B. NAYROLLES, *Quelques applications variationelles de la theorie des fonctions duales a la mecanique des solides*, J. Mecanique, **10**, 263–289, 1971.
33. Q. S. NGUYEN, *Materieu elastoplastique ecroussable. Distribution de la contrainte dans un evolution quasi-static*, Arch. Mech., **25**, 695–702, 1973.
34. B. HALPHEN, Q. S. NGUYEN, *Sur les materiaux standars generalises*, J. de Mecanique, **14**, 39–63, 1975.

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Received December 17, 1975.