Archives of Mechanics • Archiwum Mechaniki Stosowanej • 29, 2, pp. 325-338, Warszawa 1977

On wave propagation in a coupled thermo-elastic-plastic medium

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IN THIS paper the conditions for propagation of three-dimensional acceleration and strong discontinuity (shock) waves and for the amplitude attenuation in thermo-elastic viscoplastic materials are derived. The results shows that the velocities of propagation in these materials are the same as in thermoelastic material. Moreover, for Fourier's law only one longitudinal wave propagates. On the front of this wave the entropy suffers discontinuity but not temperature. The amplitude decaying depends on thermoconductivity but not on energy dissipation. For the Cattaneo relation, however, two longitudinal waves exist and all dependent variables including temperature are discontinuous on their fronts. The attenuation of the amplitude is stronger and also depends on plastic energy dissipation.

W niniejszej pracy wyprowadzono warunki na propagację trójwymiarowych fal przyspieszenia i fal silnej nieciągłości (uderzeniowych) oraz zmianę amplitudy w materiale sprężysto-plastycznym i sprężysto-lepkoplastycznym. Wykazano, że prędkości propagacji fal w tych materiałach pokrywają się z odpowiednimi prędkościami fal w materiale termosprężystym. Ponadto przy użyciu prawa Fouriera tylko jedna fala podłużna może się rozprzestrzeniać, na której czole entropia doznaje nieciągłości, a temperatura jest ciągła. Malenie amplitudy zależy od przewodnictwa cieplnego lecz nie od energii dysypacji. Natomiast w przypadku równania Cattaneo istnieją dwie fale podłużne i na ich czołach wszystkie zmienne zależne, łącznie z temperaturą, doznają skoków. Malenie amplitudy jest bardziej znaczne i zależy również od energii dysypacji.

В настоящей работе выведены уравления распространения трехмерных волн ускорения и волн сильного разрыва (ударных) и изменения амплитуды в термо-упруго-вязко-пластических материалах. Показано, что скорости распространения волн в этих материалах совпадают с соответствующими скоростями волн в термоупругом материале. Кроме этого, при использовании закона Фурье, может распространяться только одна продольная волна, на фронте которой энтропия испытывает разрыв, а температура непрерывна. Убывание амплитуды зависит от теплопроводности, но не от энергии диссипации. В случае же уравнения Каттанео существуют две продольные волны и на их фронтах все зависимые переменные, включая температуру, испытывают скачки. Убывание амплитуды более значительно и зависит от энергии диссипации.

Introduction

A PROPAGATION of three-dimensional strong discontinuities and acceleration waves of an arbitrary geometry in an elastic-plastic medium with small deformations is investigated. The model of a medium is based on general thermodynamic principles. For heat conduction the Maxwell-Cattaneo equation [1] is obtained in the same manner as it was done in [2, 3, 4] where one-dimensional waves were considered in materials with internal state variables.

The wave propagation speeds coincide with the respective speeds in a thermoelastic medium. Differential equations for determining the intensities of shock and acceleration waves were derived. Cases of Fourier's law and the Maxwell-Cattaneo relation are investigated.

There is only one longitudinal wave, in the case of Fourier's law, at the front of which entropy is discontinuous and temperature remains continuous. The attenuation of the intensity is dependent on thermoconduction but is independent of a dissipation of the energy. In the case when a speed of propagation of thermal signals is finite, we have two fronts of longitudinal waves at which all variables including temperature are discontinuous. In the last case the attenuation is stronger than in the case of Fourier's law, and it also depends on plastic dissipation energy.

An isothermal case of wave propagation in an elasto-viscoplastic medium was considered in [5, 6].

1. The constitutive equations for the thermo-elastic-viscoplastic medium

In order to build the model of a thermo-elastic-visco-plastic material we shall make use of thermodynamical principles.

We assume that the following variables determine the local state of a heat conducting elastic-plastic body: the stress tensor $\hat{\sigma}$, the elastic and plastic strain tensors $\hat{e}^e$ and $\hat{e}^p$, the specific free energy F, the specific entropy s, the absolute temperature T, the temperature gradient $\bar{g}$, the heat flux vector $\bar{q}$, the mass density ϱ and internal structural parameters χ.

Assume that there exist two groups of the internal state parameters χ, namely, the mechanical parameters χ_M and the thermal parameters χ_T. The last group is connected with a temperature gradient history [2, 3].

Let us assume the set of independent[(1)] variables $\hat{e}^e$, $\hat{e}^p$, T, χ, $\bar{g}$ in terms of which all the remaining variables may be expressed, e.g.:

$$F = F(\hat{e}^e, \hat{e}^p, T, \chi, \bar{g}), \quad s = s(\hat{e}^e, \hat{e}^p, \ldots, \bar{g}) \ldots. \tag{1.1}$$

The second law of thermodynamics is here expressed by the Claisius-Duhem inequality [7, 8]

$$-\dot{F}+\frac{1}{\varrho_0}\sigma_{ij}\varepsilon_{ij}-s\dot{T}-\frac{\bar{q}\bar{g}}{\varrho_0 T} \geqslant 0, \tag{1.2}$$

ε_{ij} is the total strain rate tensor, ϱ_0 is the initial density. Taking into account Eqs. (1.1), we hav:

$$\left(\frac{\sigma_{ij}}{\varrho_0}-\frac{\partial F}{\partial e^e_{ij}}\right)\varepsilon^e_{ij}-\left(\frac{\partial F}{\partial T}+s\right)\dot{T}-\frac{\partial F}{\partial g_i}\dot{g}_i+\underline{\left(\frac{\sigma_{ij}}{\varrho_0}-\frac{\partial F}{\partial e^p_{ij}}\right)\varepsilon^p_{ij}-\frac{\partial F}{\partial \chi_i}\dot{\chi}_i-\frac{q_i T_{,i}}{\varrho_0 T}} \geqslant 0. \tag{1.3}$$

So far as the underlined terms in inequality (1.3) are independent of ε^e_{ij}, T and $\dot{g}_i$ the following restrictions are imposed on the constitutive equations (1.1) [9]:

$$\sigma_{ij} = \varrho_0\frac{\partial F}{\partial e^e_{ij}}, \quad s = -\frac{\partial F}{\partial T}, \quad \frac{\partial F}{\partial g} = 0 \tag{1.4}$$

(1) It is possible to include $\hat{e}^p$ in internal state variables and consider a total strain tensor $\hat{e}$ instead of $\hat{e}^e$ and $\hat{e}^p$.

and also the inequality of the general dissipation must be satisfied

$$(1.5) \qquad \frac{1}{\varrho_0}\tau_{ij}\dot{\varepsilon}^p_{ij} - \frac{\partial F}{\partial \chi^i_M}\dot{\chi}^i_M - \frac{\partial F}{\partial \chi^i_T}\dot{\chi}^i_T - \frac{q_i T_{,i}}{\varrho_0 T} \geqslant 0,$$

where $\tau_{ij} = \sigma_{ij} - \varrho_0 \dfrac{\partial F}{\partial e^p_{ij}}$ is an active stress tensor.

Following [2, 3], we assume the form of evolution equations for

$$(1.6) \qquad \dot{\chi}_T = \operatorname{grad} T + \dot{\chi}_*(e, T, \chi).$$

Introducing Eq. (1.6) into inequality (1.5), we obtain

$$(1.7) \qquad -\left(\frac{\partial F}{\partial \chi^i_T} + \frac{q_i}{\varrho_0 T}\right)T_{,i} - \underline{\frac{\partial F}{\partial \chi^i_T}\dot{\chi}^i_* - \frac{\partial F}{\partial \chi^i_M}\dot{\chi}^i_M + \frac{1}{\varrho_0}\tau_{ij}\dot{\varepsilon}^p_{ij}} \geqslant 0.$$

The underlined terms in (1.7) are independent of $\bar{g}$ since χ_T is independent of $\bar{g}$ [3], and we obtain

$$(1.8) \qquad \frac{\partial F}{\partial \chi^i_T} = -\frac{q_i}{\varrho T}, \qquad D = \frac{1}{\varrho}\tau_{ij}\dot{\varepsilon}^p_{ij} - \frac{\partial F}{\partial \chi^i_M}\dot{\chi}^i_M - \frac{\partial F}{\partial \chi^i_T}\dot{\chi}^i_* \geqslant 0.$$

In our following considerations we shall use the approach based on introducing the dissipation function D and on Onsager's general principle. Instead of Onsager's principle we can postulate the equivalent but physically more evident Zigler's principle [10] of the maximum dissipation rate in a real process which requires

$$D = \frac{1}{\varrho}\tau_{ij}\dot{\varepsilon}^p_{ij} + \frac{\partial F}{\partial \chi_i}\dot{\chi}_i \geqslant \frac{1}{\varrho}\tau_{ij}\dot{\tilde{\varepsilon}}^p_{ij} + \frac{\partial F}{\partial \chi_i}\dot{\tilde{\chi}}_i.$$

It means that the dissipation reaches a maximum value in the process with the real $\dot{\varepsilon}^p_{ij}$ and $\dot{\chi}_i$, and $\dot{\tilde{\varepsilon}}^p_{ij}\ \dot{\tilde{\chi}}_i$ are variables in an arbitrary process.

This implies the following relations:

$$(1.9) \qquad \tau_{ij} = \lambda\frac{\partial D}{\partial \dot{\varepsilon}^p_{ij}}, \qquad F_{\chi_i} = \lambda\frac{\partial D}{\partial \dot{\chi}_i}.$$

If the function D and F are given, then Eqs. (1.4), (1.6), (1.8), (1.9) are the constitutive equations for an elastic-plastic medium.

Assume that the dissipation function D is represented by the expression

$$(1.10) \quad D = D_1(\dot{\varepsilon}^p_{ij}, \dot{\chi}_M) + D_2(\dot{\chi}^*_{T_i}) = \left(1 + \frac{\chi_M}{k_s}\right)I_p[k_s(T, \chi_M) + \Psi(I_p)] + \frac{k}{2\varrho T}\dot{\chi}^*_{T_i}\dot{\chi}^*_{T_i}$$

and the free energy F consists of two terms

$$(1.11) \qquad F = F_1(\hat{e}^e, T) + F_2(\hat{e}^p, T, \chi).$$

The assumption (1.11) means that plastic deformations do not give an influence on the elastic properties of the material.

In the Eq. (1.10) $I_p = (\dot{\varepsilon}^p_{ij}\dot{\varepsilon}^p_{ij})^{1/2}$ — is the second invariant of the plastic strain tensor, $\chi_M = W_p = \int\limits_0^t \tau_{ij}\dot{\varepsilon}^p_{ij}\,dt$ is the plastic deformation work of the active stresses, the relation $k_s = k_s(\chi_M, T)$ is dependence one can determines from one dimensional static loading, the function $\Psi(I_p)$ is characterising the strain rate influence on constitutive equations and one can determine it from dynamic experiments [11]. If $\Psi(I_p) \equiv 0$ we have equations for the strain rate independent plasticity.

Relatively function F_2 we assume that it is a square power function of his arguments

$$(1.12)\qquad F_2 = \frac{a}{2\varrho} e^p_{ij} e^p_{ij} + \frac{1}{2k_s \varrho} \chi^2_M + \frac{1}{2\varrho T \tau_0} \chi_{T_i} \chi_{T_i}.$$

From the assumption (1.10) follows that the inequality (1.8) can be represented, as two independent inequalities

$$D_1 = \frac{1}{\varrho} \tau_{ij} \varepsilon^p_{ij} + \frac{\partial F}{\partial \chi_M} \dot{\chi}_M \geqslant 0, \quad D_2 = \frac{\partial F}{\partial \chi_{T_i}} \dot{\chi}^*_{T_i} \geqslant 0.$$

Taking into account that $\dot{\chi}_M = \tau_{ij} \varepsilon^p_{ij}$ and using Eqs. (1.9), (1.12), one can obtain from the first inequality

$$(1.13)\qquad \left(1 + \frac{\chi_M}{k_s}\right) \tau_{ij} = \lambda \frac{\partial D_1}{\partial \varepsilon^p_{ij}}, \quad S = (\tau_{ij} \tau_{ij})^{1/2} = \lambda \frac{\partial D_1}{\partial I_p} = \frac{D_1}{I_p}.$$

That leads to

$$(1.14)\qquad \tau_{ij} = \sigma_{ij} - a e^p_{ij} = \frac{k_s(\chi_M, T) + \Psi(I_p)}{I_p} \varepsilon^p_{ij}, \quad S = k_s(\chi_M, T) + \Psi(I_p).$$

Thus the made assumptions lead us to the particular form of the constitutive equations for the visco-plastic medium with the isotropic and kinematic hardening.

It is necessary to remark that from (1.13)–(1.14) it is clear, that the kinematic hardening always is connected with a dry friction mechanism and depends only on e^p_{ij}, χ, T but not on ε^p_{ij}. At the same time isotropic hardening can depend both on the viscosity and on the dry friction mechanism.

Let us now find $\dot{\chi}_*$

$$(1.15)\qquad \dot{\chi}_* = \frac{1}{\tau_0} \chi_T$$

and, using Eqs. (1.6), (1.8), (1.12), we obtain

$$(1.16)\qquad \tau_0 \dot{q} = k \operatorname{grad} T - q,$$

where τ_0 is the thermal relaxation time and k, the coefficient of thermal conductivity.

This equation is the Maxwell-Cattaneo equation. A survey of the works where this equation is investigated can be found in [3, 12].

It is clear from Eq. (1.16) that the heat flux $\bar{q}$ can be considered as an internal state variable [3, 4].

Let us determine now the elastic part of the free energy F_1; then, the constitutive equations of the model considered will be determined completely.

Since we assume small strains and small temperature increments we can take, in the expansion of F_1, only quadratic terms

$$(1.17)\qquad F_1 = \frac{\lambda}{2\varrho} E_1^2 + \frac{\mu}{\varrho} E_2 - \frac{3\lambda + 2\mu}{\varrho} \alpha (T - T_0) E_1 - \frac{c_E}{2T_0} (T - T_0)^2 + O\left(e^3_{ij}, \frac{T^3}{T^0_3}\right),$$

where $E_1 = e^e_{ii}$, $E_2 = e^e_{ij} e^e_{ij}$ are the first and second invariants of the elastic strain tensor $\hat{e}^e$, λ and μ are the elastic constants of the material, α is the coefficient of thermal expansion and c_E is the specific heat at constant deformations.

Relations (1.4) and (1.7) give the following expressions:

(1.18) $$\sigma_{ij} = \lambda e^{e}_{kk}\delta_{ij}+2\mu e^{e}_{ij}-(3\lambda+2\mu)\alpha(T-T_0)\delta_{ij},$$

(1.19) $$s = \frac{3\lambda+2\mu}{\varrho}\alpha e^{e}_{kk}+c_E\frac{T-T_0}{T_0}.$$

Determining e^{e}_{ij} through the total strains e_{ij} and assuming $\varepsilon^{p}_{ii} = 0$, we obtain the final form of the constitutive equations

(1.20) $$\dot{\sigma}_{ij} = \lambda\dot{e}_{kk}\delta_{ij}+2\mu\dot{e}_{ij}-(3\lambda+2\mu)\alpha\dot{T}\delta_{ij}-\frac{2\mu}{\tau}\frac{\hat{\Phi}(S-k_s)}{S}s_{ij},$$
$$\hat{\Phi} = \begin{cases}\Phi(z), & z\geqslant 0,\\ 0, & z<0,\end{cases}\qquad \Phi(z) = \Psi^{-1}(z),$$

$s_{ij} = \tau_{ij}-\frac{\tau_{kk}}{3}\delta_{ij}$, τ is a constant characterizing the material viscosity dimensions [sek].

Equations (1.20) are a modification of the well-known equations for a thermo-elastic-plastic medium [11, 13]. In order to obtain a closed system of equations describing the medium considered, the equation of motion

(1.21) $$\sigma_{ij,j}-\varrho\dot{v}_i = 0$$

and the equation of heat flux in the form

(1.22) $$\varrho T\dot{s}+\operatorname{div}q = \tau_{ij}\varepsilon^{p}_{ij}$$

should be added to Eqs. (1.16) and (1.20).

2. The propagation of strong discontinuity waves

The total system of equations can be represented as the first order of differential equations with respect to the velocity vector $\bar{v}$, the stress tensor $\hat{\sigma}$, the temperature T, the heat flux $\bar{q}$ and the plastic strain tensor e^{p}_{ij}.

Using Eqs. (1.14), (1.16), (1.19)–(1.22) and Cauchy relations between ε_{ij} and the velocity vector v, we obtain the following system:

(2.1) $$\sigma_{ij,j}-\varrho\dot{v}_i = 0$$
$$\dot{\sigma}_{ij} = \lambda v_{k,k}\delta_{ij}+\mu(v_{i,j}+v_{j,i})-(3\lambda+2\mu)\alpha T\delta_{ij}-\frac{2\mu}{\tau}\frac{\hat{\Phi}(S-k_s)}{S}s_{ij},$$
$$(3\lambda+2\mu)T_0\alpha v_{k,k}+\varrho c_E\dot{T} = -q_{i,i}+\frac{1}{\tau}\hat{\Phi}(S-k_s)S,$$
$$\tau_0\dot{q}+q_i = -kT_{,i},\quad \dot{e}^{p}_{ij} = \frac{\hat{\Phi}(S-k_s)}{\tau S}s_{ij}.$$

This system of equations constitutes a quasi-linear hyperbolic first order system with the principal linear part. The right-hand sides of Eqs. (2.1) are continuous functions of their arguments. For the purpose of investigating the strong discontinuity surfaces of the system considered, a general theory developed for equations of a divergent form [14] may be used.

By a generalized solution of the system of Eqs. (2.1) which may be written in the matrix form

$$L(U) = (A^t U)_{,t} + (A^i U)_{,i} + B(U) = 0,$$

we mean a piece-wise continuous vector function U having piece-wise continuous derivatives in a region G and for which the relation

$$\int_R [(A^t \xi_{,t} + A^i \xi_{,i}) U - B(U)\xi] dt\, dx_i = 0 \tag{2.3}$$

holds for arbitrary test functions ξ for all subdomains $R \subset G$. Then, in smooth regions of the solution, Eq. (2.2) results from Eq. (2.3) and on the discontinuous surface $\varphi(x_i, t) = 0$ should be satisfied

$$(-cA^t + A^i \nu_i)\,[U] = 0. \tag{2.4}$$

Here A^t and A^i are matrices, the vector $[U] = U^+ - U^-$ denotes a jump of solution U across the surface $\varphi = 0$, c is the speed of propagation of the surface and ν is a normal of the surface.

In the case of the system of Eqs. (2.1), the relations (2.4) yield the following system of equations to be satisfied by the "jumps":

$$\begin{gathered}
[\sigma_{ij}]\nu_j + \varrho c[v_i] = 0, \\
-c[\sigma_{ij}] = \lambda[v_k]\nu_k \delta_{ij} + \mu([v_i]\nu_j + [v_j]\nu_i) + (3\lambda + 2\mu)\alpha c[T]\delta_{ij}, \\
(3\lambda+2\mu) T_0 \alpha [v_k]\nu_k - \varrho c c_E [T] = -[q_i]\nu_i, \qquad \tau_0 c[q_i] = k[T]\nu_i.
\end{gathered} \tag{2.5}$$

We can find the expressions for $[T]$ and $[q_i]\nu_i$ from the two last equations

$$[T] = \frac{3\lambda+2\mu}{\varrho} T_0 \alpha \left(c_E c - \frac{k}{\varrho c \tau_0} \right)^{-1} [v_k]\nu_k, \tag{2.6}$$

$$[q_i]\nu_i = \frac{3\lambda+2\mu}{c\varrho\tau_0} T_0 \alpha k \left(c_E c - \frac{k}{\varrho c \tau_0} \right)^{-1} [v_k]\nu_k. \tag{2.7}$$

Substituting them into Eq. (2.5), we have

$$\begin{gathered}
-c[\sigma_{ij}] = \lambda^*[v_k]\nu_k \delta_{ij} + \mu([v_i]\nu_j + [v_j]\nu_i), \\
\lambda^* = \lambda + \frac{(3\lambda+2\mu)^2}{\varrho} \alpha^2 T_0 c \left(c_E c - \frac{k}{\varrho c \tau_0} \right)^{-1}, \\
(\lambda^* + \mu)[v_k]\nu_k \nu_i + (\mu - \varrho c^2)[v_i] = 0.
\end{gathered} \tag{2.8}$$

It follows from Eqs. (2.8) that for the medium considered two types of waves, longitudinal and transverse, exist.

For transverse waves we have

$$c^2 = \mu \varrho^{-1}, \quad [v_k]\nu_k = 0, \quad -c[\sigma_{ij}] = \mu([v_i]\nu_j + [v_j]\nu_i) \tag{2.9}$$

and for longitudinal ones we obtain

$$c^2 = (\lambda^* + 2\mu)\varrho^{-1}, \quad [v_i] = W\nu_i, \quad -c[\sigma_{ij}] = (\lambda^*\delta_{ij} + 2\mu\nu_i\nu_j) W. \tag{2.10}$$

The speeds of the longitudinal waves c are determined from the equations

$$c^4-(c_A^2+c_T^2)c^2+c_0^2c_T^2=0,$$

(2.11)
$$c_A^2=c_0^2+\frac{(3\lambda+2\mu)^2}{\varrho^2c_E}\alpha^2T_0,\quad c_0^2=(\lambda+2\mu)\varrho^{-1},\quad c_T^2=\frac{k}{\varrho c_E\tau_0}.$$

The roots of the biquadratic equation (2.11) are

(2.12)
$$\bar{c}_{1,2}^2=\frac{c_{1,2}^2}{c_A^2}=\frac{1}{2}\left(1+\frac{c_T^2}{c_A^2}\right)\pm\left[\frac{1}{4}\left(1+\frac{c_T^2}{c_A^2}\right)^2-\frac{c_T^2c_0^2}{c_A^4}\right]^{1/2}.$$

Let us obtain the expansion of c_1^{-2} and $\bar{c}_2^2$ in a series for the small τ_0

(2.13)
$$\bar{c}_1^2=\frac{1}{\varepsilon}+\left(1-\frac{c_0^2}{c_A^2}\right)+O(\varepsilon),\quad \bar{c}_2^2=\frac{c_0^2}{c_A^2}-\varepsilon\beta(1-\beta)+O(\varepsilon^2),$$
$$\varepsilon=\frac{c_A^2}{c_T^2}=\frac{c_A^2\varrho c_E}{k}\tau_0,\quad \beta=\frac{c_0^2}{c_A^2}.$$

It is clear from Eqs. (2.13) that there exist two longitudinal waves: T — wave with the speed c_1 and M — wave with the speed c_2.

If the medium in uncoupled, then $\alpha\to0$, and one obtains from Eq. (2.12) that $c_1^2\to c_T^2$ for the thermal wave and $c_2^2\to c_0^2$ for the mechanical wave.

The T — wave speed, c_1 and the M — wave speed, c_2 satisfy the inequalities

$$c_1^2>c_T^2>c_0^2>c_2^2.$$

For Fourier's law we have $\tau_0\to0$ and $c_2^2\to c_0^2$, $c_1^2\to\infty$; thus there remains only one M — wave and the jump of temperature on the wave front vanishes.

Let us consider now the change of wave intensities during propagation in space.

The system of Eqs. (2.1) may be rewritten in terms of the jumps

(2.14)
$$[\sigma_{ij,j}-\varrho[\dot{v}_i]=0,$$
$$[\dot{\sigma}_{ij}]=\lambda\delta_{ij}[v_{k,k}]+\mu([v_{i,j}]+[v_{j,i}])-(3\lambda+2\mu)\alpha\delta_{ij}[\dot{T}]-2\mu[\varepsilon_{ij}^p],$$
$$(3\lambda+2\mu)T_0\alpha[v_{k,k}]+c_E\varrho[\dot{T}]=-[q_{i,i}]+\frac{1}{\tau}[\hat{\Phi}(S-k_s)S],$$
$$\tau_0[\dot{q}_i]+[q_i]=-k[T_{,i}],\quad [\varepsilon_{ij}^p]=\frac{1}{\tau}\left[\frac{\hat{\Phi}(S-k_s)s_{ij}}{S}\right].$$

Using the kinematic conditions of the compatibility of the first order [15], we obtain

(2.15)
$$\left[\frac{\partial f}{\partial t}\right]=-cF+\frac{\delta[f]}{\delta t},\quad \left[\frac{\partial f}{\partial x_i}\right]=F\nu_i+g^{\alpha\beta}\frac{\partial[f]}{\partial y_\alpha}\frac{\partial x_i}{\partial y_\beta},$$
$$F=\left[\frac{\partial f}{\partial x_k}\right]\nu_k.$$

Here we use the following notation:

$$\left[\frac{\partial\sigma_{ij}}{\partial x_k}\right]\nu_k=\Sigma_{ij},\quad \left[\frac{\partial v_i}{\partial x_k}\right]\nu_k=V_i,\quad Q_i=\left[\frac{\partial q_i}{\partial x_k}\right]\nu_k,$$
$$\theta=\left[\frac{\partial T}{\partial x_k}\right]\nu_k,\quad \frac{\delta}{\delta t}=\frac{\partial}{\partial t}+c\nu_k\frac{\partial}{\partial x_k},$$

$g^{\alpha\beta}$ is the contravariant metric tensor of wave surface, y_α, y_β are curvilinear coordinates on the surface, α, $\beta = 1, 2$. Substituting relations (2.15) into Eqs. (2.14), we obtain

(2.16)
$$\Sigma_{ij}\nu_j + g^{\alpha\beta}\frac{\partial[\sigma_{ij}]}{\partial y_\alpha}\frac{\partial x_j}{\partial y_\beta} + \varrho c V_i - \varrho\frac{\delta[v_i]}{\delta t} = 0,$$
$$-c\Sigma_{ij} + \frac{\delta[\sigma_{ij}]}{\delta t} = \lambda\left(V_k\nu_k + g^{\alpha\beta}\frac{\partial[v_k]}{\partial y_\alpha}\frac{\partial x_k}{\partial y_\beta}\right)\delta_{ij} + \mu\left[V_i\nu_j + V_j\nu_i + g^{\alpha\beta}\left(\frac{\partial[v_i]}{\partial y_\alpha}\frac{\partial x_j}{\partial y_\beta} + \frac{\partial[v_j]}{\partial y_\alpha}\frac{\partial x_i}{\partial y_\beta}\right)\right] + (3\lambda+2\mu)\alpha\left(c\theta - \frac{\delta[T]}{\delta t}\right)\delta_{ij} - \frac{2\mu}{\tau}\left[\frac{\hat{\Phi}(S-k_s)}{S}s_{ij}\right],$$
$$(3\lambda+2\mu)T_0\alpha\left(V_k\nu_k + g^{\alpha\beta}\frac{\partial[v_k]}{\partial y_\alpha}\frac{\partial x_k}{\partial y_\beta}\right) + c_E\varrho\left(-c\theta + \frac{\delta[T]}{\delta t}\right) = -Q_k\nu_k - g^{\alpha\beta}\frac{\partial[q_k]}{\partial y_\alpha}\frac{\partial x_k}{\partial y_\beta} + \frac{1}{\tau}[\hat{\Phi}S],$$
$$\tau_0\left(-cQ_i + \frac{\delta[q_i]}{\delta t}\right) + [q_i] = -k\left(\theta\nu_i + g^{\alpha\beta}\frac{\partial[T]}{\partial y_\alpha}\frac{\partial x_i}{\partial y_\beta}\right).$$

Multiplying the last equation by ν_i and performing the convolution, one determines

(2.17)
$$Q_i\nu_i = \frac{k}{\tau_0 c}\theta + \frac{1}{c}\frac{\delta[q_i\nu_i]}{\delta t} + \frac{q_i\nu_i}{\tau_0 c}.$$

Using the third equation in Eqs. (2.16), we find

(2.18)
$$\theta = \left(c_E c\varrho - \frac{k}{c\tau_0}\right)^{-1}\left\{(3\lambda+2\mu)T_0\alpha\left(V_k\nu_k + g^{\alpha\beta}\frac{\partial[v_k]}{\partial y_\alpha}\frac{\partial x_k}{\partial y_\beta}\right) + \varrho c_E\frac{\delta[T]}{\delta t} + \frac{1}{c}\frac{\delta[q_i\nu_i]}{\delta t} + \frac{[q_i]\nu_i}{\tau_0 c} + g^{\alpha\beta}\frac{\partial[q_k]}{\partial y_\alpha}\frac{\partial x_k}{\partial y_\beta} - \frac{1}{\tau}[\hat{\Phi}S]\right\}.$$

By means of Eqs. (2.6)–(2.7) and (2.18) we can exclude $[T]$, $[q_i]$ and θ from the second equation (2.16)

(2.19)
$$-c\Sigma_{ij} + \frac{\delta[\sigma_{ij}]}{\delta t} = \lambda^*\left(V_k\nu_k + g^{\alpha\beta}\frac{\partial[v_k]}{\partial y_\alpha}\frac{\partial x_k}{\partial y_\beta}\right)\delta_{ij} + \mu\left[V_i\nu_j + V_j\nu_i + g^{\alpha\beta}\left(\frac{\partial[v_i]}{\partial y_\alpha}\frac{\partial x_j}{\partial y_\beta} + \frac{\partial[v_j]}{\partial y_\alpha}\frac{\partial x_i}{\partial y_\beta}\right)\right] + \left\{a\frac{\delta W}{\delta t} + \frac{ca}{2}g^{\alpha\beta}\frac{\partial(W\nu_k)}{\partial y_\alpha}\frac{\partial x_k}{\partial y_\beta} + \frac{a}{2\tau_0}W - b[\hat{\Phi}S]\right\}\delta_{ij} - 2\mu[\varepsilon^p_{ij}],$$
$$a = \frac{2kT_0\alpha^2}{\varrho^2 c\tau_0}(3\lambda+2\mu)^2\left(c_E c - \frac{k}{\tau_0\varrho c}\right)^{-2}, \quad b = \frac{(3\lambda+2\mu)c\alpha}{\varrho\tau}\left(c_E c - \frac{k}{\varrho c\tau_0}\right)^{-1}.$$

Multiplying Eqs. (2.19) by ν_j and excluding Σ_{ij} from the first Eq. (2.16), we obtain three equations, respectively V_i

(2.20)
$$(\lambda^*+\mu)V_k\nu_k\nu_i + (\mu - \varrho c^2)V_i = cg^{\alpha\beta}\frac{\partial[\sigma_{ij}]}{\partial y_\alpha}\frac{\partial x_j}{\partial y_\beta} - \varrho c\frac{\delta[v_i]}{\delta t} + \frac{\delta[\sigma_{ij}]\nu_j}{\delta t} - \lambda^* g^{\alpha\beta}\frac{\partial[v_k]}{\partial y_\alpha}\frac{\partial x_k}{\partial y_\beta}\nu_i - \mu g^{\alpha\beta}\left(\frac{\partial[v_i]}{\partial y_\alpha}\frac{\partial x_j}{\partial y_\beta} + \frac{\partial[v_j]}{\partial y_\alpha}\frac{\partial x_i}{\partial y_\beta}\right)\nu_j - \left\{a\frac{\delta W}{\delta t} + \frac{ca}{2}g^{\alpha\beta}\frac{\partial W\nu_k}{\partial y_\alpha}\frac{\partial x_k}{\partial \nu_q} + \frac{aW}{2\tau_0} - b[\hat{\Phi}S]\right\}\nu_i + 2\mu[\varepsilon^p_{ij}]\nu_j.$$

Now we can exclude the jump $[\sigma_{ij}]$ from Eqs. (2.20) taking into account Eqs. (2.8)

$$(2.21)\qquad (\lambda^*+\mu)V_k\nu_k\nu_i+(\mu-\varrho c^2)V_i = -(\lambda^*+\mu)\left[g^{\alpha\beta}\frac{\partial[v_k]\nu_k}{\partial y_\alpha}\frac{\partial x_i}{\partial y_\beta}+g^{\alpha\beta}\frac{\partial[v_k]}{\partial y_\alpha}\frac{\partial x_k}{\partial y_\beta}\nu_i\right]$$
$$+2\mu\Omega[v_i]-2\varrho c\frac{\delta[v_i]}{\delta t}-\left\{a\frac{\delta W}{\delta t}-ca\Omega W+\frac{a}{2\tau_0}W-b[\hat{\Phi}S]\right\}\nu_i+2\mu[\varepsilon^p_{ij}]\nu_j.$$

Here we used the following relations from differential geometry [15]:

$$(2.22)\qquad \frac{\partial\nu_j}{\partial y_\alpha} = -g^{\sigma\tau}b_{\sigma\alpha}\frac{\partial x_j}{\partial y_\tau},\quad g^{\alpha\beta}g^{\sigma\tau}b_{\sigma\alpha}\frac{\partial x_j}{\partial y_\tau}\frac{\partial x_j}{\partial y_\beta}=2\Omega,\quad g^{\alpha\beta}\frac{\partial\nu_j}{\partial y_\alpha}\frac{\partial x_j}{\partial y_\beta} = -2\Omega,$$

$b_{\sigma\alpha}$ are coefficients of the second quadratic form of the surface, Ω is the mean curvature. Multiplying Eq. (2.21) by ν_i we obtain the equation for determining the intensity of strong discontinuity longitudinal waves

$$(2.23)\qquad \begin{gathered}\frac{\delta W}{\delta t} = c\Omega W-\frac{a_0 W}{2\tau_0(1+a_0)}+\frac{b_0}{1+a_0}[\hat{\Phi}S]+\frac{\mu}{\varrho c(1+a_0)}[\varepsilon^p_{ij}]\nu_i\nu_j,\\ a_0=\frac{a}{2\varrho c},\quad b_0=\frac{b}{2\varrho c}.\end{gathered}$$

For Fourier's law τ_0 vanishes and we obtain

$$(2.24)\qquad \frac{\delta W}{\delta t} = c_0\Omega W-\frac{(3\lambda+2\mu)^2\alpha^2T_0}{2\varrho k}W+\frac{\mu}{\varrho c_0}[\varepsilon^p_{ij}]\nu_i\nu_j.$$

Comparing these two equations we can find that the attenuation in Fourier's case is independent of plastic energy dissipation since the third term in Eqs. (2.23) vanishes when $\tau_0\to 0$ and the second term, which characterizes the thermal attenuation, increases. Indeed

$$\frac{a_0}{2\tau_0(1+a_0)} = \frac{c_A^2-c_0^2}{2c\tau_0}\,\frac{1+\dfrac{1}{2}\dfrac{c^2}{c_T^2}}{1+\dfrac{c_A^2-c_0^2}{c_T^2}}+O(\tau_0^2) > \frac{(3\lambda+2\mu)^2\alpha^2T_0}{2\varrho k},$$
$$c^2 > 2(c_A^2-c_0^2).$$

where $\dfrac{(3\lambda+2\mu)^2\alpha^2T_0}{2\varrho k}$ is a coefficient of the thermal attenuation for Fourier's law. As it has already been remarked, with $\tau_0\to 0$ the temperature is continuous on the M-wave, and its derivatives and the heat flux are discontinuous; the jumps can then be determined from the following relations:

$$[T_{,i}] = (3\lambda+2\mu)\frac{T_0\alpha}{k}W\nu_i,\qquad [q_i] = -(3\lambda+2\mu)T_0\alpha W\nu_i.$$

Let us now consider transverse waves. Taking into account that $c^2=\mu\varrho^{-1}$ and $[v_k]\nu_k = 0$, we obtain from Eq. (2.21)

$$(2.25)\qquad (\lambda^*+\mu)V_k\nu_k\nu_i+2\varrho c\frac{\delta[v_i]}{\delta t}+(\lambda^*+\mu)g^{\alpha\beta}\frac{\partial[v_j]}{\partial y_\alpha}\frac{\partial x_j}{\partial y_\beta}\nu_i-2\mu[v_i]\Omega-\frac{2\mu}{\tau}\left[\frac{\hat{\Phi}}{S}s_{ij}\nu_j\right]$$
$$+b[\hat{\Phi}S]\nu_i = 0.$$

Multiplying Eq. (2.25) by ν_i and performing the convolution we find

$$(2.26)\qquad (\lambda^*+\mu)V_k\nu_k-\frac{2\mu}{\tau}\left[\frac{\hat{\Phi}}{S}s_{ij}\nu_i\nu_j\right]+(\lambda+\mu)g^{\alpha\beta}\frac{\partial[v_j]}{\partial y_\alpha}\frac{\partial x_j}{\partial y_\beta}+b[\hat{\Phi}S]=0.$$

After multiplying Eq. (2.26) by ν_i and subtracting the result from Eq. (2.25), we obtain the equation for determining the intensity of jump for the transverse wave:

$$(2.27)\qquad \frac{\delta[v_i]}{\delta t}=c\left(\Omega[v_i]+\frac{1}{\tau}\left[\frac{\hat{\Phi}}{S}(s_{ij}\nu_j-s_{kj}\nu_k\nu_l\nu_i)\right]\right).$$

It is clear that attenuation of intensity of the jump depends on the geometry of the wave surface and on the visco-plasticity of the material, but is independent of the thermal effect and of the dissipation. Temperature and heat flux are continuous on transverse waves.

3. Acceleration waves

Turning to the acceleration or weak discontinuity wave, we define it as a surface $\varphi(x_i, t)=0$ on which the solution U is continuous and its derivatives are discontinuous, i.e.

$$(3.1)\qquad [U]=0,\quad \left[\frac{\partial U}{\partial x_i}\right]\neq 0,\quad \left[\frac{\partial U}{\partial t}\right]\neq 0.$$

The kinematical conditions of the compatibility of the first order can be obtained from Eqs. (2.15), taking into account conditions (3.1). Substituting them into Eqs. (2.16), we find

$$\Sigma_{ij}\nu_j+\varrho cV_i=0,$$

$$(3.2)\qquad -c\Sigma_{ij}=\lambda V_k\nu_k\delta_{ij}+\mu(V_i\nu_j+V_j\nu_i)+(3\lambda+2\mu)\alpha c\theta\delta_{ij},$$

$$(3\lambda+2\mu)T_0\alpha V_k\nu_k-\varrho c_E c\theta=-Q_i\nu_i,\qquad \tau_0 cQ_i=k\theta\nu_i.$$

After excluding θ from Eqs. (3.2), we obtain the equations

$$(\lambda^*+\mu)V_k\nu_k+(\mu-\varrho c^2)V_i=0,$$

$$(3.3)\qquad c^2=\mu\varrho^{-1},\quad V_k\nu_k=0,\quad -c\Sigma_{ij}=\mu(V_i\nu_j+V_j\nu_i),$$

$$c^2=(\lambda^*+2\mu)\varrho^{-1},\quad V_i=V\nu_i,\quad -c\Sigma_{ij}=\lambda^*V\delta_{ij}+2\mu V\nu_i\nu_j,$$

which coincide with the same equations for the strong discontinuity waves (2.8)–(2.10). Thus there are two types of acceleration waves, longitudinal and transverse, which have the same speeds as respective strong waves.

For the purpose of determining the intensities of these waves we shall use the following kinematic conditions of compatibility of the second order [15]:

$$\left[\frac{\partial^2\sigma_{ij}}{\partial x_j\partial x_l}\right] = M_{ij}\nu_j\nu_l+g^{\alpha\beta}\frac{\partial\Sigma_{ij}}{\partial y_\alpha}\left(\nu_l\frac{\partial x_j}{\partial y_\beta}+\nu_j\frac{\partial x_l}{\partial y_\beta}\right)-\Sigma_{ij}g^{\alpha\beta}g^{\sigma\tau}b_{\alpha\sigma}\frac{\partial x_j}{\partial y_\alpha}\frac{\partial x_l}{\partial y_\tau},$$

$$\left[\frac{\partial^2 v_i}{\partial x_j\partial x_l}\right] = L_i\nu_j\nu_l+g^{\alpha\beta}\frac{\partial V_i}{\partial y_\alpha}\left(\nu_l\frac{\partial x_j}{\partial y_\beta}+\nu_j\frac{\partial x_l}{\partial y_\beta}\right)-V_i g^{\alpha\beta}g^{\sigma\tau}b_{\alpha\sigma}\frac{\partial x_j}{\partial y_\alpha}\frac{\partial x_l}{\partial y_\alpha},$$

(3.4) $$\left[\frac{\partial^2 T}{\partial x_j\partial x_l}\right] = \vartheta\nu_j\nu_l+g^{\alpha\beta}\frac{\partial\theta}{\partial y_\alpha}\left(\nu_l\frac{\partial x_j}{\partial y_\beta}+\nu_j\frac{\partial x_l}{\partial y_\beta}\right)-\theta g^{\alpha\beta}g^{\sigma\tau}b_{\alpha\sigma}\frac{\partial x_j}{\partial y_\alpha}\frac{\partial x_l}{\partial y_\tau},$$

$$\left[\frac{\partial^2\sigma_{ij}}{\partial x_l\partial t}\right] = \left(\frac{\delta\Sigma_{ij}}{\delta t}-M_{ij}c\right)\nu_l-cg^{\alpha\beta}\frac{\partial\Sigma_{ij}}{\partial y_\alpha}\frac{\partial x_l}{\partial y_\beta},\quad \left[\frac{\partial^2\sigma_{ij}}{\partial t^2}\right] = M_{ij}c^2-2c\frac{\delta\Sigma_{ij}}{\delta t},$$

$$\left[\frac{\partial^2 v_i}{\partial x_l\partial t}\right] = \left(\frac{\delta V_i}{\delta t}-L_i c\right)\nu_l-cg^{\alpha\beta}\frac{\partial V_i}{\partial y_\alpha}\frac{\partial x_l}{\partial y_\beta},\quad \left[\frac{\partial^2 v_i}{\partial t^2}\right] = L_i c^2-2c\frac{\delta V_i}{\delta t},$$

$$\left[\frac{\partial^2 T}{\partial x_l\partial t}\right] = \left(\frac{\delta\theta}{\delta t}-\vartheta c\right)\nu_l-cg^{\alpha\beta}\frac{\partial\theta}{\partial y_\alpha}\frac{\partial x_l}{\partial y_\beta},\quad \left[\frac{\partial^2 T}{\partial t^2}\right] = \vartheta c^2-2c\frac{\delta\theta}{\delta t},$$

$$\left[\frac{\partial^2 T}{\partial x_k\partial x_k}\right] = \vartheta-2\Omega\theta.$$

In order to make further calculations simpler we shall restrict our considerations to Fourier's law. The way of calculating the Maxwell-Cattaneo equation is the same.

From Eqs. (2.14) and (3.4), we fi.d

(3.5) $$(3\lambda+2\mu)T_0\alpha[v_{k,k}]-cc_E\varrho\theta = k(\vartheta-2\Omega\theta)+[\tau_{ij}\varepsilon^P_{ij}].$$

Since for $\tau_0\to 0$ on the acceleration surface $\theta = 0$, we obtain

(3.6) $$\vartheta = (3\lambda+2\mu)T_0\alpha\frac{V}{k}.$$

Since the last term in Eq. (3.5) vanishes, it is clear that the dissipation of the plastic energy has no influence on the intensities of the acceleration waves in Fourier's case. But it should be remarked that the dissipation has influence on the second and higher terms of the expansions of the solution near the wave front.

The following dynamic conditions of the compatibility of the second order may be obtained from Eqs. (2.1):

(3.7)
$$\left[\frac{\partial^2\sigma_{ij}}{\partial t\partial x_l}\right] = \lambda\left[\frac{\partial^2 v_k}{\partial x_k\partial x_l}\right]\delta_{ij}+\mu\left(\left[\frac{\partial^2 v_i}{\partial x_j\partial x_l}\right]+\left[\frac{\partial^2 v_j}{\partial x_i\partial x_l}\right]\right)-(3\lambda+2\mu)\alpha\left[\frac{\partial^2 T}{\partial t\partial x_l}\right]\delta_{ij}-2\mu\left[\frac{\partial\varepsilon^P_{ij}}{\partial x_l}\right],$$

$$\left[\frac{\partial^2\sigma_{ij}}{\partial t^2}\right] = \lambda\left[\frac{\partial^2 v_k}{\partial x_k\partial t}\right]\delta_{ij}+\mu\left(\left[\frac{\partial^2 v_i}{\partial x_j\partial t}\right]+\left[\frac{\partial^2 v_j}{\partial x_i\partial t}\right]\right)-(3\lambda+2\mu)\alpha\left[\frac{\partial^2 T}{\partial t^2}\right]\delta_{ij}-2\mu\left[\frac{\partial\varepsilon^P_{ij}}{\partial t}\right];$$

(3.8) $$\left[\frac{\partial^2\sigma_{ij}}{\partial x_j\partial x_l}\right] = \varrho\left[\frac{\partial^2 v_i}{\partial t\partial x_l}\right],\quad \left[\frac{\partial^2\sigma_{ij}}{\partial x_j\partial t}\right] = \varrho\left[\frac{\partial^2 v_i}{\partial t^2}\right].$$

Substituting the expressions for jumps from Eqs. (3.4), into Eqs. (3.7), we find

$$(3.9)\quad \left(\frac{\delta\Sigma_{ij}}{\delta t}-M_{ij}c\right)\nu_l-g^{\alpha\beta}c\frac{\partial\Sigma_{ij}}{\partial y_\alpha}\frac{\partial x_l}{\partial y_\beta}=\lambda\left\{L_k\nu_k\nu_l+g^{\alpha\beta}\frac{\partial V_k}{\partial y_\alpha}\left(\nu_k\frac{\partial x_l}{\partial y_\beta}+\nu_l\frac{\partial x_k}{\partial y_\beta}\right)\right.$$

$$\left.-V_k g^{\alpha\beta}g^{\sigma\tau}b_{\alpha\sigma}\frac{\partial x_k}{\partial y_\beta}\frac{\partial x_l}{\partial y_\tau}\right\}\delta_{ij}+(3\lambda+2\mu)^2\alpha^2\frac{cT_0}{k}V\nu_l\delta_{ij}+\mu\left\{L_i\nu_j\nu_l+L_j\nu_i\nu_l\right.$$

$$+g^{\alpha\beta}\frac{\partial V_j}{\partial y_\alpha}\left(\nu_l\frac{\partial x_i}{\partial y_\beta}+\nu_i\frac{\partial x_l}{\partial y_\beta}\right)+g^{\alpha\beta}\frac{\partial V_i}{\partial y_\alpha}\left(\nu_l\frac{\partial x_j}{\partial y_\beta}+\nu_j\frac{\partial x_l}{\partial y_\beta}\right)-g^{\alpha\beta}g^{\sigma\tau}b_{\alpha\sigma}\left(V_i\frac{\partial x_j}{\partial y_\beta}\right.$$

$$\left.\left.+V_j\frac{\partial x_i}{\partial y_\beta}\right)\frac{\partial x_l}{\partial y_\tau}\right\}+\frac{\mu\nu_l}{c\tau S}\frac{\partial}{\partial S}\frac{\hat{\Phi}(S-k_s)}{S}s_{km}[\dot{s}_{km}]s_{ij}+\frac{2\mu\nu_l}{\tau c}\frac{\hat{\Phi}(S-k_s)}{S}[s_{ij}],$$

$$\left[\frac{\partial\varepsilon_{ij}^P}{\partial x_l}\right]=-\frac{\nu_l}{c}\left[\frac{\partial\varepsilon_{ij}^P}{\partial t}\right],$$

where the following relations were used:

$$[\dot{K}_s]=[\varepsilon_{ij}^P]=0,\qquad [\dot{s}_{ij}]=-c\left(\Sigma_{ij}-\frac{1}{3}\Sigma_{kk}\delta_{ij}\right).$$

Thus it is possible to obtain from Eqs. (3.7) twenty four equations, respectively M_{ij} and L_i. But only the following six equations are independent:

$$(3.10)\quad \frac{\delta\Sigma_{ij}}{\delta t}-M_{ij}c=\lambda\left(L_k\nu_k+g^{\alpha\beta}\frac{\partial V_k}{\partial y_\alpha}\frac{\partial x_k}{\partial y_\beta}\right)\delta_{ij}+(3\lambda+2\mu)^2\alpha^2\frac{cT_0}{k}V_k\nu_k\delta_{ij}$$

$$+\mu\left[L_i\nu_j+L_j\nu_i+g^{\alpha\beta}\left(\frac{\partial V_i}{\partial y_\alpha}\frac{\partial x_j}{\partial y_\beta}+\frac{\partial V_j}{\partial y_\alpha}\frac{\partial x_i}{\partial y_\beta}\right)\right]+\frac{\mu}{c\tau S}\frac{\partial}{\partial S}\frac{\hat{\Phi}(S-K_s)}{S}s_{km}[\dot{s}_{km}]s_{ij}$$

$$+\frac{2\mu}{\tau c}\frac{\hat{\Phi}(S-K_s)}{S}[s_{ij}].$$

Indeed, if we multiply Eq. (3.10) by ν_l and subtract the result from Eq. (3.9), then we find

$$g^{\alpha\beta}\frac{\partial}{\partial y_\alpha}[\Sigma_{ij}c+\lambda V_k\nu_k+\mu(V_i\nu_j+V_j\nu_i)]=0.$$

This condition is satisfied identically since Eq. (3.3) is true. For other equations it could be proved in the same way. Let us obtain three additional independent equations from twelve equations of motion (3.8). Substituting the jump expressions from Eqs. (3.4) into Eqs. (3.8), we obtain

$$(3.11)\quad \begin{aligned}&M_{ij}\nu_j\nu_l+g^{\alpha\beta}\left(\nu_l\frac{\partial x_j}{\partial y_\beta}+\nu_j\frac{\partial x_l}{\partial y_\beta}\right)\frac{\partial\Sigma_{ij}}{\partial y_\alpha}-\Sigma_{ij}g^{\alpha\beta}g^{\sigma\tau}b_{\alpha\sigma}\frac{\partial x_j}{\partial y_\beta}\frac{\partial x_l}{\partial y_\tau}\\&\qquad+\left(L_i\varrho c-\varrho\frac{\delta V_i}{\delta t}\right)\nu_l+g^{\alpha\beta}\varrho c\frac{\partial V_i}{\partial y_\alpha}\frac{\partial x_l}{\partial y_\beta}=0,\\&\left(M_{ij}c-\frac{\delta\Sigma_{ij}}{\delta t}\right)\nu_j+g^{\alpha\beta}c\frac{\partial\Sigma_{ij}}{\partial y_\alpha}\frac{\partial x_j}{\partial y_\beta}-2c\varrho\frac{\delta V_i}{\delta t}+\varrho c^2L_i=0.\end{aligned}$$

Three independent equations could be obtained after multiplying the first Eq. (3.11) by ν_l

$$(3.12)\quad M_{ij}\nu_j+\varrho cL_i+g^{\alpha\beta}\frac{\partial\Sigma_{ij}}{\partial y_\alpha}\frac{\partial x_j}{\partial y_\beta}-\varrho\frac{\delta V_i}{\delta t}=0.$$

After excluding M_{ij} from Eqs. (3.10) by means of Eq. (3.12), one finds the following system of three equations, respectively:

$$(3.13)\quad \frac{\delta\Sigma_{ij}}{\delta t}\nu_j+(\varrho c^2-\mu)L_i+cg^{\alpha\beta}\frac{\partial\Sigma_{ij}}{\partial y_\alpha}\frac{\partial x_j}{\partial y_\beta}-\varrho c\frac{\delta V_i}{\delta t}=(\lambda+\mu)L_k\nu_k\nu_i$$
$$+\lambda g^{\alpha\beta}\frac{\partial V_k}{\partial y_\alpha}\frac{\partial x_k}{\partial y_\beta}\nu_i+\mu g^{\alpha\beta}\frac{\partial V_j}{\partial y_\alpha}\nu_j\frac{\partial x_i}{\partial y_\beta}+\frac{(3\lambda+2\mu)^2\alpha^2T_0c}{K}V_k\nu_k\nu_i$$
$$+\frac{\mu}{c\tau S}s_{ij}\nu_j\frac{\partial}{\partial S}\frac{\hat{\Phi}(S-K_s)}{S}s_{km}[\dot{s}_{km}]+\frac{2\mu}{\tau c}\frac{\hat{\Phi}(S-K_s)}{S}[\dot{s}_{ij}]\nu_j.$$

Using the expressions (3.2) for Σ_{ij} through V_i we obtain

$$(3.14)\quad -(\lambda+\mu)L_k\nu_k\nu_i+(\varrho c^2-\mu)L_i=(\lambda+\mu)g^{\alpha\beta}\left(\frac{\partial V_k}{\partial y_\alpha}\frac{\partial x_k}{\partial y_\beta}\nu_i+\frac{\partial V_k\nu_k}{\partial y_\alpha}\frac{\partial x_i}{\partial y_\beta}\right)$$
$$-\mu V_i g^{\alpha\beta}g^{\sigma\tau}b_{\alpha\tau}\frac{\partial x_j}{\partial y_\tau}\frac{\partial x_j}{\partial y_\beta}+2\varrho c\frac{\delta V_i}{\delta t}+\frac{(3\lambda+2\mu)^2\alpha^2T_0c}{K}V_k\nu_k\nu_i$$
$$+\frac{\mu}{c\tau S}s_{ij}\nu_j\frac{\partial}{\partial S}\frac{\hat{\Phi}(S-K_s)}{S}s_{km}[\dot{s}_{km}]+\frac{2\mu}{\tau c}\frac{\hat{\Phi}(S-K_s)}{S}[\dot{s}_{ij}]\nu_j,$$

where $[\dot{s}_{ij}]=2\mu\left(\nu_i\nu_j-\frac{1}{3}\delta_{ij}\right)V$.

Multiplying Eqs. (3.14) by ν_i and taking into account Eqs. (2.22) and (3.3), we obtain for the longitudinal wave, the following equation for determining the intensity V:

$$(3.15)\quad \frac{\delta V}{\delta t}=c\Omega V-\frac{(3\lambda+2\mu)^2\alpha^2T_0V}{2k\varrho}-\frac{\mu}{\varrho\tau c^2}\left\{\frac{s_{ij}\nu_i\nu_j}{2S}\frac{\partial}{\partial S}\left(\frac{\hat{\Phi}}{S}\right)s_{km}[\dot{s}_{km}]+\frac{\hat{\Phi}}{S}[\dot{s}_{ij}]\nu_i\nu_j\right\}.$$

Comparing Eq. (3.15) with Eqs. (2.23) for the strong discontinuity, it is evident that the thermal attenuation for both waves is the same and the difference is connected with the terms characterizing the visco-plastic attenuation. The same conclusion is true for the Maxwell-Cattaneo case.

Assuming $c^2=\mu\varrho^{-1}$ and $V_k\nu_k=0$ in Eq. (3.14) and multiplying the result by ν_i, one finds, for transverse waves, the relation

$$(3.16)\quad -(\lambda+\mu)\left(L_k\nu_k+g^{\alpha\beta}\frac{\partial V_k}{\partial y_\alpha}\frac{\partial x_k}{\partial y_\beta}\right)=\frac{(3\lambda+2\mu)^2\alpha^2T_0c}{k}V_k\nu_k$$
$$+\frac{2\mu}{\tau c}\left(\frac{s_{ij}\nu_i\nu_j}{2S}\frac{\partial}{\partial S}\frac{\hat{\Phi}(S-k_s)}{S}s_{km}[\dot{s}_{km}]+\frac{\hat{\Phi}}{S}[\dot{s}_{ij}]\nu_i\nu_j\right).$$

After multiplying Eqs. (3.16) by ν_i and subtracting the result from Eq. (3.14), we obtain the equation for determining the intensity of the transverse wave

$$(3.17)\quad \frac{\delta V_i}{\delta t}=c\Omega V_i-\frac{1}{\tau}\left[\frac{s_{ij}\nu_j-s_{kj}\nu_k\nu_j\nu_i}{2S}\frac{\partial}{\partial S}\left(\frac{\hat{\Phi}}{S}\right)s_{km}[\dot{s}_{km}]+\frac{\hat{\Phi}}{S}([\dot{s}_{ij}\nu_j]-[\dot{s}_{kj}]\nu_k\nu_j\nu_i)\right].$$

This equation coincides with the same one for isothermal transverse acceleration waves in the elasto-visco-plastic medium considered in [6].

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Received August 9, 1976.