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BRIEF NOTES

On the existence and uniqueness of solutions in the linear theory of Cosserat elasticity. II

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IN [1] THE NOTION of generalized solution with finite energy (in the sense of VISHIK and LADYZENSKAYA [2]) to the dynamical problem of linear Cosserat elasticity is defined. The existence and uniqueness of a generalized (weak) solution in the case of zero initial conditions is proved. The present paper is concerned with some existence and uniqueness theorems in the dynamic theory of Cosserat elasticity for the general case of non-zero initial conditions. The existence theorem is derived by energy arguments. A uniqueness theorem of the classical solution for the mixed dynamical problem is given too. The approach of proof depends upon convexity arguments.

1. Introduction

LET Ω BE a bounded domain and properly regular in the sense of FICHERA [3] in the Euclidean space R^3 with orthogonal coordinates $x = (x_1, x_2, x_3)$.

By $\partial\Omega$ we denote the boundary of Ω. Let $(0, T)$ be a time interval with $0 < T < +\infty$ and Q the right-hand cylinder $Q = \Omega \times (0, T)$.

We shall consider the spaces $C^m(\bar{\Omega})$, $L_2(\Omega)$, $\mathbf{C}^m(\bar{\Omega})$, $\mathbf{L}_2(\Omega)$ of scalar and vector functions as defined in the usual way. We denote by $W_2^m(\Omega)$, and $\mathbf{W}_2^m(\Omega)$ the completions of the spaces $C^m(\bar{\Omega})$ and $\mathbf{C}^m(\bar{\Omega})$ in the norms induced by the inner products:

$$(1.1)\qquad (\varphi, \psi)_{W_2^m} = \sum_{k=0}^{m} \int_{\Omega} \varphi_{,k_1k_2k_3} \psi_{,k_1k_2k_3} dx, \quad k = k_1+k_2+k_3$$

and

$$(1.2)\qquad (\mathbf{u}, \mathbf{v})_{\mathbf{W}_2^m} = \sum_{j=1}^{4} (u_j, v_j)_{W_2^m(\Omega)},$$

respectively.

Here as well as further we employ the summation convention for repeated indices; a comma followed by a subscript indicates differentiation with respect to the space variables x_i $(i = 1, 2, 3)$.

We shall consider the spaces $\mathbf{C}^m(\Omega)$, $\mathring{\mathbf{C}}^1(\Omega)$, $\mathbf{W}_2^m(\Omega)$, $\mathring{\mathbf{W}}_2^1(\Omega)$, $\mathscr{L}_2(\Omega)$, $\mathscr{C}^1(\Omega)$, $\ddot{\mathscr{C}}^1(\Omega)$, $C^m([0, T]; H)$, $L_p([0, T]; H)$, $\mathscr{F}(Q)$, $\mathring{\mathscr{F}}(Q)$, $\mathscr{H}^1(Q)$, $\mathring{\mathscr{H}}^1(Q)$, $\mathring{\mathscr{H}}^1_{\cdot}(Q)$, defined in the same manner as in [1]. In addition, we denote by $\mathscr{W}_2^1(\Omega)$, $\mathring{\mathscr{W}}_2^1(\Omega)$ the Sobolev spaces $\mathscr{W}_2^1(\Omega) = \mathbf{W}_2^1(\Omega) \times \mathbf{W}_2^1(\Omega)$; $\mathring{\mathscr{W}}_2^1(\Omega) = \mathring{\mathbf{W}}_2^1(\Omega) \times \mathring{\mathbf{W}}_2^1(\Omega)$.

2. Formulation of the dynamical problem

The basic equations in the linear theory of non-homogeneous and anisotropic Cosserat elastic solids are:

the equations of motion[1]

$$\tau_{ji,j}+F_i = \varrho\ddot{u}_i, \qquad (2.1)$$
$$\mu_{ji,j}+\varepsilon_{ijk}+M_i = \varrho J_{ik}\varphi_k,$$

the constitutive law

$$\tau_{ij} = E_{ijkl}\gamma_{kl}+K_{ijkl}\varkappa_{kl}, \qquad (2.2)$$
$$\mu_{ij} = K_{klij}\gamma_{kl}+M_{ijkl}\varkappa_{kl},$$

the kinematic relations

$$\gamma_{ij} = u_{j,i}-\varepsilon_{ijk}\varphi_k, \quad \varkappa_{ij} = \varphi_{j,k}. \qquad (2.3)$$

In these equations, $\tau_{ij}(x, t)$ and $\mu_{ij}(x, t)$ represent the stress tensor and the couple-stress tensor, respectively; $u_i(x, t)$ — the displacement vector; $\varphi_i(x, t)$ — the microrotation vector; $F_i(x, t)$ — the body force vector; $M_i(x, t)$ — the body couple vector; $\gamma_{ij}(x, t)$ — the strain tensor; $\varkappa_{ij}(x, t)$ — the microstrain tensor; $\varrho(x)$ — the mass density; $J_{ik}(x)$ — the microinertia coefficients; $E_{ijkl}(x)$, $K_{ijkl}(x)$, $M_{ijkl}(x)$ — the characteristic constants of the material; ε_{ijk} — the unit antysymmetric tensor.

The tensors E_{ijkl}, M_{ijkl}, J_{ik} are assumed to meet in Ω the following conditions of symmetry:

$$E_{ijkl}(x) = E_{klij}(x), \quad M_{ijkl}(x) = M_{klij}(x), \quad J_{ik}(x) = J_{ki}(x). \qquad (2.4)$$

Let $\Omega \subset R^3$ be a bounded domain and C^1 — smooth. By $\partial\Omega$ we denote the boundary of Ω.

DEFINITION 2.1. *By a classical solution for the dynamical problem of the linear theory of Cosserat elasticity in the cylinder* $Q = \Omega\times(0, T)$, we mean a pair $(\mathbf{u}, \boldsymbol{\varphi}) \in [\mathbf{C}^2(Q)\cap \mathbf{C}^1(\bar{Q})]\times[\mathbf{C}^2(Q)\cap\mathbf{C}^1(\bar{Q})]$ satysfying the system (2.1)–(2.3) for $(x, t) \in Q$, together with the boundary conditions:

$$\mathbf{u} = 0, \quad \boldsymbol{\varphi} = 0 \quad \text{on} \quad \partial\Omega\times(0, T) \qquad (2.5)$$

and initial conditions

$$\big(\mathbf{u}(x, 0), \dot{\mathbf{u}}(x, 0), \boldsymbol{\varphi}(x, 0), \dot{\boldsymbol{\varphi}}(x, 0)\big) = \big(\mathbf{u}_0(x), \dot{\mathbf{u}}_0(x), \boldsymbol{\varphi}_0(x), \dot{\boldsymbol{\varphi}}_0(x)\big), \qquad (2.6)$$

where $\mathbf{u}_0(x)$, $\dot{\mathbf{u}}_0(x)$, $\boldsymbol{\varphi}_0(x)$, $\dot{\boldsymbol{\varphi}}_0(x)$ are prescribed functions on Ω.

3. The existence and uniqueness of a generalized solution

In the present section we establish the existence of a unique generalized solution for the dynamical problem of linear Cosserat elasticity (2.1)–(2.6) in the general case of non-zero initial conditions.

We make the same assumptions as in Sect. 3 [1].

We define the linear functionals $\mathscr{L}(y, z)$, $\mathscr{D}(f, z)$, $\mathscr{E}(\dot{y}_0, z)$ for $z \in \mathring{\mathscr{F}}(Q)$ as in Sect. 3 [1].

[1] A superposed dot is used for time differentiation.

We recall the definition of the generalized weak solution for the dynamical problem of linear Cosserat elasticity (2.1)–(2.3), (2.5)–(2.6) in the sense of VISHIK and LADYZENSKAYA [2].

DEFINITION 3.1. *The pair* $y = (\mathbf{u}, \boldsymbol{\varphi}) \in \mathscr{H}^1(Q)$ *will be called a finite energy solution for the dynamical problem* (2.1)–(2.3), (2.5)–(2.6) *with the initial condition* $(y_0, \dot{y}_0) \in \mathring{\mathscr{W}}{}^1_2(\Omega) \times \times \mathscr{L}_2(\Omega)$, *the body source being* $f = (\mathbf{F}, \mathbf{M}) \in L_1([0, T]; \mathscr{L}_2(\Omega))$ *if y satisfies the following conditions*:

$$(3.1) \qquad \mathscr{L}(y, z) = \mathscr{D}(y, z) + \mathscr{E}(\dot{y}_0, z), \quad z \in \mathring{\mathscr{H}}{}^1_{\cdot}(Q),$$

$$(3.2) \qquad \lim y(t) = y_0 \quad (\text{limit in } \mathscr{L}_2(Q)).$$

Now we are ready to state and prove the existence theorem for general initial conditions. It is proved that the solution is stronger than required by the definition of a finite energy solution.

THEOREM 3.1. *Let the following conditions* 1)–2) *be satisfied*:

$$1) \quad f = (\mathbf{F}, \mathbf{M}) \in \mathscr{C}^{m+1}([0, T]; \mathscr{L}_2(\Omega)),$$

$$2) \quad y_0 \in \mathscr{C}^{m+3}(\overline{\Omega}), \quad \dot{y}_0 \in \mathscr{C}^{m+2}(\Omega).$$

Then there exists a unique finite energy solution $y = (\mathbf{u}, \boldsymbol{\varphi})$ of the dynamical problem (2.1)–(2.3), (2.5)–(2.6) in Q and, furthermore, $y \in C^m([0, T]; \mathring{\mathscr{W}}{}^1_2(\Omega))$.

The proof of this theorem is based on theorem 2 [1] from the case of zero initial conditions. The proof method we have used is a recurrence one.

4. The uniqueness of a classical solution

In this section we give the definition of a classical solution for the mixed dynamical problem of Cosserat elasticity and prove a uniqueness theorem without supposing the energy of deformation is a positive definite quadratic form.

Let $\Omega \subset R^3$ be a bounded domain and C^1 — smooth. By $\partial\Omega$ and $\mathbf{n}(n_i)$ we denote the boundary of Ω and the unit outward normal on $\partial\Omega$, respectively.

DEFINITION 4.1. *By a classical solution for the mixed dynamical problem of the linear theory of Cosserat elasticity in the cylinder* $Q = \Omega \times (0, T)$ *we mean a pair* $y = (\mathbf{u}, \boldsymbol{\varphi}) \in \in [\mathbf{C}^2(Q) \cap \mathbf{C}^1(\overline{Q})] \times [\mathbf{C}^2(Q) \cap \mathbf{C}^1(\overline{Q})]$ *satisfying the system* (2.1)–(2.3) *for* $(x, t) \in Q$ *together with the boundary conditions*

$$(4.1) \qquad \begin{aligned} &\mathbf{u} = \tilde{\mathbf{u}}; \quad \boldsymbol{\varphi} = \tilde{\boldsymbol{\varphi}} \quad \text{on} \quad \partial\Omega_1 \times (0, T), \\ &\tau_{ij} n_j = \tilde{\tau}_{ni}; \quad \mu_{ij} n_j = \tilde{\mu}_{ni} \quad \text{on} \quad \partial\Omega_2 \times (0, T) \end{aligned}$$

and the initial conditions (2.6).

Here, $\partial\Omega_1$ and $\partial\Omega_2$ are fixed subsets of $\partial\Omega$ such that $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$.

Because of the linearity of the system (2.1)–(2.3) it is sufficient to prove the unicity of solution for the mixed dynamical problem with homogeneous boundary conditions and homogeneous initial conditions.

THEOREM 4.1. *Let the following conditions* 1)–2) *be satisfied*:

1) $E_{ijkl}(x)$, $M_{ijkl}(x)$, $K_{ijkl}(x)$, $\varrho(x)$, $J_{ik}(x) \in C(\Omega)$,

2) $\varrho(x) > 0$ and $J_{ik}\xi_i\xi_k \geqslant \lambda\xi_l\xi_l$ for every vector $\xi(\xi_i) \in R^3$, $\lambda > 0$.

Then, the mixed dynamical problem with homogeneous conditions has only null solution.

The proof of this theorem is derived by convexity arguments which are due to KNOPS and PAYNE and currently used in the study of improperly posed problems [4].

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