Archives of Mechanics • Archiwum Mechaniki Stosowanej • 29, 2, pp. 259-272, Warszawa 1977

Homogeneous continuum as a model of layered body

M. GOLUBIEWSKI (ŁÓDŹ)

THE PAPER contains a proposition of constructions homogeneous, continuous models for layered. materials. The layers are assumed to have constant thicknesses and consist of homogeneous, elastic materials of arbitrary anisotropy. The theory proposed proves to be more effective than the theory of effective stiffness known from the literature.

W pracy zaproponowano metodę konstruowania jednorodnych i ciągłych modeli dla ciał o strukturze warstwowej. Przyjęto, że warstwy te mają stałą grubość i składają się z materiałów jednorodnych i sprężystych o dowolnej anizotropii. Przedstawiona teoria okazuje się efektywniejsza od znanej z literatury teorii efektywnych sztywności.

В работе предложен метод построения однородных и сплошных моделей для тел со слоистой структурой. Принимается, что эти слои имеют постоянную толщину и состоят из однородных и упругих материалов с произвольной анизотропией. Представленная теория оказывается более эффективной чем известные из литературы теории эффективных жесткостей.

1. Introduction

THE PAPER puts forward a proposition of constructing homogeneous and continuous models of consecutive orders for a certain class of non-homogeneous bodies. Our considerations will be confined to bodies consisting of two different homogeneous materials which are linearly elastic and exhibit rectilinear anisotropy of the arbitrary type. Moreover, the layers which are made of two different materials have constant thicknesses and alternate with each other in the body, their interfaces ensuring perfect contact between the neighbouring layers. Thus the bodies have periodic structure, the period being equal to the sum of the two neighbouring layers. They will be called layered bodies or simply composites. It is well known that exact solutions of boundary value problems for such non-homogeneous bodies are, in the majority of cases, extremely difficult to obtain, the volume of computations increasing with the number of layers in the structure considered. Thus, precisely in the case of a densely layered medium it is of extremely importance to treat it as a certain homogeneous and continuous medium, the properties of which depend upon the properties of its ingredients and its internal geometric structure. Each medium of that kind capable of representing the actual layered body will be called a model of that body. Its practical value obviously depends on the simplicity of the equations governing its behaviour and on its ability to reflect all the details of processes occurring in actual composites subjected to loadings.

Until quite recently, the model which was used in the case of composities made of isotropic layers was based on the so-called "effective moduli" theory and represented a transversally isotropic medium known from classical elasticity; its effective elasticity constants were determined, for example, by J. V. RIZNIČENKO [1] (1949), G. W. POSTMA [2] (1955), S. M. RYTOV [3] (1956) and E. BEHRENS [4] (1967). The aim of these papers was to solve certain geophysical problems connected with the propagation of waves in layered media.

The presently observed increasing interest in layered media is due to advances in the industry of composite materials; their application to engineering structures considerably increases the demand for an effective tool in solving numerous boundary value problems.

The model proposed by the theory of effective moduli often leads to satisfactory results. However, it proves to be insufficient in some dynamical problems. It was shown by S. M. RYTOV [3], J. D. ACHENBACH, C. T. SUN and G. HERRMANN [5, 7] that layered media of this type exhibit considerable dispersion properties in the process of monochromatic wave propagation in unbounded media, even in the case of relatively long waves. Since dispersion does not occur in classical anizotropic bodies, the theory of effective moduli yields a long-wave approximation of the layered medium and may lead to considerable errors in cases where the wavelengths are comparable with the periods of internal structure of the composites.

The first authors who proposed a dispersive model of composites were C. T. SUN, J. D. ACHENBACH and G. HERRMANN; in papers [6–8], written in the period 1967–68, they presented the so-called theory of effective stiffness. Equations of the first approximation of that theory may be written [8] in terms of the notions introduced by R. D. MINDLIN [9] in connection with the bodies with microstructure. Dynamic interactions of individual layers were taken into account by assuming that at the interfaces only the field of displacements is continuous. D. S. DRUMHELLER and A. BEDFORD fulfilled the additional condition of continuity of the stress vector at those surfaces and, in 1973–74, presented the generalized effective stiffness theory [10, 11].

Other attempts to construct dispersive models of these types of bodies are due to A. BEDFORD and M. STERN [12, 13] (1972), G. A. HEGAMIER and his co-workers [14–17] 1973–74 and M. BEN-AMOZ [18, 19] (1975).

The models proposed by the both versions of the effective stiffness theory seem to be most useful from the point of view of applications, because of their simplicity. In a considerable number of papers [8, 20–25] one of the two versions mentioned above has been applied.

The method of constructing the models of consecutive orders has some elements in common with the effective stiffness theories but seems to show a certain advantage in the process of obtaining consecutive approximations. This advantage follows from our belief that such a process should:

1) lead to independent fundamental fields in all approximations;

2) be the same for all approximations;

3) ensure, in every approximation, the compatibility of displacements and stresses at the interfaces between the layers;

4) use a similar method of smoothing out the compatibility conditions at the interfaces and also the Lagrange function density.

In the papers dealing with the both versions of the effective stiffness theory there appear several remarks on the construction of models of arbitrary orders, but it may easily be verified that the processes proposed there do not satisfy the postulates 1)–4) listed above. In particular, the model proposed by A. Bedford and D. S. Drumheller makes impossible the existence of a "free" displacement field in the zero and first-order approximations, and hence the theory does not fulfil the first three postulates. In the process used by C. T. Sun, J. D. Achenbach and G. Herrmann, continuity of the stress vector field is not ensured in any of the approximations, and this contradicts the third postulate. In both versions of the effective stiffness theory the method of smoothing out the compatibility conditions differs from that used for the Lagrange functions. Smoothing of the compatibility conditions is performed according to the scheme

$$F^{bc}(t, x_\alpha, x_3^{bc}) = 0 \quad \Rightarrow F^{bc}(t, x_i) = 0, \tag{1.1}$$

where x_3^{bc} denote the coordinates of the points at the surfaces of contact. The corresponding process with respect to the Lagrange function density has, however, the form

$$l_1(t, x_\alpha, x_3^b)+l_2(t, x_\alpha, x_3^c) \Rightarrow l_1(t, x_i)+l_2(t, x_i), \tag{1.2}$$

x_3^b and x_3^c are the coordinates of the points belonging to the middle surfaces of the homogeneous layers.

In this paper the "smoothing" process follows the first scheme in both cases and this is what seems to be right from the formal point of view. Hence Eq. (1.2) is replaced with

$$l_1^*(t, x_\alpha, x_3^{bc})+l_2^*(t, x_\alpha, x_3^{bc}) \Rightarrow l_1^*(t, x_i)+l_2^*(t, x_i). \tag{1.3}$$

This uniformity of the smoothing process is achieved by representing the all fields defined in the composite in terms of their distributions on the so-called "bilayers" which are the non-homogeneous neighbourhoods of the planes of contact, and contain both materials in the proportions characteristic of the whole composite. In this approach the discrete structure is represented in the continuous medium merely by the surfaces of contact. The existence of independent fundamental fields in the zero and first approximations (under the conditions of compatibility of the displacement and stress vectors at the interfaces), is achieved by a proper definition of the strain tensors in the bilayers for models of consecutive orders. The zero order model obtained in this manner is a rigid body model, while the first order model behaves like a classical elastic anisotropic body (in statical conditions); in the process of propagation of monochromatic waves in an infinite medium, dispersion appears. In the case of waves propagating perpendicularly to the layers, the results are more accurate (especially for small wavelengths) than those resulting from the first order model used by Sun, Achenbach and Herrmann and characterized by twice as many independent fundamental fields.

2. Structure of the layered body

Let us denote by the symbols f (fiber) and m (matrix), respectively, the homogeneous material treated as reinforcement of the composite, and another homogeneous material considered as the matrix. Symbols a, b, c will also be used to denote those materials, symbol b denoting in such cases a material different from c, i.e.

$$a, b, c = f, m \quad \text{and} \quad b \neq c. \tag{2.1}$$

The voluminal share of material a in the composite is denoted by η_a, hence it follows that

$$(2.2) \qquad \eta_f + \eta_m = 1.$$

The period of the composite is denoted by the symbol Δ, and hence the layer made of material a has the thickness $\eta_a \Delta$.

The unbounded composite medium will be described in the orthogonal Cartesian coordinate system $\{x_i\} = \{x_\alpha\} \times \{x_3\}$, $i = 1, 2, 3$; $\alpha = 1, 2$, the x_3-axis being perpendicular to the planes of contact between the layers.

The composite stucture may uniquely be determined by prescribing the lattice and the base, the lattice constituting an arbitrary set of planes at distances Δ from each other and parallel to the contact planes, and the base representing an arbitrary portion of the composite bounded by two planes normal to the x_3-axis at the distance Δ from each other. Since Δ is the period of the composite, the voluminal shares of materials f and m in an arbitrary base are the same as in the entire composite. The composite structure is reconstructed from its lattice and a base in a procedure similar to that used in the case of crystalline structures (cf. [26]).

By means of the periodic structure of the composite we may introduce the equivalence relations in the set of coordinates $\{x_3\}$ in the set of planes $\{\pi\}$ parallel to the lattice planes, and in the set $\{P\}$ which is a set of closed, three-dimensional intervals with a characteristic plane π, bounded by the planes parallel to the lattice planes. The following definitions of the sets determine the correspondence between the elements of the sets $\{x_3\}$, $\{\pi\}$, $\{P\}$:

$$(2.3) \qquad \forall x_3' \in \{x_3\} \exists \pi' \in \{\pi\} : \pi' = \{(x_\alpha, x_3) \in \{x_i\} : (x_\alpha) \in \{x_\alpha\}, \quad x_3 = x_3'\};$$

$$(2.4) \qquad \forall x_3' \in \{x_3\}, d \geqslant 0, g \geqslant 0 \exists P_{dg}' \in \{P\} : P_{dg}' = \{(x_i) \in \{x_i\} : x_\alpha \in \{x_\alpha\}, \\ x_3' - d \leqslant x_3 \leqslant x_3' + g\}.$$

The equivalence relation within the set $\{x_3\}$ is introduced as follows:

$$(2.5) \qquad x_3 \sim x_3' \Leftrightarrow |x_3 - x_3'| = k\Delta, \quad k = 0, 1, 2, \ldots$$

while the corresponding relations in the sets $\{\pi\}$ and $\{P\}$ are defined by

$$(2.6) \qquad \begin{aligned} &\pi \sim \pi' \Leftrightarrow x_3 \sim x_3', \\ &P_{dg} \sim P_{d'g'}' \Leftrightarrow x_3 \sim x_3', \quad d = d', \quad g = g'. \end{aligned}$$

Here the equivalence relations in the set $\{x_3\}$ and Eqs. (2.3), (2.4) have been used.

The equivalence classes of the above relations corresponding to the representants x_3', π', P' will be denoted by symbols $\tilde{x}_3'$, $\tilde{\pi}'$, $\tilde{P}'$.

Let us now introduce the names for certain coordinates from the set $\{x_3\}$, characteristic for the composite.

x_3^a — coordinates determining the center of gravity of an arbitrarily selected layer made of material a;

$\dot{x}_3^{bc}$ — coordinates determining such a point of contact between two arbitrarily selected, neighbouring layers made of materials b and c, that $x_3^{bc} - 0$ denotes a point located within the material b, and $x_3^{bc} + 0$ — a point located within the material c.

Taking these definitions into account and using Eqs. (2.3) and (2.6) we obtain the symbol π^a for the middle plane of the layer a, and the symbol π^{bc} for the plane of contact bc.

On using Eqs. (2.4) and (2.7), let us introduce the following names for certain intervals from the set $\{P\}$: B^a — base a, B^{bc} — base bc, P^a — layer a, P^{bc} — bilayer bc.

The intervals are defined by the formulae

$$(2.7)\qquad B^a \overset{\text{def}}{=} P'_{dg}: x'_3 \sim x^a_3, \quad d = g = \frac{1}{2}\Delta,$$

$$(2.8)\qquad B^{bc} \overset{\text{def}}{=} P'_{dg}: x'_3 \sim x^{bc}_3, \quad d = \eta_b \Delta, \quad g = \eta_c \Delta,$$

$$(2.9)\qquad P^a \overset{\text{def}}{=} P'_{dg}: x'_3 \sim x^a_3, \quad d = g = \frac{1}{2}\eta_a \Delta,$$

$$(2.10)\qquad P^{bc} \overset{\text{def}}{=} P'_{dg}: x'_3 \sim x^{bc}_3, \quad d = \frac{1}{2}\eta_b \Delta, \quad g = \frac{1}{2}\eta_c \Delta.$$

If $h(P'_{dg}) = d+g$ denotes the thickness of the interval P'_{dg}, then $h(B^a) = h(B^{bc}) = \Delta$, $h(P^a) = \eta_a \Delta$ and $h(P^{bc}) = \Delta/2$.

Let $S(\mathbf{A})$ denote the generalized sum of sets of the family $\mathbf{A}$, i.e.

$$(2.11)\qquad x \in S(\mathbf{A}) \Leftrightarrow \exists X : x \in X, \quad X \in \mathbf{A}.$$

The obvious identities hold true,

$$(2.12)\qquad S(\tilde{B}^a) \equiv \{x_i\},$$

$$(2.13)\qquad S(\tilde{B}^{bc}) \equiv \{x_i\},$$

$$(2.14)\qquad S(\tilde{P}^f \cup \tilde{P}^m) \equiv \{x_i\},$$

$$(2.15)\qquad S(\tilde{P}^{fm} \cup \tilde{P}^{mf}) \equiv \{x_i\},$$

which indicate that the composite covers may consecutively be: the class of bases a, class of bases bc, two classes of layers f and m, and two classes of bilayers fm and mf.

The base intervals B^a or B^{bc} are usually assumed to be the basical cells in constructing the homogeneous models. They are not, however, the smallest possible intervals capable of covering the composite and in which the voluminal share of materials f and m is the same as in the entire composite. The smallest intervals are represented by the bilayers P^{fm} and P^{mf}.

In this paper the composite will be treated according to the notation (2.15), not only because of the property of the bilayers mentioned above but also in view of the fact that they do not require coupling (fitting). Moreover, they contain the contact planes at which the phenomena of wave reflection and refraction occur and this leads to dispersion in the process of propagation of waves in infinite composite medium. All the fields determined in the composite will be represented on bilayers by the values of certain functions and the values of their derivatives calculated at the contact planes bc. The functions will be called model fields.

3. Local coordinate systems, change of variables

In order to obtain a uniform notation of the fields determined in mutually equivalent composite intervals, let us introduce local coordinate systems for each interval belonging to the classes $\tilde{P}^a$, $\tilde{B}^{bc}$ or $\tilde{P}^{bc}$.

Denoting by $\{x_i\}|_{P_{dg}}$ the intersection of the coordinate, set $\{x_i\}$ by the interval considered P_{dg}, the set of global coordinates parametrizing each representant of the interval classes mentioned above is transformed into the set of local coordinates:

$$(3.1) \qquad \varphi^a: \{x_i\}_{|P^a} \rightarrow \{(x_\alpha, z_a)\} : (x_\alpha, z_a) = (x_\alpha, x_3 - x_3^a),$$

$$(3.2) \qquad \varphi^{bc}: \{x_i\}_{|B^{bc}} \rightarrow \{(x_\alpha, z_{bc})\} : (x_\alpha, z_{bc}) = (x_\alpha, x_3 - x_3^{bc}).$$

With respect to the representants P^b, P^c satisfying the condition

$$(3.3) \qquad P^b \cap P^c = \pi^{bc}$$

the following equality holds true

$$(3.4) \qquad P^b \cup P^c = B^{bc}.$$

In such a case it is possible to pass from the local coordinate systems determined in the layers b and c to the local system determined in the base bc by means of the following transformations

$$(3.5) \qquad \psi^{b(bc)}: \{(x_\alpha, z_b)\} \rightarrow \{(x_\alpha, z_{bc})\}_{|P^b} : (x_\alpha, z_{bc}) = \left(x_\alpha, z_b - \frac{1}{2}\eta_b \Delta\right),$$

$$(3.6) \qquad \psi^{c(bc)}: \{(x_\alpha, z_c)\} \rightarrow \{(x_\alpha, z_{bc})\}_{|P^c} : (x^\alpha, z_{bc}) = \left(x_\alpha, z_c + \frac{1}{2}\eta_c \Delta\right).$$

The mutual relations between the introduced coordinate set transformations may clearly be illustrated by the corresponding diagrams. For those pairs of P^b and P^c which satisfy the corresponding condition (3.3) the following diagrams are commutative:

$$(3.7) \qquad \begin{array}{ccc} \{(x_\alpha, z_b)\} & \xleftarrow{\varphi^b} & \{x_i\}_{|P^b} \\ & {\scriptstyle \psi^{b(bc)}} \searrow & \downarrow {\scriptstyle \varphi^{bc}_{|P^b}} \\ & & \{(x_\alpha, z_{bc})\}_{|P^b} \end{array} \qquad \begin{array}{ccc} \{x_i\}_{|P^c} & \xrightarrow{\varphi^c} & \{(x_\alpha, z_c)\} \\ {\scriptstyle \varphi^{bc}_{|P^c}} \downarrow & \swarrow {\scriptstyle \psi^{c(bc)}} & \\ \{(x_\alpha, z_{bc})\}_{|P^c} & & \end{array},$$

what means that

$$(3.8) \qquad \psi^{b(bc)} \circ \varphi^b = \varphi^{bc}_{|P^b},$$

$$(3.9) \qquad \psi^{c(bc)} \circ \varphi^c = \varphi^{bc}_{|P^c}.$$

The symbol $\varphi^{bc}|_{P^a}$ denotes the restriction of the transformation φ^{bc} to the layer P^a.

Let us finally introduce in layers P^a the local coordinate systems $\{(x_\alpha, \zeta_a)\}$, one axis of which is referred to dimensionless coordinates,

$$(3.10) \qquad \bar{\varphi}^a: \{(x_\alpha, z_a)\} \rightarrow \{(x_\alpha, \zeta_a)\} : (x_\alpha, \zeta_a) = \left(x_\alpha, \left(\frac{1}{2}\eta_a \Delta\right)^{-1} z_a\right).$$

All the transformations defined above are bijections and thus the inverse transformations must exist.

4. Representation of arbitrary fields in bilayers by means of smooth model fields

The set of coordinate systems introduced here allows for a uniform representation in equivalent intervals of the fields sought for and defined in the entire composite at any time t.

Let $Q(t, x_i)$ denote an arbitrary scalar-, vector- or tensor-valued field, determined and continuous for

$$(x_i) = (x_i^0) \in \{x_i\} \setminus S(\tilde{\pi}^{fm} \cup \tilde{\pi}^{mf}),$$

with bounded left- and right-hand derivatives

$$\frac{\partial}{\partial x_3} Q(t, x_\alpha, x_3^{bc}-0), \qquad \frac{\partial}{\partial x_3} Q(t, x_\alpha, x_3^{bc}+0).$$

Let us denote by $\bar{\bar{Q}}^{a}_{x_3^a}(t, x_\alpha, \zeta_a)$ the intersection of the field $Q(t, x_i)$ with an isolated layer a. Changing the variables according to Eqs. (3.1), (3.10)

$$\bar{\varphi}^a \circ \varphi^a : \{x_i\}_{|P^a} \rightarrow \{(x_\alpha, \zeta_a)\}, \tag{4.1}$$

we obtain

$$\bar{\bar{Q}}^{a}_{x_3^a}(t, x_\alpha, \zeta_a) = Q(t, x_i)_{|P^a}. \tag{4.2}$$

The index x_3^a replaced by the value of (x_3) at the center of gravity of the layer a uniquely determines the position of that layer in the composite.

Let us now separate the variables in $\bar{\bar{Q}}^{a}_{x_3^a}$ by expanding it with respect to ζ_a into an infinite series of any of the classical orthogonal polynomials defined in the standard interval $[-1, 1]$. In this paper Legendre polynomials will be used. In the representation shown below the right-hand side series uniformly converges to the left-hand function at $\zeta_a \in [-1+\delta, 1-\delta]$, δ being an arbitrarily small positive real number [27]. The Fourier-Legendre coefficients are calculated from the formula

$$\underset{j}{\bar{\bar{Q}}}{}^{a}_{x_3^a}(t, x_\alpha) = \frac{2j+1}{2} \int_{-1}^{1} \bar{\bar{Q}}^{a}_{x_3^a}(t, x_\alpha, \zeta_a) \underset{j}{P}(\zeta_a) d\zeta_a. \tag{4.3}$$

The representation has the form

$$\bar{\bar{Q}}^{a}_{x_3^a}(t, x_\alpha, \zeta_a) = \lim_{n\rightarrow\infty} \sum_{j=0}^{n} \underset{j}{\bar{\bar{Q}}}{}^{a}_{x_3^a}(t, x_\alpha) \underset{j}{P}(\zeta_a), \tag{4.4}$$

where $\underset{j}{P}$ is the Legendre polynomial of order j, the following representation of those polynomials being used in this paper (cf. [28])

$$\underset{j}{P}(\zeta_a) = \sum_{k=0}^{[\frac{1}{2}j]} \underset{k}{a_j} \zeta_a^{j-2k}, \tag{4.5}$$

Here $[\nu]$ denotes the integral part of ν, and

(4.6) $$\underset{k}{a_j} = (-1)^k \frac{(2j-2k)!}{2^j k!(j-k)!(j-2k)!}.$$

Changing the variables in Eq. (4.4) by means of the transform inverse to Eq. (3.1), we obtain

(4.7) $$\bar{Q}^a_{x_3^a}(t, x_\alpha, z_a) = \bar{\bar{Q}}^a_{x_3^a}(t, x_\alpha, \zeta_a),$$

where

(4.8) $$\bar{Q}^a_{x_3^a}(t, x_\alpha, z_a) = \lim_{n\to\infty} \sum_{j=0}^{n} \underset{j}{\bar{Q}^a_{x_3^a}}(t, x_\alpha) \underset{j}{W}(z_a),$$

and

(4.9) $$\underset{j}{W}(z_a) \overset{\text{def}}{=\!=} \left(\frac{1}{2}\eta_a \Delta\right)^j \underset{j}{P}(\zeta_a),$$

(4.10) $$\underset{j}{\bar{Q}^a_{x_3^a}}(t, x_\alpha) \overset{\text{def}}{=\!=} \left(\frac{1}{2}\eta_a \Delta\right)^{-j} \underset{j}{\bar{\bar{Q}}^a_{x_3^a}}(t, x_\alpha).$$

We shall use later the following representation of the field $Q(t, x_i)$ in the layer a, obtained by means of the identical transformation of the right-hand side of Eq. (4.8):

(4.11) $$\bar{Q}^a_{x_3^a}(t, x_\alpha, z_a) = \lim_{n\to\infty} \sum_{j=0}^{n} \underset{j}{\overset{n}{\bar{Q}^a_{x_3^a}}}(t, x_\alpha) z_a^j,$$

where, in accordance with Eqs. (4.9), (4.5), we have

(4.12) $$\underset{j}{\overset{n}{\bar{Q}^a_{x_3^a}}}(t, x_\alpha) \overset{\text{def}}{=\!=} \sum_{k=0}^{\left[\frac{1}{2}(n-j)\right]} \underset{j+2k}{\bar{Q}^a_{x_3^a}}(t, x_\alpha) \underset{k}{a_{j+2k}} \left(\frac{1}{2}\eta_a \Delta\right)^{2k},$$

and

(4.13) $$\underset{j}{\overset{n}{\bar{Q}^a_{x_3^a}}}(t, x_\alpha) \equiv 0 \quad \text{for} \quad j > n.$$

Let us now introduce the infinite sequence of fields

(4.14) $$\underset{j}{Q^a}(t, x_i), \quad j = 0, 1, 2, \ldots, \infty$$

defined and smooth for $(x_i) = (x_i^0) \in \{x_i\}$; they will be called model fields of the function $Q(t, x_i)$. Those fields play a fundamental role in passing from the non-homogeneous body to its homogeneous and continuous model. The following conditions are imposed on those model fields

(4.15) $$\underset{j}{Q^a}(t, x_\alpha, x_3^a) \equiv \underset{j}{\bar{Q}^a_{x_3^a}}(t, x_\alpha).$$

Using these identities we may rewrite the Eq. (4.11) in the form

(4.16) $$\bar{Q}^a_{x_3^a}(t, x_\alpha, z_a) = \lim_{n\to\infty} \sum_{j=0}^{n} \underset{j}{\overset{n}{Q^a}}(t, x_\alpha, x_3^a) z_a^j,$$

Here

$$(4.17)\qquad \overset{n}{\underset{j}{Q}}{}^a(t, x_\alpha, x_3^a) = \sum_{k=0}^{\left[\frac{1}{2}(n-j)\right]} \underset{j+2k}{Q^a}(t, x_\alpha, x_3^a)\underset{k}{a_{j+2k}}\left(\frac{1}{2}\eta_a\Delta\right)^{2k}.$$

Thus the field $\underset{j}{Q}(t, x_i)|_{P^a}$ is represented by the values of the model fields $\underset{j}{Q^a}(t, x_i)$ which they assume at the points lying at the middle plane of the layer a. In the considerations that follow an important role is played by the Taylor identity which holds in an arbitrary interval $P \supset P^a$

$$(4.18)\qquad \underset{j}{Q^a}(t, x_\alpha, x_3^a) \equiv \sum_{s=0}^{n-j} \frac{1}{S!}(x_3^a - x_3)^s \underset{j}{Q}{}^a_{,\underset{s}{3}}(t, x_\alpha, x_3) + \overset{n}{\underset{j}{r}}{}^a(t, x_\alpha, x_3, x_3^a).$$

Here

$$(4.19)\qquad \overset{n}{\underset{j}{r}}{}^a(t, x_\alpha, x_3, x_3^a) = \int_{x_3}^{x_3^a} \frac{1}{(n-j)!}(x_3^a - x_3')^{n-j} \underset{j}{Q}{}^a_{,\underset{n-j+1}{3}}(t, x_\alpha, x_3')\,dx_3'$$

is the remainder of the Taylor formula, and

$$(4.20)\qquad Q_{,\underset{s}{3}} \overset{\text{def}}{=\!=} \frac{\partial^s Q}{\partial x_3^s}.$$

By disregarding the remainder in the expansion (4.18), we obtain

$$(4.21)\qquad \underset{j}{Q^a}(t, x_\alpha, x_3^a) =^* \sum_{s=0}^{n-j} \frac{1}{s!}(x_3^a - x_3)^s \underset{j}{Q}{}^a_{,\underset{s}{3}}(t, x_\alpha, x_3).$$

The symbol $=^*$ is replaced by identity if and only if

$$(4.22)\qquad \underset{j}{Q}{}^a_{,\underset{n-j+1}{3}}(t, x_\alpha, x_3) \equiv 0$$

or $x_3 = x_3^a$.

In order to obtain the representation of $Q(t, x_i)$ at the bilayers let us consider two arbitrary layers b and c contiguous to each other so that the relations (3.3), (3.4) hold true. Thus we may use the transformations (3.5), (3.6) and write down the fields defined in the layers considered, b and c, and represented in the form (4.16), (4.17) in terms of the local coordinate system in the base bc.

The following identities will be used:

$$(4.23)\qquad \sum_{j=0}^{n} \overset{n}{\underset{j}{Q}}{}^b(t, x_\alpha, x_3^b)\left(z_{bc} + \frac{1}{2}\eta_b\Delta\right)^j \equiv \sum_{j=0}^{n}\left[\sum_{q=0}^{n-j} \overset{n}{\underset{j+q}{Q}}{}^b(t, x_\alpha, x_3^b)\binom{p+q}{q}\left(\frac{1}{2}\eta_b\Delta\right)^q\right] z_{bc}^j,$$

$$(4.24)\qquad \sum_{j=0}^{n} \overset{n}{\underset{j}{Q}}{}^c(t, x_\alpha, x_3^c)\left(z_{bc} - \frac{1}{2}\eta_c\Delta\right)^j$$

$$\equiv \sum_{j=0}^{n}\left[\sum_{q=0}^{n-j} (-1)^q \overset{n}{\underset{j+q}{Q}}{}^c(t, x_\alpha, x_3^c)\binom{p+q}{q}\left(\frac{1}{2}\eta_c\Delta\right)^q\right] z_{bc}^j.$$

Moreover, the values of model fields at the middle surfaces of the layers appearing in Eqs. (4.16), (4.17) will be written in terms of their values at the contact planes and values of their derivatives calculated at the same planes. This may be done by substituting $x_3 = x_3^{bc}$ into the formulae of the type of Eqs. (4.21). Thus we have

$$Q_j^b(t, x_\alpha, x_3^b) = {}^*\sum_{s=0}^{n-j} \frac{1}{s!}(-1)^s\left(\frac{1}{2}\eta_b\Delta\right)^s Q^b_{j\ ,3\atop s}(t, x_\alpha, x_3^{bc}), \tag{4.25}$$

$$Q_j^c(t, x_\alpha, x_3^c) = {}^*\sum_{s=0}^{n-j} \frac{1}{s!}\left(\frac{1}{2}\eta_c\Delta\right)^s Q^c_{j\ ,3\atop s}(t, x_\alpha, x_3^{bc}). \tag{4.26}$$

Taking then into account the relations (4.12) we obtain the following representation of the field $Q(t, x_i)|_{B^{bc}}$ in a base bc, and hence also in the bilayer $P^{bc} \subset B^{bc}$:

$$Q(t, x_i)_{|B^{bc}} = Q^{bc}_{x_3^{bc}}(t, x_\alpha, z_{bc}). \tag{4.27}$$

Here

$$Q^{bc}_{x_3^{bc}}(t, x_\alpha, z_{bc}) = \lim_{n\to\infty} \sum_{j=0}^{n} \overset{n}{\underset{j}{Q}}{}^{bc}(t, x_\alpha, x_3^{bc}) z_{bc}^j, \tag{4.28}$$

and

$$\overset{n}{\underset{j}{Q}}{}^{bc}(t, x_\alpha, x_3^{bc}) \overset{\text{def}}{=} \begin{cases} \overset{n}{\underset{j}{Q}}{}^{b(bc)}(t, x_\alpha, x_3^{bc}) & \text{for} \quad z_{bc} < 0, \\ \overset{n}{\underset{j}{Q}}{}^{c(bc)}(t, x_\alpha, x_3^{bc}) & \text{for} \quad z_{bc} > 0. \end{cases} \tag{4.29}$$

The following notations have been used here:

$$\overset{n}{\underset{j}{Q}}{}^{b(bc)}(t, x_\alpha, x_3^{bc}) = {}^*\sum_{q=0}^{n-j} \binom{j+q}{q}\left(\frac{1}{2}\eta_b\Delta\right)^q \sum_{k=0}^{\left[\frac{1}{2}(n-j-q)\right]} \underset{k}{a}_{j+q+2k}\left(\frac{1}{2}\eta_b\Delta\right)^{2k} \times \sum_{s=0}^{n-j-q-2k} (-1)^s \frac{1}{s!}\left(\frac{1}{2}\eta_b\Delta\right)^s Q^b_{j+q+2k\ ,3\atop s}(t, x_\alpha, x_3^{bc}), \tag{4.30}$$

$$\overset{n}{\underset{j}{Q}}{}^{c(bc)}(t, x_\alpha, x_3^{bc}) = {}^*\sum_{q=0}^{n-j} (-1)^q\binom{j+q}{q}\left(\frac{1}{2}\eta_c\Delta\right)^q \sum_{k=0}^{\left[\frac{1}{2}(n-j-q)\right]} \underset{k}{a}_{j+q+2k}\left(\frac{1}{2}\eta_c\Delta\right)^{2k} \times \sum_{s=0}^{n-j-q-2k} \frac{1}{s!}\left(\frac{1}{2}\eta_c\Delta\right)^s Q^c_{j+q+2k\ ,3\atop s}(t, x_\alpha, x_3^{bc}). \tag{4.31}$$

In the case when the field $Q(t, x_i)$ is defined and is continuous also for $(x_i) = (x_i^0) \in S(\tilde{\pi}^{fm} \cup \tilde{\pi}^{mf})$, the following condition should be fulfilled in each bilayer bc:

$$\lim_{z_{bc}\to 0^-} Q^{bc}_{x_3^{bc}}(t, x_\alpha, z_{bc}) = \lim_{z_{bc}\to 0^+} Q^{bc}_{x_3^{bc}}(t, x_\alpha, z_{bc}) \tag{4.32}$$

which, owing to the formula (4.28), is equivalent to the condition

$$\lim_{n\to\infty} \overset{n}{\underset{0}{Q}}{}^{b(bc)}(t, x_\alpha, x_3^{bc}) = \lim_{n\to\infty} \overset{n}{\underset{0}{Q}}{}^{c(bc)}(t, x_\alpha, x_3^{bc}). \tag{4.33}$$

It will be called the condition of compatibility of the field Q in the bilayer bc.

Let us observe that

$$\forall P^b \exists B^{bc},\ B^{cb}: B^{bc} \cap B^{cb} = P^b \tag{4.34}$$

and hence both compatibility conditions written for the neighbouring bilayers P^{fm} and P^{mf} and holding true in the bases B^{fm} and B^{mf} which contain those bilayers, also hold true in the layer $P^a \supset P^{fm} \cup P^{mf}$. Since $S(\tilde{P}^f \cup \tilde{P}^m) \equiv \{x_i\}$, the both conditions remain true in each composite layer. The smoothed version of compatibility condition is

$$\lim_{n\to\infty} \overset{n}{\underset{0}{Q}}{}^{b(bc)}(t, x_i) = \lim_{n\to\infty} \overset{n}{\underset{0}{Q}}{}^{c(bc)}(t, x_i). \tag{4.35}$$

5. Homogeneous, continuous model of the composite

Representation of the field Q in the bilayers may now be used in constructing the homogeneous and continuous model of the composite. All the fields sought for, like the field of deformation, stress, kinetic energy density, potential strain energy density, possess their own model fields. There is no need, however, to use them all since only the field of displacements may be assumed as the fundamental field $u_i(t, x_i)$. Basing upon it we shall define the remaining fields in the bilayers. To simplify the formalism, let us introduce the following notations for the fields of mass density ϱ and the elastic moduli tensor c_{ijkl} which are sectionally constant in the bilayers:

$$\varrho^{bc} \overset{\text{def}}{=} \begin{cases} \varrho^b & \text{for} \quad z_{bc} < 0, \\ \varrho^c & \text{for} \quad z_{bc} > 0, \end{cases} \tag{5.1}$$

$$c^{bc}_{ijkl} \overset{\text{def}}{=} \begin{cases} c^b_{ijkl} & \text{for} \quad z_{bc} < 0, \\ c^c_{ijkl} & \text{for} \quad z_{bc} > 0. \end{cases} \tag{5.2}$$

Replacing the symbol Q in Eqs. (4.28)–(4.31) with u_i, we obtain the representation of the displacement field in the selected bilayer bc. In referring to the formulae it will be understood that this change of variables has been done.

The model of order n is obtained by reducing the right-hand sequence in Eq. (4.28) to the single n-th term; thus, according to Eqs. (4.30), (4.31), the number of model displacement fields in the n-th order model equals $2(n+1)$.

In the following considerations symbol $Q|_{P^{bc}}$ will denote the representation of the field Q in the bilayer bc, and the arguments t, x_α, x_3^{bc} will not be explicitly written. For instance, Eq. (4.28) (applied to the field u_i) will take the form

$$u_{i|P^{bc}} = \lim_{n\to\infty} \sum_{j=0}^{n} \overset{n}{\underset{j}{u}}{}^{bc} z^j_{bc}. \tag{5.3}$$

In order to represent the deformation field on the composite in the bilayer bc, the classical definition of small deformations is used; the differentiation in the direction normal to the contact planes must be performed, because of the fixed value of x_3, with respect to the local coordinate z_{bc}, i.e.

$$\varepsilon_{ij|P^{bc}} \overset{\text{df}}{=} u_{(i,j)|P^{bc}} = \delta_{\alpha(j}u_{i),\alpha|P^{bc}} + \delta_{z_{bc}(j}u_{i),z_{bc}|P^{bc}}, \tag{5.4}$$

$\delta_{\alpha j}$ and $\delta_{z_{bc}j}$ being the Kronecker symbols:

(5.5) $$\delta_{\alpha j} = \begin{cases} 0, & \alpha \neq j, \\ 1, & \alpha = j \end{cases}, \quad \delta_{z_{bc}j} = \begin{cases} 0, & j = 1, 2, \\ 1, & j = 3. \end{cases}$$

The result is represented in the form

(5.6) $$\varepsilon_{ij|p^{bc}} = \lim_{n\to\infty} \sum_{s=0}^{n} \overset{n}{\underset{s}{\varepsilon}}{}^{bc}_{ij} z^s_{bc},$$

coefficients in this series being expressed in terms of the model fields of displacements and their derivatives by means of the expansion coefficients (5.3),

(5.7) $$\overset{n}{\underset{s}{\varepsilon}}{}^{bc}_{ij} \overset{\text{def}}{=} \delta_{\alpha(j} \overset{n-1}{\underset{s}{u}}{}^{bc}_{i),\alpha} + (s+1)\,\delta_{3(j} \overset{n}{\underset{s+1}{u}}{}^{bc}_{i)}.$$

All these coefficients have the same structure what makes the form of expansion (5.6) consistent with the assumed representation of the fields in the bilayers.

Let us observe that — due to Eq. (4.13),

(5.8) $$\overset{n}{\underset{j}{u}}{}^{bc}_{i} \equiv 0 \quad \text{for} \quad j > n,$$

and hence the formula (5.7) yields

(5.9) $$\overset{n}{\underset{s}{\varepsilon}}{}^{bc}_{ij} \equiv 0 \quad \text{for} \quad s \geqslant n.$$

Using the classical definition of the stress tensor τ_{ij}, kinetic energy density k, potential strain energy density w and making use of the notations (5.1), (5.2), we obtain the representations of these fields in the bilayers:

(5.10) $$\tau_{ij|p^{bc}} = c^{bc}_{ijkl}\, \varepsilon_{kl}|_{p^{bc}},$$

(5.11) $$k_{|p^{bc}} = \frac{1}{2} \varrho^{bc}\, \dot{u}_{i|p^{bc}} \dot{u}_{i|p^{bc}},$$

(5.12) $$w_{|p^{bc}} = \frac{1}{2} \tau_{ij|p^{bc}} \varepsilon_{ij|p^{bc}},$$

the dot in Eq. (5.11) denoting differentiation with respect to time.

The Lagrange function density l is assumed as the difference of the kinetic energy density and the potential strain energy density. Its representation in the bilayers has the form

(5.13) $$l_{|p^{bc}} = k_{|p^{bc}} - w_{|p^{bc}}.$$

Let us assume that the mean density of the Lagrange function in the bilayer *fm* and *mf* is calculated from the formula

(5.14) $$l^{bc}_{\text{mean}} = \frac{2}{\Delta} \int_{-\frac{1}{2}\eta_b \Delta}^{\frac{1}{2}\eta_c \Delta} l_{|p^{bc}}\, dz_{bc}$$

and let us disregard the coefficients of the odd powers of z_{bc} in Eq. (5.13) and the terms involving odd powers of Δ; it is found then that these mean values may be treated as values of certain functional l^* defined in the smooth model fields and their respective derivatives computed at the point lying at the planes of contact π^{fm} and π^{mf}. Disregarding the coefficients of odd powers of z_{bc} and Δ is connected with the invariance of Lagrange function density (5.13) with respect to the reference frame transformations $\{(x_\alpha, z_{bc})\}$ and $\{x_i\}$ which correspond to the mirror reflections in the planes $\tilde{\pi}^{bc}$ and $x_3 = 0$, respectively.

Hence we have

$$l^{bc}_{\text{mean}} = l^*(t, x_\alpha, x_3^{bc}). \tag{5.15}$$

Functional $l^*(t, x_i)$ is assumed as the Lagrange function density for the composite model. The model fields describing most accurately the behaviour of composite is that which satisfy the Hamilton principle of conditional stationary action in which the role of constraints is played by the smoothed conditions of compatibility of displacements $u_i(t, x_i)$ and stresses $\tau_{3i}(t, x_i)$ in the bilayers fm and mf. This principle may symbolically be written in the form

$$\delta \int\limits_{t_0}^{t_1} dt \int\limits_{v} l^* dv + \int\limits_{t_0}^{t_1} \delta A dt = 0, \tag{5.16}$$

$$\lim_{n\to\infty} \Big(\overset{n}{\underset{0}{u}}{}_i^{b(bc)} - \overset{n}{\underset{0}{u}}{}_i^{c(bc)}\Big) = 0, \tag{5.17}$$

$$\lim_{n\to\infty} \Big(\overset{n}{\underset{0}{\tau}}{}_{3i}^{b(bc)} - \overset{u}{\underset{0}{\tau}}{}_{3i}^{c(bc)}\Big) = 0. \tag{5.18}$$

In the above formulae v denotes a regular region constituting of an arbitrary number of bilayers, and δA is the virtual work done by external forces. Condition (5.17) will be called displacement constraints, while Eq. (5.18) — stress constraints. The magnitudes appearing in Eq. (5.18) are expressed by the model field of displacements and their derivatives according to the Eqs. (5.10), (5.6), (5.7), (5.2) and (4.28)–(4.31) after replacing in these formulae Q with u_i. There are four equations of constraints and they constitute the differential equations for the displacement model fields. In the case of the model of order zero we obtain a single equation of displacement constraints, while in the case of first order model — two equations of displacement constraints and one equation of stress constraints. In higher order models 4 mutually independent equations of constraints are found. Owing to that fact the number of independent displacement model fields in models of consecutive orders equals 1, 1, 2, 4, 6 ..., respectively.

Particular attention should be paid to the first order model which is relatively simple since it involves only a single independent field of displacements and, at the same time, exhibits the dispersion effects connected with the layered body microstructure. In the theory of effective stiffness the same effect may be obtained by using at least two independent displacement fields (cf. [8, 10]). In static problems or such dynamic problems where disregarding the composite microstructure becomes justified ($\Delta/\lambda \to 0$, λ — wavelength), the first order model represents the model of anisotropic body of the classical theory of elasticity which is a long-wave approximation of the layered body. This problem will be dealt with in a separate paper.

References

1. J. V. RIZNIČENKO, *On seismic quasi-anisotropy* [in Russian], Izv. AN SSSR, Ser. Geograf. Geofiz., **13**, 518–544, 1949.
2. G. W. POSTMA, *Wave propagation in a stratified medium*, Geophysics, **20**, 780–806, 1955.
3. S. M. RYTOV, *Acoustic properties of finely layered medium* [in Russian], Akust. Zurn., **2**, 1, 71–83, 1956.
4. E. BEHRENS, *Sound propagation in lamellar composite materials and averaged elastic constants*, J. Acoust. Soc. A., **42**, 378–383, 1967.
5. C. T. SUN, J. D. ACHENBACH, G. HERRMANN, *Time-harmonic waves in a stratified medium. Propagation in the direction of the layering*, J. Appl. Mech., **35**, 408–411, 1968.
6. C. T. SUN, J. D. ACHENBACH, G. HERRMANN, *Effective stiffness theory for laminated media*, Northwestern University, Structural Mech. Lab., Technical Report No 64–4, 1967.
7. C. T. SUN, J. D. ACHENBACH, G. HERRMANN, *Continnum theory for a laminated medium*, J. Appl. Mech., **35**, 467–475, 1968.
8. J. D. ACHENBACH, C. T. SUN, G. HERRMANN, *On the vibrations of a laminated body*, J. Appl. Mech., **35**, 689–696, 1968.
9. R. D. MINDLIN, *Micro-structure in linear elasticity*, Arch. Rat. Mech. Anal., **16**, 51–78, 1964.
10. D. S. DRUMHELLER, A. BEDFORD, *On continuum theory for a laminated medium*, J. Appl. Mech., **40**, 527–532, 1973.
11. A. BEDFORD, D. S. DRUMHELLER, *On a generalized effective stiffness theory*, J. Appl. Mech., **41**, 305–307, 1974.
12. A. BEDFORD, M. STERN, *A multi-continuum theory for composite elastic materials*, Acta Mechanica, **14**, 85–102, 1972.
13. A. BEDFORD, M. STERN, *Wave propagation in elastic laminates using a multi-continuum theory*, Acta Mechanica, **15**, 21–38, 1972.
14. G. A. HEGEAMIER, A. H. NAYFEH, *A continuum theory for wave propagation in lamined composites. Case 1: Propagation normal to the laminates*, J. Appl. Mech., **40**, 503–510, 1973.
15. G. A. HEGAMIER, T. C. BACHE, *A continuum theory for wave propagation in laminated compositites. Case 2: Propagation parallel to the laminates*, J. Elasticity, **3**, 125, 1973.
16. G. A. HEGAMIER, G. A. GURTMAN, A. H. NAYFEH, *A continuum mixture theory of wave propagation in laminated and fiber reinforced composites*, Int. J. Solids Structures, **9**, 395–414, 1973.
17. G. A. HEGAMIER, T. C. BACHE, *A general continuum theory with microstructure for wave propagation in elastic laminated composites*, J. Appl. Mech., **41**, 101–105, 1974.
18. M. BEN-AMOZ, *On wave propagation in laminated composites. I: Propagation parallel to the laminates*, Int. J. Engrg. Sci., **13**, 43–56, 1975.
19. M. BEN-AMOZ, *On wave propagation in laminated composites. II: Propagation normal to the laminates*, Int. J. Engrg. Sci., **13**, 57–67, 1975.
20. C. T. SUN, J. D. ACHENBACH, G. HERRMANN, *Dispersion of time-harmonic waves in a laminated plate*, AIAA Paper No 68–353, 1968.
21. C. T. SUN, *Surface waves in layered media*, Bull. of the Seismol. Soc. of America, **60**, 345–366, 1970.
22. C. T. SUN, *Theory of laminated plates*, J. Appl. Mech., **38**, 947–954, 1971.
23. C. T. SUN, *Microstructure theory for composite beam*, J. Appl. Mech., **38**, 947–954 1971.
24. C. SVE, G. HERRMANN, *Movin g load on a laminated composite*, J. Appl. Mech., **41**, 663–667, 1974.
25. A. BEDFORD, D. S. DRUMHELLER, *The propagation of stress waves into a laminated half space using a second order microstructure theory*, Int. Solids Struct., **11**, 841–852, 1975.
26. C. KITTEL, *Wstęp do fizyki ciala stalego*, PWN, Warszawa 1974.
27. G. E. ŠILOV, *Matematičeskij analiz. Funkcji odnogo peremennogo*. Čast, 3. Ird. Nauka, Moskwa 1970.
8. N. N. LEBIEDIEW, *Funkcje specjalne i ich zastosowania*, PWN, Warszawa 1957.

TECHNICAL UNIVERSITY OF ŁÓDŹ.

Received February 23, 1976.